

Regularization parameters for the self-force in Schwarzschild spacetime. II. Gravitational and electromagnetic cases

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We obtain all “regularization parameters” (RPs) needed for calculating the gravitational and electromagnetic self-forces for an arbitrary geodesic orbit around a Schwarzschild black hole. These RP values are required for implementing the previously introduced mode-sum method, which allows a practical calculation of the self-force by summing over contributions from individual multipole modes of the particle’s field. In the gravitational case, we provide here full details of the analytic method and results briefly reported in a recent Letter [Phys. Rev. Lett. **88**, 091101 (2002)]. In the electromagnetic case, the RPs are obtained here for the first time.

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I. INTRODUCTION

This is the second in a series of papers aimed at establishing a practical calculation scheme for the self-force acting on a point particle in orbit around a black hole. This scheme—referred to as the “mode-sum method”—stems from the general regularization prescription of Mino, Sasaki, and Tanaka (MST) [1] and Quinn and Wald (QW) [2]. In effect, the mode-sum method reformulates the MST-QW general result in the language of multipole modes, thereby making it accessible to standard numerical treatment. In practice, the application of this method involves two basic parts: (i) calculation of the “full” modes of the force, through numerical integration of the decoupled field equations, and (ii) analytical derivation of certain parameters (whose values depend on the orbit under consideration) called the “regularization parameters” (RPs). Previously, the explicit values of the RPs were derived analytically in a few special cases of orbits in Schwarzschild spacetime—specifically, for circular and radial orbits in the scalar field case [3] and for radial trajectories in the gravitational case [4,5]. In these works, the RP values were calculated through a rather cumbersome local expansion of the (l -multipole) Green’s function. The application of this technique to more general orbits appears a challenging task.

In a recent Letter [6], the joint groups of Barack and Ori (BO) and Mino, Nakano, and Sasaki (MNS) devised an alternative, more direct method for obtaining the RPs. The new method is based on a multipole decomposition of the explicit “direct” part of the force (see below). Using this method, BO and MNS were able to calculate the explicit RP values for both the scalar and gravitational self-forces, for any geodesic orbit in Schwarzschild spacetime. In a preceding paper [7] (hereafter referred to as “paper I”) BO described the full details of the new calculation technique, as applied to the toy model of the *scalar* self-force acting on a scalar charge. The current paper deals with the more interesting case of the *gravitational* self-force on a mass particle and provides full details of the RPs derivation in this case. In addition, we

shall derive here the RP values for the *electromagnetic* self-force acting on an electrically charged particle orbiting a Schwarzschild black hole. (BO and MNS applied two slightly different methods in obtaining the RPs; MNS describe their calculation in [8].)

The analysis presented in this paper relies greatly on the technique and results of paper I, to which we shall frequently refer the reader. Though the basic idea of our calculation is the same as in the scalar toy model of paper I, some unavoidable technical complexities arise when coming to consider the gravitational or electromagnetic cases. In particular, one then has to consider an extension of the particle’s four-velocity (which takes part in constructing the “direct force”—see below) off the world line and address the question of the RPs dependence on the (nonunique) choice of such extension. Another, more fundamental issue, is the gauge dependence of the gravitational self-force and its implication to the mode-sum scheme (see Ref. [9]).

Most of our paper will be concerned with the (most interesting) gravitational case. Our analytical technique is easily applicable to the (simpler) electromagnetic case, which we shall later consider in a separate section. The structure of this paper is as follows: In the rest of this introductory section we set up the physical scenario of a pointlike mass particle orbiting a Schwarzschild black hole, introduce the MST-QW prescription for calculating the gravitational self-force on this particle, and review the basics of the mode-sum method. In Sec. II we present the MST expression for the direct part of the gravitational self-force and analytically process this expression to extract information relevant for deriving the RPs. Section III contains the heart of our calculation: namely, the derivation of all RPs for any geodesic orbit, in the gravitational case. Section IV deals with the electromagnetic case. In Sec. V we summarize the prescription for calculating the gravitational and electromagnetic self-forces via the mode-sum method and give some concluding remarks.

Throughout this paper we use geometrized units (with $G = c = 1$), metric signature $-+++$, and the standard Schwarzschild coordinates t, r, θ, φ .

A. Pointlike particle model

We consider a pointlike particle of mass μ , moving freely in the vacuum exterior of a Schwarzschild black hole with mass $M \gg \mu$. (QW [2] discuss the extent to which the concept of a pointlike particle makes sense in the context of the radiation reaction problem.) In the limit $\mu \rightarrow 0$, the particle traces a geodesic $z^\mu(\tau)$ of the Schwarzschild background. As a result of angular momentum conservation, the geodesic orbit (as well as the orbit under the self-force effect) is confined to a plane, which, without loss of generality, we shall take as the equatorial plane, $\theta = \pi/2$.

When the mass μ is finite, the particle no longer moves on a geodesic. In this case, it is useful to view the particle as being subject to a self-force induced by its own gravitational field (treated as a perturbation over the background geometry).¹ The particle's equation of motion thus takes the form

$$\mu u^\alpha{}_{;\beta} u^\beta = F_{\text{self}}^\alpha, \quad (1)$$

where $u^\alpha \equiv dz^\alpha/d\tau$, a semicolon denotes covariant differentiation with respect to the background geometry, and $F_{\text{self}}^\alpha \propto O(\mu^2)$ describes the leading-order self-force effect. In the following we shall consider the self-force acting on the particle at an arbitrary point along its world line, denoted by $z = z_0 \equiv (t_0, r_0, \theta_0, \varphi_0)$ (in our setup $\theta_0 = \pi/2$). We shall use the notation $x \equiv (t, r, \theta, \varphi)$ to represent a point in the neighborhood of z_0 .

We will denote the metric of the perturbed spacetime as $g_{\alpha\beta} + h_{\alpha\beta}$, where $g_{\alpha\beta}$ is the (Schwarzschild) background metric and $h_{\alpha\beta}(\propto \mu)$ is the metric perturbation induced by the particle. Following MST-QW, we consider the metric perturbation $h_{\alpha\beta}$ specifically in the harmonic gauge. We shall denote by $\bar{h}_{\alpha\beta}$ the trace-reversed perturbation:

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}. \quad (2)$$

B. Gravitational self-force according to MST-QW

MST and QW found that the gravitational self-force on a particle freely moving in a vacuum spacetime can be formally constructed as [10]

$$F_{\text{self}}^\alpha = \lim_{x \rightarrow z_0} F_{\text{tail}}^\alpha(x), \quad (3)$$

where F_{tail}^α is the ‘‘tail’’ force, associated with the mere effect of waves scattered *inside* (rather than propagating along) the particle's past light cone. The tail force can be derived from the ‘‘tail’’ part of the metric perturbation, as defined by MST [1], through

¹One might attempt an alternative point of view, which regards the particle as freely moving in a ‘‘perturbed spacetime.’’ This picture, though, is somewhat problematic, as the perturbed spacetime is singular at the point-particle limit and, hence, is not strictly defined at the particle's location. See, however, Ref. [10].

$$F_{\text{tail}}^\alpha(x) = \mu k^{\alpha\beta\gamma\delta}(x) \bar{h}_{\beta\gamma;\delta}^{\text{tail}}(x). \quad (4)$$

Here, the tensor $k^{\alpha\beta\gamma\delta}(x)$ is any (sufficiently regular) extension of the tensor

$$k_0^{\alpha\beta\gamma\delta} = \frac{1}{2} g^{\alpha\delta} u^\beta u^\gamma - g^{\alpha\beta} u^\gamma u^\delta - \frac{1}{2} u^\alpha u^\beta u^\gamma u^\delta + \frac{1}{4} u^\alpha g^{\beta\gamma} u^\delta + \frac{1}{4} g^{\alpha\delta} g^{\beta\gamma}, \quad (5)$$

defined at $x = z_0$, where u^α and $g^{\alpha\delta}$ refer to the values of the four-velocity and the metric tensor at z_0 . [Later we shall employ a specific extension of $k_0^{\alpha\beta\gamma\delta}$; note that the choice of extension does not affect the physical self-force F_{self}^α , though, obviously, it does affect the field $F_{\text{tail}}^\alpha(x)$ off the world line.]

The (singular) difference between the ‘‘full’’ perturbation $\bar{h}_{\alpha\beta}(x)$ and the tail part $\bar{h}_{\alpha\beta}^{\text{tail}}(x)$ is associated with the instantaneous effect of waves propagating directly along the particle's light cone. This part is referred to as the ‘‘direct’’ perturbation:

$$\bar{h}_{\alpha\beta}^{\text{dir}}(x) \equiv \bar{h}_{\alpha\beta}(x) - \bar{h}_{\alpha\beta}^{\text{tail}}(x). \quad (6)$$

Correspondingly, we define the ‘‘direct’’ force as

$$F_{\text{dir}}^\alpha(x) \equiv \mu k^{\alpha\beta\gamma\delta}(x) \bar{h}_{\beta\gamma;\delta}^{\text{dir}}(x). \quad (7)$$

Defining also the ‘‘full’’ force

$$F_{\text{full}}^\alpha(x) \equiv \mu k^{\alpha\beta\gamma\delta}(x) \bar{h}_{\beta\gamma;\delta}(x), \quad (8)$$

we then have

$$F_{\text{tail}}^\alpha(x) = F_{\text{full}}^\alpha(x) - F_{\text{dir}}^\alpha(x). \quad (9)$$

The explicit form of the direct perturbation has been derived by MST [1] (see also [11,8]). It is given below in Eq. (13) and serves as the starting point for our analysis.

Note that both the direct force and full force, which were defined above as vector fields in the neighborhood of z_0 , diverge as $x \rightarrow z_0$. However, their difference, yielding the tail force, admits a perfectly regular limit $x \rightarrow z_0$, which, according to MST-QW, represents the physical self-force. In this respect, notice also the freedom one has in choosing the extension $k^{\alpha\beta\gamma\delta}(x)$, as long as this extension is regular enough and reduces to $k_0^{\alpha\beta\gamma\delta}$ in the limit $x \rightarrow z_0$. One has to make sure, though, that *the same extension* $k^{\alpha\beta\gamma\delta}(x)$ is applied to both the direct and full forces.

C. Mode-sum method

The mode-sum method was previously introduced [3,4] as a practical method for calculating the MST-QW self-force given in Eq. (3) [12,13]. The method is reviewed in paper I; here, we merely describe the basic prescription (as applied to the gravitational case) and introduce the relevant notation.

In the mode-sum scheme, one first formally expands the gravitational tail force, as well as the full and direct forces,

into multipole l modes, in the form

$$F_{\text{tail,full,dir}}^\alpha(x) = \sum_{l=0}^{\infty} F_{\text{tail,full,dir}}^{\alpha l}(x). \quad (10)$$

Here, precisely as in the scalar case, the modes $F_{\text{tail}}^{\alpha l}$, $F_{\text{full}}^{\alpha l}$, and $F_{\text{dir}}^{\alpha l}$ are obtained by decomposing (each of the vectorial components of) the corresponding quantities F_{tail}^α , F_{full}^α , and F_{dir}^α into standard scalar spherical harmonics and then, for any given multipole number l , summing over all azimuthal numbers m . It is important to emphasize here that the various l modes introduced in Eq. (10) are defined in our scheme through a *scalar* harmonic decomposition. In this regard, recall that the (full) metric perturbation in Schwarzschild spacetime is usually decomposed into *tensor* harmonic modes in actual calculations. The construction of the full-force scalar-harmonic modes $F_{\text{full}}^{\alpha l}$ from the full perturbation tensor-harmonic modes can be prescribed in a straightforward manner (as, e.g., in [5,14]).

The basic prescription for constructing the gravitational self-force via the mode-sum scheme is given by [3,4]

$$F_{\text{self}}^\alpha = \sum_{l=0}^{\infty} \left[\lim_{x \rightarrow z_0} F_{\text{full}}^{\alpha l}(x) - A^\alpha L - B^\alpha - C^\alpha/L \right] - D^\alpha, \quad (11)$$

where $L \equiv l + 1/2$ and the (l -independent) coefficients A^α , B^α , C^α , and D^α are the *regularization parameters*. The RPs A^α , B^α , and C^α may be defined by the demand that the sum in Eq. (11) converge. Equivalently (and more practically), one may define these parameters by requiring convergence of the sum

$$\sum_{l=0}^{\infty} \left[\lim_{x \rightarrow z_0} F_{\text{dir}}^{\alpha l}(x) - A^\alpha L - B^\alpha - C^\alpha/L \right] \equiv D^\alpha. \quad (12)$$

This sum then defines the fourth parameter D^α . From the above definitions it is clear that the RPs values may be derived through analysis of the direct-force modes $F_{\text{dir}}^{\alpha l}(x)$.

Equation (11) constitutes a practical prescription for constructing the gravitational self-force, given (i) the values of all necessary RPs, and (ii) the full-force modes $F_{\text{full}}^{\alpha l}$. In this paper we derive all RPs for any (equatorial) geodesic orbit in Schwarzschild spacetime, hence setting an analytical basis for calculations of the gravitational self-force for all such orbits.

II. ANALYZING THE DIRECT GRAVITATIONAL FORCE

A. Direct part of the metric perturbation

The direct part of the trace-reversed metric perturbation was obtained by MST—see Eq. (2.27) of Ref. [1]. In the Appendix we process the expression obtained by MST and bring it to the form

$$\bar{h}_{\beta\gamma}^{\text{dir}}(x) = 4\mu\epsilon^{-1}\hat{u}_\beta^1\hat{u}_\gamma^1 + \epsilon^{-1}P_{\beta\gamma}^{(2)}(x, z_0). \quad (13)$$

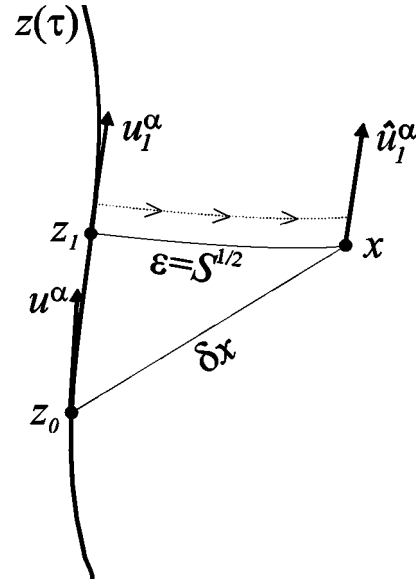


FIG. 1. Geometric setup and notation: The self-force is calculated at the point $z = z_0$ on the geodesic $z(\tau)$. x is an off-world-line point in the neighborhood of z_0 , and $\delta x \equiv x - z_0$. $\epsilon \equiv S^{1/2}$ denotes the length of the short geodesic section connecting x to the world line and normal to it, and z_1 is the point where this short geodesic intersects the world line. u^α and u_1^α denote the four-velocities at z_0 and z_1 , respectively. \hat{u}^α and \hat{u}_1^α (the former not shown in the sketch) are vectors at x , generated by parallelly propagating the vectors u^α and u_1^α along the short geodesic section from z_0 or z_1 , respectively, to x .

Here, ϵ is the spatial geodesic distance from the point x to the geodesic $z(\tau)$ (i.e., the length of the short geodesic connecting x to the world line and normal to it), z_1 denotes the intersection of this short normal geodesic with the world line, and \hat{u}_1^α is the four-velocity parallelly propagated (PP) from z_1 to x . (See Fig. 1 for an illustration of the geometric setup described here.) The function $P_{\beta\gamma}^{(2)}$ is a regular function of x , of order δx^2 (and higher orders), where $\delta x^\mu \equiv x^\mu - z_0^\mu$. The explicit form of $P_{\beta\gamma}^{(2)}$ will not be needed in our analysis.

For later convenience, we first reexpress $\bar{h}_{\beta\gamma}^{\text{dir}}$ in terms of the four-velocity PP from z_0 to x (rather than from z_1 to x), which we denote by \hat{u}_α (or \hat{u}^α). Both \hat{u}_α and \hat{u}_1^α are regular functions of x , and the difference between them is proportional to δx^2 (and to the Riemann tensor). Absorbing this difference in the function $P_{\beta\gamma}^{(2)}$, we may rewrite the direct metric perturbation as

$$\bar{h}_{\beta\gamma}^{\text{dir}}(x) = 4\mu S^{-1/2}\hat{u}_\beta\hat{u}_\gamma + S^{-1/2}\bar{P}_{\beta\gamma}^{(2)}(x, z_0), \quad (14)$$

where $S \equiv \epsilon^2$, and the new function $\bar{P}_{\beta\gamma}^{(2)}$ has the same features as $P_{\beta\gamma}^{(2)}$; namely, it is a regular function, of order δx^2 .

B. Extending the tensor $k^{\alpha\beta\gamma\delta}$ off the world line

Given the above expression for the direct perturbation, the direct force is constructed as a vector field through Eq. (7). In this equation, recall, $k^{\alpha\beta\gamma\delta}(x)$ is an extension off the point

z_0 of the tensor $k_0^{\alpha\beta\gamma\delta}$ defined at z_0 in terms of $u^\alpha(z_0)$ and $g^{\alpha\beta}(z_0)$ [see Eq. (5)]. In our analysis we decompose the components of the field $F_{\text{dir}}^\alpha(x)$ in spherical harmonics. Since this decomposition is nonlocal (it involves an integration over the two-sphere $r, t = \text{const}$), it will generally depend on the extension of $k_0^{\alpha\beta\gamma\delta}$, which we now have to specify.

A natural extension of $k_0^{\alpha\beta\gamma\delta}$ was prescribed by MST [1] (also MNS [8]) by setting in the right-hand side of Eq. (5), $u^\alpha \rightarrow \hat{u}_1^\alpha(x)$ and $g^{\alpha\beta} \rightarrow g^{\alpha\beta}(x)$ —namely, by PP the four-velocity u^α from z_1 to x and assigning to the metric function its actual value at x . However, for our analysis, we found it useful to apply a different extension: one in which all (contravariant) tensorial components of $k^{\alpha\beta\gamma\delta}(x)$ are assigned fixed values—the same values they have at $x \rightarrow z_0$:

$$k^{\alpha\beta\gamma\delta}(x) \equiv k_0^{\alpha\beta\gamma\delta}. \quad (15)$$

Note that this definition is coordinate dependent; here, we refer to (contravariant components in) the *Schwarzschild coordinates*. Throughout the rest of this paper, $k^{\alpha\beta\gamma\delta}(x)$ will denote specifically the extension defined in Eq. (15), to which we shall refer as the “fixed-components” extension. This extension turns out to be most convenient for the numerical determination of the modes of the full force (recall that the same choice of extension must be made for both the direct and full forces).

C. Constructing the direct force

To analyze the direct force $F_{\text{dir}}^\alpha(x) = \mu k^{\alpha\beta\gamma\delta} \bar{h}_{\beta\gamma,\delta}^{\text{dir}}$, we first use Eq. (14) to obtain

$$\begin{aligned} \bar{h}_{\beta\gamma,\delta}^{\text{dir}} = & -2\mu\epsilon^{-3} S_{,\delta} \hat{u}_\beta \hat{u}_\gamma + 4\mu\epsilon^{-1} (\hat{u}_{\beta,\delta} \hat{u}_\gamma + \hat{u}_\beta \hat{u}_{\gamma,\delta}) \\ & - \epsilon^{-3} S_{,\delta} \tilde{P}_{\beta\gamma}^{(2)}/2 + \epsilon^{-1} P_{\beta\gamma\delta}^{(1)}, \end{aligned} \quad (16)$$

where $P_{\beta\gamma\delta}^{(1)} \equiv \tilde{P}_{\beta\gamma,\delta}^{(2)}$ is a regular function, of order δx (and higher orders). Since \hat{u}_α is PP (from z_0 to x), its covariant derivatives are proportional to δx . Therefore, the second term in the above expression may be absorbed in the fourth term: this merely amounts to modifying the explicit form of $P_{\beta\gamma\delta}^{(1)}$. Considering next the third, the ϵ^{-3} term in Eq. (16), and recalling $S_{,\delta} \propto O(\delta x)$, we write this term in the form $\epsilon^{-3} P_{\beta\gamma\delta}^{(3)}$, where $P_{\beta\gamma\delta}^{(3)}$ is a regular quantity of $O(\delta x^3)$ (and higher orders). Absorbing then the term $\epsilon^{-1} P_{\beta\gamma\delta}^{(1)}$ = $\epsilon^{-3} (\epsilon^2 P_{\beta\gamma\delta}^{(1)})$ in the term $\epsilon^{-3} P_{\beta\gamma\delta}^{(3)}$ (which amounts to redefining $P_{\beta\gamma\delta}^{(3)}$), we finally write $\bar{h}_{\beta\gamma,\delta}^{\text{dir}}$ as

$$\bar{h}_{\beta\gamma,\delta}^{\text{dir}} = -2\mu\epsilon^{-3} S_{,\delta} \hat{u}_\beta \hat{u}_\gamma + \epsilon^{-3} P_{\beta\gamma\delta}^{(3)}. \quad (17)$$

Consequently, the direct force takes the form

$$F_{\text{dir}}^\alpha(x) = \mu^2 \left(-\frac{1}{2} K^{\alpha\delta} \epsilon^{-3} S_{,\delta} + \epsilon^{-3} P_{(3)}^\alpha \right), \quad (18)$$

where

$$K^{\alpha\delta} \equiv 4k^{\alpha\beta\gamma\delta} \hat{u}_\beta \hat{u}_\gamma \quad (19)$$

and $\mu P_{(3)}^\alpha \equiv k^{\alpha\beta\gamma\delta} P_{\beta\gamma\delta}^{(3)}$ (the factor of 4 is introduced for later convenience). Note that the quantity $P_{(3)}^\alpha$, which is regular at $x = z_0$, generally contains also terms of order δx^4 and higher. However, the contribution from such higher-order terms to F_{dir}^α vanishes at $\delta x = 0$, and so these terms may be ignored in our analysis. We shall indeed drop these higher-order terms and take $P_{(3)}^\alpha$ to be a polynomial in δx of *homogeneous* order δx^3 .

The coefficients of the tensor $K^{\alpha\delta}$ are not constant, as the field \hat{u}_β is a PP field and not a field of constant components. It will prove convenient to expand $K^{\alpha\delta}$ in δx and express it as

$$K^{\alpha\delta} = K_0^{\alpha\delta} + K_1^{\alpha\delta} + K_2^{\alpha\delta} + \dots, \quad (20)$$

where $K_0^{\alpha\delta} [= K^{\alpha\delta}(x \rightarrow z_0)]$ is a field of constant components, $K_1^{\alpha\delta}$ is proportional to δx , and so on. (Note that the terms $K_n^{\alpha\delta}$ —unlike $K_0^{\alpha\delta}$ —depend on the extension.) Considering now the first term in the expression for the direct force, Eq. (18), and recalling $S_{,\delta} \propto \delta x$, we observe that the contribution from the term $K_3^{\alpha\delta}$ and higher-order terms of $K^{\alpha\delta}$ to the direct force vanishes at $x \rightarrow z_0$. We hence drop these terms. In addition, we observe that the term $K_2^{\alpha\delta} \epsilon^{-3} S_{,\delta}$ may be absorbed in the term $\epsilon^{-3} P_{(3)}^\alpha$ of the direct force, which merely amounts to redefining $P_{(3)}^\alpha$. Thus, the direct force takes the form

$$F_{\text{dir}}^\alpha = \mu^2 \left(\frac{1}{2} K_0^{\alpha\delta} \epsilon^{-3} S_{,\delta} + \frac{1}{2} K_1^{\alpha\delta} \epsilon^{-3} S_{,\delta} + \epsilon^{-3} P_{(3)}^\alpha \right). \quad (21)$$

As in paper I, we now expand S in powers of δx , in the form

$$S = S_0 + S_1 + S_2 + \dots, \quad (22)$$

where S_0 is the leading order ($\propto \delta x^2$) term of S , S_1 is the correction term of homogeneous order δx^3 , and so on. In this work we will need only the explicit form of S_0 :

$$S_0 = (g_{\mu\nu} + u_\mu u_\nu) \delta x^\mu \delta x^\nu. \quad (23)$$

The factor ϵ^{-3} appearing in the last expression for the direct force, Eq. (21), is then expanded as

$$\begin{aligned} \epsilon^{-3} = & S^{-3/2} \\ = & S_0^{-3/2} - \frac{3}{2} S_0^{-5/2} S_1 + \left(\frac{15}{8} S_0^{-7/2} S_1^2 - \frac{3}{2} S_0^{-5/2} S_2 \right) + \dots \\ = & \epsilon_0^{-3} - \frac{3}{2} \epsilon_0^{-5} S_1 + \epsilon_0^{-7} \left(\frac{15}{8} S_1^2 - \frac{3}{2} \epsilon_0^2 S_2 \right) + \dots, \end{aligned} \quad (24)$$

where $\epsilon_0 \equiv S_0^{1/2}$. In this expansion, the first term scales as δx^{-3} , the second as δx^{-2} , and so on. The terms included in the ellipses scale as δx^0 or higher powers of δx .

Next we expand the direct force in powers of δx , using the above expansions of S and ϵ^{-3} . Based on Eq. (21), this expansion takes the form

$$F_{\text{dir}}^\alpha = \mu^2(\epsilon_0^{-3}P_{(1)}^\alpha + \epsilon_0^{-5}P_{(4)}^\alpha + \epsilon_0^{-7}P_{(7)}^\alpha), \quad (25)$$

in which $P_{(n)}^\alpha$ denote polynomials of homogeneous order n in δx and where we have omitted higher-order terms that vanish at $x \rightarrow z_0$. Notice that the term $\epsilon^{-3}P_{(3)}^\alpha$ of Eq. (21) has been absorbed here in the term $\epsilon_0^{-7}P_{(7)}^\alpha$ (with higher-order corrections that vanish at $x \rightarrow z_0$ and are thus omitted). Also absorbed in $\epsilon_0^{-7}P_{(7)}^\alpha$ are other terms like $K_0^{\alpha\delta}\epsilon_0^{-3}S_{2,\delta}$, $K_0^{\alpha\delta}\epsilon_0^{-5}S_1S_{1,\delta}$, $K_1^{\alpha\delta}\epsilon_0^{-3}S_{1,\delta}$, etc. The functions $P_{(1)}^\alpha$ and $P_{(4)}^\alpha$ are given explicitly by

$$P_{(1)}^\alpha = -\frac{1}{2}K_0^{\alpha\delta}S_{0,\delta}, \quad (26a)$$

$$P_{(4)}^\alpha = -\frac{1}{2}K_0^{\alpha\delta}S_0S_{1,\delta} + \frac{3}{4}K_0^{\alpha\delta}S_1S_{0,\delta} - \frac{1}{2}K_1^{\alpha\delta}S_0S_{0,\delta} \quad (26b)$$

(the explicit form of $P_{(7)}^\alpha$ will not be needed). Note that the leading-order term of the direct force, $\mu^2\epsilon_0^{-3}P_{(1)}^\alpha$, emerges exclusively from the leading-order term $\propto K_0^{\alpha\delta}$ in Eq. (21), whereas the next-order term $\mu^2\epsilon_0^{-5}P_{(4)}^\alpha$ is composed of contributions coming from both terms $\propto K_0^{\alpha\delta}$ and $\propto K_1^{\alpha\delta}$. The $\propto K_0^{\alpha\delta}$ contributions (and thus the entire leading-order term) are all analogous to ones that occur in the scalar model [see Eq. (23) of paper I], whereas the $\propto K_1^{\alpha\delta}$ contribution has no counterpart in the scalar case considered therein. We also point out that, since $K_0^{\alpha\delta}$ does not depend on the extension of $k^{\alpha\beta\gamma\delta}$ (unlike $K_{n>0}^{\alpha\delta}$), one finds that the leading-order term of F_{dir}^α is extension independent, whereas the explicit form of the higher-order terms does depend, in general, on the choice of extension.

III. DERIVATION OF THE REGULARIZATION PARAMETERS: GRAVITATIONAL CASE

In principle, the derivation of the RPs will now involve expanding the direct-force components F_{dir}^α in scalar spherical harmonics and then taking the limit $x \rightarrow z_0$ [just as in the scalar case analysis—cf. Eq. (27) in paper I]. This will yield the l -mode contribution to the direct force, $F_{\text{dir}}^{\alpha l}$, from which one may deduce the values of all RPs. However, at this point we may exploit the remarkable analogy between the expression derived here for the gravitational direct force, Eq. (25), and the corresponding expression obtained in the scalar toy model [see Eq. (22) of paper I]: these expressions differ only in the explicit form of the three coefficients $P_{(1,4,7)}^\alpha$. Conveniently, this analogy will now allow us to base most of our analysis on results already obtained in paper I.

We begin by recalling the expression obtained for the direct force in the scalar case² [see Eq. (27) in paper I]:

$$F_\delta^{(\text{dir},\text{sca})} = \epsilon_0^{-3}P_\delta^{(1,\text{sca})} + \epsilon_0^{-5}P_\delta^{(4,\text{sca})} + \epsilon_0^{-7}P_\delta^{(7,\text{sca})}, \quad (27)$$

²For later convenience, we use here a redefinition of $F_\delta^{(\text{dir},\text{sca})}$, with the factor q^2 omitted (q is the scalar charge).

where $P_\delta^{(1,\text{sca})} = -\frac{1}{2}S_{0,\delta}$, $P_\delta^{(4,\text{sca})} = -\frac{1}{2}S_0S_{1,\delta} + \frac{3}{4}S_1S_{0,\delta}$, and $P_\delta^{(7,\text{sca})}$ is a polynomial in δx , of homogeneous order δx^7 , whose explicit value will not be needed here. We use the label “sca” to distinguish quantities associated with the scalar case from their gravitational-case counterparts.

Comparing Eqs. (25) and (27) and taking into account also the explicit form of the coefficients $P_\delta^{(1,4,\text{sca})}$ and $P_\delta^{(1,4)}$, we now express the gravitational direct force as a sum of three terms in the form

$$F_{\text{dir}}^\alpha = \mu^2(F_1^\alpha + F_2^\alpha + F_3^\alpha), \quad (28)$$

where

$$F_1^\alpha \equiv K_0^{\alpha\delta}F_\delta^{\text{dir},\text{sca}},$$

$$F_2^\alpha \equiv -\frac{1}{2}\epsilon_0^{-5}K_1^{\alpha\delta}S_0S_{0,\delta}$$

$$= -\frac{1}{2}\epsilon_0^{-3}K_1^{\alpha\delta}S_{0,\delta},$$

$$F_3^\alpha \equiv \epsilon_0^{-7}(P_{(7)}^\alpha - K_0^{\alpha\delta}P_\delta^{(7,\text{sca})}). \quad (29)$$

We proceed by considering separately the contributions to the RPs from each of the three terms $F_{1,2,3}^\alpha$.

A. Contribution to the RPs from the term F_1^α

Consider first the term F_1^α of the gravitational direct force. This term is just the scalar direct force, contracted with $K_0^{\alpha\delta}$ —an array of constant coefficients [recall $K_0^{\alpha\delta} \equiv K^{\alpha\delta}(\delta x \rightarrow 0)$, where $K^{\alpha\delta}$ is the tensor defined in Eq. (19)]. Since the constant array $K_0^{\alpha\delta}$ does not interfere with the multipole decomposition, one may immediately conclude that the contribution from the term F_1^α to any of the RPs, in the gravitational case, would be precisely the same as in the scalar case—multiplied by $K_0^{\alpha\delta}$. Denoting by R_i^α ($i = 1, 2, 3$) the contribution of the term F_i^α to any of the RPs, we thus simply have

$$R_1^\alpha = K_0^{\alpha\delta}R_\delta^{(\text{sca})}, \quad (30)$$

where the scalar-case values $R_\delta^{(\text{sca})}$ are those given explicitly in paper I. (We have made here the obvious replacement $q \rightarrow \mu$.) In particular, since $C_\delta^{(\text{sca})} = D_\delta^{(\text{sca})} = 0$, we find $C_1^\alpha = D_1^\alpha = 0$.

B. Contribution to the RPs from the term F_2^α

We next consider the term F_2^α in Eq. (28). This term has the form $\propto \epsilon_0^{-5}P_{(4)}^\alpha$. As shown in paper I, in evaluating the contribution of this kind of terms to the l mode of the direct force at z_0 , one is allowed to take their limit $\delta t, \delta r \rightarrow 0$ before applying the multipole decomposition. This is true regardless of the explicit form of the polynomial $P_{(4)}^\alpha$. We hence proceed by considering $F_2^\alpha(\delta t = \delta r = 0)$; we show that this quantity actually vanishes, even before applying the multipole decomposition.

Examine the form of F_2^α , as defined in Eq. (29): The quantity $K_1^{\alpha\delta}$, recall, is the first-order variation of the tensor $K^{\alpha\delta} \equiv 4k^{\alpha\beta\gamma\delta} \hat{u}_\beta \hat{u}_\gamma$ with respect to δx . Recalling that $k^{\alpha\beta\gamma\delta}$ is a tensor of constant components, we have $K_1^{\alpha\delta} = 4k^{\alpha\beta\gamma\delta} (\delta u_\beta u_\gamma + u_\beta \delta u_\gamma)$, where δu_β is the first-order variation in the PP four-velocity $\hat{u}_\beta(x)$ (namely, $\delta u_\beta = \Gamma_{\beta\rho}^\lambda \mu_\lambda \delta x^\rho$, with $\Gamma_{\beta\rho}^\lambda$ being the connection coefficients at z_0). Thus,

$$F_2^\alpha \equiv -2\epsilon_0^{-3} k^{\alpha\beta\gamma\delta} (\delta u_\beta u_\gamma + u_\beta \delta u_\gamma) S_{0,\delta}. \quad (31)$$

Consider now the explicit form of $k^{\alpha\beta\gamma\delta}$, given in Eq. (5). Three of the five terms of $k^{\alpha\beta\gamma\delta}$ are proportional to u^δ . These three terms will contribute nothing to F_2^α as

$$u^\delta S_{0,\delta} = 2u^\delta (g_{\mu\delta} + u_\mu u_\delta) \delta x^\mu = 0.$$

Consider next the first and fifth terms of $k^{\alpha\beta\gamma\delta}$, proportional to $u^\beta u^\gamma$ and $g^{\beta\gamma}$, respectively. Both terms yield contributions to F_2^α which are proportional to $\delta u_\beta u^\beta$. This quantity, in fact, vanishes for our orbital setup: To see that, first recall $\hat{u}^\beta \hat{u}_\beta = u^\beta u_\beta = -1$, as the length of the four velocity is preserved when PP. Then, observe that the linear variation of this equality with respect to δx yields

$$0 = \delta(\hat{u}^\beta \hat{u}_\beta) = \delta u^\beta \hat{u}_\beta + \hat{u}^\beta \delta u_\beta = 2\delta u_\beta u^\beta + \delta g^{\alpha\beta}(x) \hat{u}_\alpha \hat{u}_\beta \quad (32)$$

(to linear order in δx), where $\delta g^{\alpha\beta}(x) = g^{\alpha\beta}{}_{,\gamma}(z_0) \delta x^\gamma$ is the linear variation in $g^{\alpha\beta}(x)$. Since in our setup the trajectory is equatorial and since $g^{\alpha\beta}{}_{,\theta} = g^{\alpha\beta}{}_{,\varphi} = 0$ at the equatorial plane, the linear variation $\delta g^{\alpha\beta}$ vanishes (recall that in considering F_2^α we reduce δx to just $\delta\theta, \delta\varphi$). Consequently, we obtain, from Eq. (32), $\delta u_\beta u^\beta = 0$.

To conclude the above discussion, we find $F_2^\alpha = 0$ (in the limit $\delta t = \delta r = 0$). Hence, obviously, this term yields no contribution to any of the RPs:

$$R_2^\alpha = 0. \quad (33)$$

Note that this result may no longer be valid when using k extensions other than the ‘‘fixed-components’’ extension employed here: Usually, there will arise additional terms in Eq. (31), corresponding to first-order variations of the tensor $k^{\alpha\beta\gamma\delta}(x)$. Also, notice that the result (33) will generally not hold when considering nonequatorial orbits, as the variation $\delta g^{\alpha\beta}$ in Eq. (32) will generally fail to vanish. In both cases (namely, a different k extension and/or a nonequatorial orbit), our calculation would lead, in general, to a nonvanishing contribution R_2^α .

C. Contribution to the RPs from the term F_3^α

We finally turn to the term F_3^α in Eq. (28). Recalling that $K_0^{\alpha\delta}$ is just an array of constants, we may write this term as

$$F_3^\alpha = \epsilon_0^{-7} \tilde{P}_{(7)}^\alpha, \quad (34)$$

where $\tilde{P}_{(7)}^\alpha \equiv P_{(7)}^\alpha - K_0^{\alpha\delta} P_\delta^{(7,\text{sca})}$ is once again a polynomial in δx , of order δx^7 .

The contribution of the term F_3^α to the l -mode direct force is obtained by carrying out the (Legendre) integration over a two-sphere $r=t=\text{const}$ and then taking the limits $\delta t, \delta r \rightarrow 0$. As shown in paper I, in evaluating the contribution of a term of the form (34) (regardless of the explicit form of $\tilde{P}_{(7)}^\alpha$) one may interchange the integration and the limits and set $\delta t = \delta r = 0$ before integrating over the two-sphere—just as with the term F_2^α considered above. To carry out the Legendre integration, it proves especially convenient—as in paper I—to use a new set of spherical coordinates (θ', φ') , in which the particle is located at the polar axis, $\theta' = 0$.³ The contribution from F_3^α to the l -mode direct force can then be expressed as

$$\mu^2 \frac{L}{2\pi} \int \hat{\epsilon}_0^{-7} \hat{P}_{(7)}^\alpha P_l(\cos \theta') d(\cos \theta') d\varphi', \quad (35)$$

where P_l is the Legendre polynomial and $\hat{\epsilon}_0, \hat{P}_{(7)}^\alpha$ are the reductions of $\epsilon_0, \tilde{P}_{(7)}^\alpha$, respectively, to $r=r_0$ and $t=t_0$. (Note that, conveniently, in the θ', φ' system the contribution to any l mode at $x \rightarrow z_0$ comes only from the axially symmetric, $m=0$ mode.) From Eq. (23), recalling $\epsilon_0 = S_0^{1/2}$ and $u_\theta = 0$, we obtain, explicitly,

$$\hat{\epsilon}_0 = [r_0^2 (\delta\theta^2 + \delta\varphi^2) + u_\varphi^2 \delta\varphi^2]^{1/2}. \quad (36)$$

To implement the integral (35), it proves convenient, as in paper I, to introduce Cartesian-like coordinates x, y on the two-sphere, which we define here by

$$x \equiv \theta' \cos \varphi', \quad y \equiv \theta' \sin \varphi'. \quad (37)$$

Note $x=y=0$ at z_0 , and hence we have simply $\delta x^x = x$ and $\delta x^y = y$. It is simple to show that a choice of transformation $(\theta, \varphi) \rightarrow (\theta', \varphi')$ can be made such that the coordinates x, y would relate to the original coordinates θ, φ through

$$x = \delta\varphi + O(\delta x^2), \quad y = \delta\theta + O(\delta x^2). \quad (38)$$

Expressed in terms of the new coordinates, the polynomial $\hat{P}_{(7)}^\alpha(\delta\theta, \delta\varphi)$ in Eq. (35) becomes $\bar{P}_{(7)}^\alpha(y, x) + O(\delta x^8)$, where $\bar{P}_{(7)}^\alpha$ is a polynomial of homogeneous order 7 in x, y . The contribution from the $O(\delta x^8)$ corrections to the direct force vanishes at $x \rightarrow z_0$ and can therefore be omitted. From Eq. (36) we also get

$$\hat{\epsilon}_0 = \bar{\epsilon}_0(x, y) + O(\delta x^2), \quad (39)$$

³We should emphasize here that we do *not* regard θ', φ' as new spacetime coordinates—namely, all vectorial and tensorial quantities are still taken with respect to the original coordinates θ, φ . That is, θ', φ' are merely used here as new variables for implementing the Legendre integral. The same holds for the coordinates x, y introduced below.

where $\bar{\epsilon}_0 \equiv [r_0^2(x^2 + y^2) + u_\varphi^2 x^2]^{1/2}$. Again, only the leading-order term here contributes to the direct force at $x \rightarrow z_0$, and we are allowed to drop the $O(\delta x^2)$ correction. Hence, as far as the calculation of the RPs is concerned, we may express the contribution from the term F_3^α to the l -mode direct force, Eq. (35), as

$$\mu^2 \frac{L}{2\pi} \int \bar{\epsilon}_0^{-7} \bar{P}_{(7)}^\alpha(x, y) P_l(\cos \theta') dx dy. \quad (40)$$

Note that the Jacobian of the transformation $(\cos \theta', \varphi') \rightarrow (x, y)$, which is actually given by $1 + O(\delta x^2)$, has been set here to just 1: The higher-order corrections are once again omitted, as they vanish at $x \rightarrow z_0$.

Examine now the integral in Eq. (40): $\bar{\epsilon}_0$ is an even function of both x and y and so is the function $\cos \theta'$. However, each of the possible individual terms in the polynomial $\bar{P}_{(7)}^\alpha$ (such as $\propto xy^6$ or $\propto x^4 y^3$, for instance) is necessarily an *odd* function of either x or y . Consequently, we observe that the entire integrand in Eq. (40) is odd in either x or y . Therefore, obviously, the integration over the two-sphere would vanish. As a consequence, no contribution to the RPs will arise from the term F_3^α :

$$R_3^\alpha = 0. \quad (41)$$

Notice that this last result is valid for any (sufficiently regular) k extension. A modification of the extension would only affect the explicit form of the polynomials $\bar{P}_{(7)}^\alpha$, but would not alter the odd-parity structure of the integrand in Eq. (40).

D. Summary: RPs values in the gravitational case

Let us now collect the above results: We have found that neither of the terms F_2^α and F_3^α contributes to the l -mode direct force. The sole contribution to the RPs comes from the term F_1^α —this contribution is given in Eq. (30). The RPs in the gravitational case are therefore given by

$$R^\alpha = K_0^{\alpha\delta} R_\delta^{(\text{sca})}, \quad (42)$$

where, recall, R^α stands for any of the RPs and the scalar-case values $R_\delta^{(\text{sca})}$ are those given explicitly in paper I. We now need only to provide the explicit form of $K_0^{\alpha\delta}$: Recalling $K_0^{\alpha\delta} = K^{\alpha\delta}(x \rightarrow z_0)$, one easily gets, from Eqs. (19) and (5),

$$K_0^{\alpha\delta} = g^{\alpha\delta} + u^\alpha u^\delta, \quad (43)$$

where, recall, u^α and $g^{\alpha\delta}$ denote the values of these quantities at z_0 . Note that $K_0^{\alpha\delta}$ is just the spatial projection operator at z_0 —namely, $(K_0^{\alpha\delta} V_\delta) u_\alpha = 0$ for any vector V_δ .

Let us finally write Eq. (42) more explicitly: First, recalling (see paper I) that the scalar parameter $A_\delta^{(\text{sca})}$ has no component tangent to u^α (i.e., $u^\delta A_\delta^{(\text{sca})} = 0$), we simply obtain

$$A^\alpha = A_{(\text{sca})}^\alpha \quad (44)$$

(with the obvious substitution $q \rightarrow \mu$). Unlike the situation with the parameter A^α , the quantity $u^\delta B_\delta^{(\text{sca})}$ does not vanish [see Eq. (85) in paper I], and we leave the expression for B^α in the form

$$B^\alpha = K_0^{\alpha\delta} B_\delta^{(\text{sca})} \quad (45)$$

(again, with $q \rightarrow \mu$). Finally, as $C_\delta^{(\text{sca})} = D_\delta^{(\text{sca})} = 0$, we shall have, in the gravitational case alike,

$$C^\alpha = D^\alpha = 0. \quad (46)$$

IV. DERIVATION OF THE REGULARIZATION PARAMETERS: ELECTROMAGNETIC CASE

In this section we consider the *electromagnetic* self-force acting on an electrically charged particle: we prescribe the mode-sum scheme in this case and construct all required RPs for an arbitrary (equatorial) geodesic orbit in Schwarzschild spacetime. The same analytic calculation used for deriving the gravitational-case RPs will prove directly applicable also to the electromagnetic case, with only minor adaptations required.

We shall consider a particle carrying an electric charge e (with $|e| \ll M$) and assume the same orbital configuration as in the gravitational case (namely, the particle is taken to move along an equatorial orbit, which in the limit $e \rightarrow 0$ becomes a geodesic). We shall also maintain here the notation for the various quantities z_0 , x , ϵ , S , \hat{u}_α , and so on. We shall denote by $\phi_\alpha(x)$ the vector potential associated with the “full” electromagnetic field induced by the particle. In this section we ignore the gravitational self-force.

A formal expression for the electromagnetic self-force in curved spacetime was obtained long ago by DeWitt and Brehne [15] (and was reproduced recently by QW [2] using a different method). For a geodesic in vacuum spacetime, the electromagnetic self-force is obtained from an electromagnetic “tail” force, just as in the gravitational case:

$$F_{\text{self}}^{\alpha(\text{EM})} = \lim_{x \rightarrow z_0} F_{\text{tail}}^{\alpha(\text{EM})}(x), \quad (47)$$

where hereafter we use the label “EM” to signify quantities associated with the electromagnetic case. The formal construction of the vector field $F_{\text{tail}}^{\alpha(\text{EM})}(x)$ is described in [15,2]. As in the gravitational case, the electromagnetic tail force can be written as the difference between a “full” force and a “direct” force—just as in Eq. (9). In the electromagnetic case, these two vector fields are given by [8]

$$F_{\text{full}}^{\alpha(\text{EM})}(x) = e k^{\alpha\beta\gamma} \phi_{\beta;\gamma}, \quad F_{\text{dir}}^{\alpha(\text{EM})} = e k^{\alpha\beta\gamma} \phi_{\beta;\gamma}^{\text{dir}}, \quad (48)$$

where $\phi_\beta^{\text{dir}}(x)$ is the “direct” part of the vector potential (given explicitly below) and $k^{\alpha\beta\gamma}(x)$ is a (sufficiently regular) extension of the tensor

$$k_0^{\alpha\beta\gamma} \equiv g^{\alpha\gamma} u^\beta - g^{\alpha\beta} u^\gamma, \quad (49)$$

defined at z_0 . As in the gravitational case, we shall adopt here the “fixed-components” extension, defined (in Schwarzschild coordinates) through $k^{\alpha\beta\gamma}(x) \equiv k_0^{\alpha\beta\gamma}$.

The mode-sum prescription for the electromagnetic self-force is completely analogous to the one prescribed in the gravitational and scalar cases: Given the (scalar harmonic) l modes $F_{\text{full}}^{\alpha l(\text{EM})}$ of the electromagnetic full force, the electromagnetic self-force is constructed through

$$F_{\text{self}}^{\alpha(\text{EM})} = \sum_{l=0}^{\infty} \left[\lim_{x \rightarrow z_0} F_{\text{full}}^{\alpha l(\text{EM})}(x) - A^{\alpha(\text{EM})} L - B^{\alpha(\text{EM})} - C^{\alpha(\text{EM})}/L \right] - D^{\alpha(\text{EM})}, \quad (50)$$

where the various electromagnetic-case RPs are to be obtained, again, by analyzing the multipole modes of the direct force. The rest of this section is devoted to calculating these electromagnetic RPs.

As in the gravitational case, our starting point would be the expression for the direct part of the particle’s field—this time the direct part of the vector potential—as obtained by MNS [see Eq. (B3) of Ref. [8]]. In precisely the same manner as in the gravitational case, this expression can be brought to the form

$$\phi_{\beta}^{\text{dir}}(x) = eS^{-1/2} \hat{u}_{\beta} + S^{-1/2} P_{\beta}^{(2)}(x) \quad (51)$$

[in analogy with Eq. (14)], where $P_{\beta}^{(2)}$ is a (regular) function of $O(\delta x^2)$. The derivatives of the direct vector potential then take the form

$$\phi_{\beta;\gamma}^{\text{dir}} = -(e/2)S^{-3/2} S_{,\gamma} \hat{u}_{\beta} + eS^{-1/2} \hat{u}_{\beta;\gamma} - S^{-3/2} S_{,\gamma} P_{\beta}^{(2)}/2 + S^{-1/2} P_{\beta;\gamma}^{(2)}, \quad (52)$$

which [as in obtaining Eq. (17)] can be put into the form

$$\phi_{\beta;\gamma}^{\text{dir}} = -\frac{1}{2} e \epsilon^{-3} S_{,\gamma} \hat{u}_{\beta} + \epsilon^{-3} P_{\beta\gamma}^{(3)}, \quad (53)$$

with $P_{\beta\gamma}^{(3)}$ being a (regular) function of order δx^3 (and higher orders). Consequently, the direct electromagnetic force takes precisely the same form as in Eq. (18),

$$F_{\text{dir}}^{\alpha(\text{EM})}(x) = e^2 \left(-\frac{1}{2} K_{\text{EM}}^{\alpha\delta} \epsilon^{-3} S_{,\delta} + \epsilon^{-3} P_{(3)}^{\alpha} \right), \quad (54)$$

where, this time,

$$K_{(\text{EM})}^{\alpha\delta} \equiv k^{\alpha\beta\delta} \hat{u}_{\beta} \quad (55)$$

and $eP_{(3)}^{\alpha} \equiv k^{\alpha\beta\gamma} P_{\beta\gamma}^{(3)}$. Again, we may drop all terms of $P_{(3)}^{\alpha}$ which are of order δx^4 and higher (as they do not contribute to the direct force at $x \rightarrow z_0$) and take $P_{(3)}^{\alpha}$ to be of homogeneous order δx^3 .

Thanks to the complete analogy between the forms of the electromagnetic and gravitational direct forces [compare Eqs. (54) and (18)], our analysis now proceeds precisely as in the gravitational case: We expand $K_{(\text{EM})}^{\alpha\delta}$ and S in δx , as in

Eqs. (20) and (22), and consequently write the direct force as a sum of three terms—in precise analogy with Eqs. (25) and (28):

$$F_{\text{dir}}^{\alpha(\text{EM})} = e^2 (\epsilon_0^{-3} P_{(1)}^{\alpha(\text{EM})} + \epsilon_0^{-5} P_{(4)}^{\alpha(\text{EM})} + \epsilon_0^{-7} P_{(7)}^{\alpha(\text{EM})}) \equiv e^2 (F_1^{\alpha(\text{EM})} + F_2^{\alpha(\text{EM})} + F_3^{\alpha(\text{EM})}), \quad (56)$$

where $F_{i=1,2,3}^{\alpha(\text{EM})}$ are defined in Eq. (29), with the replacements $K_n^{\alpha\delta} \rightarrow K_{n(\text{EM})}^{\alpha\delta}$ and $P_{(7)}^{\alpha} \rightarrow P_{(7)}^{\alpha(\text{EM})}$. The only point at which our current analysis differs from the gravitational case is in the explicit values taken by the various coefficients $K_n^{\alpha\delta}$ (and consequently in the explicit values of the terms $P_{(n)}^{\alpha}$).

Consider first the contribution to the l -mode direct force coming from the term $F_1^{\alpha(\text{EM})}$: In a complete analogy with Eq. (30), we obtain

$$R_{1(\text{EM})}^{\alpha} = K_{0(\text{EM})}^{\alpha\delta} R_{\delta}^{\text{(sca)}}(q \rightarrow e), \quad (57)$$

where $R_{i(\text{EM})}^{\alpha}$ ($i=1,2,3$) stands for the contribution of the term $F_i^{\alpha(\text{EM})}$ to any of the RPs, and the array of constant coefficients $K_{0(\text{EM})}^{\alpha\delta} \equiv K_{(\text{EM})}^{\alpha\delta}(x \rightarrow z_0)$ is now given by

$$K_{0(\text{EM})}^{\alpha\delta} = -(g^{\alpha\delta} + u^{\alpha} u^{\delta}). \quad (58)$$

Noticing $K_{0(\text{EM})}^{\alpha\delta} = -K_0^{\alpha\delta}$ [compare Eq. (58) to Eq. (43)] and recalling Eqs. (30) and (42), we then conclude

$$R_{1(\text{EM})}^{\alpha} = -R_{1(\text{grav})}^{\alpha}(\mu \rightarrow e) = -R_{(\text{grav})}^{\alpha}(\mu \rightarrow e), \quad (59)$$

where hereafter we use the label “grav” to signify the gravitational-case values.

Next, consider the term $F_2^{\alpha(\text{EM})} \equiv -\frac{1}{2} \epsilon_0^{-3} K_{1(\text{EM})}^{\alpha\delta} S_{0,\delta}$. Here, the coefficient $K_{1(\text{EM})}^{\alpha\delta}$ (the first-order correction in $K_{(\text{EM})}^{\alpha\delta}$) is given by

$$K_{1(\text{EM})}^{\alpha\delta} = k^{\alpha\beta\delta} \delta u_{\beta}. \quad (60)$$

As in the gravitational case, it is easy to show that $F_2^{\alpha(\text{EM})}$ (evaluated at $\delta t = \delta r = 0$) actually vanishes, even before taking its multipole decomposition: From Eq. (49) we observe that $k^{\alpha\beta\delta}$ is composed of two terms, one proportional to u^{β} and the other proportional to u^{δ} . The $\propto u^{\beta}$ term contributes nothing to $K_{1(\text{EM})}^{\alpha\delta}$, since $u^{\beta} \delta u_{\beta} = 0$ (as explained when discussing the gravitational case). The $\propto u^{\delta}$ term will yield a zero contribution as well, by virtue of $S_{0,\delta} u^{\delta} = 0$. We thus find that in the electromagnetic case—just as in the gravitational case—the term $F_2^{\alpha(\text{EM})}$ contributes nothing to the RPs—namely, $R_{2(\text{EM})}^{\alpha} = 0$.

As to the last term in the electromagnetic direct force, $F_3^{\alpha(\text{EM})}$, using the same parity considerations as in the gravitational case, one shows that the contribution from this term to any of the RPs will vanish—namely, $R_{3(\text{EM})}^{\alpha} = 0$. This vanishing is irrespective of the explicit form of the polynomial $P_{(7)}^{\alpha(\text{EM})}$ in Eq. (56).

In conclusion, thus, we find that the sole contribution to the RPs in the electromagnetic case comes from the term

$F_1^{\alpha(\text{EM})}$, Eq. (59). We hence obtain

$$R_{(\text{EM})}^{\alpha} = -R_{(\text{grav})}^{\alpha}(\mu \rightarrow e). \quad (61)$$

V. SUMMARY AND CONCLUDING REMARKS

Let us summarize our mode-sum prescription for constructing the gravitational and electromagnetic self-forces. We start with the gravitational case.

(1) For a given trajectory, compute the tensor-harmonic modes of the metric perturbation, $\bar{h}^{(i)l'm}(r,t)$, by numerically integrating the separable field equations (e.g., in the harmonic gauge [4]).

(2) Given $\bar{h}^{(i)l'm}(r,t)$, construct the full modes $F_{\text{full}}^{\alpha l}$ at the particle's location. This is done by applying the operator in Eq. (8) to $\bar{h}^{(i)l'm}(r,t)$, using the “fixed-components” extension described above and then expanding the resultant field into scalar spherical harmonics (and summing over $il'm$ for a given l). This procedure is implemented in Refs. [5,14].

(3) Use Eqs. (62) below [along with Eqs. (83) of paper I] to obtain the RPs values corresponding to the trajectory under consideration.

(4) Finally, apply the mode-sum formula, Eq. (11).

This prescription is now being implemented by Barack and Lousto for radial [5] and circular [14] orbits in Schwarzschild spacetime.

The prescription for constructing the electromagnetic self-force is similar: First, one has to compute the vector-harmonic modes of the (full-field) vector potential for the given orbital configuration. Then, one constructs the full-force modes $F_{\text{full}}^{\alpha l(\text{EM})}$ —this construction is carried out by applying Eq. (48) to each of the full-field vector-harmonic modes and then decomposing each of these modes in scalar spherical harmonics. Finally, one applies the mode-sum formula (50), with the electromagnetic RPs values given in Eqs. (62) below.

The values of the RPs in the gravitational and electromagnetic cases are summarized as follows:

$$A_{\alpha}^{(\text{grav})} = -A_{\alpha}^{(\text{EM})} = A_{\alpha}^{(\text{sca})}, \quad (62a)$$

$$B_{\alpha}^{(\text{grav})} = -B_{\alpha}^{(\text{EM})} = (\delta_{\alpha}^{\beta} + u_{\alpha}u^{\beta})B_{\beta}^{(\text{sca})}, \quad (62b)$$

$$C_{\alpha}^{(\text{grav,EM})} = D_{\alpha}^{(\text{grav,EM})} = 0 \quad (62c)$$

(with the obvious replacements $q \rightarrow \mu$ or $q \rightarrow e$), where the quantities labeled “sca” are the scalar-field parameters given explicitly in Eqs. (83) of paper I. The gravitational RPs were calculated previously in a different method, using the l -mode Green's function expansion technique [4,5], in the special case of radial orbits. The results agree with the values (62).

As we discussed above, there is a certain ambiguity in the values of the RPs, which arises from the freedom in choosing the extension of the tensor k off the evaluation point z_0 (the choice of this extension affects, of course, the multipole decomposition of the force). However, our mode-sum scheme produces no ambiguity in the eventual value of the self-force—one only has to make sure that the full-force modes $F_{\text{full}}^{\alpha l}$ in Eq. (11) [or in Eq. (50)] are calculated using the same

extension as the one used in calculating the RPs. It is thus essential to recall here that the RPs values summarized above are those referring to the “fixed-components” extension of the tensors $k^{\alpha\beta\gamma\delta}$ or $k^{\alpha\beta\gamma}$ (expressed in Schwarzschild coordinates). This extension is most easily applicable in the numerical computation of the full-force modes [5,14].

Based on our above analysis, we may phrase the following general statements concerning the extension dependence of the RPs in the electromagnetic and gravitational cases: (i) The RPs A^{α} , C^{α} , and D^{α} are *insensitive* to the extension of $k^{\alpha\beta\gamma\delta}$ (provided it is regular enough). (ii) The value of B^{α} *does* depend, in general, on the choice of extension; however, all sufficiently regular extensions which differ from the “fixed-components” extension $k^{\alpha\beta\gamma\delta}$ by an amount of only $O(\delta x^2)$ will admit the same value of B^{α} —the one given in Eq. (62b). It is interesting to refer here to the MNS analysis [8], in which a different extension has been employed: MNS extended the tensor $k^{\alpha\beta\gamma\delta}$ by PP the four-velocity from z_0 to x and just assigning to $g^{\alpha\beta}$ the actual value it has at x . Interestingly, within this extension [differing from the “fixed-components” extension already at $O(\delta x)$], all RPs attain precisely the same values as in the scalar case [6] (except that in the electromagnetic case all RPs are to be multiplied by -1).

Finally, it is important to recall that the gravitational self-force is a *gauge-dependent* notion, as discussed in Ref. [9]. The prescription described in this paper applies to the self-force associated with the *harmonic gauge* (in which the original MST-QW scheme has been formulated). It also applies, with the same RPs values, to any other gauge related to the harmonic gauge through a regular gauge transformation [9]. However, for other, nonregular gauges, the mode-sum scheme is not guaranteed to be valid in its present form. A method for overcoming this gauge problem has been sketched in [9] and is currently being implemented for circular orbits in Schwarzschild spacetime [14]. A different strategy (applicable in the Schwarzschild case) would be to calculate the self-force directly in the harmonic gauge [16].

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APPENDIX: DIRECT METRIC PERTURBATION

In this appendix we obtain Eq. (13) for the direct metric perturbation by processing the expression given by MST in Ref. [1].

By considering the Hadamard expansion of the (full) metric perturbation, MST obtained the following expression for the (retarded) trace-reversed perturbation [see Eq. (2.27) of Ref. [1]]:

$$\bar{h}_{\beta\gamma}(x) = 2\mu \left[\frac{2\beta(x, z_0)}{\epsilon} \hat{u}_\beta^1 \hat{u}_\gamma^1 - u_1^\sigma S^{:\lambda} u_1^\rho R_{\sigma\lambda\rho}^1(\hat{\beta} \hat{u}_\gamma^1) + 2\epsilon R_{\hat{\beta}\lambda\hat{\gamma}\sigma}^1 u_1^\lambda u_1^\sigma \right] + \text{tail term} + O(\epsilon^2). \quad (\text{A1})$$

Here we use the notation of our Sec. II (see Fig. 1): namely, x is a point in the neighborhood of the force evaluation point z_0 , $\epsilon \equiv S^{1/2}$ is the length of the short geodesic section connecting x to the world line and normal to it, z_1 denotes the intersection of this geodesic with the world line, u^α and u_1^α (or u_α^1) denote the four-velocities at z_0 and z_1 , respectively, and \hat{u}^α and \hat{u}_1^α (or \hat{u}_α^1) are their PP to x . In addition, indices in parentheses denote symmetrization and $R_{\alpha\beta\gamma\delta}^1$ represents the Riemann tensor PP from z_1 to x with respect to any of its indices carrying a caret. The ‘‘tail term’’ represents a nonlocal contribution to the full perturbation, with its form given explicitly in [1]. The function β (denoted κ^{-1} in [1]) is a regular function satisfying $\beta = 1 + O(\delta x^2)$ [see Eq. (A14) therein]. Note that the correction term proportional to the four-acceleration in Eq. (2.27) of Ref. [1] can be omitted, since, for geodesic orbits, it contributes to the self-force only at order higher than $O(\mu^2)$. For the same reason, we omit here the $O(\tau_r^{-1}\epsilon)$ term indicated therein. Finally, notice the notational change $\sigma \rightarrow S/2$.

The direct part of the metric perturbation is now taken as the difference between the full perturbation given in Eq. (A1) and the tail term. The terms included in $O(\epsilon^2)$ do not contribute to the direct perturbation at the limit $x \rightarrow z_0$ ($\epsilon \rightarrow 0$); nor do they contribute to the direct force, whose construction involves only first-order derivatives of $\bar{h}_{\alpha\beta}(x)$. We thus redefine the direct perturbation by ignoring these $O(\epsilon^2)$ terms, which leaves us with only the three terms in the square brackets, scaling as ϵ^{-1} , ϵ^1 , and ϵ^1 , respectively.

Consider now the second and third terms in the square brackets: First, note that the (coordinate components of the) two vectors u_1^α and u^α differ only at $O(\delta x)$. Hence [recalling $\epsilon, S^{:\lambda} \propto O(\delta x)$], this difference contributes only to $O(\delta x^2)$ in Eq. (A1). We may thus ignore this correction and just replace u_1^α with u^α in the second and third terms in the square brackets. Likewise, we replace \hat{u}_1^α with \hat{u}^α in the second term. Similarly, we may ignore the $O(\delta x)$ difference between $R_{\alpha\beta\gamma\delta}^1$ and $R_{\alpha\beta\gamma\delta}$ (the latter denoting the coordinate value of the Riemann tensor at z_0) as it contributes only to $O(\delta x^2)$ in Eq. (A1). The direct metric perturbation thus takes the form

$$\bar{h}_{\beta\gamma}^{\text{dir}}(x) = 2\mu \left[\frac{2\beta(x, z_0)}{\epsilon} \hat{u}_\beta^1 \hat{u}_\gamma^1 - u^\sigma S^{:\lambda} u^\rho R_{\sigma\lambda\rho}(\hat{\beta} \hat{u}_\gamma^1) + 2\epsilon R_{\hat{\beta}\lambda\hat{\gamma}\sigma} u^\lambda u^\sigma \right]. \quad (\text{A2})$$

Examine now more closely the second term in Eq. (A2): Since $S^{:\lambda} \propto O(\delta x)$, the only contribution to the direct force which does not vanish at $x \rightarrow z_0$ arises from differentiating $S^{:\lambda}$. Recalling Eq. (23), we have

$$S^{:\lambda}_{;\delta} = 2(\delta_\delta^\lambda + u^\lambda u_\delta) + O(\delta x).$$

Note that $S^{:\lambda}_{;\delta} u^\delta = 0$ (at $x \rightarrow z_0$). Note also that the second term in Eq. (A2) (unlike the other two terms) is perfectly regular at $x = z_0$. This allows us to evaluate its contribution directly at z_0 , which we do by just ‘‘removing’’ the carets from $R_{\sigma\lambda\rho\hat{\beta}}$ and \hat{u}_γ^1 . Recalling Eq. (7), the contribution from this term to the direct force at z_0 then reads

$$-2\mu^2 k^{\alpha\beta\gamma\delta} u^\sigma S^{:\lambda}_{;\delta} u^\rho R_{\sigma\lambda\rho}(\beta u^\mu_\gamma)$$

(evaluates at $x = z_0$). Examining the form of the tensor $k^{\alpha\beta\gamma\delta}$, given in Eq. (5), we observe that three of its five terms are proportional to u^δ and thus vanish when contracted with $S^{:\lambda}_{;\delta}$. The first term of $k^{\alpha\beta\gamma\delta}$ is proportional to $u^\beta u^\gamma$ and thus yields a vanishing contribution when contracted with either $u^\rho R_{\sigma\lambda\rho\beta}$ or $u^\rho R_{\sigma\lambda\rho\gamma}$. Likewise, the last term of $k^{\alpha\beta\gamma\delta}$, proportional to $g^{\gamma\beta}$, is found to vanish when contracted with either $u^\rho R_{\sigma\lambda\rho\beta} u^\mu_\gamma$ or $u^\rho R_{\sigma\lambda\rho\gamma} u^\mu_\beta$. We conclude that the contribution of this regular term to the self-force vanishes—even before taking the harmonic decomposition.

Finally, consider the third term in Eq. (A2). Noticing that this term has the form $\epsilon \times$ (a regular function of x), we may write it as

$$\epsilon^{-1} P_{\beta\gamma}^{(2)}(x, z_0), \quad (\text{A3})$$

where $P_{\beta\gamma}^{(2)}$ is a certain regular function, of order δx^2 . (This form will be convenient for our analysis in Sec. II.) We further notice that the terms of $O(\delta x^2)$ included in the function β contribute to $\bar{h}_{\beta\gamma}$ an amount of precisely the form (A3). We may thus absorb this contribution in the contribution (A3) coming from the third term and replace the function β with just 1. The explicit form of the regular function $P_{\beta\gamma}^{(2)}(x)$ will not be needed in our analysis.

In conclusion, we find that the direct part of the metric perturbation is effectively given by Eq. (13). This expression is used as a starting point for the analysis in this paper.

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