

BMN Gauge Theory as a Quantum Mechanical System

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ABSTRACT

We rigorously derive an effective quantum mechanical Hamiltonian from $\mathcal{N} = 4$ gauge theory in the BMN limit. Its eigenvalues yield the exact one-loop anomalous dimensions of scalar two-impurity BMN operators for all genera. It is demonstrated that this reformulation vastly simplifies computations. E.g. the known anomalous dimension formula for genus one is reproduced through a one-line calculation. We also efficiently evaluate the genus two correction, finding a non-vanishing result. We comment on multi-trace two-impurity operators and we conjecture that our quantum-mechanical reformulation could be extended to higher quantum loops and more impurities.

In [1] a modified, simpler version of the AdS/CFT duality between IIB string theory and $\mathcal{N} = 4$ supersymmetric gauge theory was considered. It involves taking a limit on both sides of the correspondence. On the string side the Penrose limit of the $\text{AdS}_5 \times \text{S}^5$ background results in the so-called plane-wave background [2,3,4], while the limit on the gauge theory side leads to the consideration of operators of large R -charge J in conjunction with a large N limit. After identification of the correct operators [1] it is possible to relate and match, for various massless and massive states, the mass spectrum of free plane-wave string excitations [3,4] to the planar scaling dimensions of the corresponding Berenstein-Maldacena-Nastase (BMN) operators [1,5,6].

It is natural to study the extension of the BMN correspondence to the case of interacting strings. On the gauge theory side, this involves the consideration of corrections to the planar limit. It was demonstrated in [7,8,9,10] that such corrections indeed survive the BMN limiting procedure. As far as the scaling dimensions of BMN operators are concerned, the corrections turned out to be finite. Much work [11,12,13,14,15,16,17,18,19] has also been undertaken to develop a string field theory approach, which should eventually permit to obtain the string corrections to the free mass formula of plane-wave states. A recent effort to derive the gauge theory results [7,8,9,10] for genus one from string theory has been reported in [20]. An alternative, more heuristic approach, which is conceptionally and methodologically interpolating between string and gauge theory has been pursued in refs. [21,22,23,24].

In order to unearth its relationship to string theory, it is important to understand the structure of BMN gauge theory as completely as possible. It is widely suspected (see most of the above references) that the formalism underlying the limit is essentially one-dimensional and should therefore resemble the description of a quantum mechanical system. In this note we shall demonstrate, in the example of two impurities and on the one-loop level, but to arbitrary order in the topological expansion, that this is indeed the case. In fact, we will *derive* the parts of the Hamiltonian relevant to our situation directly from the gauge theory. Aside from providing this conceptual insight, we shall also show how the rather laborious computations of [7,8,9,10] can be significantly simplified and extended. An important first hint of this hidden simplicity was discovered by Janik [25], and we will complete and considerably extend his arguments. For steps towards directly deriving a Hamiltonian from $\mathcal{N} = 4$ gauge theory by dimensional reduction, see [26].

Recall [7,8,9,10] that the BMN suggestion leads to a double-scaling limit of the $\mathcal{N} = 4$ gauge theory: The R -charge J and the number of colors N tend to infinity such that the effective quantum loop counting parameter $\lambda' = g_{\text{YM}}^2 N / J^2$ and the effective genus counting parameter $g_2^2 = J^4 / N^2$ stay finite. The quantities of physical interest are the anomalous dimensions of the

BMN operators. In [8] it was conjectured that *physical* quantities involving string interactions should not depend on g_2 but instead on an *effective* string coupling constant $g_2 \sqrt{\lambda}$. This also appears to be implicit in the work of [21, 22, 23]. It would seem to imply that at the one-loop ($\mathcal{O}(\lambda')$) level no corrections from genera higher than one should be allowed. Our new procedure allows us to easily perform a genus two calculation of the one-loop anomalous dimensions, which would have been horrendously complicated with the previous methods. We will find a non-vanishing result below, which appears to disprove the just mentioned conjecture.

The scalar BMN operators are made from the three complex scalar fields Z, ϕ, ψ of the gauge theory. They are obtained by modifying charge J BPS $(k+1)$ -trace operators $\text{Tr} Z^{J_0} \text{Tr} Z^{J_1} \dots \text{Tr} Z^{J_k}$ (where $J = J_0 + J_1 + \dots + J_k$) by doping them with a number of ‘‘impurities’’ [1] ϕ, ψ :

$$\mathcal{O}_p^{J_0, J_1, \dots, J_k} = \text{Tr}(\phi Z^p \psi Z^{J_0-p}) \text{Tr} Z^{J_1} \dots \text{Tr} Z^{J_k} \quad (1)$$

where $0 \leq p \leq J_0$. Here we will only consider the simplest case of two impurities. At the classical level the doped operators have conformal dimension $J + 2$. This dimension gets corrected once interactions are turned on: The operators are no longer protected if the impurities are inserted into the same trace. However, it is possible to find linear combinations of the \mathcal{O}_α (α is a multi-index) which have a definite conformal dimension. Conformal scaling operators are characterized by being eigenstates of the dilatation operator D with eigenvalue Δ , which equals their conformal dimension. In the planar limit it is possible to consider single trace operators alone and the corrections are given by the BMN prediction [1]. For a precise definition of these operators at finite J see [27]. When one takes into account non-planar contributions single-trace operators no longer suffice [28, 9, 10] and we have to find the appropriate linear combinations in the class of multi-trace operators in eq.(1).

The matrix elements of the dilatation operator in the BMN basis can be read off from the two-point functions of the BMN operators. With α and β multi-indices we can write a one-loop two-point function of BMN operators as

$$\langle \mathcal{O}_\alpha(x) \bar{\mathcal{O}}_{\bar{\beta}}(0) \rangle = \frac{1}{|x|^{2(J+2)}} (S_{\alpha\bar{\beta}} + T_{\alpha\bar{\beta}} \log(|x\Lambda|^{-2})) \quad (2)$$

where Λ is a (divergent) renormalization constant (see e.g. [9]). Here $S_{\alpha\bar{\beta}}$ is the tree level mixing matrix while $T_{\alpha\bar{\beta}}$ encodes the interactions. As pointed out in references [29, 25] the one-loop dilatation operator matrix element, D_α^β , defined by $D \mathcal{O}_\alpha = D_\alpha^\beta \mathcal{O}_\beta$, can be expressed as

$$D_\alpha^\beta = (J + 2) \delta_\alpha^\beta + T_{\alpha\bar{\gamma}} (S^{-1})^{\bar{\gamma}\beta}. \quad (3)$$

The matrix elements $S_{\alpha\bar{\beta}}$ can be identified as expectation values in a zero-dimensional Gaussian complex matrix model and efficiently evaluated using matrix model techniques [7, 30]. We write:

$$S_{\alpha\bar{\beta}} = \langle \mathcal{O}_\alpha \bar{\mathcal{O}}_{\bar{\beta}} \rangle \quad (4)$$

where all fields appearing in $\mathcal{O}_\alpha(x)$ have been replaced with space-time independent fields. At the one-loop level a similar simplification can be achieved in the case of $T_{\alpha\bar{\beta}}$ by making use of an effective vertex inside the matrix model correlator which captures the sum of all contributing one-loop Feynman diagrams [7, 8]. More precisely, one has

$$T_{\alpha\bar{\beta}} = \langle \mathcal{O}_\alpha H \bar{\mathcal{O}}_{\bar{\beta}} \rangle \quad (5)$$

where

$$H = -\frac{g_{\text{YM}}^2}{8\pi^2} :(\text{Tr}[\bar{Z}, \bar{\phi}][Z, \phi] + \text{Tr}[\bar{Z}, \bar{\psi}][Z, \psi] + \text{Tr}[\bar{\phi}, \bar{\psi}][\phi, \psi]): \quad (6)$$

In the correlator, two of the legs of H connect to \mathcal{O}_α and the other two to $\bar{\mathcal{O}}_{\bar{\beta}}$. The normal ordering symbol $: :$ means that pairwise contractions between fields inside H are to be omitted. The remaining fields are contracted among each other by free matrix model correlators. The order in which the contractions are performed is irrelevant, so one may first contract the effective vertex with \mathcal{O}_α . At this point one notices that the fields which are generated after the contraction are again given by a linear combination of the operators \mathcal{O} :

$$H \circ \mathcal{O}_\alpha = H_\alpha^\gamma \mathcal{O}_\gamma. \quad (7)$$

By making use of (4) the one-loop correlator (5) is readily evaluated

$$T_{\alpha\bar{\beta}} = \langle (H \circ \mathcal{O}_\alpha) \bar{\mathcal{O}}_{\bar{\beta}} \rangle = H_\alpha^\gamma \langle \mathcal{O}_\gamma \bar{\mathcal{O}}_{\bar{\beta}} \rangle = H_\alpha^\gamma S_{\gamma\bar{\beta}}. \quad (8)$$

Comparing this to eq.(3) we find

$$D_\alpha^\beta = (J + 2) \delta_\alpha^\beta + H_\alpha^\beta. \quad (9)$$

We see that the matrix H_α^β is the one-loop part of the dilatation matrix D_α^β and conclude that the two matrices have the same eigenvectors. Thus diagonalizing the matrix H will immediately give us the anomalous dimension of the BMN operators. In particular, we will never need to know the explicit form of the tree level mixing matrix $S_{\alpha\beta}$. Furthermore, the eigenvectors $\hat{\mathcal{O}}_\alpha$ of H can be identified with the BMN operators up to normalization constants C_α . It is obvious from conformal field theory that the eigenoperators of the dilatation operator have orthogonal correlators

$$\langle \hat{\mathcal{O}}_\alpha \hat{\mathcal{O}}_{\bar{\beta}} \rangle = \delta_{\alpha\bar{\beta}} |C_\alpha|^2. \quad (10)$$

In other words they are orthogonal with respect to the inner product induced by the mixing matrix $S_{\alpha\bar{\beta}}$ at tree-level. To obtain the normalization constants C_α , however, would require the knowledge of the tree-level mixing matrix.

Let us, as described above, consider the action of the effective vertex H on the general state eq.(1); performing the Wick contractions one finds that $H := H_0 + H_+ + H_-$ with

$$\begin{aligned}
H_0 \circ \mathcal{O}_p^{J_0, J_1, \dots, J_k} &= -\frac{g_{\text{YM}}^2 N}{4\pi^2} \left[\mathcal{O}_{p+1}^{J_0, J_1, \dots, J_k} - 2 \mathcal{O}_p^{J_0, J_1, \dots, J_k} + \mathcal{O}_{p-1}^{J_0, J_1, \dots, J_k} \right], \quad (11) \\
H_+ \circ \mathcal{O}_p^{J_0, J_1, \dots, J_k} &= \frac{g_{\text{YM}}^2}{4\pi^2} \sum_{J_{k+1}=1}^{p-1} \left(\mathcal{O}_{p-J_{k+1}}^{J_0 - J_{k+1}, J_1, \dots, J_{k+1}} - \mathcal{O}_{p-1-J_{k+1}}^{J_0 - J_{k+1}, J_1, \dots, J_{k+1}} \right) \\
&\quad - \frac{g_{\text{YM}}^2}{4\pi^2} \sum_{J_{k+1}=1}^{J_0 - p - 1} \left(\mathcal{O}_{p+1}^{J_0 - J_{k+1}, J_1, \dots, J_{k+1}} - \mathcal{O}_p^{J_0 - J_{k+1}, J_1, \dots, J_{k+1}} \right), \\
H_- \circ \mathcal{O}_p^{J_0, J_1, \dots, J_k} &= \frac{g_{\text{YM}}^2}{4\pi^2} \sum_{i=1}^k J_i \left(\mathcal{O}_{J_i+p}^{J_0+J_i, J_1, \dots, \mathfrak{J}_i, \dots, J_k} - \mathcal{O}_{J_i+p-1}^{J_0+J_i, J_1, \dots, \mathfrak{J}_i, \dots, J_k} \right) \\
&\quad - \frac{g_{\text{YM}}^2}{4\pi^2} \sum_{i=1}^k J_i \left(\mathcal{O}_{p+1}^{J_0+J_i, J_1, \dots, \mathfrak{J}_i, \dots, J_k} - \mathcal{O}_p^{J_0+J_i, J_1, \dots, \mathfrak{J}_i, \dots, J_k} \right).
\end{aligned}$$

Here we have neglected boundary terms and interactions involving both impurities at the same time. In the BMN limit these will not contribute. The states $\mathcal{O}_p^{J_0, J_1, \dots, J_k}$ are not orthogonal in the gauge theory, but this is not necessary for finding the spectrum of H . We observe that the special multi-trace states $\mathcal{O}_p^{J_0, J_1, \dots, J_k}$ form a complete set as far as the action of the operator H is concerned. H_0 is trace-number conserving, and H_+ and H_- respectively increase and decrease the number of traces by one. (Clearly we have $H_- \circ \mathcal{O}_p^{J_0} = 0$). Interestingly, we see that the set of BPS type two-impurity operators $\text{Tr}(\phi Z^{J_1}) \text{Tr}(\psi Z^{J_2}) \text{Tr} Z^{J_3} \dots$ completely decouples. We can now imagine the J_i to be very large so that we can view $x := p/J$ and $r_i := J_i/J$ as continuous variables, allowing us to formulate the spectral problem directly in the BMN limit. We therefore replace the discrete states eq.(1) by a set of continuum states

$$\mathcal{O}_p^{J_0, J_1, \dots, J_k} \rightarrow |x; r_1, \dots, r_k\rangle \quad (12)$$

spanning a Hilbert space, where

$$x \in [0, r_0], \quad r_0, r_i \in [0, 1] \quad \text{and} \quad r_0 = 1 - (r_1 + \dots + r_k). \quad (13)$$

Clearly the states eq.(12) are invariant under arbitrary permutations $\pi \in S_k$ of the trace numbers: $|x; r_1, \dots, r_k\rangle = |x; r_{\pi(1)}, \dots, r_{\pi(k)}\rangle$. Now it is easy to

derive the continuum limit of eqs.(11). Defining

$$H = \frac{\lambda'}{4\pi^2} h, \quad \lambda' = \frac{g_{\text{YM}}^2 N}{J^2} \quad \text{and} \quad g_2 = \frac{J^2}{N} \quad (14)$$

the action of $h := h_0 + g_2 h_+ + g_2 h_-$ on the continuum states in eq.(12) can be written as

$$\begin{aligned} h_0 |x; r_1, \dots, r_k\rangle &= -\partial_x^2 |x; r_1, \dots, r_k\rangle, \\ h_+ |x; r_1, \dots, r_k\rangle &= \int_0^x dr_{k+1} \partial_x |x - r_{k+1}; r_1, \dots, r_{k+1}\rangle \\ &\quad - \int_0^{r_0-x} dr_{k+1} \partial_x |x; r_1, \dots, r_{k+1}\rangle, \\ h_- |x; r_1, \dots, r_k\rangle &= \sum_{i=1}^k r_i \partial_x |x + r_i; r_1, \dots, \mathfrak{X}_i, \dots, r_k\rangle \\ &\quad - \sum_{i=1}^k r_i \partial_x |x; r_1, \dots, \mathfrak{X}_i, \dots, r_k\rangle. \end{aligned} \quad (15)$$

We note that this Hamiltonian manifestly terminates at $\mathcal{O}(g_2)$. There are no contact terms.

We will now first diagonalize our states w.r.t. the trace-number conserving, free Hamiltonian h_0 , which is exact iff $g_2 = 0$. The $(k+1)$ -trace eigenstates (with n integer) are

$$|n; r_1, \dots, r_k\rangle = \frac{1}{\sqrt{r_0}} \int_0^{r_0} dx e^{2\pi i n x / r_0} |x; r_1, \dots, r_k\rangle, \quad (16)$$

they obey

$$h_0 |n; r_1, \dots, r_k\rangle = E_{|n; r_1, \dots, r_k\rangle}^{(0)} |n; r_1, \dots, r_k\rangle \quad (17)$$

with “energy” eigenvalues (i.e. anomalous dimensions)

$$E_{|n; r_1, \dots, r_k\rangle}^{(0)} = 4\pi^2 \frac{n^2}{r_0^2}. \quad (18)$$

For multi-trace states the spectrum is continuous, while single-trace states (where $k = 0$ and $r_0 = 1$), corresponding to the original BMN operators [1], have a purely discrete spectrum. Now we can proceed to evaluate the topological corrections to the energies by standard quantum mechanical perturbation theory.

Let us evaluate the action of the interaction piece $h' = h_+ + h_-$ of our Hamiltonian on the free eigenstates. From eqs.(15),(16) we find for the trace-creation and trace-annihilation operators

$$\begin{aligned}
h_+ |n; r_1, \dots, r_k\rangle &= \int_0^{r_0} dr_{k+1} \sum_{m=-\infty}^{\infty} \frac{4m \sin^2\left(\pi n \frac{r_{k+1}}{r_0}\right)}{\sqrt{r_0} \sqrt{r_0 - r_{k+1}} \left(m - n \frac{r_0 - r_{k+1}}{r_0}\right)} |m; r_1, \dots, r_{k+1}\rangle, \\
h_- |n; r_1, \dots, r_k\rangle &= \sum_{i=1}^k \sum_{m=-\infty}^{\infty} \frac{4r_i m \sin^2\left(\pi m \frac{r_i}{r_0 + r_i}\right)}{\sqrt{r_0} \sqrt{r_0 + r_i} \left(m - n \frac{r_0 + r_i}{r_0}\right)} |m; r_1, \dots, \mathfrak{r}_i, \dots, r_k\rangle. \quad (19)
\end{aligned}$$

Note that $r_0 = 1 - (r_1 + \dots + r_k)$ is defined to be the size of the first trace of the operator on the left-hand side of the equations.

The one-loop anomalous dimensions of the BMN operators equal, up to a factor of $\lambda'/4\pi^2$, the eigenvalues of the operator h :

$$h |\hat{n}; r_1, \dots, r_k\rangle = E_{|n; r_1, \dots, r_k\rangle} |\hat{n}; r_1, \dots, r_k\rangle. \quad (20)$$

As discussed above, the exact eigenstates $|\hat{n}; r_1, \dots, r_k\rangle$ correspond, up to normalization, to the diagonalized BMN operators. At $g_2 = 0$ we have already diagonalized the Hamiltonian, therefore we can proceed to evaluate the energies as an expansion in the genus counting parameter. As our perturbation h' is entirely off-diagonal, energies determined by non-degenerate perturbation theory will be given as a series in the square of the perturbation parameter g_2 in accordance with the nature of the gauge theory genus expansion

$$E_{|n; r_1, \dots, r_k\rangle} = \sum_{h=0}^{\infty} g_2^{2h} E_{|n; r_1, \dots, r_k\rangle}^{(h)}. \quad (21)$$

Correspondingly, the exact eigenstates $|\hat{n}; r_1, \dots, r_k\rangle$ are linear combinations of the bare states $|m; s_1, \dots, s_l\rangle$. The mixing coefficients are power series in g_2 . The free Hamiltonian h_0 immediately gives us the energies at the spherical level, *cf.* (17). Higher genera contributions can be obtained by quantum mechanical perturbation theory.

At this point it seems convenient to introduce a scalar product on the space of states.

$$\langle n; s_1, \dots, s_l | m; r_1, \dots, r_k \rangle = \delta_{k,l} \delta_{m,n} \sum_{\pi \in S_k} \prod_{i=1}^k \delta(s_i - r_{\pi(i)}) \quad (22)$$

This will be used only as a tool to isolate certain matrix elements of the Hamiltonian by the following representation of the unit operator

$$1 = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{0 \leq r_0, r_1, \dots, r_k \leq 1} d^k r \sum_{n=-\infty}^{\infty} |n; r_1, \dots, r_k\rangle \langle n; r_1, \dots, r_k|. \quad (23)$$

It enables us to write compact expressions for the corrections to the eigenvalues. With this scalar product our Hamiltonian is not Hermitian. The scaling dimensions, however, have to be real and it is not hard to see that they are: Letting H act on $\bar{\mathcal{O}}_{\bar{\beta}}$ instead of \mathcal{O}_{α} in (8) we get $T = SH^{\dagger}$ and thus $H = SH^{\dagger}S^{-1}$ which implies that $H = \lambda/4\pi^2 \cdot h$ has real eigenvalues. Clearly, the eigenvalues and eigenvectors of h are independent of the definition of a scalar product. Using the scalar product and eq. (19) one easily writes down the matrix elements of the interaction $h' = h_+ + h_-$ in the momentum basis, for instance

$$\begin{aligned} \langle m; r | h' | n \rangle &= -\frac{1}{(1-r)^{3/2}} \frac{4m}{n - \frac{m}{1-r}} \sin^2(\pi nr), \\ \langle n | h' | m; r \rangle &= \frac{r}{(1-r)^{1/2}} \frac{4n}{n - \frac{m}{1-r}} \sin^2(\pi nr). \end{aligned} \quad (24)$$

The corrections to the energy of a general state $|\alpha\rangle$ are then given by standard formulas of quantum mechanical perturbation theory. The first two non-vanishing orders of non-degenerate perturbation theory give

$$\begin{aligned} E_{|\alpha\rangle}^{(1)} \langle \alpha | \alpha \rangle &= \langle \alpha | h' \Delta_{|\alpha\rangle} h' | \alpha \rangle, \\ E_{|\alpha\rangle}^{(2)} \langle \alpha | \alpha \rangle &= \langle \alpha | h' \Delta_{|\alpha\rangle} h' \Delta_{|\alpha\rangle} h' \Delta_{|\alpha\rangle} h' | \alpha \rangle \\ &\quad - E_{|\alpha\rangle}^{(1)} \langle \alpha | h' (\Delta_{|\alpha\rangle})^2 h' | \alpha \rangle. \end{aligned} \quad (25)$$

Here the free propagator $\Delta_{|\alpha\rangle}$ is defined as

$$\Delta_{|\alpha\rangle} = \frac{1 - |\alpha\rangle \langle \alpha|}{E_{|\alpha\rangle}^{(0)} - h_0}. \quad (26)$$

Some technical complications potentially arise due to the fact that the free spectrum eq.(18) is degenerate as was first noticed in [10] (*cf.* discussion after eq.(31)).

We now calculate the energy shift at genus one and two. Determining the energy shift at genus one is a one-line computation

$$E_{|n\rangle}^{(1)} = \int_0^1 dr \sum_{m=-\infty}^{\infty} \langle n | h' | m; r \rangle \frac{1}{4\pi^2 \left(n^2 - \frac{m^2}{(1-r)^2} \right)} \langle m; r | h' | n \rangle = \frac{1}{12} + \frac{35}{32 \pi^2 n^2} \quad (27)$$

and implies a correction to the anomalous dimension in agreement with the result of references [9,10]. This computation was originally performed in [25], however, it relied on the assumption that there are no $\mathcal{O}(g_2^2)$ contact-terms in h ; this fact is manifest in our formalism.

Having convinced ourselves of the simplicity of this method we proceed to genus two, *cf.* eq.(25). Using (24) one readily computes

$$\langle n|h'(\Delta_{|n})^2 h'|n\rangle = -\frac{1}{120} + \frac{1}{32\pi^2 n^2} - \frac{21}{256\pi^4 n^4}. \quad (28)$$

The computation of the first term of (25) splits into two parts, namely a contribution from a single-trace channel and a contribution from a triple-trace channel (corresponding to the number of traces of the BMN operators changing as 1-2-1-2-1 and 1-2-3-2-1 respectively).

For the evaluation of the contribution from the single-trace channel term one needs the matrix elements:

$$\begin{aligned} \langle l|h'\Delta_{|n} h'|n\rangle &= \frac{l}{n} \langle n|h'\Delta_{|n} h'|l\rangle, \\ \langle n|h'\Delta_{|n} h'|l\rangle &= \frac{n}{6(n-l)} + \frac{l^6 - l^4 n^2 + 6l^2 n^4 - 2n^6}{4\pi^2 n(l-n)^3 l^2 (n+l)^2}. \end{aligned} \quad (29)$$

Using these yields

$$E_{|n}^{(2),\text{single}} = -\frac{1}{2688} + \frac{107}{9216\pi^2 n^2} - \frac{695}{12288\pi^4 n^4} - \frac{3785}{8192\pi^6 n^6}. \quad (30)$$

Turning to the final computation of the triple-trace channel contribution we note the intermediate formulas

$$\begin{aligned} \langle l; s_1, s_2|h'\Delta_{|n} h'|n\rangle &= \frac{l}{n(1-s_1-s_2)s_1 s_2} \langle n|h'\Delta_{|n} h'|l; s_1, s_2\rangle, \\ \langle n|h'\Delta_{|n} h'|l; s_1, s_2\rangle &= -\frac{8}{\pi^2} \frac{s_1 s_2 (1-s_2)^3}{\sqrt{1-s_1-s_2}} n \sin^2(\pi n s_2) \left(n + \frac{l}{1-s_1-s_2} \right) \\ &\quad \times \sum_{m=1}^{\infty} \frac{m^2 \sin^2(\pi m \frac{s_1}{1-s_2})}{\left(m^2 - n^2(1-s_2)^2 \right)^2 \left(m^2 - l^2 \left(\frac{1-s_2}{1-s_1-s_2} \right)^2 \right)} \\ &\quad + (s_1 \leftrightarrow s_2). \end{aligned} \quad (31)$$

The single-trace state $|n\rangle$ and the triple-trace state $|l; s_1, s_2\rangle$ are degenerate if $n = \pm \frac{l}{(1-s_1-s_2)}$ which can happen only for $n > 1$. The matrix element (31) is easily seen to vanish for $n = -\frac{l}{1-s_1-s_2}$. However for $n = \frac{l}{1-s_1-s_2}$ it is finite. For $n > 1$ this observation casts some doubt on the use of non-degenerate

perturbation theory; however, for $n = 1$ there clearly is no problem [10]. The above sum can be performed with the help of the formula

$$\sum_{k=1}^{\infty} \frac{\sin^2(\pi k x)}{k^2 - b^2} = \frac{\pi}{2b} \frac{\sin(\pi b x) \sin(\pi b(1-x))}{\sin(\pi b)} \quad (32)$$

for $0 \leq x \leq 1$ and b non-integer. After some algebra using *Mathematica* one obtains

$$\begin{aligned} E_{|n\rangle}^{(2),\text{triple}} &= \frac{1}{2} \sum_{l=-\infty}^{\infty} \int_{0 \leq s_1 + s_2 \leq 1} ds_1 ds_2 \frac{\langle n | h' \Delta_{|n}\rangle h' | l; s_1, s_2 \rangle \langle l; s_1, s_2 | h' \Delta_{|n}\rangle h' | n \rangle}{4\pi^2 \left(n^2 - \frac{l^2}{(1-s_1-s_2)^2} \right)} \\ &= -\frac{13}{40320} - \frac{47}{2560 \pi^2 n^2} + \frac{97}{768 \pi^4 n^4} + \frac{385}{16384 \pi^6 n^6} + \int_0^1 ds_0 f_n(s_0). \end{aligned} \quad (33)$$

The function that remains to be integrated is

$$f_n(s_0) = -\frac{(1-s_0)^2 s_0 (15 + 4\pi^2 s_0^2) \cot(\pi n s_0)}{256 \pi^3 n^3}. \quad (34)$$

For $n = 1$ the integral is finite and easily evaluated to be

$$\int_0^1 dr_0 f_1(r_0) = -\frac{\zeta(3)}{128 \pi^4} - \frac{45 \zeta(3)}{512 \pi^6} + \frac{15 \zeta(5)}{128 \pi^6} \quad (35)$$

For $n > 1$ there are poles at $s_0 = \frac{m}{n}$, $1 \leq m \leq n-1$. These are related to the above discussed degeneracy of the single-trace state $|n\rangle$ with the triple-trace states $|m; s_1, s_2\rangle$ where $s_0 = 1 - s_1 - s_2 = \frac{m}{n}$. Deferring this problem to future work, we proceed by regulating the integral by a principle value prescription. We find the same result as eq.(35) except for the replacement $\pi \rightarrow \pi n$. In total we get for the one-loop two-torus contribution

$$\begin{aligned} E_{|n\rangle}^{(2)} &= -\frac{11}{46080} \frac{1}{\pi^2 n^2} + \left(\frac{521}{12288} - \frac{\zeta(3)}{128} \right) \frac{1}{\pi^4 n^4} \\ &\quad + \left(-\frac{5715}{16384} - \frac{45 \zeta(3)}{512} + \frac{15 \zeta(5)}{128} \right) \frac{1}{\pi^6 n^6}. \end{aligned} \quad (36)$$

Our derivation of $E_{|n\rangle}^{(2)}$ is rigorous only for $n = 1$; it would be important to more carefully examine the validity of this formula for $n > 1$.

As we discussed in the beginning, the various terms *do not cancel*, and thus $E_{|n\rangle}^{(2)} \neq 0$, disproving the idea that the genus counting parameter g_2^2 should always be accompanied by a factor of λ' . However, interestingly, the

n -independent, constant terms do cancel. This suggests, together with the result of the sphere and the torus, that the exact energy might scale as

$$E_{|n\rangle} \sim 4\pi^2 n^2 \sum_{h=0}^{\infty} c_h \left(\frac{g_2^2}{4\pi^2 n^2} \right)^h \quad \text{for } n \rightarrow \infty, \quad (37)$$

where the c_h are numerical constants: $c_0 = 1, c_1 = \frac{1}{12}, c_2 = -\frac{11}{6!2^4}, \dots$

We can also find the scaling dimensions of multi-trace operators from our formalism. The result (38) was first derived, using different methods, in [31]. Some extra care has to be taken due to the continuous spectrum of multi-trace operators, *cf.* eq.(18). In particular, multi-trace operators are delta-function normalized, see eq.(22). A straightforward repetition of perturbation theory, as in eq.(25), for multi-trace operators reveals that the relevant contribution, proportional to the normalizing delta function, results solely from the disconnected channels. The latter are defined as the contributions where the external traces without impurities do not participate in the interaction. Assuming that no subtleties arise from divergences of connected channels (as we have explicitly verified for genus one) we find

$$E_{|n;r_1, \dots, r_k\rangle}^{(h)} = r_0^{4h-2} E_{|n\rangle}^{(h)}. \quad (38)$$

It would be important to more carefully investigate this issue in the framework of our formalism.

Clearly it should be interesting to extract further information from our Hamiltonian formulation. In particular, it would be exciting to solve the eigenvalue problem in the WKB limit, as this might lead to non-perturbative insights into the genus expansion, and, in consequence, to non-perturbative results on plane-wave strings, *cf.* eq.(37). It will also be very interesting to understand the terms one needs to add to the Hamiltonian in order to include the effects of radiative corrections beyond one loop, and of more than two impurities. We do not see any reason why this should not be possible, and therefore conjecture that BMN gauge theory can quite generally be reformulated as a quantum mechanical system.

Many of the above methods and insights are already present in the more difficult case of finite J . The novel ‘‘BMN’’ way of looking at $\mathcal{N} = 4$ gauge theory is beginning to lead to startling progress beyond the admittedly somewhat artificial limit of infinitely large R -charges. In particular, in [27] it was shown that finite J versions of BMN operators can be rigorously defined, and should be considered to be generalizations of the simplest unprotected field in $\mathcal{N} = 4$, the so-called Konishi scalar. They are the ‘‘next to best thing to BPS operators’’, as they are, in a rather precise sense, ‘‘almost protected’’. This discovery has led to a fresh look at the representation theory of $\mathcal{N} = 4$

operators, and to explicit results for anomalous dimensions of whole families of operators [27]. In this context we should mention a very interesting, recent paper by Minahan and Zarembo [32]. In a spirit conceptually very close to the present work they are considering the planar, one-loop diagonalization of more complicated operators with many impurities, uncovering some of the structures (e.g. Bethe-Ansatz) found in integrable spin chains. One could therefore hope that currently known results are just the beginning of the discovery of an integrable sector in $\mathcal{N} = 4$ gauge theory. As is frequently the case for two-dimensional integrable systems, such structures might exist both on the lattice (here: finite J) and in the continuum limit (here: $J \rightarrow \infty$).

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