Static, Self-Gravitating Elastic Bodies

Robert Beig
Institut für Theoretische Physik der Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria

Bernd G. Schmidt
Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1, D-14476 Golm, Germany

January 19, 2004

Abstract

There is proved an existence theorem, in the Newtonian theory, for static, self–gravitating, isolated bodies composed of elastic material. The theorem covers the case where these bodies are small, but allows them to have arbitrary shape.

Keyword: self-gravitating elastic bodies

1 Introduction

Most solutions of the Einstein field equations — whether known explicitly or given by existence theorems — describing static isolated bodies are spherically symmetric. The reason for this is the choice of matter model which usually is that of a perfect fluid - and such models are necessarily spherically symmetric. This latter statement has been proved by Lichtenstein [7] in the Newtonian theory and - in the same generality - is still a conjecture in General Relativity.(For the best results available, see [3], [8]). The only nonspherical solutions known to us are the axially symmetric ones constructed by Rein for Vlasov Matter [11]. In the present work we pursue another way to describe nonspherical gravitational fields by choosing as our matter model elastic bodies, coupled to the static Einstein equations. In the spherically symmetric case this has been done by Park [10]. In the nonspherical case nothing is known in the Einstein theory nor — to the best of our knowledge — in the Newtonian theory. Thus, as a first step, in the present paper we prove an existence theorem, purely in the Newtonian theory, for static self-gravitating bodies composed of elastic material. The theorem allows these bodies to have arbitrary shape.

The main limitation of the present work is that we restrict ourselves to solutions close to the natural state of the body, which, in physical terms, means that we require the body to be "sufficiently small". The main technical tool, then, is the implicit function theorem near that natural state. For "pure traction problems" such as the one studied here, there often occur phenomena of non-uniqueness beyond the trivial one stemming from invariance under Euclidean motions. These phenomena, which have been thoroughly studied (see Chillingworth et al [4]) do not happen in the problem at hand. The reason is that traction problems require, e.g. in the case of vanishing traction, certain compatibility conditions ("equilibration conditions") on the load, namely that the total force and total torque it exerts on the body be zero. In our case, where the load is given by the pull of the body's own gravitational field, these quantities are a priori zero.

We also point out that we are not able to make statements on the global problem, i.e. what happens far away from the natural state. For this one would invoke variational techniques, in particular the powerful methods introduced into the subject by Ball (see e.g. [2]).

Let $\mathbf{B}$ be an open, bounded, connected subset of $\mathbf{R^3}$ with smooth boundary. The domain $\mathbf{B}$ ("body") is our reference configuration. We also consider 1-1 maps $\phi : \mathbf{B} \to \mathbf{R^3}$ ("physical space"), $x_i = \phi_i(X_A)$. Let $\mathbf{B}^\phi \subset \mathbf{R^3}$ be the image of $\mathbf{B}$ under ϕ. Then the basic field equations are as follows:

$$- \, div_x T^\phi = \rho \; grad_x U \qquad \text{in } \mathbf{B}^\phi \tag{1}$$

$$\Delta_x U = 4\pi G \rho \qquad \text{in } \mathbf{R^3} \tag{2}$$

Here G is the Newton constant, U is the gravitational potential, T^ϕ is the symmetric Cauchy stress tensor, the mass density ρ satisfies $\rho = n\rho_0$ with ρ_0 a positive constant in $\mathbf{B}^\phi$ and $\rho_0 \equiv 0$ in $\mathbf{R^3} \setminus \mathbf{B}^\phi$. Finally, the number density $n : \mathbf{B}^\phi \to \mathbf{R^3}$ is given by $n(x) = \det \nabla f(x)$, where f is the inverse map of ϕ. We assume $n > 0$ in $\overline{\mathbf{B}^\phi}$. Let us remark that Equ.(1) and Equ.(2) also describe perfect fluids, namely if it is assumed that $T^\phi = pI$, where the pressure p is a function just of n and $I_{ij} = \delta_{ij}$.

We first make some observations on the equations (1) and (2) separately. The divergence structure of (1) implies that its right-hand side, say b, satisfies a compatibility condition, as follows: Let $\xi_i(x)$ be a Killing vector of $\mathbf{R^3}$, considered as flat Euclidean space, i.e. of the form

$$\xi_i(x) = c_i + \omega_{ij} x_j \Leftrightarrow \partial_i \xi_j + \partial_j \xi_i = 0, \tag{3}$$

where c_i and $\omega_{ij} = \omega_{[ij]}$ are constants. Then, upon scalar multiplication of (1) with ξ and integrating over $\mathbf{B}^\phi$, we easily find that

$$\int_{\mathbf{B}^\phi} b\cdot\xi + \int_{\partial\mathbf{B}^\phi} t^\phi\cdot\xi = 0, \tag{4}$$

where $t^\phi_i = T^\phi_{ij}\nu^\phi_j$ with ν^ϕ the outward unit-normal of $\partial\mathbf{B}^\phi$.
On the other hand the "load" b inserted on the r.h.side of (1) has the property that it gives no contribution to Equ.(4). This is seen as follows. Define a symmetric 2-tensor Θ by

$$\Theta = \frac{1}{4\pi G}(grad_x U\otimes grad_x U - \frac{1}{2}I\ grad_x U\cdot grad_x U). \tag{5}$$

Then Equ. (2) implies

$$div_x\Theta = \rho\ grad_x U \tag{6}$$

Suppose, in addition, that U satisfies

$$U = O(\frac{1}{|x|}) \qquad\qquad \text{on } \mathbf{R^3}\setminus\mathbf{B}^\phi \tag{7}$$

Operating with ξ on (6) as before on (1), but with integration over $\mathbf{R^3}$, we find that the load b used in Equ.(1) is "automatically equilibrated" in the above sense. Put differently, choosing for ξ the three translation Killing vectors, this statement amounts to saying that the force exerted on the body by its own gravitational field is zero. Similarly, using three rotational Killing vectors, implies the vanishing of the gravitational self-torque.
We want to solve the coupled system (1) and (2) subject to no-traction boundary conditions, namely that t^ϕ_i be zero on $\partial\mathbf{B}^\phi$, which is a free boundary. To make the problem tractable it is thus important to write the above equations as PDE's on $\mathbf{B}$, rather than $\mathbf{B}^\phi$, using the Piola transform. With the definition $T_{iA} = n^{-1}f_{A,j}T^\phi_{ij}$, one finds (see e.g. [5]) that

$$-\ div_X T = \rho_0\ grad_x U. \tag{8}$$

If T^ϕ_{ij} is solely a function of $f_{A,i}$, T_{iA} can be viewed as a function of $(\nabla\phi)_{i,A} = F_{i,A}$. This follows from the chain rule for differentiation. The potential $U(x)$, satisfying (2) and (7), is given in physical space by

$$U(x) = -\ G\rho_0 \int_{\mathbf{B}^\phi}\frac{n(x')}{|x-x'|}\ d^3x'. \tag{9}$$

Consequently (8) takes the form

$$-\ \partial_A T_{iA} = G\rho_0\int_{\mathbf{B}}\frac{\phi_i(X)-\phi_i(X')}{|\phi(X)-\phi(X')|^3}d^3X' \tag{10}$$

and the compatibility conditions (4), using (3), result in

$$\int_{\partial \mathbf{B}} t = 0 \tag{11}$$

$$\int_{\partial \mathbf{B}} t \wedge \phi(x) = 0, \tag{12}$$

where $t_i = T_{iA}\nu_A$. Our aim is to solve (10) for ϕ, subject to the boundary conditions

$$t|_{\partial \mathbf{B}} = 0. \tag{13}$$

We assume that

A1 $T_{iA}(\nabla\phi) = 0$, when $\nabla\phi = I$

A2 The linearization-at-($\phi = id$) of the operator $div_X T$ is strongly elliptic, in other words, $a_{iAjB} = \frac{\partial T_{iA}(F)}{\partial F_{jB}}$ satisfies $a_{iAjB} = a_{jBiA}$ and $a_{iAjB}|_{F=I} v_i v_j V_A V_B \geq 0$

The physical meaning of condition (A2) is as follows: The natural state is usually supposed to be such that $a_{iAjB}|_{F=I} = \mu(\delta_{ij}\delta_{AB} + \delta_{Aj}\delta_{iB}) + \lambda\delta_{iA}\delta_{jB}$ for constants μ and λ, the Lamé moduli. The ellipticity condition (A2) is then equivalent to the inequalities $\mu > 0, 2\mu + \lambda > 0$. A different interpretation of (A2) would be by saying that plane waves propagating according to the linearized-at-$F = I$ time dependent equations have real frequency.
We note that (at least) condition (A2) rules out fluids. And, indeed, the theorem of the next section stating the existence of bodies of arbitrary shape, can not possibly apply to perfect fluids, as noted in the Introduction.

2 The Main Theorem

We now state our precise assumptions. As configuration space $\mathcal{C}$ we take maps $\phi : \mathbf{B} \to \mathbf{R^3}$ with $\phi_i \in W^{2,p}(\mathbf{B})^3$, $p > 3$, and in it $\mathcal{C}_\epsilon \subset \mathcal{C}$ of maps $\phi_i = X_i + h_i$ with $\| h \|_{2,p} < \epsilon$. For ϵ sufficiently small, ϕ is C^1 - map close to the identity with C^1 - inverse (see Appendix). For the stress tensor $T_{iA}(\nabla\phi)$ we assume that it is in $C^2(\mathbf{R^9}, \mathbf{R^9})$ and that it satifies conditions $(A1, 2)$ of section 1, wherefrom it follows [13] that the operator $\phi \mapsto T(\nabla\phi)$ is a C^1 - mapping from $W^{2,p}(\mathbf{B})^3$ to $W^{1,p}(\mathbf{B})^9$. Our main result is
Theorem: For sufficiently small G there is a solution $\phi \in \mathcal{C}_\epsilon$ of (10) subject to (13). This solution is unique provided

$$h_i(\vec{0}) = 0,\ \partial_{[i} h_{j]}(\vec{0}) = 0. \tag{14}$$

(We assume that $\vec{0} \in \mathbf{B}$.)

We remark that the smallness-condition for G can of course, by scaling, be rephrased by $G\rho_0 \ll \frac{|T|}{L^2}$, where L is a typical length scale of $\mathbf{B}$ and $|T|$ an upper bound for the stress tensor. Our method of proof follows the geometrical treatment of the Stoppelli theorem [12] due to LeDret [6].
Proof: Consider the map $E : \mathcal{C} \mapsto \mathcal{Y} = \{(b,t) \in W^p(\mathbf{B})^3 \times W^{1-1/p,p}(\partial\mathbf{B})^3, p > 3\}$, the operator of nonlinear elasticity given by

$$\phi \mapsto (-div_X T(\nabla\phi), T(\nabla\phi)\nu) \tag{15}$$

The operator E is well-defined and C^1 (see [6]). Now recall from the discussion of Section 1 that elements $(b,t) \in \mathcal{Y}$ lying in $E(\phi)$ satisfy the compatibility ("equilibration") conditions, namely

$$\int_{\mathbf{B}} b + \int_{\partial\mathbf{B}} t = 0 \tag{16}$$

$$\int_{\mathbf{B}} b \wedge \phi(X) + \int_{\partial\mathbf{B}} t \wedge \phi(X) = 0 \tag{17}$$

The set $\mathcal{Y}_\phi$ of pairs $(b,t) \in \mathcal{Y}$ satisfying (16) and (17), for given ϕ, is a vector subspace of $\mathcal{Y}$ of codimension 6, when $\phi \in \mathcal{C}_\epsilon$. (More precisely, it follows from the results of LeDret, and is easy to check, that the only elements $\phi \in \mathcal{C}$, at which $\mathcal{Y}_\phi$ fails to have codimension 6, are those for which the image $\phi(\mathbf{B})$ is parallel to a fixed direction $v \in \mathbf{R^3}$ - which is impossible if $\phi \in \mathcal{C}_\epsilon$.) Let us choose some complement S of $L_e \subset \mathcal{Y}_e$, where $L_e = \mathcal{Y}_{id}$ and define a projection $P : \mathcal{Y}_\phi \mapsto L_e$. The linear maps $P : \mathcal{Y}_\phi \mapsto L_e$ are isomorphisms and C^1 (see [6], proof of Proposition 1.4). Next consider the ("live") load afforded by the gravitational force, i.e. $b = G\bar{U}(\phi)$ with $G\bar{U}_i(\phi)$ given by the right-hand side of (10). By explicit calculation, or from the discussion of section1, it follows that $\bar{U}(\phi) \subset \mathcal{Y}_\phi$. Note that this requires $\mathbf{B}$ to be connected. If $\mathbf{B}$ had several connected components, $\bar{U}(\phi)$ would be automatically equilibrated only with respect to the whole of $\mathbf{B}$, whereas (16) and (17) would for the operator E be required to hold separately for each connected component of $\mathbf{B}$. It is thus important that we have only one body.
We want to solve the equation

$$E(\phi) = G\bar{U}(\phi) \text{ on } \mathbf{B}, \qquad t_i = 0 \text{ on } \partial\mathbf{B} \tag{18}$$

for small G. We know from A1 that $\phi = id$ is a solution for $G = 0$. We write

$$F(G,\phi) = E(\phi) - G\bar{U}(\phi), \tag{19}$$

with F viewed as a function $\mathbf{R} \times \mathcal{C}_\epsilon \mapsto \mathcal{Y}$. In the Appendix we show that $\bar{U}$, whence F, maps $\mathcal{C}_\epsilon$ into $\mathcal{Y}$ in a C^1 - fashion. If we now compute the linearization of F at $\phi = id$ for $G = 0$, we find that this is a map from $\mathcal{C}$ to $\mathcal{Y}$ which is not surjective, due to the presence of the equilibration conditions. To get round this difficulty, consider the modified operator

$$F'(G,\phi) = P(E(\phi)) - GP(\bar{U}(\phi), \tag{20}$$

with F' viewed as a map $\mathbf{R} \times \mathcal{C}_\epsilon \mapsto L_e$. Clearly every solution of $F' = 0$ is also a solution of $F = 0$. If, in addition, we eliminate the translational and rotational freedom by replacing $\mathcal{C}$ by $\mathcal{C}_{sym}$, consisting of all elements $\phi_i = X_i + h_i$ in $\mathcal{C}$ for which $u_i(\vec{0}) = 0$ and $\partial_{[i}u_{j]}(\vec{0}) = 0$, it follows from standard linear theory (see [9], Lemma 3.17 of Chap.7), that the linearization-at-($\phi = id$) of F' at $G = 0$ is an isomorphism $\mathcal{C}_{sym} \mapsto L_e$. Hence our claim follows from the implicit function theorem.

3 Appendix

The Poisson integral used in the body of the paper is given by

$$(\bar{U}_i[\phi])(X) = \int_{\mathbf{B}} \frac{\phi_i(X) - \phi_i(X')}{|\phi(X) - \phi(X')|^3} d^3X' \tag{21}$$

Here $\mathbf{B} \subset \mathbf{R^3}$ is bounded, not empty, open and connected with $\partial\mathbf{B}$ smooth and $\phi_i \in W^{2,p}(\mathbf{B})^3$, $p > 3$. Furthermore $\phi_i(X) = X_i + h_i(X)$, and we assume that $\parallel h \parallel_{2,p} < \epsilon$, ϵ small. It follows from Sobolev embedding that $\partial_j h_i$ is small , in particular bounded in $\bar{\mathbf{B}}$. By the mean–value theorem we infer that

$$|h(X) - h(X')| < C'|X - X'| \tag{22}$$

This is immediate for $\mathbf{B}$ convex, otherwise see [5], p. 224. Making ϵ smaller, if necessary, we have that

$$|\partial h(X)| < \frac{1}{2} \tag{23}$$

Consequently, there exist positive constants E, E' such that

$$E|X - X'| \leq |\phi(X) - \phi(X')| \leq E'|X - X'| \tag{24}$$

It immediately follows that $\bar{U}_i[\phi] \in C^0(\bar{\mathbf{B}})^3$. Thus $\bar{U}_i$ is a bounded map of $\mathcal{U}_\epsilon(id_\mathbf{B}) \subset W^{2,p}(\mathbf{B})^3 \to C^0(\bar{\mathbf{B}})^3$, whence to to $W^{0,p}(\mathbf{B})^3$. Here $\mathcal{U}_\epsilon(id_\mathbf{B})$ denotes the set of ϕ's in the ϵ–ball centered at the identity map. We want to show $\bar{U}_i$ is actually C^1. We first compute the Gateaux–derivative (directional derivative) of $\bar{U}_i$, namely

$$\left[\frac{d}{dt}\bar{U}_i[\phi^t]\right]_{t=0} = D\bar{U}_i[\phi^0] \cdot v \tag{25}$$

where $\phi_i^t(X) = X_i + tv_i(X), v_i \in W^{2,p}(\mathbf{B})^3$. First observe that the expression

$$\frac{1}{t}\left\{\frac{\phi_i^t(X)-\phi_i^t(X')}{|\phi^t(X)-\phi^t(X')|^3}-\frac{\phi_i^0(X)-\phi_i^0(X')}{|\phi^0(X)-\phi^0(X')|^3}\right\} \tag{26}$$

for $X \neq X'$, converges pointwise for $t \to 0$ to

$$\frac{v_i(X)-v_i(X')}{|\phi^0(X)-\phi^0(X')|^3}-\frac{3(\phi_i^0(X)-\phi_i^0(X'))(\phi_j^0(X)-\phi_i^0(X'))(v_j(X)-v_j(X'))}{|\phi^0(X)-\phi^0(X')|^5} \tag{27}$$

Next note the following chain of elementary inequalities: b_1, b_0 vectors $\in \mathbf{R^3}$ (or $\mathbf{R^n}$)

$$\left|\ \frac{1}{|b_1|}-\frac{1}{|b_0|}\ \right| \leq \frac{|b_1-b_0|}{|b_1||b_0|} \tag{28}$$

$$\left|\ \frac{1}{|b_1|^2}-\frac{1}{|b_0|^2}\ \right| \leq \left(\frac{1}{|b_1|^2|b_0|}+\frac{1}{|b_0|^2|b_1|}\right)|b_1-b_0| \tag{29}$$

$$\left|\ \frac{1}{|b_1|^3}-\frac{1}{|b_0|^3}\ \right| \leq \left(\frac{1}{|b_1|^3|b_0|}+\frac{1}{|b_1|^2|b_0|^2}+\frac{1}{|b_1||b_0|^3}\right)|b_1-b_0| \tag{30}$$

a_1, b_1, a_0, a_1 vectors $\in \mathbf{R^n}$

$$\left|\frac{a_1}{|b_1|^3}-\frac{a_0}{|b_0|^3}\right| \leq \left(\frac{1}{|b_1|^3|b_0|}+\frac{1}{|b_1|^2|b_0|^2}+\frac{1}{|b_1||b_0|^3}\right)|a_1||b_1-b_0| \tag{31}$$

$$+\frac{|a_1-a_0|}{|b_0|^3} \tag{32}$$

setting

$$a_1 = b_1 = \phi^t(X)-\phi^t(X') \tag{33}$$

$$a_0 = b_0 = \phi^0(X)-\phi^0(X') \tag{34}$$

we find

$$\begin{aligned}&\left|\frac{\phi_i^t(X)-\phi_i^t(X')}{|\phi^t(X)-\phi^t(X')|^3}-\frac{\phi_i^0(X)-\phi_i^0(X')}{|\phi^0(X)-\phi^0(X')|^3}\right| \leq \\ &\quad\left(\frac{2}{|\phi^0(X)-\phi^0(X')|^3}+\frac{1}{|\phi^t(X)-\phi^t(X')|^2|\phi^0(X')-\phi^0(X)|}\right. \\ &\quad\left.+\frac{1}{|\phi^t(X')-\phi^t(X)||\phi^0(X')-\phi^0(X)|^2}\right)t\ |v(X')-v(X)|\end{aligned} \tag{35}$$

It follows, using (24), that the sequence in (26) is bounded by a positive, t–independent function, whose integral over $X' \in \mathbf{B}$ is a bounded function of X

in $\bar{\mathbf{B}}$. So, by dominated convergence, the previous limit, whence the Gateaux derivative actually exists. But the linear operator defined by the directional derivative $v \in W^{2,p}(\mathbf{B})^3 \rightarrow W^{0,p}(\mathbf{B})^3$, is clearly bounded. So, by a standard theorem (see e.g. [1], Corollary 2.4.10), $\bar{U}(\phi)$ is a C^1–functional.

Acknowledgments: The authors thank Horst Beyer and Alan Rendall for useful discussions.

References

[1] Abraham,R., Marsden,J.E., Ratiu,T. 1988 *Manifolds, tensor analysis, and applications* New York: Springer

[2] Ball,J.M. 1998 The calculus of variations and material science. Quarterly Appl.Math. **56**, 719-740

[3] Beig,R., Simon,W. 1992 On the Uniqueness of Static Perfect-Fluid Solutions in General Relativity. Commun.Math.Phys. **144**, 373-390

[4] Chillingworth,D.R.J., Marsden,J.E., Wan,Y.H. 1982 Symmetry and bifurcation in three dimensional elasticity Part 1. Arch. Rat.Mech.An. **80** 295-331; Part2. ibid. **83** 363-395; Part3. ibid. **84** 203-233.

[5] Ciarlet,P.G. 1988 *Mathematical Elasticity Vol.1:Three-Dimensional Elasticity.* Amsterdam:North-Holland

[6] LeDret,H. 1987 Structure of the set of equilibrated loads in nonlinear elasticity and applications to existence and nonexistence. J.Elasticity **17**, 123-141

[7] Lichtenstein,L. 1933 *Gleichgewichtsfiguren rotierender Flüssigkeiten* Berlin:Springer

[8] Lindblom,L., Masood-ul-Alam,A.K.M. 1994 On the Spherical Symmetry of Static Stellar Models. Commun.Math.Phys.**162** 123-145

[9] Marsden,J.E., Hughes,T.J.R. 1983 *Mathematical Foundations of Elasticity* New York:Dover

[10] Park, J., 2000 Spherically Symmetric Static Solutions of the Einstein Equations with Elastic Matter Source. General Rel. Grav. **32**, 235-252

[11] Rein,G. 2000 Stationary and static stellar dynamical models with axial symmetry. Nonlinear Analysis **41**, 313-344.

[12] Stoppelli,F. 1954 Un teorema di esistenza e di unicita relativo alle equazioni dell'elastostatica isoterma per deformazioni finite. Ricerche Matematiche **3**, 247-267

[13] Valent,T. 1978 Sulla differentiabilita dell'operatore di Nemytsky. Rend.Acc.Naz.Lincei **65**, 15-26