

# On the Einstein-Vlasov system with hyperbolic symmetry

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## Abstract

It is shown that a spacetime with collisionless matter evolving from data on a compact Cauchy surface with hyperbolic symmetry can be globally covered by compact hypersurfaces on which the mean curvature is constant and by compact hypersurfaces on which the area radius is constant. Results for the related cases of spherical and plane symmetry are reviewed and extended. The prospects of using the global time coordinates obtained in this way to investigate the global geometry of the spacetimes concerned are discussed.

## 1 Introduction

One of the key mathematical problems in general relativity is the determination of the global properties of solutions of the Einstein equations coupled to various matter fields. We investigate this problem in the case of collisionless matter described by the Vlasov equation. The underlying strategy is to first establish the existence of a suitable global time coordinate  $t$  and then to study the asymptotic behaviour of the solution when  $t$  tends to its limiting values, which might correspond to the approach to a singularity or to a phase of unending expansion. The following is almost exclusively concerned with the first of these two steps. Since the general case is beyond the range of current mathematical techniques we study a case with symmetry.

There are several existing results on global time coordinates for solutions of the Einstein-Vlasov system. In the spatially homogeneous case it is natural to

choose a Gaussian time coordinate based on a homogeneous hypersurface. The maximal range of a Gaussian time coordinate in a globally hyperbolic solution of the Einstein-Vlasov system evolving from data on a compact manifold which are homogeneous (or locally homogeneous) was determined in [20]. It is finite for models of Bianchi IX and Kantowski-Sachs types and finite in one time direction and infinite in the other for the other Bianchi types. All other results presently available on the subject concern spacetimes which admit a group of isometries acting on two-dimensional spacelike orbits, at least after passing to a covering manifold. The group may be two-dimensional (local  $U(1) \times U(1)$  symmetry) or three-dimensional (spherical, plane or hyperbolic symmetry). In all these cases the quotient of spacetime by the symmetry group has the structure of a two-dimensional Lorentzian manifold  $Q$ . The orbits of the group action (or appropriate quotients in the case of a local symmetry) are called surfaces of symmetry. Thus there is a one-to-one correspondence between surfaces of symmetry and points of  $Q$ .

Three types of time coordinates which have been studied in the inhomogeneous case are CMC, areal and conformal coordinates. A CMC time coordinate  $t$  is one where each hypersurface of constant time has constant mean curvature and on each hypersurface of this kind the value of  $t$  is the mean curvature of that slice. In the case of areal coordinates the time coordinate is a function of the area of the surfaces of symmetry. In some papers in the literature it is taken to be proportional to the area. In this paper it is taken to be proportional to the square root of the area. In the case of conformal coordinates the metric on the quotient manifold  $Q$  is conformally flat. The properties of the last two kinds of time coordinates will be described in more detail later.

Next the known results concerning these time coordinates in solutions of the Einstein-Vlasov system will be summarized briefly. For the detailed statements the reader is referred to the original papers. In the case of spherical symmetry the existence of one compact CMC hypersurface implies that the whole spacetime can be covered by a CMC time coordinate which takes all real values [19, 6]. The existence of one compact CMC hypersurface in this case was proved later by Henkel [11], thus providing a complete picture in the spherically symmetric case. His proofs use the concept of prescribed mean curvature (PMC) foliation which may be useful more generally. A general local existence theorem for PMC foliations was proved in [10]. In the case of  $U(1) \times U(1)$  symmetry (which includes plane symmetry as a special case) it was proved in [21] that the existence of one compact CMC hypersurface (without loss of generality with negative mean curvature) implies that there is a foliation by compact CMC hypersurfaces where the mean curvature takes on all negative values. It was also shown that the foliation covers the whole spacetime between the initial hypersurface and the singularity, but it was left open whether it covers the whole spacetime in the other time direction. The latter question will be discussed further in Section 6. In the special case of Gowdy symmetry (which includes plane symmetry) the existence of one compact CMC hypersurface follows from [12]. Finally, in the case of hyperbolic symmetry, the results of [19] imply that the existence of one compact CMC hypersurface leads to the same conclusions as in

the  $U(1) \times U(1)$ -symmetric case, provided the Hawking mass is positive on the initial hypersurface. (The question of whether the CMC foliation includes all negative numbers as values of the mean curvature was not discussed in [19]; it will be settled in Section 6.) Under the same positivity condition the existence of one compact CMC hypersurface is proved in [11].

The main concern of this paper is with the areal and conformal time coordinates. In fact it is the first of these which is of fundamental interest—conformal coordinates serve as a convenient tool in an intermediate step in the proofs. Note that the areal time coordinate is automatically positive so that the largest possible time interval on which the solution could exist is  $]0, \infty[$ . It was shown in [16] for spacetimes with spherical symmetry that if there is one symmetric Cauchy surface of constant areal time and if the data on that hypersurface satisfy a certain inequality then the past of the initial hypersurface is covered by an areal time coordinate and this coordinate takes on all values in the range  $]0, R_1]$ , where  $R_1$  is its value on the initial hypersurface. A result improving this time interval cannot be expected since these spacetimes tend to recollapse, and then the areal coordinate will break down at some point in the time direction corresponding initially to expansion. The results on the areal coordinate in [16] for the case of plane symmetry are superseded by those on Gowdy symmetry in [2].

For Gowdy symmetry the spacetime is globally covered by an areal time coordinate which takes all values in the range  $]R_0, \infty[$  for some  $R_0 \geq 0$ . In some cases, such as the vacuum Gowdy spacetimes, it is known that  $R_0 = 0$ , cf. [14], but it is not understood in what generality this holds. For hyperbolic symmetry a direct analogue of the result for restricted data in spherical symmetry stated above was also proved in [16].

In the case of hyperbolic symmetry two main questions were left open by the work up to now. Firstly, is there always one symmetric Cauchy surface of constant areal time? Secondly, is there an areal time coordinate whose level surfaces are compact and symmetric which covers the whole spacetime, and does it take on arbitrarily large values? These questions will be answered in the affirmative in the following (Theorem 6.1). The proofs are modelled on the approach of [2], which in turn was inspired by the work of [3] on the vacuum case. A conformal time coordinate is used in an intermediate step.

In Section 2 the definitions of hyperbolic symmetry and the Einstein-Vlasov system are recalled. The local existence of conformal coordinates in a neighbourhood of the initial hypersurface is demonstrated. In Section 3 the evolution of the spacetime in the expanding direction is analysed. A global existence theorem in areal time is proved which answers the second main question modulo the first one. In the section after that a long-time existence result in the contracting direction is proved using conformal coordinates. In Section 5 the existence of a symmetric compact Cauchy surface of constant area radius is demonstrated, thus answering the first main question. All these results are combined to produce the main results of the paper in Section 6. Theorem 6.1 asserts the existence of a global foliation by hypersurfaces of constant areal time in solutions of the Einstein-Vlasov system with hyperbolic symmetry, and Theorem 6.2 makes an

analogous statement for hypersurfaces of constant mean curvature. The last section discusses possible extensions of these results and looks at what is known about the asymptotic behaviour of solutions.

## 2 Hyperbolic symmetry

In [19] a definition of spacetimes with surface symmetry was given. This comprised three cases, namely spherical, plane and hyperbolic symmetry. In the following the case of most interest is that of hyperbolic symmetry. The spacetime  $M$  is diffeomorphic to  $\mathbb{R} \times S^1 \times F$  where  $F$  is a compact orientable surface of genus greater than one. The manifold  $M$  has a covering space  $\tilde{M}$  diffeomorphic to  $\mathbb{R} \times S^1 \times \tilde{F}$  with projection  $\hat{p}$  induced by the projection  $p$  from the universal cover  $\tilde{F}$  to  $F$  according to  $\hat{p}(x, y, z) = (x, y, p(z))$ . The surface  $F$  admits metrics of constant negative curvature, and  $\tilde{F}$  endowed with the pull-back of any one of these metrics is isometric to the hyperbolic plane. Let  $G$  be the identity component of the isometry group of the hyperbolic plane. If  $\tilde{F}$  is identified with the hyperbolic plane then an action  $\phi$  of  $G$  on  $\tilde{F}$  is obtained. Define an associated action  $\hat{\phi}$  on  $\tilde{M}$  by  $\hat{\phi}(x, y, z) = (x, y, \phi(z))$ . A spacetime with underlying manifold  $M$  defined by a metric  $g_{\alpha\beta}$  and matter fields is said to have hyperbolic symmetry if the pull-back  $\hat{g}_{\alpha\beta}$  of the metric and the pull-back of the matter fields to  $\tilde{M}$  via  $\hat{p}$  are invariant under  $\hat{\phi}$ . The surfaces in  $M$  diffeomorphic to  $F$  defined by the product decomposition will be called surfaces of symmetry. A Cauchy surface  $S$  will be called symmetric if it is a union of surfaces of symmetry. Then  $\tilde{S} = p^{-1}(S)$  is invariant under the action of the group  $G$ . It is now clear how to define abstract Cauchy data for the Einstein-matter equations with hyperbolic symmetry. They should be defined on a manifold  $S$  of the form  $S^1 \times F$  and should be such that the pull-back of the data to  $\tilde{S} = S^1 \times \tilde{F}$  under the natural covering map is invariant under the natural action of  $G$ . The quotient of  $\tilde{S}$  by the action of  $G$  is diffeomorphic to  $S^1$ .

Consider now a choice of matter fields for which the Cauchy problem for the Einstein-matter equations is well-posed. The example of interest in the following is that of collisionless matter satisfying the Vlasov equation. Corresponding to initial data for the Einstein-matter equations with hyperbolic symmetry there are data on the covering manifold  $\tilde{S}$  which have a maximal Cauchy development on a manifold  $\tilde{M}$ . We will now construct a certain coordinate system on a neighbourhood of the initial hypersurface in  $\tilde{M}$ . As remarked in [19] coordinates can be introduced on the initial hypersurface  $\tilde{S}$  so that the metric takes the form  $A^2(\tilde{\theta})d\tilde{\theta}^2 + B^2(\tilde{\theta})d\Sigma^2$ , where  $d\Sigma^2 = dx^2 + \sinh^2 x dy^2$ . Correspondingly there are local coordinates on  $S$  (local in  $x$  and  $y$ ) where the metric takes this form. On a neighbourhood of the initial hypersurface in  $\tilde{M}$  we can introduce Gauss coordinates based on the given coordinates on  $\tilde{S}$ . The group  $G$  acts as a symmetry group on the initial data. On general grounds the action of  $G$  on  $\tilde{S}$  extends uniquely to an action on the maximal Cauchy development  $\tilde{M}$  by symmetries (see e.g. [7]). The hypersurfaces of constant Gaussian time are invariant under the action of  $G$ , as are the hypersurfaces of constant  $\tilde{\theta}$ . Using the

isotropy group of any point it can be seen that the  $\tilde{\theta}x$  and  $\tilde{\theta}y$  components of the metric of these hypersurfaces must vanish. In the same way, the restriction of the second fundamental form of the hypersurfaces to the surfaces of symmetry must be proportional to the metric with a factor depending only on  $t$  and  $\tilde{\theta}$ . By integration in  $t$  it then follows that the spacetime metric takes the form

$$ds^2 = -dt^2 + A^2(t, \tilde{\theta})d\tilde{\theta}^2 + B^2(t, \tilde{\theta})d\Sigma^2 \quad (2.1)$$

in the region covered by Gauss coordinates. From this we see that we can form the quotient of this metric by the action of a discrete group to get a spacetime on a subset of  $M$  which is a Cauchy development of the original data on  $S$ . Furthermore we can form the quotient by  $G$  to get the Lorentz metric  $-dt^2 + A^2(t, \tilde{\theta})d\tilde{\theta}^2$  on a subset  $Q$  of  $\mathbb{R} \times S^1$  referred to in the introduction. On  $Q$  we can pass to double null coordinates  $(u, v)$  on a neighbourhood of the quotient of the initial hypersurface. Defining new coordinates by  $t = \frac{1}{2}(u - v)$  and  $\theta = \frac{1}{2}(u + v)$  puts the metric on  $Q$  in conformally flat form. By pull-back these define new coordinates on  $M$  where the metric takes the form

$$ds^2 = e^{2\eta}(-dt^2 + d\theta^2) + R^2(dx^2 + \sinh^2 x dy^2) \quad (2.2)$$

where  $\eta$  and  $R$  are functions of  $t$  and  $\theta$  which are periodic in  $\theta$ . This proves the existence of a conformal coordinate system close to the initial hypersurface. It is possible to choose the double null coordinates in such a way that the initial hypersurface coincides with  $t = 0$  and the period of the functions  $\eta$  and  $R$  is one.

To conclude this section we formulate the Einstein-Vlasov system which governs the time evolution of a self-gravitating collisionless gas in the context of general relativity; for the moment we do not assume any symmetry of the spacetime. All the particles in the gas are assumed to have the same rest mass, normalized to unity, and to move forward in time so that their number density  $f$  is a non-negative function supported on the mass shell

$$PM := \{g_{\alpha\beta}p^\alpha p^\beta = -1, p^0 > 0\},$$

a submanifold of the tangent bundle  $TM$  of the space-time manifold  $M$  with metric  $g_{\alpha\beta}$ . We use coordinates  $(t, x^a)$  with zero shift and corresponding canonical momenta  $p^\alpha$ ; Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On the mass shell  $PM$  the variable  $p^0$  becomes a function of the remaining variables  $(t, x^a, p^b)$ :

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ab}p^a p^b}.$$

The Einstein-Vlasov system now reads

$$\partial_t f + \frac{p^a}{p^0} \partial_{x^a} f - \frac{1}{p^0} \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0,$$

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta},$$

$$T^{\alpha\beta} = \int p^\alpha p^\beta f |g|^{1/2} \frac{dp^1 dp^2 dp^3}{-p_0}$$

where  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols,  $|g|$  denotes the determinant of the metric,  $G^{\alpha\beta}$  the Einstein tensor, and  $T^{\alpha\beta}$  is the energy-momentum tensor. The pull-back of the number density  $f$  to  $T\tilde{M}$  is assumed to be invariant under the action induced on  $T\tilde{M}$  by the action of  $G$  on  $\tilde{M}$ , a fact which can be used to reduce the number of independent variables.

### 3 The expanding direction

In this section we want to investigate the Einstein-Vlasov system with hyperbolic symmetry in the expanding direction. We write the system in areal coordinates, i.e., the coordinates are chosen such that  $R=t$ . The circumstances under which coordinates of this type exist are discussed later. We prove that for initial data on a hypersurface of constant time corresponding solutions exist for all future time with respect to the areal time coordinate. It should be noted that our time coordinate has the geometric meaning of the curvature radius of the hyperbolic spaces which form the orbits of the symmetry action. Since it requires little additional effort we include for the sake of comparison the case of plane symmetry and write the metric in the form

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} d\theta^2 + t^2(dx^2 + \sin_\epsilon^2 x dy^2), \quad (3.1)$$

where  $\mu$  and  $\lambda$  are functions of  $t$  and  $\theta$ , periodic in  $\theta$  with period 1, and

$$\sin_\epsilon x := \begin{cases} \sin x & \text{for } \epsilon = 1, \\ 1 & \text{for } \epsilon = 0, \\ \sinh x & \text{for } \epsilon = -1. \end{cases}$$

For the case  $\epsilon = 1$  the orbits of the symmetry action are two-dimensional spheres. In this spherically symmetric case the global result below is easily seen to be false, cf. [16], so this case will not be considered further. For the case  $\epsilon = 0$  the orbits of the symmetry action are flat tori, and the coordinates  $x$  and  $y$  range in the interval  $[0, 2\pi]$ .

**Lemma 3.1** *Assume that (the pull-back of)  $f$  is invariant under the group action on  $T\tilde{M}$  associated to hyperbolic symmetry. Then  $f$  depends only on*

$$t, \theta, p^1, (p^2)^2 + \sinh^2 x (p^3)^2.$$

Actually, we will write

$$f = f(t, \theta, w, L)$$

where

$$w := e^\lambda p^1, \quad L := t^4((p^2)^2 + \sinh^2 x (p^3)^2).$$

In these variables

$$p^0 = e^{-\mu} \sqrt{1 + w^2 + L/t^2} =: e^{-\mu} \langle p \rangle,$$

and  $L$  is a conserved quantity along particle orbits. The analogous result holds in the case of plane symmetry  $\epsilon = 0$ .

*Proof of Lemma 3.1.* To see the above assertion on the form of  $f$  we proceed as follows. In the coordinates

$$\bar{x}^0 = \theta, \bar{x}^1 = t \cosh x, \bar{x}^2 = t \sinh x \cos y, \bar{x}^3 = t \sinh x \sin y$$

on the covering manifold  $\tilde{M}$  the symmetry orbits are given as  $\{\bar{x}^0 = \text{const}\} \times H^2(t)$  where

$$H^2(t) := \{(\bar{x}^1, \bar{x}^2, \bar{x}^3) \in \mathbb{R}^3 \mid \bar{x}^1 > 0, -(\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2 = -t^2\},$$

a hyperbolic space of radius  $t$ , cf. [15, p. 108 ff]. The reason for using these coordinates is that the isometry group of the hyperbolic spaces now has a nice representation: Its elements are given by the restriction to  $H^2(t)$  of the linear maps  $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)^{\text{tr}} \mapsto S\bar{x}$  with matrices  $S \in \text{GL}(3, \mathbb{R})$  which have the property that

$$\langle Su, Sv \rangle = \langle u, v \rangle, \quad u, v \in \mathbb{R}^3$$

where

$$\langle u, v \rangle := -u^1 v^1 + u^2 v^2 + u^3 v^3$$

is the usual inner product on three-dimensional Minkowski space. In addition, the matrices  $S$  are required to preserve orientation, i. e.,

$$S = \begin{pmatrix} a_T & b \\ c & a_S \end{pmatrix}, \quad a_T > 0, a_S \in \mathbb{R}^{2 \times 2} \text{ with } \det a_S > 0, b, c \in \mathbb{R}^2.$$

That these matrices represent the isometries of the hyperbolic spaces can be found in [15, p. 239 ff] or can be seen directly: In the new coordinates

$$\bar{g}_{00} = e^{2\lambda}, \bar{g}_{0a} = 0, \bar{g}_{ab} = \epsilon_{ab} + (1 - e^{2\mu}) \frac{\delta_{ac} \bar{x}^c \delta_{bd} \bar{x}^d}{t^2} \gamma_{ab}$$

where

$$\epsilon_{11} = -1, \epsilon_{22} = \epsilon_{33} = 1, \epsilon_{ab} = 0, a \neq b,$$

and

$$\gamma_{ab} = \gamma_{ba} = -1, a = 1, b = 2, 3, \gamma_{ab} = 1 \text{ otherwise,}$$

and since for the matrices introduced above one has  $-S_{1a} S_{1b} + S_{2a} S_{2b} + S_{3a} S_{3b} = \epsilon_{ab}$ , i. e., their columns are orthonormal with respect to  $\langle \cdot, \cdot \rangle$ , their isometry property is easily seen. Now denote by  $\bar{p}^\alpha$  the canonical momenta corresponding to the new coordinates  $\bar{x}^\alpha$ , i. e.,

$$\bar{p}^\alpha = p^0 \frac{\partial \bar{x}^\alpha}{\partial t} + p^1 \frac{\partial \bar{x}^\alpha}{\partial \theta} + p^2 \frac{\partial \bar{x}^\alpha}{\partial x} + p^3 \frac{\partial \bar{x}^\alpha}{\partial y}, \quad (3.2)$$

in particular  $\bar{p}^0 = p^1$ . Fix some  $\theta = \bar{x}^0, t > 0, \bar{x} \in H^2(t), \bar{p}^0 \in \mathbb{R}$ , and  $\bar{p} = (\bar{p}^1, \bar{p}^2, \bar{p}^3)^{\text{tr}} \in \mathbb{R}^3$ . Let  $S$  be an isometry such that  $S\bar{x} = (t, 0, 0)^{\text{tr}}$  and  $T$  a rotation in the  $(\bar{x}^2, \bar{x}^3)$ -plane such that  $TS\bar{p} = (q^1, q^2, 0)^{\text{tr}}$  with  $q^2 > 0$ . Since  $TS$  represents an isometry under which  $f$  should be invariant,

$$f(\theta, \bar{x}, \bar{p}^0, \bar{p}) = f(\theta, TS\bar{x}, p^1, TS\bar{p}) = f(\theta, t, 0, 0, p^1, q^1, q^2, 0)$$

and

$$q^1 = -\frac{1}{t} \langle TS\bar{x}, TS\bar{p} \rangle = -\frac{1}{t} \langle \bar{x}, \bar{p} \rangle,$$

$$(q^2)^2 = (q^1)^2 - (\bar{p}^1)^2 + (\bar{p}^2)^2 + (\bar{p}^3)^2.$$

When these quantities are re-expressed in the old momentum variables via (3.2) we find

$$q^1 = p^0, \quad (q^2)^2 = t^2(p^2)^2 + t^2 \sinh^2 x (p^3)^2$$

which proves the assertion in the hyperbolic case. For the plane case the argument is even easier.

In the variables which we have now introduced the complete Einstein-Vlasov system reads as follows:

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\langle p \rangle} \partial_\theta f - (\lambda_t w + e^{\mu-\lambda} \mu_\theta \langle p \rangle) \partial_w f = 0, \quad (3.3)$$

$$e^{-2\mu} (2t\lambda_t + 1) + \epsilon = 8\pi t^2 \rho, \quad (3.4)$$

$$e^{-2\mu} (2t\mu_t - 1) - \epsilon = 8\pi t^2 p, \quad (3.5)$$

$$\mu_\theta = -4\pi t e^{\mu+\lambda} j, \quad (3.6)$$

$$e^{-2\lambda} (\mu_{\theta\theta} + \mu_\theta (\mu_\theta - \lambda_\theta)) - e^{-2\mu} (\lambda_{tt} + (\lambda_t + 1/t)(\lambda_t - \mu_t)) = 4\pi q, \quad (3.7)$$

where

$$\rho(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \langle p \rangle f(t, \theta, w, L) dL dw = e^{-2\mu} T_{00}(t, \theta), \quad (3.8)$$

$$p(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\langle p \rangle} f(t, \theta, w, L) dL dw = e^{-2\lambda} T_{11}(t, \theta), \quad (3.9)$$

$$j(t, \theta) := \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, \theta, w, L) dL dw = -e^{\lambda+\mu} T_{01}(t, \theta), \quad (3.10)$$

$$q(t, \theta) := \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{\langle p \rangle} f(t, \theta, w, L) dL dw = \frac{2}{t^2} T_{22}(t, \theta); \quad (3.11)$$

recall that

$$\langle p \rangle = \sqrt{1 + w^2 + L/t^2}.$$

We prescribe initial data at some time  $t = t_0 > 0$ ,

$$f(t_0, \theta, w, L) = \mathring{f}(\theta, w, L), \quad \lambda(t_0, \theta) = \mathring{\lambda}(\theta), \quad \mu(t_0, \theta) = \mathring{\mu}(\theta)$$

and want to show that the corresponding solution exists for all  $t \geq t_0$ . To this end we make use of the continuation criterion in the following local existence result:



**Proposition 3.2** Let  $\mathring{f} \in C^1(\mathbb{R}^2 \times \mathbb{R}_0^+)$  with  $\mathring{f}(\theta+1, w, L) = \mathring{f}(\theta, w, L)$  for  $(\theta, w, L) \in \mathbb{R}^2 \times \mathbb{R}_0^+$ ,  $\mathring{f} \geq 0$ , and

$$\begin{aligned} \sup \left\{ |w| \mid (\theta, w, L) \in \text{supp} \mathring{f} \right\} &< \infty, \\ \sup \left\{ L \mid (\theta, w, L) \in \text{supp} \mathring{f} \right\} &< \infty. \end{aligned}$$

Let  $\mathring{\lambda}, \mathring{\mu} \in C^1(\mathbb{R})$  with  $\mathring{\lambda}(\theta) = \mathring{\lambda}(\theta+1)$ ,  $\mathring{\mu}(\theta+1) = \mathring{\mu}(\theta)$  for  $\theta \in \mathbb{R}$  and

$$\mathring{\mu}_\theta(\theta) = -4\pi e^{\mathring{\lambda} + \mathring{\mu}} \mathring{j}(\theta), \quad \theta \in \mathbb{R}.$$

Then there exists a unique, right maximal, regular solution  $(f, \lambda, \mu)$  of (3.3)–(3.7) with  $(f, \lambda, \mu)(t_0) = (\mathring{f}, \mathring{\lambda}, \mathring{\mu})$  on a time interval  $[t_0, T[$  with  $T \in ]t_0, \infty]$ . If

$$\sup \left\{ \mu(t, \theta) \mid \theta \in \mathbb{R}, t \in [t_0, T[ \right\} < \infty$$

then  $T = \infty$ .

This is the content of [16, Thms. 3.1 and 6.2]. In fact this was only stated in the case  $t_0 = 1$  in that reference, but that was an arbitrary choice which makes no essential difference. For a *regular solution* all derivatives which appear in the system exist and are continuous by definition, cf. [16].

We now establish a series of estimates which will result in an upper bound on  $\mu$  and will therefore prove that  $T = \infty$ . Similar estimates were used in [2] for the Einstein-Vlasov system with Gowdy symmetry. In what follows constants denoted by  $C$  will be positive, may depend on the initial data and may change their value from line to line.

Firstly, integration of (3.5) with respect to  $t$  and the fact that  $p$  is non-negative imply that

$$e^{2\mu(t, \theta)} = \left[ \frac{t_0(e^{-2\mathring{\mu}(\theta)} + \epsilon)}{t} - \epsilon - \frac{8\pi}{t} \int_{t_0}^t s^2 p(s, \theta) ds \right]^{-1} \geq \frac{t}{C - \epsilon t}, \quad t \in [t_0, T[. \quad (3.12)$$

Next we claim that

$$\int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta \leq C t^{-1+\epsilon}, \quad t \in [t_0, T[. \quad (3.13)$$

A lengthy computation shows that

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta = -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[ 2\rho + q - \frac{\rho+p}{2} (1 + \epsilon e^{2\mu}) \right] d\theta.$$

Now  $q \geq 0$  and  $p \leq \rho$  so that for  $\epsilon = 0$ ,

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta \leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta$$

and integrating this with respect to  $t$  yields (3.13) for  $\epsilon = 0$ . For  $\epsilon = -1$  we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, \theta) d\theta &\leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[ 2\rho + q - \frac{\rho+p}{2} \right] d\theta - \frac{1}{C+t} \int_0^1 e^{\mu+\lambda} \frac{\rho+p}{2} d\theta \\ &\leq -\frac{1}{C+t} \int_0^1 e^{\mu+\lambda} (2\rho+q) d\theta \leq -\frac{2}{C+t} \int_0^1 e^{\mu+\lambda} \rho d\theta, \end{aligned}$$

and integrating this inequality with respect to  $t$  yields (3.13) for  $\epsilon = -1$ . Using (3.13) and (3.6) we find

$$\begin{aligned} \left| \mu(t, \theta) - \int_0^1 \mu(t, \sigma) d\sigma \right| &= \left| \int_0^1 \int_\sigma^\theta \mu_\theta(t, \tau) d\tau d\sigma \right| \leq \int_0^1 \int_0^1 |\mu_\theta(t, \tau)| d\tau d\sigma \\ &\leq 4\pi t \int_0^1 e^{\mu+\lambda} |j(t, \tau)| d\tau \leq 4\pi t \int_0^1 e^{\mu+\lambda} \rho(t, \tau) d\tau \\ &\leq Ct^\epsilon, \quad t \in [t_0, T[, \quad \theta \in [0, 1]. \end{aligned} \quad (3.14)$$

Next we show that

$$e^{\mu(t, \theta) - \lambda(t, \theta)} \leq Ct^{1+\epsilon}, \quad t \in [t_0, T[, \quad \theta \in [0, 1]. \quad (3.15)$$

To see this observe that by (3.4), (3.5) and (3.12)

$$\begin{aligned} \frac{\partial}{\partial t} e^{\mu-\lambda} &= e^{\mu-\lambda} [4\pi t e^{2\mu} (p-\rho) + (\epsilon e^{2\mu} + 1)/t] \leq e^{\mu-\lambda} (\epsilon e^{2\mu} + 1)/t \\ &\leq \left[ \frac{1}{t} + \frac{\epsilon}{C-\epsilon t} \right] e^{\mu-\lambda}, \end{aligned}$$

and integrating this inequality with respect to  $t$  yields (3.15).

We now estimate the average of  $\mu$  over the interval  $[0, 1]$  which in combination with (3.14) will yield the desired upper bound on  $\mu$ :

$$\begin{aligned} \int_0^1 \mu(t, \theta) d\theta &= \int_0^1 \mu(t_0, \theta) d\theta + \int_{t_0}^t \int_0^1 \mu_t(s, \theta) d\theta ds \\ &\leq C + \int_{t_0}^t \frac{1}{2s} \int_0^1 [e^{2\mu} (8\pi s^2 p + \epsilon) + 1] d\theta ds \\ &= C + \frac{1}{2} \ln(t/t_0) + 4\pi \int_{t_0}^t s \int_0^1 e^{\mu-\lambda} e^{\mu+\lambda} p d\theta ds + \epsilon \int_{t_0}^t \frac{1}{2s} \int_0^1 e^{2\mu} d\theta ds \\ &\leq C + \frac{1}{2} \ln(t/t_0) + C \int_{t_0}^t s^{2+\epsilon} s^{\epsilon-1} ds + \frac{\epsilon}{2} \int_{t_0}^t \frac{ds}{C-\epsilon s}, \end{aligned}$$

where we used (3.12), (3.15) and (3.13). With (3.14) this implies

$$\mu(t, \theta) \leq C \begin{cases} t^2 & , t \in [t_0, T[, \quad \theta \in [0, 1], \quad \epsilon = 0, \\ 1 + \ln t & , t \in [t_0, T[, \quad \theta \in [0, 1], \quad \epsilon = -1, \end{cases} \quad (3.16)$$

which by Proposition 3.2 implies  $T = \infty$ . Thus we have proven:

**Theorem 3.3** *For initial data as in Proposition 3.2 the corresponding solution exists for all  $t \in [t_0, \infty[$  where  $t$  denotes the area radius of the surfaces of symmetry of the induced spacetime. The solution satisfies the estimates (3.13), (3.15) and (3.16).*

## 4 The contracting direction

In this section we consider the Einstein-Vlasov system in the contracting direction and use conformal coordinates in which the metric takes the form (2.2). As in the case of areal coordinates used in the expanding direction  $f$  depends only on

$$t, \theta, p^1, (p^2)^2 + \sinh^2 x (p^3)^2.$$

We will write

$$f = f(t, \theta, w, L)$$

where

$$w := e^\eta p^1, \quad L := R^4 ((p^2)^2 + \sinh^2 x (p^3)^2).$$

As above,

$$p^0 = e^{-\eta} \sqrt{1 + w^2 + L/R^2} =: e^{-\eta} \langle p \rangle,$$

and  $L$  is a conserved quantity along particle orbits.

In the variables introduced above the Einstein-Vlasov system can be written in the following form:

$$\partial_t f + \frac{w}{\langle p \rangle} \partial_\theta f + \left[ -\eta_t w - \eta_\theta \left( \langle p \rangle + \frac{w^2}{\langle p \rangle} \right) + e^{-2\eta} R_\theta \frac{L}{R^3 \langle p \rangle} \right] \partial_w f = 0. \quad (4.1)$$

$$-R_{\theta\theta} + \eta_t R_t + \eta_\theta R_\theta + \frac{1}{2R} [R_t^2 - R_\theta^2 - e^{2\eta}] = 4\pi R e^{2\eta} \rho, \quad (4.2)$$

$$R_{t\theta} - \eta_t R_\theta - \eta_\theta R_t = 4\pi R e^{2\eta} j, \quad (4.3)$$

$$R_{tt} - R_{\theta\theta} + \frac{1}{R} [R_t^2 - R_\theta^2 - e^{2\eta}] = 4\pi R e^{2\eta} (\rho - p), \quad (4.4)$$

$$\eta_{tt} - \eta_{\theta\theta} - \frac{1}{R^2} [R_t^2 - R_\theta^2 - e^{2\eta}] = 4\pi e^{2\eta} (p - \rho - q), \quad (4.5)$$

where

$$\rho(t, \theta) := \frac{\pi}{R^2} \int_{-\infty}^{\infty} \int_0^{\infty} \langle p \rangle f(t, \theta, w, L) dL dw = e^{-2\eta} T_{00}(t, \theta), \quad (4.6)$$

$$j(t, \theta) := \frac{\pi}{R^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, \theta, w, L) dL dw = -e^{-2\eta} T_{01}(t, \theta), \quad (4.7)$$

$$p(t, \theta) := \frac{\pi}{R^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\langle p \rangle} f(t, \theta, w, L) dL dw = e^{-2\eta} T_{11}(t, \theta), \quad (4.8)$$

$$q(t, \theta) := \frac{\pi}{R^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{\langle p \rangle} f(t, \theta, w, L) dL dw = \frac{2}{R^2} T_{22}(t, \theta). \quad (4.9)$$

The equations (4.2), (4.3) are the constraints and (4.4), (4.5) are the evolution equations; from the relation of their right hand sides with the energy momentum tensor it is obvious how this form of the field equations is obtained.

Let a smooth solution of the system (4.1)–(4.9) on some time interval  $]t_-, t_0]$  be given. We want to show that if this interval is bounded and if  $R$  is bounded away from zero on this interval then  $f$ ,  $R$ ,  $\eta$  and all their derivatives are bounded as well, with bounds depending on the data at  $t = t_0$  and the lower bound on  $R$ . To this end we define auxiliary variables

$$\tau := \frac{1}{\sqrt{2}}(t - \theta), \quad \xi := \frac{1}{\sqrt{2}}(t + \theta)$$

so that

$$\partial_\tau = \frac{1}{\sqrt{2}}(\partial_t - \partial_\theta), \quad \partial_\xi = \frac{1}{\sqrt{2}}(\partial_t + \partial_\theta).$$

The analysis which follows is modeled on the one in [2], cf. also [3].

*Step 1:  $C^1$ -bounds on  $R$ .* A short computation using (4.2) and (4.3) shows that

$$\begin{aligned} \partial_\theta R_\xi &= \frac{1}{\sqrt{2}}(R_{t\theta} + R_{\theta\theta}) \\ &= \left(\sqrt{2}\eta_\xi + \frac{1}{\sqrt{2}R}R_\tau\right)R_\xi - \frac{1}{2\sqrt{2}R}e^{2\eta} - 2\sqrt{2}\pi R e^{2\eta}(\rho - j) \\ &< \left(\sqrt{2}\eta_\xi + \frac{1}{\sqrt{2}R}R_\tau\right)R_\xi; \end{aligned}$$

observe that by (4.6) and (4.7),  $|j| \leq \rho$ . Assume that  $R_\xi(t, \theta) = 0$  for some  $t \in ]t_-, t_0]$  and  $\theta \in \mathbb{R}$ . Then by the periodicity of  $R$  with respect to  $\theta$ ,

$$0 = R_\xi(t, \theta + 1) < R_\xi(t, \theta) \exp\left(\int_\theta^{\theta+1} \left(\sqrt{2}\eta_\xi + \frac{1}{\sqrt{2}R}R_\tau\right) d\theta'\right) = 0,$$

a contradiction. Thus  $R_\xi \neq 0$  on  $]t_-, t_0] \times S^1$ . Similarly,

$$\partial_\theta R_\tau > -\left(\sqrt{2}\eta_\tau + \frac{1}{\sqrt{2}R}R_\xi\right)R_\tau$$

which yields the same assertion for  $R_\tau$ . This implies that the quantity

$$g^{\alpha\beta}\partial_{x^\alpha}R\partial_{x^\beta}R = e^{-2\eta}(-R_t^2 + R_\theta^2) = -\frac{1}{2}e^{-2\eta}R_\xi R_\tau$$

does not change sign. Since  $R$  is periodic in  $\theta$ , there must exist points where  $R_\theta = 0$ , hence the quantity above is negative everywhere, and by our choice of time direction,

$$R_t > 0, \quad |R_\theta| < R_t \quad \text{on } ]t_-, t_0] \times S^1. \quad (4.10)$$

By (4.4),

$$\frac{1}{2}R_{\xi\tau} = -\frac{1}{2R}R_\xi R_\tau + e^{2\eta} \left(\frac{1}{R} + 4\pi R(\rho - p)\right) > -\frac{1}{2R}R_\xi R_\tau;$$

observe that by (4.6) and (4.8)  $p \leq \rho$ . Now we fix some  $(t, \theta) \in ]t_-, t_0[ \times \mathbb{R}$ . Then for  $s \in [t, t_0]$ ,

$$\frac{d}{ds} R_\xi(s, \theta + t - s) = \sqrt{2} R_{\xi\tau} > - \frac{d}{ds} [\ln R(s, \theta + t - s)] R_\xi(s, \theta + t - s).$$

Integrating this differential inequality yields

$$R_\xi(t, \theta) < \frac{R(t_0, \theta + t - t_0)}{R(t, \theta)} R_\xi(t_0, \theta + t - t_0).$$

Similarly,

$$R_\tau(t, \theta) < \frac{R(t_0, \theta - t + t_0)}{R(t, \theta)} R_\tau(t_0, \theta - t + t_0),$$

and both estimates together imply that  $R_t$  is bounded from above on  $]t_-, t_0[ \times S^1$  with a bound of the desired sort. Together with (4.10) this provides bounds for  $R_t$  and  $R_\theta$ . Note that this argument shows that the spacetime gradient of  $R^2$  is bounded even without the assumption that  $R$  is bounded away from zero.  
*Step 2:  $C^1$ -bounds on  $\eta$ .* From (4.4) and (4.5) we find

$$\eta_{\xi\tau} = - \frac{1}{2R} R_{\xi\tau} - 2\pi e^{2\eta} q.$$

We fix some  $(t, \theta) \in ]t_-, t_0[ \times \mathbb{R}$ . Then for  $s \in [t, t_0]$ ,

$$\frac{d}{ds} \eta_\xi(s, \theta + t - s) = - \frac{1}{R} \frac{d}{ds} R_\xi(s, \theta + t - s) - 2\sqrt{2}\pi e^{2\eta} q(s, \theta + t - s).$$

Integrating this and integrating the term containing  $R_\xi$  by parts yields

$$\begin{aligned} \eta_\xi(t, \theta) &= \eta_\xi(t_0, \theta + t - t_0) - \frac{R_\xi(t_0, \theta + t - t_0)}{R(t_0, \theta + t - t_0)} + \frac{R_\xi(t, \theta)}{R(t, \theta)} \\ &\quad + \sqrt{2} \int_t^{t_0} \left( 2\pi e^{2\eta} q - \frac{R_\tau R_\xi}{R^2} \right) (s, \theta + t - s) ds. \end{aligned} \quad (4.11)$$

We already know that

$$\int_t^{t_0} R_{\xi\tau}(s, \theta + t - s) ds = \frac{1}{\sqrt{2}} (R_\xi(t_0, \theta + t - t_0) - R_\xi(t, \theta))$$

is bounded, and on the other hand by (4.4) the left hand side can be written as

$$2 \int_t^{t_0} \left[ \frac{1}{R} (-R_t^2 + R_\theta^2) + \frac{1}{R} e^{2\eta} + 4\pi R e^{2\eta} (\rho - p) \right] (s, \theta + t - s) ds.$$

Here the first term is bounded by Step 1, and the second and third terms are non-negative. Thus by (4.9), (4.6), (4.8),

$$\int_t^{t_0} e^{2\eta} q(s, \theta + t - s) ds \leq \int_t^{t_0} \frac{1}{R^2} e^{2\eta} (\rho - p)(s, \theta + t - s) ds$$

is bounded as well; recall that  $R$  is bounded away from zero by assumption. Thus (4.11) implies that  $\eta_\xi$  is bounded on  $]t_-, t_0] \times S^1$ . Analogously to (4.11) we have

$$\begin{aligned} \eta_\tau(t, \theta) &= \eta_\tau(t_0, \theta - t + t_0) - \frac{R_\tau(t_0, \theta - t + t_0)}{R(t_0, \theta - t + t_0)} + \frac{R_\tau(t, \theta)}{R(t, \theta)} \\ &\quad + \sqrt{2} \int_t^{t_0} \left( 2\pi e^{2\eta} q - \frac{R_\tau R_\xi}{R^2} \right) (s, \theta - t + s) ds \end{aligned} \quad (4.12)$$

from which we can conclude that  $\eta_\tau$  is bounded on  $]t_-, t_0] \times S^1$  as well. Thus also  $\eta_t, \eta_\theta$  and  $\eta$  itself are bounded there.

*Step 3: Bounds on matter quantities.* We have for any  $t \in ]t_-, t_0]$ ,  $\theta \in \mathbb{R}$ ,  $w \in \mathbb{R}$ ,  $L > 0$ ,

$$f(t, \theta, w, L) = f(t_0, \Theta(t_0, t, \theta, w, L), W(t_0, t, \theta, w, L), L)$$

where  $\Theta(\cdot, t, \theta, w, L), W(\cdot, t, \theta, w, L)$  is the solution of the characteristic system

$$\begin{aligned} \dot{\theta} &= \frac{w}{\langle p \rangle}, \\ \dot{w} &= -\eta_t w - \eta_\theta \left( \langle p \rangle + \frac{w^2}{\langle p \rangle} \right) + e^{-2\eta} R_\theta \frac{L}{R^3 \langle p \rangle} \end{aligned}$$

of the Vlasov equation with  $\Theta(t, t, \theta, w, L) = \theta$ ,  $W(t, t, \theta, w, L) = w$ . This representation of  $f$  implies immediately that  $f$  remains non-negative and bounded by its maximum at  $t = t_0$ . By Steps 1 and 2 the right hand side of the second equation in the characteristic system is linearly bounded in  $w$ , and hence if the  $w$ -support of  $f$  is compact initially,

$$\sup\{|w| | (\theta, w, L) \in \text{supp} f(t), t \in ]t_-, t_0]\} < \infty.$$

This immediately implies that  $\rho, p, j, q$  are bounded on  $]t_-, t_0] \times S^1$ .

*Step 4: Bounds on second order derivatives of  $R$  and  $\eta$ .* Steps 1, 2 and 3 together with (4.2), (4.3) and (4.4) imply that  $R_{\theta\theta}, R_{t\theta}$  and  $R_{tt}$  are bounded as claimed on  $]t_-, t_0] \times S^1$ . The bounds on the second order derivatives of  $\eta$  are a bit less trivial: We add the equations (4.11) and (4.12) to obtain a formula for  $\eta_\theta$ . When this formula is differentiated with respect to  $\theta$  there results a number of terms which are bounded by the previous steps and the terms

$$2\pi \int_t^{t_0} e^{2\eta} \frac{\pi}{R^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{\langle p \rangle} \partial_\theta f(s, \theta + t - s, w, L) dL dw ds, \quad (4.13)$$

$$2\pi \int_t^{t_0} e^{2\eta} \frac{\pi}{R^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{L}{\langle p \rangle} \partial_\theta f(s, \theta - t + s, w, L) dL dw ds. \quad (4.14)$$

To deal with the first term we introduce the differential operators

$$W = \sqrt{2} \partial_\tau = \partial_t - \partial_\theta, \quad S = \partial_t + \frac{w}{\langle p \rangle} \partial_\theta$$

so that

$$\partial_\theta = \frac{\langle p \rangle}{\langle p \rangle + w} (S - W).$$

By the Vlasov equation

$$Sf = - \left[ -\eta_t w - \eta_\theta \left( \langle p \rangle + \frac{w^2}{\langle p \rangle} \right) + e^{-2\eta} R_\theta \frac{L}{R^3 \langle p \rangle} \right] \partial_w f.$$

When this is substituted into (4.13) the resulting term can be integrated by parts with respect to  $w$ , and all the terms which then appear are bounded by the previous steps. As to the  $W$ -contribution,

$$(Wf)(s, \theta + t - s, w, L) = \frac{d}{ds} [f(s, \theta + t - s, w, L)]$$

so that the corresponding term in (4.13) can be integrated by parts with respect to  $s$  which again results in bounded terms. In order to deal with (4.14) we replace  $w$  by  $-w$  in the integral and redefine

$$W = \partial_t + \partial_\theta, \quad S = \partial_t - \frac{w}{\langle p \rangle} \partial_\theta;$$

the rest of the argument should then be obvious, and  $\eta_{\theta\theta}$  is seen to be bounded on  $]t_-, t_0] \times S^1$ . This way of dealing with derivatives in connection with the Vlasov equation was introduced for the Vlasov-Maxwell system in [8]. Now that  $\eta_{\theta\theta}$  is bounded the same is true for  $\eta_{tt}$  by (4.5), and  $\eta_{t\theta}$  can be dealt with like  $\eta_{\theta\theta}$ : Taking the difference of (4.11) and (4.12) gives the necessary formula for  $\eta_t$ .

*Step 5: Higher order derivatives.* Via the characteristic system the  $C^2$ -bounds on  $R$  and  $\eta$  give bounds on the first order derivatives of  $\Theta(\cdot, t, \theta, w, L)$  and  $W(\cdot, t, \theta, w, L)$  with respect to  $\theta, w, L$ . This yields corresponding  $C^1$ -bounds first on  $f$  and then as in Step 3 on  $\rho, p, j, q$ . These in turn imply  $C^3$ -bounds on  $R$ . The third order derivatives of  $\eta$  then have to be dealt with by repeating the argument of Step 4, and although the details would be tedious it should be clear that this process can be iterated to bound any desired derivative on  $]t_-, t_0] \times S^1$  in terms of the data at  $t = t_0$  and the positive lower bound on  $R$ .

Later we will require a slight generalization of these results. The essence of the first part of Step 1 above is to show that the gradient of  $R$  is timelike. The argument is carried out on a region which is covered by Cauchy surfaces of constant conformal time. However this fact is also true for any solution of the Einstein-Vlasov system with hyperbolic symmetry and a compact Cauchy surface. This follows from [19, Lemma 2.5]. Once this has been established the estimates in the later steps hold for any subset  $Z$  of the half-plane  $t \leq t_0$  provided  $Z$  is a future set. By definition this means that any future directed causal curve in the region  $t \leq t_0$  starting at a point of  $Z$  remains in  $Z$ . (For information on concepts such as this concerning causal structures see e.g. [9].) Thus if  $R$  is bounded away from zero on  $Z$  and  $t$  is bounded on  $Z$  then all the unknowns and their derivatives can be controlled on  $Z$ .

Now consider a special choice of the subset  $Z$ , namely that which is defined by the inequalities  $t_1 < t \leq t_0$  and  $\theta_1 + t_0 - t < \theta < \theta_2 - t_0 + t$  for some numbers  $\theta_1$ ,  $\theta_2$  and  $t_1$  satisfying the inequalities  $\theta_1 < \theta_2$  and  $t_1 > t_0 - (1/2)(\theta_2 - \theta_1)$ . Suppose a solution of the Einstein-Vlasov system in conformal coordinates defined on this region is such that  $R$  is bounded away from zero. Then the functions defining the solution extend smoothly to the boundary of  $Z$  at  $t = t_1$ . They define smooth Cauchy data for the Einstein-Vlasov system. Applying the standard local existence theorem (without symmetry) allows the solution to be extended through that boundary. Repeating the construction of conformal coordinates in Section 2 then shows that we get an extension of the solution written in conformal coordinates through that boundary.

## 5 Existence of an areal time coordinate

In Section 2 it was shown that the maximal Cauchy development of initial data for the Einstein-Vlasov system with hyperbolic symmetry admits conformal coordinates in a neighbourhood of the initial hypersurface. The aim of this section is to show that a spacetime of this kind always admits a symmetric compact Cauchy surface of constant areal time. The basic strategy follows that of [3]. The fundamental object of study is a spacetime in conformal coordinates which develops (to the past) from initial data on the hypersurface  $t = 0$ . The solution will exist on a subset of the region  $t \leq 0$  of  $\mathbb{R}^2$ . We will consider solutions defined on open subsets of the half-plane  $t \leq 0$  containing  $t = 0$  which are future sets. The union of all regions admitting solutions of this type is an open future set. Denote it by  $\mathcal{D}$ . By definition  $R$  is a positive function defined everywhere on  $\mathcal{D}$ .

Consider first the case that the past boundary of  $\mathcal{D}$  is empty, i.e., that  $\mathcal{D} = ]-\infty, 0] \times S^1$ . Let  $\Sigma_\rho$  be the level set defined by the equation  $R(t, \theta) = \rho$  for some  $\rho$  less than the minimum of  $R$  on  $t = 0$ . Since the gradient of  $R$  is everywhere timelike the level sets  $\Sigma_\rho$  are smooth spacelike submanifolds. These level sets can be represented in the form  $t = f(\theta)$ , and since the coordinates are conformal and the level sets spacelike it follows that  $|f'| < 1$ . Hence the  $\Sigma_\rho$  are compact and their pull-backs define compact spacelike hypersurfaces in a Cauchy development of the initial data of interest. By [5] these are Cauchy surfaces. As a consequence the maximal Cauchy development of those data contains symmetric compact Cauchy surfaces of constant areal time.

It remains to consider the case where  $\mathcal{D}$  has a non-empty past boundary. Call this boundary  $\mathcal{B}$ . It is achronal. Hence for any given  $\theta$  there is only one  $t$  such that  $(t, \theta) \in \mathcal{B}$ . If  $p$  is a point of  $\mathcal{B}$  then all points of  $\mathcal{B}$  close to  $p$  are outside the light cone of  $p$ . Hence  $\mathcal{B}$  is a Lipschitz curve with Lipschitz constant one. In other words it can be represented in the form  $t = h(\theta)$  for a function  $h$  with Lipschitz constant one. This is a special case of a general property of achronal boundaries, cf. [9, p. 187]. As a consequence  $\mathcal{B}$  is compact. Since the gradient of  $R^2$  is bounded on the whole region where the solution is defined (as shown in Section 4) it follows that  $R^2$  is uniformly continuous. Hence the function  $R^2$



extends to a continuous function on the closure  $\bar{\mathcal{D}}$  of  $\mathcal{D}$ . Its square root  $\bar{R}$  is a continuous extension of  $R$  to  $\bar{\mathcal{D}}$ . It will now be proved that  $\bar{R}=0$  on  $\mathcal{B}$ . From this point on it is possible to argue as above to obtain the existence of a Cauchy surface on which  $R$  is constant.

Consider first a point  $\theta_0$  where  $h$  has a local maximum and let  $t_0 = h(\theta_0)$ . Then there exists  $\epsilon > 0$  such that for any  $t$  greater than  $t_0$  the points  $(t, \theta)$  with  $\theta_0 - \epsilon < \theta < \theta_0 + \epsilon$  lie in  $\mathcal{D}$ . If  $\bar{R}(t_0, \theta_0) > 0$  then the solution has a smooth limit at the points  $(t_0, \theta)$  with  $\theta_0 - \epsilon < \theta < \theta_0 + \epsilon$  provided  $\epsilon$  is small enough. This gives Cauchy data at  $t = t_0$ . The corresponding local solution of the evolution equations can be used to extend the original solution, thus contradicting the assumed maximality of  $\mathcal{D}$ . It follows that in fact  $\bar{R}$  vanishes at any point of  $\mathcal{B}$  where  $h$  has a local maximum.

Let  $\mathcal{D}_t$  be the intersection of  $\mathcal{D}$  with the hypersurface where the time coordinate takes the value  $t$ . It is an open subset of  $S^1$ . It is either the whole of  $S^1$ , the empty set or a disjoint union of open intervals which are its connected components. Consider now a point  $(t_0, \theta_0)$  of  $\mathcal{B}$  which is an endpoint of a component of  $\mathcal{D}_{t_0}$ . Suppose further that  $\bar{R}(t_0, \theta_0) > 0$ . Without loss of generality we can assume that  $\theta_0$  is a right endpoint of a component; the argument for a left endpoint is strictly analogous. For some  $\epsilon > 0$  the solution extends smoothly to the points  $(t_0, \theta)$  with  $\theta_0 - \epsilon < \theta \leq \theta_0$ . This can be extended to smooth data at  $t = t_0$  on the interval  $\theta_0 - \epsilon < \theta < \theta_0 + \epsilon$ . There is a corresponding local solution. Using the domain of dependence we see that this new solution extends the original solution at those points outside the past light cone of the point  $(t_0, \theta_0)$ . In this way we obtain an extension of the original solution and a contradiction unless the part of  $\mathcal{B}$  immediately to the left of  $\theta_0$  is a null curve. If  $\mathcal{B}_1$  is the set of points of  $\mathcal{B}$  which are endpoints of components of some  $\mathcal{D}_t$  and which have  $\bar{R} > 0$  then it has now been shown that  $\mathcal{B}_1$  is a union of null segments. Returning to the point  $(t_0, \theta_0)$  we can ask what happens to the interval of which it is an endpoint as  $t$  increases. The endpoint could vanish if the interval coalesces with another while  $\bar{R}$  remains positive. However that would mean that  $\theta_0$  would have to be a local maximum of  $h$ , a case already excluded. Hence as we move to the right in  $\theta$  the corresponding point of  $\mathcal{B}$  must continue to be an endpoint. Eventually  $\bar{R}$  must become zero. But in that case the future-pointing character of the gradient of  $R$  can be used to obtain a contradiction. It can be concluded that  $\mathcal{B}_1$  is empty.

It remains to consider the case that there is a point  $(t_0, \theta_0)$  of  $\mathcal{B}$  which is not an endpoint of a component of some  $\mathcal{D}_t$ . In this case  $(t, \theta_0)$  belongs to  $\mathcal{D}$  for every  $t > t_0$ . Let  $(a(t), b(t))$  be the component of  $\mathcal{D}_t$  containing  $\theta_0$ . The functions  $a(t)$  and  $b(t)$  are decreasing and increasing, respectively. Let  $a_0$  and  $b_0$  be their limits as  $t \rightarrow t_0$  from above. If  $a_0 < b_0$  then  $(t_0, \theta_0)$  corresponds to a local maximum of  $f$  and so  $\bar{R} = 0$  there. Otherwise it is a limit of the points  $(t, b(t))$  which are right endpoints of components of  $\mathcal{D}_t$ , and so by continuity  $\bar{R} = 0$ .

A natural question arising in this context is whether a spacetime which exists globally in one time direction with respect to a conformal time coordinate cannot be extended in that direction to a strictly larger globally hyperbolic spacetime.

It will be shown that this is impossible. As an aid in doing this we consider the action of the symmetry group on  $\tilde{M}$ . We know exactly what the action looks like near the initial hypersurface. There is a two-dimensional subgroup  $H$  of  $G$  which acts transitively on the orbits of the action of  $G$ . Corresponding to  $H$  there are two Killing vector fields which are spacelike and linearly independent near the initial hypersurface. We claim that they are spacelike and linearly independent on the whole of  $\tilde{M}$ , so that the orbits of the action of  $H$  are everywhere spacelike and two-dimensional. To show this it suffices to show that a Killing vector generated by the action of  $H$  can neither vanish nor become null anywhere on  $\tilde{M}$ , cf. [1, Prop. 2.4]. Suppose that a Killing vector of this kind vanishes at a point  $p$ . Without loss of generality we can assume that  $p$  lies to the future of the initial hypersurface and that the Killing vector is spacelike and non-zero on the chronological past of  $p$ . The integral curve of the Killing vector through  $p$  is null and if we extend it maximally it must either reach the initial hypersurface or tend to a zero of the Killing vector. The first of these possibilities contradicts the fact that the Killing vector is spacelike on the initial hypersurface. In the second case we see that the Killing vector has a zero to the future of the initial surface.

Now it will be shown that the existence of zero of the Killing vector at a point  $p$  to the future of the initial hypersurface leads to a contradiction. The flow generated by the Killing vector leaves  $p$  invariant. Its linearization maps the space of null vectors at  $p$  to itself and must have a fixed point. In other words the linearization leaves a null direction invariant. The null geodesic with this initial direction is then also invariant. It must intersect the initial hypersurface, and the point of intersection is invariant under the flow. This gives the desired contradiction.

Now we come back to the question of inextendibility. Suppose that the spacetime exists for all conformal time in one time direction, without loss of generality the future. Suppose that there is a point  $p$  belonging to a future development of the initial data but not in the region covered by the conformal coordinates. It lies on an orbit of  $H$ . Let  $\gamma_1$  and  $\gamma_2$  be the two null geodesics through  $p$  orthogonal to the orbit. They are orthogonal to all orbits they meet. They must enter the region covered by conformal coordinates and then they coincide with the curves with  $t = \pm\theta + C$  for some constant  $C$  and constant values of the other coordinates. Without loss of generality we can assume that  $p$  is on the boundary of the region covered by the conformal coordinates. In the approach to  $p$  the geodesic  $\gamma_2$  intersects the geodesic  $\gamma_1$  infinitely many times. This contradicts the strong causality and hence the global hyperbolicity of the extension. The conclusion is that an interval of conformal time which is infinite in one time direction proves that the maximal Cauchy development is exhausted in that time direction.

There is also another criterion for inextendibility which can conveniently be discussed here. It is related to an argument given in [3]. Consider the action of the group  $H$  on  $\tilde{M}$  introduced above and the corresponding Killing vectors. Since they are always linearly independent with spacelike orbits, the  $2 \times 2$  matrix of inner products of these vectors always has positive determinant  $\Delta$ . Suppose

that a solution is given which exists globally towards the future in areal time. It will now be shown that it has no proper globally hyperbolic extension to the future. If there was such an extension there would be a point  $p$  of the extension not belonging to the original spacetime. Let  $\gamma_1$  be a null geodesic through  $p$  as above. We may assume without loss of generality that it immediately enters the original spacetime on leaving  $p$  towards the past. In the region covered by the areal time coordinate it has constant values of the coordinates  $x$  and  $y$ . Along the part of  $\gamma_1$  belonging to the original spacetime  $\Delta$  is proportional to  $R^4$ . As a point on  $\gamma_1$  approaches  $p$  the areal time coordinate  $t$  tends to infinity, which means that  $\Delta$  tends to infinity. But  $\Delta$  is a smooth function at the point  $p$ . Hence no point  $p$  of this kind can exist.

## 6 Main results

In this section the analytical and geometrical information obtained in previous sections is combined to obtain the main results of the paper.

**Theorem 6.1** *Let  $(M, g_{\alpha\beta}, f)$  be the maximal globally hyperbolic development of initial data for the Einstein-Vlasov system with hyperbolic symmetry. Then  $M$  can be covered by symmetric compact hypersurfaces of constant area radius. The area radius of these hypersurfaces takes all values in the range  $]R_0, \infty[$  where  $R_0$  is a non-negative number.*

The quantity  $R_0$  depends on the solution, but there is no good understanding of how many solutions have  $R_0 \neq 0$ .

*Proof of Theorem 6.1.* For a spacetime satisfying the hypotheses of the theorem we know from Section 2 that a conformal coordinate system can be introduced on a neighbourhood of the initial hypersurface  $S_0$  corresponding to the original data. By the results of Section 5 it follows that this region can be extended to the past so as to include a Cauchy surface  $S_A$  of constant areal time. Moreover, either the conformal time coordinate extends to all negative values, or  $R$  tends to zero as the past boundary  $\mathcal{B}$  of the region covered by conformal coordinates is approached. In the first of these cases the region covered by the conformal time coordinate includes the entire past of the initial hypersurface in the maximal Cauchy development, as shown in Section 5. Also the past of  $S_A$  in that region admits a foliation by hypersurfaces of constant  $R$ . In that region we can transform to areal coordinates. For we can choose the spatial coordinate  $\theta$  so that its coordinate lines in  $Q$  are orthogonal to that foliation. In the second case (where  $R$  tends to zero on a boundary  $\mathcal{B}$ ) the past of  $S_A$  is also covered by areal coordinates. It exhausts the past of  $S_A$  in the maximal Cauchy development, as will now be shown. A well-known argument related to Hawking's singularity theorem shows that it is enough to check that the mean curvature of the foliation tends uniformly to infinity as  $R$  tends to zero. The mean curvature is given in areal coordinates by [16]

$$\text{tr}k = -e^{-\mu}(\dot{\lambda} + 2/t) \tag{6.1}$$

Substituting the field equation for  $\dot{\lambda}$  and using the fact that  $\rho \geq 0$  gives the inequality

$$\text{tr}k \leq -(1/2t)(3e^{-\mu} + e^{\mu}) \leq -\sqrt{3}/t \quad (6.2)$$

and hence the desired result. Thus in both cases the entire past of  $S_0$  in the maximal Cauchy development is covered. It follows from the results of Section 3 that the spacetime and the areal time coordinate can be extended so that the time coordinate covers the interval  $]R_0, \infty[$ . It can then be concluded by the argument at the end of Section 5 that the entire future of  $S_0$  in the maximal Cauchy development is covered.

**Theorem 6.2** *Let  $(M, g_{\alpha\beta}, f)$  be the maximal globally hyperbolic development of initial data for the Einstein-Vlasov system with hyperbolic symmetry. Then  $M$  can be covered by compact hypersurfaces of constant mean curvature. The mean curvature of these hypersurfaces takes all values in the range  $] -\infty, 0[$ .*

*Proof.* It follows from (6.2) that there are Cauchy surfaces with everywhere negative mean curvature. Under these circumstances it was shown by Henkel [11] that the initial singularity is a crushing singularity and thus a neighbourhood of it can be foliated by CMC hypersurfaces. Next the statement about the range of the CMC time coordinate will be proved. The important observation is that the argument used to prove the corresponding statement in [21] extends straightforwardly to the case of hyperbolic symmetry. It is more powerful than the arguments used to extend the CMC foliation in [19]. It remains to see that the CMC foliation covers the entire future of the initial hypersurface. It is enough to show that if  $p$  is any point of the spacetime there is a compact CMC hypersurface which contains  $p$  in its past. Let  $S_1$  be the Cauchy surface of constant areal time passing through  $p$ . Equation (6.2) shows that the mean curvature of  $S_1$  is strictly negative. Hence it has a maximum value  $H_1 < 0$ . Let  $S_2$  be the compact CMC hypersurface with mean curvature  $H_1/2$ . Then the infimum of the mean curvature of  $S_2$  is greater than the supremum of the mean curvature of  $S_1$  and a standard argument [13] shows that  $S_2$  is strictly to the future of  $S_1$ . Hence  $p$  is in the past of  $S_2$ , as required.

*Remark.* In the case of Gowdy symmetry the mean curvature of the hypersurfaces of constant areal time is negative and the corresponding argument applies, thus showing that the CMC foliation covers the spacetime in that case too.

The considerations of this paper were confined to the case where the matter content of spacetime is described by the Vlasov equation, but in fact many of the arguments do not depend on the details of the matter model. It would be worth to investigate which parts of the conclusions extend to which matter models. Some relevant estimates for scalar fields, wave maps and electromagnetic fields can be found in [19], [6], [21] and [11].

## 7 Possible further developments

In this paper we have seen how it is possible to get rather complete information on the existence of global geometrically defined time coordinates in spacetimes

with hyperbolic symmetry and, more generally, with surface symmetry. The motivation for being interested in this question was that its answer could provide a tool for understanding the global geometry of these spacetimes. To do so it is necessary to understand something about the asymptotic behaviour of the solutions as expressed with respect to these time coordinates. The kind of questions which we would like to answer are the following. Is the spacetime geodesically complete towards the future? Is there a curvature singularity in the past? Is the singularity in the past velocity dominated? Does the solution become homogeneous in the future in some sense? Can we obtain detailed asymptotic expansions for the solution in these regimes?

The one strong and rather general result concerning questions like this which is available is the following theorem of [16]. If there is a foliation by Cauchy surfaces of constant  $R$  such that  $R$  approaches zero as the singularity is approached and if, in the hyperbolic case, the Hawking mass is positive near the singularity then the Kretschmann scalar blows up as the singularity is approached. That some restriction is required is shown by the pseudo-Schwarzschild solution (cf. [19]). For negative values of the mass parameter this solution, which has hyperbolic symmetry, has negative Hawking mass, and  $R$  does not approach zero at the boundary of the maximal Cauchy development. The Kretschmann scalar remains finite near that boundary. A guess, which is not contradicted by any known results, is that the pseudo-Schwarzschild solutions are the only solutions of the Einstein-Vlasov system with hyperbolic symmetry which do not have a singularity where the Kretschmann scalar blows up. It would be very interesting to have a rigidity theorem of this kind.

Apparently the only other source of information about the asymptotics of surface symmetric solutions of the Einstein-Vlasov system comes from the study of spatially homogeneous solutions. The spatially homogeneous solutions with spherical, plane and hyperbolic symmetry are the spacetimes of Kantowski-Sachs, Bianchi I and Bianchi III types respectively. There are exceptional cases (all vacuum) which are the Schwarzschild solution (appropriately identified), the flat Kasner solution and the pseudo-Schwarzschild solution. These were presented in Appendix B to [19]. All other solutions are such that curvature invariants blow up in both time directions (Kantowski-Sachs) or that curvature invariants blow up in one time direction and the spacetime is geodesically complete in the other time direction (Bianchi types I and III) [20].

More detailed results on the asymptotics have been obtained for Bianchi types I and III [22]. Translating the theorems obtained there into the notation of this paper leads to the following results. All non-vacuum solutions of either of these two Bianchi types are such that  $R \rightarrow 0$  as the past boundary of the maximal Cauchy development is approached. Generically  $\mu \sim (1/2)\ln t$  and  $\lambda \sim -(1/2)\ln t$  as  $t \rightarrow 0$ , where we only indicate the leading term in an asymptotic expansion. There is a smaller class of solutions (not explicitly known) where  $\mu \sim (1/2)\ln t$  and  $\lambda \sim \text{const.}$  as  $t \rightarrow 0$ . Finally there is an even smaller class where  $\mu \sim \ln t$  and  $\lambda \sim \ln t$  as  $t \rightarrow 0$ . In the expanding direction all solutions of both Bianchi types are geodesically complete. For type I we have  $\mu \sim (1/2)\ln t$  and  $\lambda \sim \ln t$  as  $t \rightarrow \infty$ . The behaviour of the type III solutions was not completely determined in [22].

It was shown that (in the notation of the present paper)  $\lambda$  is increasing for large  $t$  but probably it grows slower than any positive multiple of  $\ln t$ .

Clearly the next step is to extend these results on asymptotic behaviour to the inhomogeneous case. It would be convenient if the inhomogeneous solutions behaved essentially like the homogeneous ones in their asymptotic regimes. There are, however, phenomena which may prevent this. These are the possibility of the occurrence of analogues of the spikes found in Gowdy spacetimes [23] in the contracting direction and the Jeans instability [4] in the expanding direction.

*Acknowledgement:* GR acknowledges support by the Wittgenstein 2000 Award of P. A. Markowich.

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