Input-output relations for a 3-port grating coupled Fabry-Perot cavity

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We analyze an optical 3-port reflection grating by means of a scattering matrix formalism. Amplitude and phase relations between the 3 ports, i.e. the 3 orders of diffraction are derived. Such a grating can be used as an all-reflective, low-loss coupler to Fabry-Perot cavities. We derive the input output relations of a 3-port grating coupled cavity and find distinct properties not present in 2-port coupled cavities. The cavity relations further reveal that the 3-port coupler can be designed such that the additional cavity port interferes destructively. In this case the all-reflective, low-loss, single-ended Fabry-Perot cavity becomes equivalent to a standard transmissive, 2-port coupled cavity.

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In a recent experiment a 3-port reflection grating coupled Fabry-Perot cavity with high Finesse was demonstrated.[1] The experiment was motivated by the idea that a 3-port reflection grating should be able to provide two important features for advanced interferometry: low overall optical loss and no light transmission through optical substrates.[2] In advanced interferometers, such as for gravitational wave detectors, these couplers will be crucial for achieving the optimal combination of extremely high power laser fields, materials of high mechanical quality factors for suspended optics and cryogenic temperatures to reduce optics and suspension thermal noise.[3] Previously, a different concept for all-reflective linear Fabry-Perot cavities based on a two-port reflection grating was experimentally demonstrated.[4] In this approach the reflection grating was used in a 1st order Littrow mount where the input-output relations of the cavity are analogous to those of a conventional cavity with transmissive mirrors. The major disadvantage of this concept is, however, that it relies on high 1st order diffraction efficiency requiring deep grating structures that are associated with high scattering losses. Contrary to that, the concept demonstrated in[1] used a 2nd order Littrow mount and relies on low 1st order diffraction efficiency which can be achieved by very shallow grating structures with smaller scattering losses. The latter approach is therefore better suited for low-loss couplers to high-finesse cavities, a stringent requirement in high-power laser interferometry. A grating used in 2nd order Littrow mount, however, has 3 coupled ports in contrast to mirrors where one input port is only coupled to 2 output ports. Knowledge of the phase relations of the three ports is essential for the derivation of the input-output relations of the cavity.

In this letter we derive the amplitude and phase relations of an optical 3-port device by means of the scattering matrix formalism. We restrict ourselves to a symmetric coupling between port 2 and the other two ports 1 and 3 described by η_1, see Fig. 1. Generally, optical devices like mirrors and beam splitters can be described by a complex valued $n \times n$ scattering matrix $\boldsymbol{S}$,[5] where n input ports are represented by a vector $\boldsymbol{a}$ with the components a_i which are the complex amplitudes of the incoming waves at the ith port. The outgoing amplitudes b_i are represented by the vector $\boldsymbol{b}$. The coupling of input and output ports is given by the following equation

$$\boldsymbol{b} = \boldsymbol{S} \times \boldsymbol{a}\,. \tag{1}$$

For a loss-less device $\boldsymbol{S}$ must be unitary. Reciprocity of the device demands $|S_{ij}| \equiv |S_{ji}|$, where S_{ij} denotes an element of the matrix $\boldsymbol{S}$. The magnitudes of the scattering coefficients are unique for a given device. The phase angles of the matrix elements however can be changed by choosing different reference planes in the various input and output arms. One can therefore derive different scattering matrices for the same device. Nevertheless, certain phase relationships between the different coefficients must be maintained. Transmissive mirrors are commonly used to couple light into Fabry-Perot cavities. The input output relations of such cavities are well understood. Essential for their derivation is the knowledge of the phase relations at the mirrors for the reflected and transmitted beams. A conventional two-coupled-port mirror with amplitude reflectance ρ and transmittance τ for example is generally described by

$$\boldsymbol{S}_{2p} = \begin{pmatrix} \rho & \tau \\ \tau & -\rho \end{pmatrix} \quad \text{or} \quad \boldsymbol{S}_{2p} = \begin{pmatrix} \rho & i\tau \\ i\tau & \rho \end{pmatrix}. \tag{2}$$

Using either one of these matrices one can derive the amplitude reflectance r_{FP} and transmittance t_{FP} of a cavity consisting of 2 partially transmitting mirrors of reflectivities ρ_0, ρ_1. The length of the cavity is expressed by the tuning parameter $\phi = \omega L/c$, where ω is the angular frequency of the light and c the speed of light, thus one obtains

$$r_{\mathrm{FP}} = [\rho_0 - \rho_1 \exp(2i\phi)]d\,, \tag{3}$$

$$t_{\mathrm{FP}} = -\tau_0\tau_1 \exp(-i\phi)d \tag{4}$$

where $\rho_{0,1}$ and $\tau_{0,1}$ denote the reflectance and transmittance of the two cavity mirrors, respectively, and we have introduced the resonance factor

$$d = [1 - \rho_0\rho_1 \exp(2i\phi)]^{-1}\,. \tag{5}$$

The power gain $g_{\rm FP}$ inside the cavity is given by

$$g_{\rm FP} = |\tau_0 d|^2. \tag{6}$$

The 3-port coupler used in reference [1] can be represented by the following scattering matrix

$$\boldsymbol{S}_{3p} = \begin{pmatrix} \eta_2\exp(i\phi_2) & \eta_1\exp(i\phi_1) & \eta_0\exp(i\phi_0) \\ \eta_1\exp(i\phi_1) & \rho_0\exp(i\phi_0) & \eta_1\exp(i\phi_1) \\ \eta_0\exp(i\phi_0) & \eta_1\exp(i\phi_1) & \eta_2\exp(i\phi_2) \end{pmatrix}. \tag{7}$$

As stated above the grating is assumed to be symmetrical with respect to the grating normal. The grating period and wavelength of light is chosen in such a way that for normal incidence only the 0th and 1st order diffraction are present. The magnitudes of their amplitude reflection coefficients are denoted with ρ_0 and η_1 respectively. For incidence at the 2nd order Littrow angle the 0th, 1st, and 2nd diffraction orders are present with the magnitudes of the reflection coefficients η_0, η_1 and η_2 as depicted in Fig. 1. From the unitarity condition of $\boldsymbol{S}$ we find the energy conservation law

$$\begin{aligned} \rho_0^2 + 2\eta_1^2 &= 1\,, && (8) \\ \eta_0^2 + \eta_1^2 + \eta_2^2 &= 1\,. && (9) \end{aligned}$$

We denote the phase shift associated with the 0th, 1st,

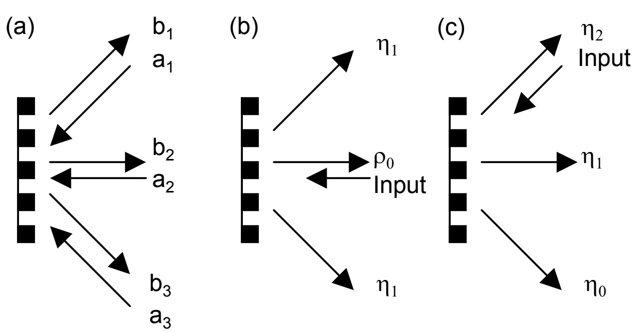

Fig. 1. A 3-port reflection grating: (a) labelling of the input and output ports, (b) amplitudes of reflection coefficients for normal incidence, (c) for 2nd order Littrow incidence.

and 2nd diffraction orders with ϕ_0, ϕ_1 and ϕ_2, respectively. As for mirrors the values for the phases are not unique. Reflection from a mirror is equivalent to 0th order diffraction of a grating. In analogy to the right matrix of equation (2) we demand no phase shift for 0th order diffraction and therefore set $\phi_0 = 0$. From the unitarity requirement of $\boldsymbol{S}$ the remaining phases can be calculated yielding the following possible set of phases

$$\begin{aligned} \phi_0 &= 0\,, && (10) \\ \phi_1 &= -(1/2)\arccos[(\eta_1^2 - 2\eta_0^2)/(2\rho_0\eta_0)]\,, && (11) \\ \phi_2 &= \arccos[-\eta_1^2/(2\eta_2\eta_0)]. && (12) \end{aligned}$$

We emphasize that the phases ϕ_1 and ϕ_2 are functions of the diffraction efficiencies and therefore vary depending on the properties of the grating. This contrasts to the properties of mirrors, where the phase shift between transmitted and reflected beams is independent of the transmittance and reflectance coefficients. Since the phase ϕ_2 is a real number, the modulus of the argument of the arccos in equation (12) must be smaller or equal to one and the following upper and lower limits for η_0 and η_2 for a given reflectivity ρ_0 can be derived, namely

$$\eta_{0,{\max \atop \min}} = \eta_{2,{\max \atop \min}} = (1 \pm \rho_0)/2. \tag{13}$$

It should be noted that these limits are fundamental in the sense that a reflection grating can only be designed and manufactured having diffraction efficiencies within these boundaries. Equations (8) - (13) provide a full set of 3-port coupling relations.

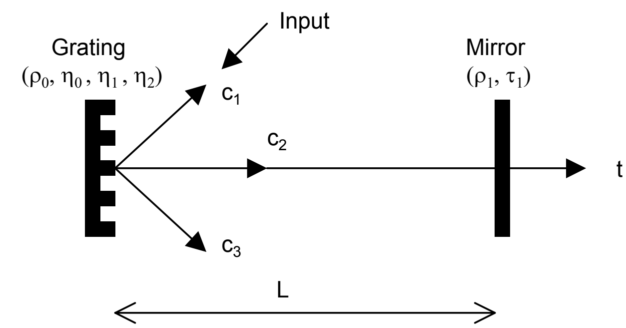

Fig. 2. A Fabry-Perot cavity with a 3-port grating coupler and a conventional end mirror. The amplitudes of the fields of interest (c_1, c_2, c_3, t) are indicated by arrows.

Knowledge of the scattering matrix $\boldsymbol{S}$ in Eq. (7) enables the calculation of input-output relations of interferometric topologies. Here we consider the 3-port grating coupled Fabry-Perot cavity. The grating cavity is formed by placing a mirror with amplitude reflectivity ρ_1 at a distance L parallel to the grating surface as it is illustrated in Fig. 2. To characterize the cavity, the amplitudes c_1, c_3 for the two waves reflected from the cavity and the intra-cavity amplitude c_2 are calculated as a function of the cavity length. Assuming unity input and no input at port 3 the cavity is described by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boldsymbol{S}_{3p} \times \begin{pmatrix} 1 \\ \rho_1 c_2 \exp(2i\phi) \\ 0 \end{pmatrix}. \tag{14}$$

Solving for the amplitudes yields

$$\begin{aligned} c_1 &= \eta_2\exp(i\phi_2) + \eta_1^2\exp[2i(\phi_1+\phi)]d, && (15) \\ c_2 &= \eta_1\exp(i\phi_1)d, && (16) \\ c_3 &= \eta_0 + \eta_1^2\exp[2i(\phi_1+\phi)]d\,, && (17) \\ t &= i\tau_1 c_2\exp(i\phi)\,. && (18) \end{aligned}$$

where $\phi = \omega L/c$ is the tuning parameter, d is given according to equation (5), and t is the amplitude of the light transmitted through the cavity. The light power at the different ports is proportional to the squared moduli of the amplitudes. The power gain inside the cavity is given by $|c_2|^2 = |\eta_1 d|^2$ analogous to equation (6) for a conventional cavity. In contrast to the power gain the power in the two reflecting ports $|c_1|^2$ and $|c_3|^2$ depend on η_2 and η_0. Fig. 3 illustrates how the power out of

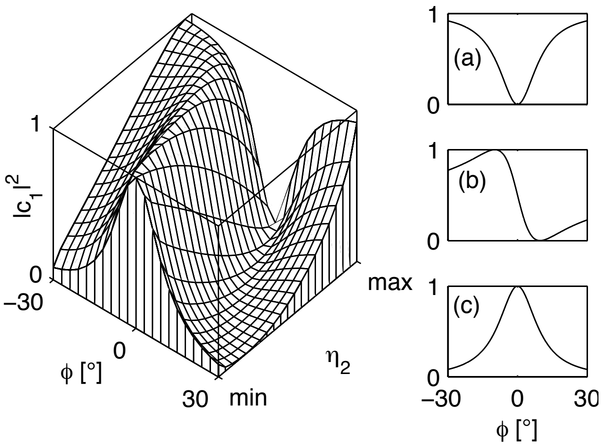

Fig. 3. Power $|c_1|^2$of cavity back reflecting port for gratings of different values of η_2. Left: Power as a function of ϕ and η_2. Right: Power as a function of ϕ for (a) $\eta_2 = \eta_{2,\max}$; (b) $\eta_2 = [(\eta_{2,\max}^2 + \eta_{2,\min}^2)/2]^{1/2}$; (c) $\eta_2 = \eta_{2,\min}$. Cavity parameters: $\rho_0^2 = 0.5$, $\rho_1 = 1$.

the back reflecting port varies as a function of η_2 and the tuning ϕ of the cavity. For simplicity a cavity with a perfect end mirror $\rho_1 = 1$ is assumed. For a coupler with $\eta_2 = \eta_{2,\max}$, the cavity does not reflect any light back to the laser for a tuning of $\phi = 0$. This corresponds to an impedance matched cavity that transmits all the light on resonance. For a coupler with $\eta_{2,\min}$, the situation is reversed and all the light is reflected back to the laser. For all other values of η_2 the back-reflected intensity has intermediate values and as a significant difference to conventional cavities: the intensity as a function of cavity-tuning is no longer symmetric to the $\phi = 0$ axis.

Finally, we investigate the influence of loss in the cavity for a coupler with $\eta_{2,\min}$. Fig. 4 illustrates the effect of an end mirror with transmittance $\tau_1 > 0$ to the power of the two reflecting ports of the cavity on resonance. As a result, apart from the intra-cavity field, losses affect mainly the back-reflecting port (dashed dotted line). The effect on the dark port (solid line) is minor as it stays essentially dark as long as the loss τ_1^2 is small compared to the coupling η_1^2.

In conclusion, we have investigated the three-port reflection grating and have derived its coupling relations. A three-port device can be used to couple light into a Fabry-Perot cavity. The input output relations of such a 3-port coupled cavity have revealed substantial differ-

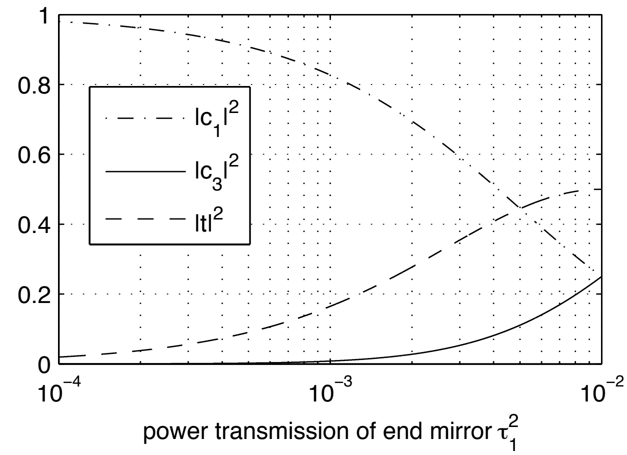

Fig. 4. Powers of the two reflected ports and the transmitting port as a function of end mirror transmittance τ_1^2 for a coupler with $\rho_0^2 = 0.99$ and $\eta_2 = \eta_{2,\min}$ for a tuning of $\phi = 0$.

ences from a conventional cavity. A grating with minimal η_2 is suitable for a coupler to an arm cavity (single ended cavity) of a gravitational wave Michelson interferometer. On resonance all power is reflected back to the beam splitter of the interferometer. Hence no power is lost to the additional port. This enables power recycling which is used in all first and probably also in second and third generation detectors. Furthermore we can calculate the phase signals carried by the fields in equations (15) and (17) when changing the cavity length L and find that the additional port splits a cavity strain signal. However, the complete strain signal is still accessible to detection. This will be the subject of a more detailed investigation in an upcoming paper.

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