

# Mapping the vacuum structure of gauged maximal supergravities: an application of high-performance symbolic algebra

**Thomas Fischbacher**

*Max-Planck-Institut für Gravitationsphysik  
Albert-Einstein-Institut  
Mühlenberg 1  
14476 Golm, Germany  
tf@aei.mpg.de*

## Abstract

The analysis of the extremal structure of the scalar potentials of gauged maximally extended supergravity models in five, four, and three dimensions, and hence the determination of possible vacuum states of these models is a computationally challenging task due to the occurrence of the exceptional Lie groups  $E_6$ ,  $E_7$ ,  $E_8$  in the definition of these potentials. At present, the most promising approach to gain information about nontrivial vacua of these models is to perform a truncation of the potential to submanifolds of the  $G/H$  coset manifold of scalars which are invariant under a subgroup of the gauge group and of sufficiently low dimension to make an analytic treatment possible.

New tools are presented which allow a systematic and highly effective study of these potentials up to a previously unreached level of complexity. Explicit forms of new truncations of the potentials of four- and three-dimensional models are given, and for  $N = 16$ ,  $D = 3$  supergravities, which are much more rich in structure than their higher-dimensional cousins, a series of new nontrivial vacua is identified and analysed.



Dedicated to the memory of  
Yvonne Schießler  
1976–2002

A great woman, a brilliant dentist, and my closest friend.



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# Chapter 1

## Introduction

While three of the four fundamental interactions – namely the strong, weak, and electromagnetic force – can be cast into the framework of a unified quantum theory, one immediate problem that has to be overcome to also include the last interaction, gravity, is that the machinery of non-abelian gauge theory, which serves so well for the other three forces, cannot be applied directly. From the particle physicist’s point of view, one important point is that the quantum of the field mediating gravity, the graviton, is not a spin-1 (vector) particle, as for the other interactions, but a spin-2 (traceless symmetric tensor) particle. Since an unified ‘theory of everything’ ultimately should not only quantitatively describe the observed particles and forces, but also give a good explanation why just this spectrum of matter and interaction particles is found in nature, theoretical approaches that unify particles of different spin deserve special interest.

While a continuous symmetry whose generators form a Lie algebra cannot connect particles of different spin in an interacting theory (due to the Coleman-Mandula theorem [6]), this can be achieved by employing a symmetry that contains *anticommuting* spinorial generators [29, 31, 49, 53]; such a symmetry, which connects bosons with fermions, has been dubbed *supersymmetry*. Soon after the advent of supersymmetry, it has been realized that a supersymmetric version of Einstein’s theory of gravity exists [24, 10]<sup>1</sup>. This can just as well be regarded as a theory with local (i.e. gauged) supersymmetry – superficially, since the anticommutator of two supersymmetry transformations is a translation, local supersymmetry gives rise to spacetime diffeomorphisms. Vice versa, since supersymmetry is not an internal symmetry, it must be made a local symmetry when going to curved spacetime.

As a single supersymmetry transformation maps bosons to fermions and

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<sup>1</sup>see e.g. [48] for a comprehensive introduction

vice versa by changing the spin by  $1/2$ , and has vanishing square due to the anticommuting property, particles are grouped into supermultiplets containing a fermionic superpartner for every boson and vice versa. Now, it is possible to introduce more than one independent supersymmetry transformation [16], and hence make these multiplets spawn a larger range of spins, up to a maximal number of  $N = 8$  independent supersymmetry transformations [8] combining the helicity  $+2$  graviton with the helicity  $-2$  graviton in one unique supermultiplet that entirely fixes the particle content of the theory.

Two observations deserve special attention here: first, the total number of spinorial components of eight four-dimensional real (Majorana, hence four-component) spinors is just the same as that of a fundamental eleven-dimensional spinor, indicating the possibility to construct a theory of an ‘extremal’ eleven-dimensional  $N = 1$  supergravity [7], which gives rise to various other supergravity theories upon dimensional reduction, but does not produce massless particles with spin greater than 2 (which, as common lore tells, cannot consistently be coupled to gravity [1]) upon reduction to four dimensions. Second, the equations of motion of maximal extended supergravity in  $D = 4$  possesses a global  $E_{7(+7)}$  symmetry, and in fact, this symmetry conjectured from the study of dimensional reduction of  $D = 11$  supergravity to various dimensions considerably facilitated the construction of maximal  $N = 8$   $D = 4$  supergravity. In general, the supergravity obtained from toroidal compactification of  $D = 11$  supergravity on a  $d$ -torus will, after dualization of all emerging  $p$ -forms to lowest possible rank, exhibit a global  $E_{11-d(11-d)}$  symmetry [9]. For the particular case of  $d = 8$ , this gives the largest finite-dimensional exceptional group  $E_8$ .

Soon after the construction of extended supergravity, it was realized that part of the extra global symmetry can be made local[17]; in particular, in  $D = 4$ , one can introduce a local  $SO(8)$  gauge symmetry under which the 28 vector fields of the supermultiplet transform in the adjoint representation [12, 13]. The construction is quite remarkable in many aspects: first, it contains gravity as well as nonabelian gauge symmetry (albeit with a gauge group too small to accommodate the fundamental interactions of the standard model), second, it is, up to the size of the nonabelian coupling constant, a totally rigid construction (just as  $D = 11$  supergravity itself) that does not feature any additional arbitrary parameters, like the number of particles per type, or their couplings, third, re-establishing supersymmetry after gauging introduces a potential for the scalars with a very rich extremal structure, giving rise to a large number of different possible vacua for such a theory.

While the quantization of general relativity suffers from the fundamental problem that in the summation over all histories leading from a given initial to a given final state, contributions from intermediate (virtual) particles



with arbitrarily high momentum cannot be brought under control by the adjustment of finitely many parameters, as is required for a physical theory to have predictive power; that is, general relativity suffers from nonrenormalizability. Since intermediate bosons and fermions give similar high-energy contributions of opposite sign, it was hoped that supergravity may also provide a renormalizable, or rather, *finite* quantum theory of gravity, due to cancellation of positive and negative divergences.

The state of affairs reported so far roughly mirrors the focus of very fruitful research on quantum gravity and unified theories during the first half of the 1980's; in particular the study of siblings of  $D = 11$   $N = 1$  supergravity obtained by dimensional reduction to all possible spaces of lower dimension turned out to provide valuable insights into the general underlying structure of such theories. With the advent of superstring theory, which contains supergravity as a low-energy limit and seemingly bears a better prospect to eventually lead to a viable 'Theory of Everything', research activity on supergravity generally declined, but was revived by recent major discoveries; of prime importance is of course the observation that the five possible consistent ten-dimensional superstring theories do not stand in isolation, as was initially believed, but are interrelated by a 'web of dualities' which also connects them to *eleven-dimensional* supergravity [54] and is regarded as compelling evidence for the existence of a fundamental eleven-dimensional 'mother' theory which has supergravity as its low energy limit and was preliminarily named *M-theory*. At present, it is hard to tell what will emerge from the up to now fragmentary knowledge of the structure of M-theory, but it is conjectured that discrete remnants of the hidden global  $E_{11-d(11-d)}$  symmetries that emerge in toroidal compactifications of  $D = 11$  supergravity are already present in the original theory, and hence, obtaining a better understanding of these exceptional symmetries in supergravity is of major importance.

Results on vacua of gauged supergravities presented in this thesis, which mainly deals with  $D = 3$ , are of considerable direct relevance to the celebrated conjectured AdS/CFT duality originally proposed by Maldacena [37] for which a large body of evidence has been collected by now. It is claimed that supergravity in Anti-deSitter (AdS) space has a dual description in terms of a conformal field theory (CFT) living on the boundary of this AdS space. Thus, for example, supergravity solutions interpolating between different vacua are in particular believed to encode a renormalization group flow for the corresponding CFT [25, 27, 3, 2].

Even by itself, the three-dimensional maximal gauged supergravity models on which we focus here are quite remarkable. First, the underlying exceptional symmetry is the maximal one,  $E_{8(+8)}$ , second, and in marked contrast

to higher-dimensional supergravity theories, the vector fields appear in these models via non-abelian Chern-Simons terms rather than the usual Yang-Mills terms. Since these terms which correspond to a non-abelian duality between vectors and scalars that does not have an analogue in higher dimensions cannot be obtained by any known type of dimensional reduction, it is impossible to get the  $N = 16$   $D = 3$  gauged supergravity models from  $D = 11$  supergravity, in contrast to higher-dimensional supergravity as well as the ungauged and half-maximal ( $N = 8$ )  $D = 3$  models. This might hint at a new supergravity theory beyond the known  $D = 11$  one, cf. [43]. Furthermore, these models, which are the ‘most symmetric’ ones in three dimensions known, exhibit a vast richness in structure; since the choice of a subgroup of  $E_{8(+8)}$  as gauge group is far less constrained in  $D = 3$  than in higher dimensions, as an arbitrary number of unwanted vectors can be dualized away into scalars, this theory allows many more gaugings than its higher-dimensional cousins, some of which, like  $G_{2(-14)} \times F_{4(-20)}$  or  $E_{7(+7)} \times SL(2)$  even possess exceptional group factors<sup>2</sup>, none of which are possible isometry groups of an eight-manifold and hence could emerge in dimensional reduction of  $D = 11$  supergravity. Furthermore, in contrast to e.g.  $N = 8$ ,  $D = 4$ , even for all the noncompact gauge groups, there is a vacuum of maximal supersymmetry where all scalar VEVs vanish, and the corresponding background supergroups also encompass the exceptional supergroups  $G(3)$  and  $F(4)$ . Turning on scalar fields, the landscape of the corresponding potentials of the various models on the coset  $E_{8(+8)}/SO(16)$  are probably the most intricate analytical potentials encountered so far in supergravity and beyond. Indeed, from the computational aspect, there is good evidence that, while conceptually simple, it is probably entirely impossible to write down an explicit analytic expression for the potential on the whole 128-dimensional manifold of scalars, since the number of terms produced would exceed the number of particles available in the accessible universe by many orders of magnitude!

All this makes the study of the extremal structure of these models an interesting and challenging subject, despite, or maybe even because, at present there is only a fragmentary understanding how they fit into the bigger scheme of things.

This work is organized as follows: the present introductory chapter is intended to provide background information and lay out the conventions for subsequent chapters and was written with the intention to also give non-experts in the field of supergravity/superstring theory who are interested in this work due to the group-theoretical tools developed here at least a conceptual overview over the underlying physical motivation. In chapter 2, we recite

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<sup>2</sup>and in fact, all exceptional groups occur here

some important steps of the constructions of ungauged and gauged maximal three-dimensional supergravity models. Chapter 3 is dedicated to the study of the compact semi-simple gauge group  $SO(8) \times SO(8)$ , while noncompact gaugings are investigated in chapter 4. In order to demonstrate the generality of our tools, we present a single instructive example from four-dimensional supergravity in chapter 5 which increases the level of complexity to which the corresponding supergravity scalar potential has been studied by three orders of magnitude. The novel approaches to symbolic algebra employed by the tools developed to make such a deep investigation possible are discussed in chapter 6. Chapter 7 concludes. Restricted potentials which would have been too lengthy to be given in the main text have been moved to appendix *A*; since numerous new results on the vacuum structure of supergravity theories are claimed by this work which might appear quite bold, being well out of reach of previously existing technology, appendix *B* was added giving explicit LISP code which should enable the reader (in conjunction with the `LambdaTensor` software package) to redo the calculations leading to the main results of this work and serve as a starting point for further investigations.

## 1.1 General Conventions

Interpretation of conventions plays a key role for this work for two reasons: first, among our main results there are ‘charts’ where to find nontrivial vacua of gauged maximal supergravity theories that are given as elements of exceptional Lie groups (not algebras); since it is difficult to develop an intuitive understanding of these groups, it is much easier to overlook mistakes that can be traced back to a mis-understanding of conventions than in many other branches of supergravity/string theory where enough structure is visible to give additional hints at correct interpretations of formulae. Second, due to the richness of structure in three-dimensional supergravity theories, it is attractive to employ computer aid in the quest for a systematic and exhaustive treatment, which naturally favours a low-level presentation and interpretation of formulae, since all conventions eventually have to be spelled out in detail for the machine. For the sake of reproducibility of the results presented, a separate appendix is devoted to `LambdaTensor` definitions corresponding to key formulae given in the main text.

Basically, one could – at least in principle – for the sake of definiteness resort to abandoning all conventions, including the Einstein summation convention, explicitly spelling out all summations, symmetrization factors, and subgroup embeddings. However, it is easy to convince oneself that doing so would clutter many formulae so badly that they become totally unpalat-

able. Hence, finding the right level of abbreviation hidden in conventions is a nontrivial issue for the presentation of results.

The set of rules given in the following should provide a complete and unambiguous definition how to interpret every new result in the main text. Although this may seem unusual and maybe even unnecessary at first, in particular since many of these rules are widely used and well known, it is nevertheless hard to draw a line between those for which there is general consensus and those for which there is not.

- (*Einstein summation convention*) When in a product, an index name appears twice, the product is to be prefixed with a sum of this index over all possible values of the index with this name. Index counting starts at 1<sup>3</sup> (Note that we do not require this index to show up once as upper and once as lower; for example, in a sum like  $P^j Q^j$  over  $SU(8)$  indices, applying these rules will yield a non- $SU(8)$ -covariant (but  $SO(8)$  covariant) quantity.) Indices which are explicitly bound otherwise, e.g. by a summation sign, are of course exempt from this summation convention.
- (*Index raising and lowering*) When a tensor that was defined with an upper (lower) index  $I$  is used with the corresponding index as lower (upper), the corresponding group metric (or its inverse) is used implicitly to lower (raise) this index.
- (*Index interpretation and index splitting*) Index names are (potentially composed) glyphs from various alphabets. The assignment of index names to various group representations is explained in the text; whenever (with one sole exception given below) a tensor that is defined with some index  $J$  corresponding to transformation under the group  $G$  occurs in a formula where in place of this index  $J$ , an index  $A$  of a different set of glyphs corresponding to a representation of a subgroup  $H \subset G$  shows up, and if there is a decomposition of the  $G$ -representation labeled by indices of the type of  $J$  with respect to the subgroup  $H$  such that the range of index values for  $A$  directly corresponds to a sub-range of index values of the index  $J$ , then the  $A$  index is implicitly promoted to a  $J$  index by offsetting; that is, one implicitly inserts an extra embedding tensor factor  $H_{JA}$  which contains entries 1 for  $J = A + \langle offset \rangle$ , zero for all other values of  $J$  and  $A$ .

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<sup>3</sup>Here, we follow usual conventions although it would be more natural to start counting at zero in the context of index splitting.

- (*One-to-many index splitting*) If a  $G$ -index decomposes to a group of  $H$ -indices in such a way that there is a one-to-one correspondence between every possible combination for this group of  $H$ -indices and a corresponding  $G$ -index, and a group of  $H$ -indices appears on a tensor in a place where this tensor was defined with the corresponding  $G$  index, then this group of indices is implicitly promoted to the corresponding  $G$  index by the insertion of an appropriate extra embedding tensor factor  $H_{g_{h_1 h_2 \dots h_n}}$  whose entries are all from the set  $\{0; 1\}$ .<sup>4</sup>
- ( $\epsilon$  and  $\delta$ ) The fully antisymmetric rank- $N$  tensor is denoted by  $\epsilon_{i_1 i_2 \dots i_N}$ . Its entries are  $+1$  for even index permutations,  $-1$  for odd permutations, and  $0$  otherwise. The tensors  $\delta_{j_1}^{i_1}$ ,  $\delta_{j_1 j_2}^{i_1 i_2}$ ,  $\delta_{j_1 j_2 \dots j_k}^{i_1 i_2 \dots i_k}$ , etc. are completely antisymmetric in their upper, resp. lower indices and have entries  $1/k!$ , resp.  $-1/k!$  if the sequence of indices  $i_1 \dots i_k$  can be obtained from  $j_1 \dots j_k$  by an even, resp. odd, index permutation, and  $0$  otherwise. Hence,  $\delta_{i_1 i_2 \dots i_k}^{j_1 j_2 \dots j_k} \delta_{j_1 j_2 \dots j_k}^{m_1 m_2 \dots m_k} = \delta_{i_1 i_2 \dots i_k}^{m_1 m_2 \dots m_k}$ .
- (*Index symmetrization*) Wherever a pair of indices is enclosed by antisymmetrizing brackets  $[\mathbf{ab}]$ ,<sup>5</sup> they are to be substituted by otherwise unbound indices  $\mathbf{cd}$  and an extra factor  $\delta_{\mathbf{cd}}^{\mathbf{ab}}$  is to be introduced into the corresponding summand. Likewise for antisymmetrizing in more than two indices. (Since the translation of index symmetrization between typographical and machine representation is a little bit awkward, and hence error-prone if lots of formulae have to be translated manually, we frequently do not make use of it even if it could be applied.)
- ( *$SO(N)$  adjoint indices*) If  $g_1, g_2$  are glyphs from the alphabet used to designate  $SO(N)$  vector indices, then composed glyphs of the form  $\underline{[g_1 g_2]}$  will designate  $SO(N)$  adjoint indices. These  $N(N-1)/2$  glyphs are consecutively labeled either by  $1, 2, \dots, N(N-1)/2$  or equivalently by  $\underline{[12]}, \underline{[13]}, \dots, \underline{[1N]}, \underline{[23]}, \dots, \underline{[(N-1)N]}$ .
- (*Index mapping*) Whenever index embedding has to be performed that cannot be achieved by simple index-range splitting (like, for example, the split of an  $SO(N)$  adjoint index into a pair of  $SO(N)$  vector indices), then an explicit embedding tensor is given. Conventionally, all

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<sup>4</sup>Note that these index splitting rules may generate ambiguities if one is not careful in the choice of alphabets. This has to be avoided. Furthermore, index groups obtained by implicit splitting will usually be offset a bit from other indices not belonging to the group to aid visual distinction. Note that this extra convention is not well suited for recursive application.

<sup>5</sup> $\mathbf{a}$  and  $\mathbf{b}$  are not indices, but names for indices, and hence typeset in boldface here

embedding tensors are named  $H$  and distinguished by the types of indices they are carrying. (For example, if indices  $I, J$  designate  $SO(16)$  vector indices, and an index  $[IJ]$  a  $SO(16)$  adjoint index in the range  $1 \dots 120$ , then  $H_{IJ}^{[IJ]}$  is the corresponding embedding tensor.) Hence, such embedding tensors  $H$  themselves are exempt from the implicit index promotion rules given above. Unless stated otherwise, embedding tensors  $H$  are defined in such a way that when they are used to map a collection of index values to a unique other collection of index values, the corresponding entry is  $\pm 1$ ; for related entries, the lexicographically first one will be  $+1$ . (For example, an  $SO(16)$  adjoint index corresponds to two pairs of different  $SO(16)$  vector indices, but a pair of different  $SO(16)$  vector indices is mapped to a single adjoint index, so the corresponding  $H_{IJ}^{[IJ]}$  has  $H_{12}^{[12]} = -H_{21}^{[12]} = 1$ .)

- (*Gamma matrices*)  $SO(16)$  resp.  $SO(8)$  Gamma matrices (to be defined in the next section) are denoted by  $\Gamma$ , resp.  $\gamma$ . For both types,  $\gamma_{\alpha\beta}^{i_1 \dots i_n} = \gamma_{\alpha\gamma_1}^{j_1} \gamma_{\gamma_2\gamma_1}^{j_2} \gamma_{\gamma_2\gamma_3}^{j_3} \dots \gamma_{\beta\gamma_{(n-1)}}^{j_n} \delta_{j_1 \dots j_n}^{i_1 \dots i_n}$ , and likewise,  $\gamma_{\dot{\alpha}\dot{\beta}}^{i_1 \dots i_n} = \gamma_{\gamma_1\dot{\alpha}}^{j_1} \dots \gamma_{\gamma_{(n-1)}\dot{\beta}}^{j_n} \delta_{j_1 \dots j_n}^{i_1 \dots i_n}$ . Furthermore, we define  $\gamma_{\gamma\delta}^{\alpha\beta} := \delta_{\rho\lambda}^{\alpha\beta} \gamma_{\rho\dot{\alpha}}^i \gamma_{\lambda\dot{\beta}}^i$ , as well as  $\gamma_{\dot{\alpha}\dot{\beta}}^{\alpha\beta\gamma\delta} := \delta_{\alpha'\beta'\gamma'\delta'}^{\alpha\beta\gamma\delta} \gamma_{\alpha'\dot{\alpha}}^i \gamma_{\beta'\dot{\beta}}^i \gamma_{\gamma'\dot{\gamma}}^k \gamma_{\delta'\dot{\delta}}^k$  and  $\gamma_{\dot{\alpha}\dot{\beta}\gamma\delta}^{ij} := \delta_{\alpha'\beta'\gamma'\delta'}^{\alpha\beta\gamma\delta} \gamma_{\alpha'\dot{\alpha}}^i \gamma_{\beta'\dot{\beta}}^k \gamma_{\gamma'\dot{\gamma}}^k \gamma_{\delta'\dot{\delta}}^j$ ; obviously, since  $\gamma_{\dot{\alpha}\dot{\beta}}^{\alpha\beta}$  and  $\gamma_{\dot{\alpha}\dot{\beta}}^{ij}$  are then discerned by the types of label only, we must not use ambiguous expressions like  $\gamma_{\dot{\alpha}\dot{\beta}}^{78}$ , but rather write either  $\gamma_{\dot{\alpha}\dot{\beta}}^{ij} \delta_i^7 \delta_j^8$  or  $\gamma_{\dot{\alpha}\dot{\beta}}^{\alpha\beta} \delta_\alpha^7 \delta_\beta^8$ .

## 1.2 $E_{8(+8)}$ conventions

The Lie algebra  $E_{8(+8)}$  plays a key role for this work. Hence we spell out all the conventions in full detail. Following [41], we define

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma_e &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \quad (1.1)$$

from which we obtain  $SO(8)$   $\gamma$ -matrices using the tensor  $G_{i\lambda\mu\rho}$  implementing the  $2 \times 2 \times 2 \rightarrow 8$  mapping<sup>6</sup>

$$\begin{aligned} G_{1111} &= 1 & G_{2112} &= 1 & G_{3121} &= 1 & G_{4122} &= 1 \\ G_{5211} &= 1 & G_{6212} &= 1 & G_{7221} &= 1 & G_{8222} &= 1 \end{aligned} \quad (1.2)$$

<sup>6</sup>Note that this matrix tensoring convention, which seems to be more widespread, accidentally is just the opposite of that implicitly used in [21]

as well as the abbreviation

$$Z_{\alpha\dot{\beta}}(\sigma_{(A)}; \sigma_{(B)}; \sigma_{(C)}) = \sigma_{(A)\alpha_1\dot{\beta}_1} \sigma_{(B)\alpha_2\dot{\beta}_2} \sigma_{(C)\alpha_3\dot{\beta}_3} G_{\alpha\alpha_1\alpha_2\alpha_3} G_{\dot{\beta}\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3} \quad (1.3)$$

via

$$\begin{aligned} \gamma^1 &= Z(\sigma_e; \sigma_e; \sigma_e) & \gamma^2 &= Z(\sigma_1; \sigma_z; \sigma_e) \\ \gamma^3 &= Z(\sigma_e; \sigma_1; \sigma_z) & \gamma^4 &= Z(\sigma_z; \sigma_e; \sigma_1) \\ \gamma^5 &= Z(\sigma_1; \sigma_x; \sigma_e) & \gamma^6 &= Z(\sigma_e; \sigma_1; \sigma_x) \\ \gamma^7 &= Z(\sigma_x; \sigma_e; \sigma_1) & \gamma^8 &= Z(\sigma_1; \sigma_1; \sigma_1) \end{aligned} \quad (1.4)$$

from which we form  $SO(16)$   $\Gamma$ -matrices using to the splitting  $J \rightarrow (j, \bar{k})$  of  $SO(16)$  vector and  $A \rightarrow (\alpha\beta, \dot{\gamma}\dot{\delta})$ ,  $\dot{A} \rightarrow (\alpha\dot{\beta}, \dot{\gamma}\dot{\delta})$  of Majorana-Weyl spinor and co-spinor indices by

$$\begin{aligned} \Gamma_{A\dot{A}}^I &= H_i^I H_{\alpha\beta}^A H_{\dot{\gamma}\dot{\delta}}^{\dot{A}} \delta_{\alpha\dot{\gamma}} \gamma_{\beta\dot{\delta}}^i + H_i^I H_{\dot{\alpha}\dot{\beta}}^A H_{\dot{\gamma}\dot{\delta}}^{\dot{A}} \delta_{\dot{\alpha}\dot{\gamma}} \gamma_{\dot{\beta}\dot{\delta}}^i \\ &\quad + H_i^I H_{\alpha\beta}^A H_{\dot{\gamma}\dot{\delta}}^{\dot{A}} \delta_{\beta\dot{\delta}} \gamma_{\alpha\dot{\gamma}}^i - H_i^I H_{\dot{\alpha}\dot{\beta}}^A H_{\dot{\gamma}\dot{\delta}}^{\dot{A}} \delta_{\dot{\beta}\dot{\delta}} \gamma_{\dot{\gamma}\dot{\alpha}}^i, \\ \Gamma_{AB}^{IJ} &= \delta_{KL}^{IJ} \Gamma_{A\dot{C}}^K \Gamma_{B\dot{C}}^L \end{aligned} \quad (1.5)$$

If we denote  $SO(16)$  adjoint indices running from 1 to 120 by  $[IJ]$ , which naturally decompose into  $SO(16)$  vector indices  $I, J$  and split  $E_{8(8)}$  adjoint indices  $\mathcal{A} \rightarrow (A, [IJ])$ , then  $E_{8(8)}$  structure constants are given by

$$\begin{aligned} f_{AB}^C &= -H_{IJ}^A H_{KL}^B H_C^{MN} \delta_{I'J'}^{IJ} \delta_{K'L'}^{KL} \delta^{I'K'} \delta_{L'J'}^{MN} \\ &\quad + \frac{1}{4} \Gamma_{AB}^{IJ} (H_{IJ}^A H_A^B H_C^B + H_B^A H_{IJ}^B H_C^A - H_A^A H_B^B H_C^C). \end{aligned} \quad (1.6)$$

In the literature, the intermediate  $SO(16)$  adjoint index in the splitting chain  $\mathcal{A} \rightarrow [IJ] \rightarrow IJ$  frequently is not displayed explicitly. Instead, a modification of the usual Einstein summation convention is introduced where one has to include an extra factor  $1/2$  whenever a sum is performed over a pair of antisymmetric indices, correcting  $IJ, JI$  double-counting. Since it is not entirely clear that this rule will not leave room for interpretation in some subtle cases (like the definition of  $E_{8(+8)}$  tensors in index-split notation), we will try to avoid it by explicitly including index splitting projection tensors in our formulae just as above when presenting new results.

The conventionally normalized Cartan-Killing metric for  $E_{8(+8)}$

$$\eta_{AB} = \frac{1}{2g} f_{A\mathcal{P}}^{\mathcal{Q}} f_{B\mathcal{Q}}^{\mathcal{P}} \quad (1.7)$$

where  $g$  is the dual Coxeter number of the Lie algebra (30 for  $E_8$ ) is then given by

$$\eta_{AB} = -\delta_{AB}, \quad \eta_{[IJ][KL]} = \delta_{[IJ][KL]}. \quad (1.8)$$

$E_{8(+8)}$  is the only simple Lie algebra for which the fundamental and adjoint representation are the same and this property plays an important role in the construction of  $N = 16$   $D = 3$  supergravity. It can be expressed by the relation

$$\mathcal{V}^{-1}t^{\mathcal{M}}\mathcal{V} = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}t^{\mathcal{A}} \Leftrightarrow \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} = \frac{1}{60}\text{tr} (t^{\mathcal{M}}\mathcal{V}t_{\mathcal{A}}\mathcal{V}^{-1}). \quad (1.9)$$

Furthermore, we define compact and noncompact  $E_{8(+8)}$  generators  $X_{\underline{[IJ]}}$ ,  $Y_A$  by

$$\left(X_{\underline{[IJ]}}\right)^{\mathcal{C}}{}_{\mathcal{B}} = f_{\underline{[IJ]}\mathcal{B}}{}^{\mathcal{C}}, \quad (Y_A)^{\mathcal{C}}{}_{\mathcal{B}} = f_{A\mathcal{B}}{}^{\mathcal{C}}. \quad (1.10)$$

It is convenient to take as a Cartan subalgebra the compact generators  $X_{\underline{[12]}}$ ,  $X_{\underline{[34]}}$ ,  $\dots$ ,  $X_{\underline{[15\,16]}}$ ; this gives the conventional choice for the set of  $E_8$  root vectors as 120 vectors of the form  $\{\pm e_i \pm e_j | i, j \in \{1, \dots, 8\}\}$  plus 128 vectors of the form  $\{\frac{1}{2}(\pm e_1 \pm e_2 \dots \pm e_8)\}$  where the total number of minus signs is even. Most explicitly, the ladder operators corresponding to the simple roots<sup>7</sup> are then given by  $((t_A)^{\mathcal{C}}{}_{\mathcal{B}} := f_{A\mathcal{B}}{}^{\mathcal{C}})$ :

$$\begin{aligned} T_{+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}} &= +t_{84} + i t_{86} - i t_{100} + t_{102} \\ T_{0+1-1\ 0\ 0\ 0\ 0\ 0} &= +t_{159} + i t_{160} - i t_{171} + t_{172} \\ T_{0\ 0+1-1\ 0\ 0\ 0\ 0} &= +t_{184} + i t_{185} - i t_{194} + t_{195} \\ T_{0\ 0\ 0+1-1\ 0\ 0\ 0} &= +t_{205} + i t_{206} - i t_{213} + t_{214} \\ T_{0\ 0\ 0\ 0+1-1\ 0\ 0} &= +t_{222} + i t_{223} - i t_{228} + t_{229} \\ T_{0\ 0\ 0\ 0\ 0+1-1\ 0} &= +t_{235} + i t_{236} - i t_{239} + t_{240} \\ T_{0\ 0\ 0\ 0\ 0\ 0+1+1} &= -t_{244} + i t_{245} + i t_{246} + t_{247} \\ T_{0\ 0\ 0\ 0\ 0\ 0+1-1} &= +t_{244} + i t_{245} - i t_{246} + t_{247} \end{aligned} \quad (1.11)$$

where our complexity conventions are such that for these compact generators, we have e.g.

$$[X_{\underline{[12]}}, T_{+1+1\ 0\ 0\ 0\ 0\ 0\ 0}] = +1 \cdot i T_{+1+1\ 0\ 0\ 0\ 0\ 0\ 0}. \quad (1.12)$$

Note that if we substitute the ladder operator of the first simple root by

$$T_{+1-1\ 0\ 0\ 0\ 0\ 0\ 0} = +t_{130} + i t_{131} - i t_{144} + t_{145} \quad (1.13)$$

then the new set will spawn the maximal compact subalgebra  $SO(16)$ . The ladder operator for the lowest  $E_8$  root is

$$T_{-1-1\ 0\ 0\ 0\ 0\ 0\ 0} = -t_{130} - i t_{131} - i t_{144} + t_{145}. \quad (1.14)$$

---

<sup>7</sup>w.r.t. lexicographical ordering



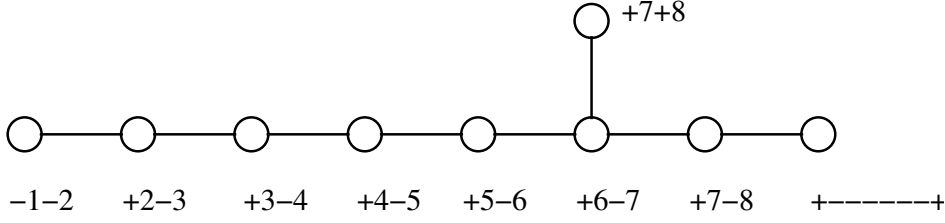


Figure 1.1: The extended Dynkin diagram of  $E_8$

From here on, we will use an abbreviating notation of  $E_8$  roots; for the spinorial roots like  $(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2})$ , we just write a sequence of eight signs, hence ‘+-----+’ in this example, while for roots of the form  $\pm e_i \pm e_j$ , we write  $\pm i \pm j$ , e.g.  $+3 - 4$  for  $(0, 0, +1, -1, 0, 0, 0, 0)$ .

Sometimes, as in the construction of the  $E_{6(+6)} \times SL(3)$  embedding tensor, a noncompact Cartan subalgebra is more useful. Here, we may take the generators  $Y_{\alpha\beta} \delta_\alpha^n \delta_\beta^n, n \in 1, \dots, 8$ . With respect to this Cartan subalgebra, simple roots are given by

$$\begin{aligned}
 T_{+-----+} &= -t_{72} - t_{79} - t_{86} + t_{93} - t_{100} + t_{107} + t_{114} + t_{121} \\
 &\quad - t_{143} + t_{156} + t_{167} + t_{180} - t_{188} - t_{199} - t_{206} + t_{213} \\
 T_{+2-3} &= 2t_{11} + 2t_{18} + t_{130} - t_{145} + t_{185} + t_{194} + t_{222} - t_{229} + t_{245} + t_{246} \\
 T_{+3-4} &= -2t_{20} - 2t_{27} + t_{134} - t_{149} + t_{160} + t_{171} + t_{226} - t_{233} + t_{236} + t_{239} \\
 T_{+4-5} &= -2t_{29} - 2t_{36} + t_{135} + t_{148} - t_{159} + t_{172} + t_{227} + t_{232} - t_{235} + t_{240} \\
 T_{+5-6} &= -2t_{38} - 2t_{45} - t_{134} + t_{149} - t_{160} + t_{171} - t_{226} + t_{233} - t_{236} + t_{239} \\
 T_{+6-7} &= -2t_{47} - 2t_{54} - t_{130} - t_{145} - t_{185} + t_{194} - t_{222} + t_{229} - t_{245} + t_{246} \\
 T_{+7-8} &= +2t_{56} + 2t_{63} + t_{134} + t_{149} - t_{160} + t_{171} + t_{226} + t_{233} - t_{236} + t_{239} \\
 T_{+7+8} &= -2t_{56} + 2t_{63} - t_{134} - t_{149} + t_{160} - t_{171} + t_{226} + t_{233} - t_{236} + t_{239}
 \end{aligned}$$

and complexity conventions are such that

$$[Y_{11}, T_{+1+1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0}] = +1 T_{+1+1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0}. \quad (1.15)$$



# Chapter 2

## Maximal three-dimensional supergravity

### 2.1 Ungauged $N = 16$ $D = 3$ SUGRA

In the following presentation, we closely follow [43]. It is convenient to introduce gauged maximal three-dimensional supergravity theories via incremental definitions that proceed through ungauged maximal three-dimensional supergravity, which was first constructed in [38]; the physical fields of this theory form an irreducible supermultiplet of 128 bosons and 128 fermions which transform as spinors and co-spinors of  $SO(16)$ . In three dimensions, the dreibein  $e_\mu^\alpha$  as well as the gravitini  $\psi_\mu^I$  do not carry propagating degrees of freedom. Due to the hidden invariance of the ungauged theory under global  $E_{8(+8)}$  and local  $SO(16)$  transformations [9], scalar fields can be described by an element  $\mathcal{V}$  of the non-compact coset manifold  $E_{8(+8)}/SO(16)$ . Using the 248-dimensional fundamental  $E_{8(+8)}$  representation,  $\mathcal{V}$  transforms as

$$\mathcal{V}(x) \rightarrow g\mathcal{V}(x)h^{-1}(x), \quad g \in E_{8(+8)}, h(x) \in SO(16). \quad (2.1)$$

Following the conventions of [38] and [43, 42], the scalar fields couple to the fermions via the currents

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \frac{1}{2}Q_\mu^{IJ}X^{IJ} + P_\mu^AY^A. \quad (2.2)$$

Defining the  $SO(16)$  field strength  $Q_{\mu\nu}^{IJ}$  obtained from the connection  $Q_\mu^{IJ}$  via

$$Q_{\mu\nu}^{IJ} := 2(\partial_{[\mu}Q_{\nu]}^{IJ} + Q_\mu^{K[I}Q_\nu^{J]K}), \quad (2.3)$$

as well as the covariant derivative  $D_\mu$

$$D_\mu\psi_\nu^I := \partial_\mu\psi_\nu^I + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\psi_\nu^I + Q_\mu^{IJ}\psi_\nu^J \quad (2.4)$$

$$D_\mu \chi^{\dot{A}} := \partial_\mu \chi^A + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \chi^{\dot{A}} + \frac{1}{4} Q_\mu^{IJ} \Gamma_{\dot{A}\dot{B}}^{IJ} \chi^{\dot{B}} \quad (2.5)$$

this implies the integrability relations

$$Q_{\mu\nu}^{IJ} + \frac{1}{2} \Gamma_{AB}^{IJ} P_\mu^A P_\nu^B = 0, \quad D_{[\mu} P_{\nu]}^A = 0. \quad (2.6)$$

If we define the supercovariant current

$$\hat{P}_\mu^A := P_\mu^A - \bar{\psi}_\mu^I \chi^{\dot{A}} \Gamma_{\dot{A}\dot{A}}^I, \quad (2.7)$$

then the Lagrangian

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{4} e R + \frac{1}{4} e P^{\mu A} P_\mu^A + \frac{1}{2} e^{\lambda\mu\nu} \bar{\psi}_\lambda^I D_\mu \psi_\nu^I \\ & - \frac{i}{2} e \bar{\chi}^{\dot{A}} \gamma^\mu D_\mu \chi^{\dot{A}} - \frac{1}{2} e \bar{\chi}^{\dot{A}} \gamma^\mu \gamma^\nu \psi_\mu^I \Gamma_{\dot{A}\dot{A}}^I P_\nu^A \\ & - \frac{1}{8} e (\bar{\chi} \gamma_\rho \Gamma^{IJ} \chi (\bar{\psi}_\mu^I \gamma^{\mu\nu\rho} \psi_\nu^J - \bar{\psi}_\mu^I \gamma^\rho \psi^{\mu J})) + \bar{\chi} \chi \bar{\psi}_\mu^I \gamma^\nu \gamma^\mu \psi_\nu^I \\ & + e \left( \frac{1}{8} (\bar{\chi} \chi) (\bar{\chi} \chi) - \frac{1}{96} \bar{\chi} \gamma^\mu \Gamma^{IJ} \chi \bar{\chi} \gamma_\mu \Gamma^{IJ} \chi \right) \end{aligned} \quad (2.8)$$

is invariant under the supersymmetry variations

$$\begin{aligned} \delta e_\mu^\alpha &= i \bar{\epsilon}^I \gamma^\alpha \psi_\mu^I \\ \mathcal{V}^{-1} \delta \mathcal{V} &= \Gamma_{\dot{A}\dot{A}}^I \bar{\chi}^{\dot{A}} \epsilon^I Y^A \\ \delta \psi_\mu^I &= D_\mu \epsilon^I - \frac{i}{4} \gamma^\mu \epsilon^J \bar{\chi} \Gamma^{IJ} \gamma_{\mu\nu} \chi \\ \delta \chi^{\dot{A}} &= \frac{i}{2} \gamma^\mu \epsilon^I \Gamma_{\dot{A}\dot{A}}^I \hat{P}_\mu^A. \end{aligned} \quad (2.9)$$

The equation of motion for scalars obtained from this Lagrangian is

$$\begin{aligned} D_\mu \left( e \left( P^{\mu A} - \bar{\psi}_\nu^I \gamma^\mu \gamma^\nu \chi^{\dot{A}} \Gamma_{\dot{A}\dot{A}}^I \right) \right) \\ = \frac{1}{2} \epsilon^{\mu\nu\rho} \bar{\psi}_\mu^I \psi_\nu^J \Gamma_{AB}^{IJ} P_\rho^B + \frac{1}{8} i e \bar{\chi} \gamma^\mu \Gamma^{IJ} \chi \Gamma_{AB}^{IJ} P_\mu^B \end{aligned} \quad (2.10)$$

which can be written in the form (by using the Rarita-Schwinger and Dirac equation for the gravitini and fermions)

$$\partial^\mu (e \mathcal{J}_\mu^{\mathcal{M}}) = 0 \quad (2.11)$$

with  $\mathcal{J}_\mu^{\mathcal{M}}$  the conserved Noether current corresponding to the global  $E_{8(8)}$  symmetry:

$$\begin{aligned} e \mathcal{J}^{\mu\mathcal{M}} = & 2 \mathcal{V}^{\mathcal{M}}{}_{\dot{B}} \hat{P}^{\mu B} - \frac{i}{2} \mathcal{V}_{IJ}^{\mathcal{M}} \bar{\chi} \gamma^\mu \Gamma^{IJ} \chi \\ & - 2e^{-1} \epsilon^{\mu\nu\rho} \left( \mathcal{V}_{IJ}^{\mathcal{M}} \bar{\psi}_\nu^I \psi_\rho^J - i \Gamma_{\dot{A}\dot{A}}^I \mathcal{V}^{\mathcal{M}}{}_{\dot{A}} \bar{\psi}_\nu^I \gamma_\rho \chi^{\dot{A}} \right). \end{aligned} \quad (2.12)$$

Since this current is conserved, we can introduce 248 abelian vector fields  $B_\mu^{\mathcal{M}}$ , defined up to gauge transformations, whose field strength  $B_{\mu\nu}^{\mathcal{M}} =$

$2\partial_{[\mu}B_{\nu]}^{\mathcal{M}}$  obeys  $\epsilon^{\mu\nu\rho}B_{\nu\rho}^{\mathcal{M}} = e\mathcal{J}^{\mu\mathcal{M}}$ . In order to find supersymmetry transformation rules for these vectors, one can generalize the supersymmetry transformations of the 36 vector fields that are obtained by direct dimensional reduction of  $D = 11$  supergravity

$$\delta B_{\mu}^{\mathcal{M}} = -2\mathcal{V}^{\mathcal{M}}_{IJ}\bar{\epsilon}^I\psi_{\mu}^J + i\Gamma_{AA}^I\mathcal{V}_A^{\mathcal{M}}\bar{\epsilon}^I\gamma_{\mu}\chi^A \quad (2.13)$$

which of course has to be compatible with the duality relation for the vectors and the Noether current, which can be re-written in supercovariant form by using the supercovariant field strength

$$\hat{B}_{\mu\nu}^{\mathcal{M}} := B_{\mu\nu}^{\mathcal{M}} + 2\mathcal{V}^{\mathcal{M}}_{IJ}\bar{\psi}_{\mu}^I\psi_{\nu}^J - 2i\Gamma_{AA}^I\mathcal{V}_A^{\mathcal{M}}\bar{\psi}_{[\mu}^I\gamma_{\nu]}\chi^A \quad (2.14)$$

so that it is given by

$$\epsilon^{\mu\nu\rho}\hat{B}_{\nu\rho}^{\mathcal{M}} = 2e\mathcal{V}^{\mathcal{M}}_A\hat{P}^{\mu A} - \frac{i}{2}e\mathcal{V}^{\mathcal{M}}_{IJ}\bar{\chi}\gamma^{\mu}\Gamma^{IJ}\chi. \quad (2.15)$$

This equation defines 248 vector fields as nonlocal, nonlinear functions of the original 128 physical scalars +120  $SO(16)$  gauge degrees of freedom (as long as the equations of motion are obeyed, i.e. the Noether current is divergence-free). By using the integrability equation (2.6), we can obtain the vector equation of motion from the derivative of the duality equation (2.15):

$$\partial_{\nu}B^{\mu\nu\mathcal{M}} = -\frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{V}_{IJ}^{\mathcal{M}}Q_{\nu\rho}^{IJ} + \langle \text{fermionic terms}(\mathcal{V}) \rangle. \quad (2.16)$$

## 2.2 Gauged $N = 16$ $D = 3$ SUGRA

Part of the  $E_{8(+8)}$  symmetry of the theory described in the previous section can be made local; let  $G_0 \subset E_{8(+8)}$  be a subgroup of the global symmetry group  $E_{8(+8)}$  that can be promoted to a gauge group (we will see later that there is a simple group-theoretical restriction for which groups this is possible). Then, (2.1) is replaced by

$$\mathcal{V}(x) \rightarrow g_0(x)\mathcal{V}(x)h^{-1}(x), \quad g_0(x) \in G_0, h(x) \in SO(16). \quad (2.17)$$

The subgroup  $G_0$  of  $E_{8(+8)}$  that will be promoted to a gauge group is characterized by its embedding tensor  $\Theta_{\mathcal{MN}}$ , which is the restriction of the Cartan-Killing metric  $\eta_{\mathcal{MN}}$  to the algebra corresponding to  $G_0$ , and thus is given as a linear combination of projectors onto the simple factors of  $G_0$  (where the relative coefficients in this linear combination turn out to be fixed by group theory, so that only a single gauge coupling parameter survives).

Labeling  $G_0$  adjoint indices by  $m, n, \dots$ , there is an embedding tensor  $\Omega^A_m$  that will map these indices to  $E_8$  indices. By choice of an appropriate basis for the  $E_8$  algebra, we can make these  $\nu = \dim G_0$  indices the first block of indices in a split  $248 = \nu + \dots$ . As usual with such index splitting, we would then use the convention that use of a  $G_0$  index  $m, n, \dots$  in place of an  $E_8$  index  $\mathcal{A}, \mathcal{B}, \dots$  corresponds to the silent omission of some embedding tensor  $\Omega$ . Although this way of splitting indices will in general not be compatible with the index split suitable for the  $SO(16)$  decomposition of  $E_{8(+8)}$  explained in section 1.2, we can nevertheless keep the convention that usage of a  $m, n, \dots$  indices corresponds to the silent omission of such embedding tensors. In particular, our convention shall be that silent indices are to be interpreted as in  $B_\mu^m t_m \equiv B_\mu^A \Theta_{AB} t^B$ .

From now on, let  $g$  be the gauge coupling constant; we first gauge-covariantize derivatives

$$\begin{aligned} \mathcal{V}^{-1} \partial_\mu \mathcal{V} &= \frac{1}{2} Q_\mu^{IJ} X^{IJ} + P_\mu^A Y^A \\ \Rightarrow \mathcal{V}^{-1} \mathcal{D}_\mu \mathcal{V} &= \frac{1}{2} \mathcal{Q}_\mu^{IJ} X^{IJ} + \mathcal{P}_\mu^A Y^A \\ &= \mathcal{V}^{-1} \partial_\mu \mathcal{V} + g B_\mu^m \mathcal{V}^{-1} t_m \mathcal{V} \end{aligned} \quad (2.18)$$

The non-abelian field strength associated with this connection is

$$\mathcal{B}_{\mu\nu}^m = 2\partial_{[\mu} B_{\nu]}^m + g f^m_{np} B_\mu^n B_\nu^p \quad (2.19)$$

and integrability conditions (2.6) become

$$\begin{aligned} \mathcal{Q}_{\mu\nu}^{IJ} + \frac{1}{2} \Gamma_{AB}^{IJ} \mathcal{P}_\mu^A \mathcal{P}_\nu^B &= g \mathcal{B}_{\mu\nu}^m \mathcal{V}_{mIJ} \\ 2D_{[\mu} \mathcal{P}_{\nu]}^A &= g \mathcal{B}_{\mu\nu}^m \mathcal{V}_{mA} \end{aligned} \quad (2.20)$$

Supersymmetry variations for the vectors still are given by (2.13). For the modified currents, they are

$$\begin{aligned} \delta \mathcal{Q}_\mu^{IJ} &= \frac{1}{2} (\Gamma^I J \Gamma^K)_{A\dot{A}} \mathcal{P}_\mu^A \bar{\chi}^{\dot{A}} \epsilon^K + g (\delta B_\mu^m) \mathcal{V}_{mIJ} \\ \delta \mathcal{P}_\mu^A &= \Gamma_{A\dot{A}}^I D_\mu (\bar{\chi}^{\dot{A}} \epsilon^I) + g (\delta B_\mu^m) \mathcal{V}_{mA} \end{aligned} \quad (2.21)$$

(Note that the new currents also depend on  $g$ .) The modifications introduced here violate the supersymmetry of the Lagrangian (2.8) with currents replaced by the covariantized definitions. Restoring local supersymmetry will therefore require additional modifications to the Lagrangian as well as the supersymmetry variations. The extra terms that have to be added to the Lagrangian can be obtained by the Noether procedure and turn out to be (at first order in  $g$ ) Chern-Simons couplings for the vectors and scalar-fermion couplings of Yukawa type as well as (at second order in  $g$ ) a scalar field potential. These couplings as well as the scalar potential are functions of tensors

formed from the scalar matrix  $\mathcal{V}^{\mathcal{M}}{}_A$  as well as the gauge group embedding tensor  $\Theta_{\mathcal{MN}}$ ; one finds that these are  $SO(16)$  representations that can be obtained via suitable linear projections from the tensor

$$T_{AB} := \mathcal{V}^{\mathcal{M}}{}_A \mathcal{V}^{\mathcal{N}}{}_B \Theta_{\mathcal{MN}} \quad (2.22)$$

which is analogous to the T-tensors given in [13] and [30], but (as a consequence of the equivalence of the  $E_8$  fundamental and adjoint representation) is quadratic and not cubic in  $\mathcal{V}$ . From  $T_{AB}$ , we form the tensors<sup>1</sup>

$$\theta = \frac{1}{248} \eta^{\mathcal{MN}} \Theta_{\mathcal{MN}} \quad (2.23)$$

$$A_1^{IJ} = \frac{8}{7} \theta \delta_{IJ} + \frac{1}{7} T_{AB} H_{[IK]}^A H_{[JK]}^B H_{IK}^{[IK]} H_{JK}^{[JK]} \quad (2.24)$$

$$A_2^{I\dot{A}} = -\frac{1}{7} T_{AB} H_{[IJ]}^A H_{IJ}^{[IJ]} H_A^B \Gamma_{\dot{A}\dot{A}}^J \quad (2.25)$$

$$A_3^{\dot{A}\dot{B}} = 2\theta \delta_{\dot{A}\dot{B}} + \frac{1}{48} \Gamma_{\dot{A}\dot{B}}^{IJKL} T_{AB} H_{[IJ]}^A H_{[KL]}^B H_{IJ}^{[IJ]} H_{KL}^{[KL]} \quad (2.26)$$

The full Lagrangian of gauged  $N = 16$   $D = 3$  supergravity is then

$$\begin{aligned} \mathcal{L}^{(CS)} &= \frac{1}{4} g \epsilon^{\mu\nu\rho} B_\mu^m (\partial_\nu B_{\rho m} + \frac{1}{3} g f_{mnp} B_\nu^n B_\rho^p) \\ \mathcal{L}^{(Y)} &= \frac{1}{2} g e A_1^{IJ} \bar{\psi}_\mu^I \gamma^{\mu\nu} \psi_\nu^J + i g e A_2^{I\dot{A}} \bar{\chi}^{\dot{A}} \gamma^\mu \psi_\mu^I + \frac{1}{2} g e A_3^{\dot{A}\dot{B}} \bar{\chi}^{\dot{A}} \chi^{\dot{B}} \\ \mathcal{L}^{(V)} &= \frac{1}{8} g^2 e \left( A_1^{IJ} A_1^{IJ} - \frac{1}{2} A_2^{I\dot{A}} A_2^{I\dot{A}} \right) \\ \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}^{(CS)} + \mathcal{L}^{(Y)} + \mathcal{L}^{(V)} \end{aligned} \quad (2.27)$$

where  $\mathcal{L}_0$  is the Lagrangian of the ungauged theory, but with modified covariantized currents,  $\mathcal{L}^{(CS)}$  are the Chern-Simons couplings of the vectors,  $\mathcal{L}^{(Y)}$  Yang-Mills couplings between scalars and fermions, and  $\mathcal{L}^{(V)}$  is the scalar potential term. Supersymmetry variations of the fermions must be further modified by terms

$$\delta_g \psi_\mu^I = i g A_1^{IJ} \gamma_\mu \epsilon^J, \quad \delta_g \chi^{\dot{A}} = g A_2^{I\dot{A}} \epsilon^I. \quad (2.28)$$

Explicit calculation then shows that this Lagrangian is supersymmetric under the given transformations and the superalgebra closes if and only if the T-tensor satisfies a series of linear relations which state that it is entirely determined by  $A_1, A_2, A_3$ , as well as a set of differential and quadratic identities on  $A_1, A_2, A_3$ . Remarkably, all these constraints can be recast into a single group-theoretic constraint on the gauge group embedding tensor  $\Theta$  which

<sup>1</sup>The index splitting is given in most explicit form here, since these quantities are of prime importance for all subsequent investigations

states that under the decomposition into  $E_8$  irreps  $\mathbf{248} \times \mathbf{248}_{\text{sym}} = \mathbf{1} + \mathbf{3875} + \mathbf{27000}$  the  $\mathbf{27000}$  component vanishes.

Under the maximal compact subgroup  $SO(16)$ , these  $E_8$  irreps decompose into ( $SO(16)$  Dynkin labels given in parentheses):

$$\begin{aligned} \mathbf{3875} &\Rightarrow \mathbf{135}_{(20000000)} + \mathbf{1920}_{(10000001)} + \mathbf{1820}_{(00010000)} \\ \mathbf{27000} &\Rightarrow \mathbf{5304}_{(02000000)} + \mathbf{13312}_{(01000010)} + \mathbf{6435}_{(00000020)} \\ &\quad + \mathbf{1820}_{(00010000)} + \mathbf{128}_{(00000010)} + \mathbf{1}_{(00000000)} \end{aligned} \quad (2.29)$$

The main subject of this work is the extremal structure of the scalar potential term

$$V := -\frac{1}{8}g^2 \left( A_1^{IJ} A_1^{IJ} - \frac{1}{2} A_2^{I\dot{A}} A_2^{I\dot{A}} \right) \quad (2.30)$$

which determines possible vacua of these gauged maximal three-dimensional supergravity theories.

Of the quadratic constraints on the  $T$ -tensor which eventually are subsumed under the gauge group embedding tensor projection condition, Eq. (3.25) of [43]

$$A_1^{IK} A_1^{KJ} - \frac{1}{2} A_2^{I\dot{A}} A_2^{J\dot{A}} = \frac{1}{16} \delta^{IJ} \left( A_1^{KL} A_1^{KL} - \frac{1}{2} A_2^{K\dot{A}} A_2^{K\dot{A}} \right) \quad (2.31)$$

deserves special attention for this work, since it provides a nontrivial consistency check that the relative factor between the  $A_1$  and  $A_2$  contributions to the scalar potential has been chosen correctly in the machine code transliteration of (2.22), (2.23), (2.30).

Taking derivatives of the  $A$ -tensors with respect to an invariant vector field,  $\delta \mathcal{V}^{\mathcal{M}}_{\mathcal{B}} / \delta \Sigma^A = f_{\mathcal{B}}^{CA} \mathcal{V}^{\mathcal{M}}_{\mathcal{C}}$ , we get [43]

$$\frac{\delta A_1^{IJ}}{\delta \Sigma^A} = \frac{1}{14} (\Gamma_{AB}^{IK} T_{KJB} \Gamma_{AB}^{JK} T_{KIB}) \quad (2.32)$$

$$\frac{\delta A_2^{I\dot{A}}}{\delta \Sigma^A} = \frac{1}{14} \Gamma_{B\dot{A}}^J \left( \Gamma_{AC}^{IJ} T_{BC} + \frac{1}{2} \Gamma_{AB}^{MN} T_{IJMN} \right) \quad (2.33)$$

$$\frac{\delta A_3^{\dot{A}\dot{B}}}{\delta \Sigma^A} = -\frac{1}{48} \gamma_{\dot{A}\dot{B}}^{IJKL} \Gamma_{AB}^{KL} T_{IJB} \quad (2.34)$$

from which we obtain by re-writing projections of the  $T$ -tensor as the corresponding  $A$ -tensors the scalar mass matrix of second derivatives for an arbitrary vacuum [19]

$$-4g^{-2} \mathcal{M}_{AB} = -8g^{-2} \frac{\delta^2 V}{\delta \Sigma^A \delta \Sigma^B}$$



$$\begin{aligned}
&= \frac{3}{4} \left( \Gamma_{A\dot{A}}^I A_2^{J\dot{A}} A_2^{J\dot{B}} \Gamma_{B\dot{B}}^I + \Gamma_{A\dot{A}}^I A_2^{J\dot{A}} A_2^{I\dot{B}} \Gamma_{B\dot{B}}^J \right) \\
&\quad + \frac{3}{4} A_1^{IJ} A_1^{IJ} \delta_{AB} - \frac{3}{4} A_1^{II} T_{AB} \\
&\quad + \frac{1}{2} \Gamma_{A\dot{A}}^I A_1^{IJ} A_3^{\dot{A}\dot{B}} \Gamma_{B\dot{B}}^J - \frac{1}{4} \Gamma_{A\dot{A}}^I A_3^{\dot{A}\dot{C}} A_3^{\dot{C}\dot{B}} \Gamma_{B\dot{B}}^I \\
&\quad + \frac{1}{4} \Gamma_{A\dot{A}}^I A_3^{\dot{A}\dot{C}} \Gamma_{C\dot{C}}^I T_{CB}
\end{aligned} \tag{2.35}$$

while the scalar kinetic term is uniformly normalized as

$$\mathcal{L}_{\text{kin}} = \frac{1}{4} e \partial^\mu \Sigma^A \partial_\mu \Sigma^A + \dots, \tag{2.36}$$

independent of the particular vacuum.

While the vectors do not carry propagating degrees of freedom when mass terms are absent, the vacuum will spontaneously break a gauge group  $G_0$  to a compact subgroup  $H_0$ , causing the vectors associated with the broken generators to absorb the corresponding Goldstone bosons and hence become massive by a topological three-dimensional variant of the Brout-Englert-Higgs effect. As explained in [19], the vector mass matrix is obtained by restricting  $\mathcal{V}^M_A \mathcal{V}^{\mathcal{L}}_A \Theta_{\mathcal{N}\mathcal{L}}$  to  $G_0$ , and this information may be extracted directly from

$$\mathcal{M}_{AB}^{\text{vec}} = g T_{AB}. \tag{2.37}$$

The fermionic analogue of this transfer of degrees of freedom from matter fields to previously nonpropagating gauge fields is realized by some of the gravitini absorbing the Goldstinos produced by supersymmetry breaking via the super-Higgs effect [11].

## 2.3 Supergravity in $AdS_3$

In three dimensions, it is possible to define a dual spin connection

$$A_\mu^\alpha := -\frac{1}{2} \epsilon^{abc} \omega_{\mu bc} \tag{2.38}$$

whose field strength

$$F_{\mu\nu}^a := 2\partial_{[\mu} A_{\nu]}^a + \epsilon^a{}_{bc} A_\mu^b A_\nu^c \tag{2.39}$$

can be used to re-write the Einstein-Hilbert term in the action in Chern-Simons form

$$-\frac{1}{4} e R = \frac{1}{4} \epsilon^{\mu\nu\rho} e_\mu{}^a F_{\nu\rho a}. \tag{2.40}$$

Generically, stationary points of the supergravity potentials considered here are not true minima, but either maxima (as is e.g. the origin of the  $D = 4$   $N = 8$  potential) or saddle points. Nevertheless, as has been shown by Breitenlohner and Freedman[5], such stationary points can nevertheless possess at least perturbative stability for negative cosmological constant if none of the second derivatives get too large with respect to the corresponding AdS radius; vacua with remaining supersymmetry will always be stable, since the positive energy argument of Poincare supersymmetry can be generalized to the corresponding supergroups. For nonsupersymmetric vacua, an expansion to second order in gravitational and scalar perturbations shows that, since allowed fluctuations are required to have finite energy and hence fall off to infinity sufficiently fast in AdS, the corresponding positive kinetic energy term can overcompensate a negative second derivative of the potential as long as it is not too large. In  $d$  dimensions, one finds the bound for the scalar mass eigenvalues  $m$  [39]

$$4m^2L^2 \geq (d-1)^2 \quad (2.41)$$

hence, in our case,  $m^2L^2 \geq -1$ . The AdS scale  $L$  is given by

$$L^{-2} = -2V_0 \quad (2.42)$$

where  $V_0$  is the value of the potential at the vacuum. The Ricci tensor is given by

$$R_{\mu\nu} = 4V_0g_{\mu\nu} = \Lambda g_{\mu\nu} \quad (2.43)$$

and the corresponding  $AdS_3$  covariant derivative is given by

$$\mathcal{D}_\mu^\pm := \partial_\mu + \frac{i}{2}\gamma_a (A_\mu^a \pm L^{-1}e_\mu^a) \quad (2.44)$$

with commutator

$$[\mathcal{D}_\mu^\pm, \mathcal{D}_\nu^\pm] = \frac{1}{2}i\gamma_a (F_{\mu\nu}^a + L^{-2}\epsilon^{abc}e_{\mu b}e_{\nu c}) \quad (2.45)$$

Concerning the supersymmetries of a given vacuum, it has been shown [43] by using arguments from [30] that the number of unbroken supersymmetries is the number of eigenvalues  $\alpha$  of  $A_1$  for which

$$16\alpha^2 = A_1^{IJ}A_1^{IJ} - \frac{1}{2}A_2^{I\dot{A}}A_2^{I\dot{A}} = \frac{4}{g^2L^2} \quad (2.46)$$

with  $L$  being the AdS scale, given by the value of the potential at a stationary point via

$$4g^{-2}L^{-2} = -8g^{-2}V_0. \quad (2.47)$$

(Note that unbroken supersymmetry cannot be realized with positive cosmological constant; maximal supersymmetry is equivalent to the vanishing of  $A_2$ .)

# Chapter 3

## $N = 16$ $D = 3$ Supergravity with compact gauge group

### 3.1 On compact gauge groups

The maximal compact subgroup of  $E_{8(+8)}$  is  $SO(16)$ . For the obvious embedding of  $SO(16)$  into  $E_{8(+8)}$ , we have  $\Theta_{[IJ]A} = 0$  and  $\Theta_{AB} = 0$ , hence the projection condition  $P^{27000}\Theta = 0$  for subgroups of this  $SO(16)$  reduces to the condition that  $\Theta$  may only carry the **135** representation of  $SO(16)$ .<sup>1</sup> (Of the other possible pieces, the **5304** is part of the **27000** of  $E_8$ , and the **1** and **1820** have to coincide with the corresponding vanishing pieces of  $\Theta_{AB}$ .) Hence, due to tracelessness, there is no simple compact gauge group. One can show (cf. [43]) that of the maximal subgroups of  $SO(16)$ , only for  $SO(8) \times SO(8)$  there is a choice of the relative gauge coupling constants for which  $P^{27000}\Theta = 0$  (namely  $g_2/g_1 = -1$ ). If we split  $SO(16)$  vector indices via  $I \rightarrow (i, \bar{i})$ , the corresponding embedding tensor is given by<sup>2</sup>

$$\Theta_{AB} = H_A^{IJ} H_B^{KL} \left( H_i^I H_j^J H_k^K H_l^L \delta_{kl}^{ij} - H_{\bar{i}}^I H_{\bar{j}}^J H_{\bar{k}}^K H_{\bar{l}}^L \delta_{\bar{k}\bar{l}}^{\bar{i}\bar{j}} \right). \quad (3.1)$$

The question whether a group embedding tensor  $\Theta$  with vanishing **27000** component can be constructed from a nonmaximal semisimple subgroup of  $SO(16)$  has not been answered yet.

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<sup>1</sup>We may of course apply a boost to rotate (conjugate) this  $SO(16)$  inside  $E_{8(+8)}$ , rendering the  $\Theta_{[IJ]A} = 0$  and  $\Theta_{AB} = 0$  components nonzero. This observation plays an important role for the construction of nonsemisimple gaugings [20].

<sup>2</sup>that is, its (129, 129), (130, 130), ..., (135, 135); (143, 143), ..., (149, 149); ...; (204, 204) components are 2, while its (221, 221), ... (248, 248) components are  $-2$ . Note that with our splitting conventions,  $H_A^{[IJ]} H_{[IJ]}^{IJ} = H_A^{IJ}$ .

## 3.2 Techniques for finding nontrivial vacua

The most fruitful technique for a study of the extremal structure of these potentials known so far appears to be that introduced in [51]: first, choose a subgroup  $H$  of the gauge group  $G$  ( $SO(8)$  in forementioned analysis of  $N = 8, D = 4$  gauged supergravity,  $SO(8) \times SO(8)$  for the case considered here); then, determine a parametrization of the submanifold  $M$  of  $H$ -singlets of the manifold of physical scalars  $P$ . Every point of this submanifold for which all derivatives within  $M$  vanish must also have vanishing derivatives within  $P$ . The reason is that, with the potential  $V$  being invariant under  $G$  and hence also under  $H$ , the power series expansion of a variation  $\delta z$  of  $V$  around a stationary point  $z_0$  in  $M$  where  $\delta z$  points out of the submanifold  $M$  of  $H$ -singlets can not have a  $\mathcal{O}(\delta z)$  term, since each term of this expansion must be invariant under  $H$  and it is not possible to form a  $H$ -singlet from just one  $H$ -nonsinglet. All the stationary points found that way will break the gauge group down to a symmetry group that contains  $H$ .

The general tendency is that, with  $H$  getting smaller, the number of  $H$ -singlets among the supergravity scalars will increase. For  $H$ -singlet spaces of low dimension, it easily happens that the scalar potential does not feature any nontrivial stationary points at all, while for higher-dimensional singlet spaces, the potential soon becomes intractably complicated. For the maximal gauged  $N = 8, D = 4$  model, using the embedding of  $SU(3) \subset SO(8)$  under which the scalars, vectors and co-vectors of  $SO(8)$  decompose into  $\mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}$  gives a case of manageable complexity with six-dimensional scalar manifold for which five nontrivial extrema were given in a complete analysis in [51]. (It seems reasonable to expect further yet undiscovered extrema breaking  $SO(8)$  down to groups smaller than  $SU(3)$ ; cf. chapter 5.)

## 3.3 The potential on submanifolds

### 3.3.1 Partial results for residual symmetry of $SU(3)_{\text{diag}}$

Since it is interesting to see how the extremal structure of  $N = 8, D = 4$  gauged  $SO(8)$  supergravity is related to  $N = 16, D = 3$   $SO(8) \times SO(8)$  gauged supergravity, especially since common lore tells that stationary points in higher dimensions have corresponding counterparts in lower dimensions, it is reasonable to try to lift the construction given in [51] to this case via an embedding of  $E_{7(+7)}$  in  $E_{8(+8)}$ , which works as follows: under the  $SO(8)_L \times SO(8)_R$  gauge group considered here,  $E_{8(+8)}$  decomposes into  $\mathbf{248} \rightarrow (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{28}) + (\mathbf{8}_v, \mathbf{8}_v) + (\mathbf{8}_s, \mathbf{8}_s) + (\mathbf{8}_c, \mathbf{8}_c)$  (where we could apply further triality

rotations to each of the  $SO(8)$  that generate permutations of the left and right  $v, s, c$  labels). Note that the compact generators from  $(\mathbf{8}_v, \mathbf{8}_v)$  extend  $SO(8) \times SO(8)$  to the maximal compact subgroup  $SO(16)$  of  $E_{8(+8)}$ , while the noncompact ones form a  $\mathbf{128}$  of  $SO(16)$ .<sup>3</sup>

When taking the diagonal of both  $SO(8)$  rotations in this scheme, each of the  $(\mathbf{8}, \mathbf{8})$  decompose into  $\mathbf{1} + \mathbf{28} + \mathbf{35}_{v,s,c}$ ; the three singlets (two noncompact, one compact) form an  $SL(2)$ , while the compact generators from  $\mathbf{35}_v$  may be used to extend the diagonal  $SO(8)$  to  $SU(8)$  which is further extended by the  $\mathbf{35}_s$  and  $\mathbf{35}_c$  to  $E_{7(+7)}$  which commutes with forementioned  $SL(2)$ . Hence, under this  $E_{7(+7)} \times SL(2)$ ,  $\mathbf{248} \rightarrow (\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{56}, \mathbf{2})$ .

Re-identifying the  $E_{8(+8)}$  generators corresponding to the  $SU(3)$  singlets, resp. the  $SU(8)$  rotations used to parametrize the singlet manifold given in [51] is straightforward; exponentiating them, however, is not. Looking closely at explicit  $248 \times 248$  matrix representations of these generators reveals that, after suitable re-ordering of coordinates, they decompose into blocks of maximal size  $8 \times 8$  and are (by using a computer) sufficiently easy to diagonalize.<sup>4</sup>

Considerable simplification of the task of computing explicit analytic expressions for the scalar potential by making use of as much group theoretical structure as possible is expected, but nowadays computers are powerful enough to not only allow a head-on approach using explicit 248-dimensional component notation and symbolic algebra on sparsely occupied tensors, but also make this the preferable route when the aim is to investigate high-dimensional singlet spaces. This is explained in detail in chapter 6.

One important complication arises from the fact that the manifold  $M$  of  $SU(3) \subset SU(4) \subset SO(8)_{\text{diag}} \subset SO(8) \times SO(8)$  singlets from the  $\mathbf{128}$  is not six-dimensional, as in the  $D = 4$   $E_{7(+7)}$  case, but twelve-dimensional, since there are two additional singlets from the noncompact directions of  $SL(2)$  as well as four more from the  $(\mathbf{56}, \mathbf{2})$ . While explicit analytic calculation of the potential on submanifolds of  $M$  reveals that the full 12-dimensional potential is<sup>5</sup> out of reach of a complete analysis using standard techniques, it is never-

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<sup>3</sup>This decomposition is particularly easy to understand starting from the set of roots of  $E_8$  given in conventional notation by  $120 \pm e_i \pm e_j$  and  $128 \sum_k \pm 1/2 e_k$  with an even number of minus signs. This decomposition just corresponds to the split of 8-dimensional root vectors into pairs of four-dimensional root vectors, where the Cartan subalgebra of  $E_{8(+8)}$  is taken to be the sum of the conventional Cartan subalgebras of both  $SO(8)$ , and for both  $SO(16)$  and  $SO(8)$ , spinors are those vectors with entries  $\pm 1/2$  and an even number of minus signs (cospinors odd).

<sup>4</sup>One advantage of this brute-force approach in comparison to more sophisticated considerations involving the octonionic structure of  $E_{8(+8)}$  is that no extra work is necessary to handle the same problem for  $E_{7(+7)}$  and  $E_{6(+6)}$ .

<sup>5</sup>by now probably just

theless possible to make progress by making educated guesses at the possible locations of extrema; for example, one notes that for four of the five stationary points given in [51], the angular parameters are just such that the sines and cosines appearing in the potential all have values  $\{-1; 0; +1\}$ . Hence it seems reasonable to try to search for stationary points by letting these compact coordinates of this particular parametrization run through a discrete set of special values only, thereby reducing the number of coordinates.

The immediate problem with the investigation of only proper submanifolds  $M'$  of the full manifold  $M$  of singlets is that, aside from the fundamental inability to prove the exhaustiveness of the list of stationary points with remaining symmetry of at least  $H$  thus obtained, the vanishing of derivatives within  $M'$  does not guarantee to have a stationary point of the full potential. A sieve for true solutions is given by the stationarity condition (4.12) in [43]

$$3 A_1^{IM} A_2^{MA} = A_2^{IB} A_3^{\dot{A}\dot{B}}. \quad (3.2)$$

which has to be checked anyway, since coordinate singularities may also create ‘fake’ stationary points.<sup>6</sup>

By doing the calculation, one finds that on the six-dimensional  $SU(3) \subset SO(8)$  invariant subspace of  $E_{7(+7)}/SU(8)$  (which one can identify as the space  $(SU(2, 1)/(SU(2) \times U(1))) \times (SL(2)/U(1))$ ) given in [51], the  $E_8$  supergravity potential does not feature any true nontrivial stationary points, hence the plan here is to extend the calculation to an eight-dimensional subspace of the full singlet manifold by also parametrizing the noncompact directions of  $SL(2)$ . Explicitly, if we define an embedding of the  $su(8)$  algebra (here with indices  $i, j$  into the  $so(16)$  algebra via

$$H^{IJ}{}_{i j} = \begin{cases} +1/2 & \text{if } I = i, \quad J = j \\ +1/2 & \text{if } I = i + 8, \quad J = j + 8 \\ +1/2 & \text{if } I = i + 8, \quad J = j \\ -1/2 & \text{if } I = i, \quad J = j + 8 \\ 0 & \text{otherwise} \end{cases} \quad (3.3)$$

then this parametrization is given by<sup>7</sup>

$$G_1^+{}^c{}_B := \frac{1}{4} f_{\alpha\beta}{}^C (\gamma_{\alpha\beta}^{1234} + \gamma_{\alpha\beta}^{1256} + \gamma_{\alpha\beta}^{1278})$$

<sup>6</sup>For the purpose of illustration, imagine a 2-sphere conventionally parametrized by  $\theta, \phi$  and a scalar field  $\Phi$  on the sphere whose gradient at the north pole is parallel to the  $\phi = \pi/2$  great circle. Then for the scalar field given in  $\theta, \phi$  coordinates,  $\partial_\theta \Phi|_N = \partial_\phi \Phi|_N = 0$ , but this point is not a stationary point.

<sup>7</sup>Note that  $G_1^+$  and  $G_2^+$  commute.

$$\begin{aligned}
G_2^+{}^C{}_B &:= \frac{1}{4} f_{\alpha\beta}{}^C (-\gamma_{\alpha\beta}^{1357} + \gamma_{\alpha\beta}^{1368} + \gamma_{\alpha\beta}^{1458} + \gamma_{\alpha\beta}^{1467}) \\
X^{(a)}{}^C{}_B &:= f_{[IJ]}{}^C H_{IJ}^{[IJ]} H^{IJ}{}_i{}^j (\delta_j^i - 4 \delta_7^i \delta_j^7 - 4 \delta_8^i \delta_j^8) \\
X^{(\phi)}{}^C{}_B &:= f_{[IJ]}{}^C H_{IJ}^{[IJ]} H^{IJ}{}_i{}^j (i \delta_7^i \delta_j^7 - i \delta_8^i \delta_j^8) \\
X^{(\vartheta)}{}^C{}_B &:= f_{[IJ]}{}^C H_{IJ}^{[IJ]} H^{IJ}{}_i{}^j (\delta_8^i \delta_j^7 - \delta_7^i \delta_j^8) \\
X^{(\psi)}{}^C{}_B &:= X^{(\vartheta)}{}^C{}_B \\
W^C{}_B &:= \frac{1}{4} f_{[IJ]}{}^C H_{IJ}^{[IJ]} H_i^I H_j^J \delta^{i\bar{j}} \\
Z^C{}_B &:= \frac{1}{4} f_{\alpha\beta}{}^C \delta^{\alpha\beta} \\
\mathcal{V} &= \exp(wW) \exp(zZ) \exp(-wW) \\
&\quad \exp(aX^{(a)}) \exp(\phi X^{(\phi)}) \exp(\vartheta X^{(\vartheta)}) \exp(\psi X^{(\psi)}) \\
&\quad \exp(\lambda_1 G_1^+ + \lambda_2 G_2^+) \\
&\quad \exp(-\psi X^{(\psi)}) \exp(-\vartheta X^{(\vartheta)}) \exp(-\phi X^{(\phi)}) \exp(-aX^{(a)})
\end{aligned} \tag{3.4}$$

and the corresponding analytic form of the potential reads<sup>8</sup>

$$\begin{aligned}
-8g^{-2}V &= \\
&\frac{189}{8} + \frac{3}{8} \cos(4\phi) + 3 \cos(2a) \cos(2\phi) + \frac{7}{2} \cosh(2\lambda_2) \\
&+ \frac{1}{8} \cosh(4\lambda_2) - \frac{1}{2} \cosh(2\lambda_2) \cos(4\phi) \\
&+ \frac{1}{8} \cosh(4\lambda_2) \cos(4\phi) - 3 \cosh(2\lambda_2) \cos(2a) \cos(2\phi) \\
&+ 3 \cosh(2\lambda_1) - 3 \cosh(2\lambda_1) \cos(2a) \cos(2\phi) + 9 \cosh(2\lambda_1) \cosh(2\lambda_2) \\
&+ 3 \cosh(2\lambda_1) \cosh(2\lambda_2) \cos(2a) \cos(2\phi) + \frac{405}{32} \cosh(z) \cosh(\lambda_1) \\
&+ \frac{23}{32} \cosh(z) \cosh(3\lambda_1) - \frac{9}{32} \cosh(z) \cosh(\lambda_1) \cos(4\phi) \\
&- \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cos(4\phi) \\
&- \frac{1}{32} \cosh(z) \cosh(\lambda_1) \cos(2a) \cos(2\phi) \\
&+ \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cos(2a) \cos(2\phi) \\
&+ \frac{3}{32} \cosh(z) \cosh(\lambda_1) \cosh(2\lambda_2) \\
&- \frac{1}{32} \cosh(z) \cosh(\lambda_1) \cosh(4\lambda_2) \\
&- \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cosh(2\lambda_2) \\
&- \frac{19}{32} \cosh(z) \cosh(3\lambda_1) \cosh(4\lambda_2) \\
&+ \frac{1}{32} \cosh(z) \cosh(\lambda_1) \cosh(2\lambda_2) \cos(4\phi) \\
&- \frac{1}{32} \cosh(z) \cosh(\lambda_1) \cosh(4\lambda_2) \cos(4\phi) \\
&+ \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cosh(2\lambda_2) \cos(4\phi) \\
&- \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cosh(4\lambda_2) \cos(4\phi) \\
&+ \frac{1}{32} \cosh(z) \cosh(\lambda_1) \cosh(4\lambda_2) \cos(2a) \cos(2\phi) \\
&- \frac{1}{32} \cosh(z) \cosh(3\lambda_1) \cosh(4\lambda_2) \cos(2a) \cos(2\phi)
\end{aligned}$$

<sup>8</sup>Due to right  $SO(16)$  invariance, this calculation can be greatly simplified by omitting the  $\exp(-\dots)$  factors in this expression. The computation of this potential took slightly more than two hours of CPU time on a 1.7 GHz Pentium-IV.

$$\begin{aligned}
& + \frac{15}{32} \cos(w - 3a) \sinh(\lambda_1) \sinh(z) + \frac{99}{16} \cos(w + a) \sinh(\lambda_1) \sinh(z) \\
& - \frac{3}{32} \cos(w - 3a) \sinh(3\lambda_1) \sinh(z) - \frac{9}{16} \cos(w + a) \sinh(3\lambda_1) \sinh(z) \\
& - \frac{3}{32} \cos(w - 3a) \cos(4\phi) \sinh(\lambda_1) \sinh(z) \\
& + \frac{4}{32} \cos(w - a) \cos(2\phi) \sinh(\lambda_1) \sinh(z) \\
& + \frac{3}{32} \cos(w - 3a) \cos(4\phi) \sinh(3\lambda_1) \sinh(z) \\
& - \frac{3}{32} \cos(w - a) \cos(2\phi) \sinh(3\lambda_1) \sinh(z) \\
& - \frac{3}{8} \cos(w - 3a) \cosh(2\lambda_2) \sinh(\lambda_1) \sinh(z) \\
& - 6 \cos(w + a) \cosh(2\lambda_2) \sinh(\lambda_1) \sinh(z) \\
& - \frac{3}{32} \cos(w - 3a) \cosh(4\lambda_2) \sinh(\lambda_1) \sinh(z) \\
& - \frac{3}{16} \cos(w + a) \cosh(4\lambda_2) \sinh(\lambda_1) \sinh(z) \\
& + \frac{3}{32} \cos(w - 3a) \cosh(2\lambda_2) \sinh(3\lambda_1) \sinh(z) \\
& + \frac{3}{32} \cos(w - 3a) \cosh(4\lambda_2) \sinh(3\lambda_1) \sinh(z) \\
& + \frac{3}{16} \cos(w + a) \cosh(4\lambda_2) \sinh(3\lambda_1) \sinh(z) \\
& + \frac{3}{8} \cos(w - 3a) \cosh(2\lambda_2) \cos(4\phi) \sinh(\lambda_1) \sinh(z) \\
& - 6 \cos(w - a) \cosh(2\lambda_2) \cos(2\phi) \sinh(\lambda_1) \sinh(z) \\
& - \frac{3}{32} \cos(w - 3a) \cosh(4\lambda_2) \cos(4\phi) \sinh(\lambda_1) \sinh(z) \\
& + \frac{3}{32} \cos(w - a) \cosh(4\lambda_2) \cos(2\phi) \sinh(\lambda_1) \sinh(z) \\
& - \frac{3}{8} \cos(w - 3a) \cosh(2\lambda_2) \cos(4\phi) \sinh(3\lambda_1) \sinh(z) \\
& + \frac{3}{32} \cos(w - 3a) \cosh(4\lambda_2) \cos(4\phi) \sinh(3\lambda_1) \sinh(z) \\
& + \frac{3}{8} \cos(w - a) \cosh(4\lambda_2) \cos(2\phi) \sinh(3\lambda_1) \sinh(z)
\end{aligned}$$

Note that again, just as in four dimensions, the dependency on the angular coordinates  $\vartheta, \psi$  drops out. Parametrizing six dimensions, this potential is slightly more general than the one given in [21] (which furthermore uses a different normalization of the embedding tensor and hence has to be re-scaled by a factor  $1/4$  to compare it with this result); at first, it looks a bit surprising that the extra  $SL(2)$  angle  $w$  combines so nicely with the  $SU(8)$  angle  $a$ , where one a priori might have expected this extra angle to at least double the complexity of the potential.

### 3.3.2 Residual symmetry of $G_{2 \text{diag}}$

Before we turn to the discussion of stationary points of the potential of  $N = 16$   $D = 3$   $SO(8) \times SO(8)$  gauged SUGRA, let us look at parametrizations of some other particularly interesting submanifolds. The subgroup of  $SO(8)_{\text{diag}}$  that leaves both the vector and the spinor whose coordinates are  $(0, 0, \dots, 0, 1)$  invariant is  $G_2$  which in our conventions (1.4) also leaves the co-spinor  $(0, 0, \dots, 0, 1)$  invariant. Hence, we get one  $G_2$  singlet from each of the  $(\mathbf{8}, \mathbf{8})$   $SO(8) \times SO(8)$  representations, and the group commuting with  $G_{2 \text{diag}}$  is  $SL(2) \times SL(2)$ , yielding a four-parameter submanifold of  $E_{8(+8)}/SO(16)$  on which the potential can be calculated without further truncation, so one can hope to deduce a statement about *all* stationary points with at least a



remaining symmetry of  $G_{2 \text{diag}}$ .

The simplest case featuring a nontrivial stationary point is obtained by restriction to the submanifold of  $SO(7)$  singlets where  $SO(7)$  is chosen in such a way that either  $\mathbf{8}_s \rightarrow \mathbf{7} + \mathbf{1}$  or (here, equivalently)  $\mathbf{8}_c \rightarrow \mathbf{7} + \mathbf{1}$ . When using a suggestive parametrization for the  $G_{2 \text{diag}}$  case, this simpler case follows by omitting one angular parameter.

The manifold of  $G_{2 \text{diag}}$  singlet scalars is parametrized by

$$\begin{aligned}
V^c_{\mathcal{B}} &:= f_{[IJ]\mathcal{B}}^c H_{IJ}^{[IJ]} H_i^I H_j^J \left( 2\delta_8^i \delta_8^j - \frac{1}{4} \delta^{ij} \right) \\
S^c_{\mathcal{B}} &:= \frac{1}{4} f_{\alpha\beta\mathcal{B}}^c \left( 8\delta_8^\alpha \delta_8^\beta - \delta^{\alpha\beta} \right) \\
W^c_{\mathcal{B}}, Z^c_{\mathcal{B}} &\text{ as in (3.4)} \\
\mathcal{V} &= \exp(wW) \exp(zZ) \exp(-wW) \\
&\quad \exp(vV) \exp(sS) \exp(-vV)
\end{aligned} \tag{3.5}$$

yielding the potential

$$\begin{aligned}
-8g^{-2}V &= \frac{243}{8} + \frac{7}{2} \cosh(2s) + \frac{49}{8} \cosh(4s) + \frac{21}{8} \cos(4v) \\
&\quad - \frac{7}{2} \cos(4v) \cosh(2s) + \frac{7}{8} \cos(4v) \cosh(4s) \\
&\quad + \frac{1141}{64} \cosh(z) \cosh(s) + \frac{427}{64} \cosh(z) \cosh(3s) \\
&\quad - \frac{7}{64} \cosh(z) \cosh(5s) - \frac{25}{64} \cosh(z) \cosh(7s) \\
&\quad - \frac{21}{64} \cosh(z) \cos(4v) \cosh(s) \\
&\quad + \frac{21}{64} \cosh(z) \cos(4v) \cosh(3s) \\
&\quad + \frac{7}{64} \cosh(z) \cos(4v) \cosh(5s) \\
&\quad - \frac{7}{64} \cosh(z) \cos(4v) \cosh(7s) \\
&\quad - \frac{1645}{128} \cos(w-v) \sinh(s) \sinh(z) \\
&\quad + \frac{651}{128} \cos(w-v) \sinh(3s) \sinh(z) \\
&\quad + \frac{7}{128} \cos(w-v) \sinh(5s) \sinh(z) \\
&\quad - \frac{49}{128} \cos(w-v) \sinh(7s) \sinh(z) \\
&\quad - \frac{315}{64} \cos(w+3v) \sinh(s) \sinh(z) \\
&\quad + \frac{133}{64} \cos(w+3v) \sinh(3s) \sinh(z) \\
&\quad - \frac{7}{64} \cos(w+3v) \sinh(5s) \sinh(z) \\
&\quad - \frac{7}{64} \cos(w+3v) \sinh(7s) \sinh(z) \\
&\quad + \frac{35}{128} \cos(w+7v) \sinh(s) \sinh(z) \\
&\quad - \frac{21}{128} \cos(w+7v) \sinh(3s) \sinh(z) \\
&\quad + \frac{7}{128} \cos(w+7v) \sinh(5s) \sinh(z) \\
&\quad - \frac{1}{128} \cos(w+7v) \sinh(7s) \sinh(z)
\end{aligned} \tag{3.6}$$

which reduces to the  $SO(7)$  potential by setting  $v = 0$ , giving

$$\begin{aligned}
-8g^{-2}V &= 33 + 7 \cosh(4s) + \frac{35}{2} \cosh(z) \cosh(s) + 7 \cosh(z) \cosh(3s) \\
&\quad - \frac{1}{2} \cosh(z) \cosh(7s) - \frac{35}{2} \cos(w) \sinh(s) \sinh(z) \\
&\quad + 7 \cos(w) \sinh(3s) \sinh(z) - \frac{1}{2} \cos(w) \sinh(7s) \sinh(z)
\end{aligned} \tag{3.7}$$

### 3.3.3 Residual symmetry of $SO(6)_{\text{diag}}$

For noncompact gauge groups (to be discussed in the next chapter), any vacuum will break the gauge group down to a subgroup of its maximal compact subgroup, thus when studying gaugings like  $SO(6, 2) \times SO(6, 2)$ , it is natural to consider breaking to groups like  $H = SO(6)_{\text{diag}}$ . Once a suitable parametrization of the submanifold of  $H$ -singlets is obtained, this can often be re-cycled for other gauge groups by re-doing the calculation with a different embedding tensor  $\Theta$ . Hence, of all the possible ways to embed  $SO(6)$  into  $SO(8) \times SO(8)$ , we will consider here only the case of the diagonal  $SO(6)$  of the  $SO(6, 2) \times SO(6, 2)$  subgroup of  $SO(8, 8)$  which itself is formed from the  $(\mathbf{1}, \mathbf{28}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{8}_s, \mathbf{8}_s)$  of the  $E_8 \rightarrow SO(8) \times SO(8)$  decomposition  $\mathbf{248} = (\mathbf{1}, \mathbf{28}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{8}_v) + (\mathbf{8}_s, \mathbf{8}_s) + (\mathbf{8}_c, \mathbf{8}_c)$ . The particular choice we make is that our  $SO(6)_{\text{diag}}$  shall leave fixed the last two spinor coordinates. Using such a diagonal embedding, there are five singlets under  $SO(6) \times SO(2)$ , seven singlets under  $SO(6)$ , and twelve singlets under  $SU(3)$ , so from the point of complexity, so it is reasonable to try to calculate the potential on the  $SO(6)$  invariant seven-manifold, which consists of the noncompact directions of the  $SL(3) \times SL(2)$  group commuting with  $SO(6)_{\text{diag}}$ . The generators of this  $SL(3) \times SL(2)$  are given as follows, in the same notation as in [19]:

$$\begin{aligned}
p_1^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \left( \delta_1^i \delta_7^j + \delta_2^i \delta_8^j - \delta_3^i \delta_6^j + \delta_4^i \delta_5^j \right) \left( \delta_i^I \delta_j^{J-8} - \delta_i^{J-8} \delta_j^I + \delta_i^{I-8} \delta_j^J - \delta_i^J \delta_j^{I-8} \right) f_{[IJ]\mathcal{B}}^{\mathcal{C}} \\
p_2^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \delta^{ij} \left( \delta_i^{I-8} \delta_j^J - \delta_i^I \delta_j^{J-8} \right) f_{[IJ]\mathcal{B}}^{\mathcal{C}} \\
p_3^{\mathcal{C}\mathcal{B}} &= -\frac{1}{2} \left( \delta_1^i \delta_7^j + \delta_2^i \delta_8^j - \delta_3^i \delta_6^j + \delta_4^i \delta_5^j \right) \left( \delta_i^I \delta_j^J - \delta_i^J \delta_j^I - \delta_i^{I-8} \delta_j^{J-8} + \delta_i^{J-8} \delta_j^{I-8} \right) f_{[IJ]\mathcal{B}}^{\mathcal{C}} \\
p_4^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \left( \delta_1^{\dot{\gamma}} \delta_2^{\dot{\delta}} - \delta_3^{\dot{\gamma}} \delta_4^{\dot{\delta}} + \delta_5^{\dot{\gamma}} \delta_6^{\dot{\delta}} + \delta_7^{\dot{\gamma}} \delta_8^{\dot{\delta}} \right) \left( \delta_{\dot{\gamma}}^{\dot{\alpha}} \delta_{\dot{\delta}}^{\dot{\beta}} - \delta_{\dot{\gamma}}^{\dot{\beta}} \delta_{\dot{\delta}}^{\dot{\alpha}} \right) f_{\dot{\alpha}\dot{\beta}\mathcal{B}}^{\mathcal{C}} \\
p_5^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \delta^{\dot{\alpha}\dot{\beta}} f_{\dot{\alpha}\dot{\beta}\mathcal{B}}^{\mathcal{C}} & p_6^{\mathcal{C}\mathcal{B}} &= \left( \delta_7^{\alpha} \delta_8^{\beta} - \delta_8^{\alpha} \delta_7^{\beta} \right) f_{\alpha\beta\mathcal{B}}^{\mathcal{C}} \\
p_7^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \delta^{\alpha\beta} f_{\alpha\beta\mathcal{B}}^{\mathcal{C}} & p_8^{\mathcal{C}\mathcal{B}} &= p_7^{\mathcal{C}\mathcal{B}} - 2 \left( \delta_7^{\alpha} \delta_7^{\beta} + \delta_8^{\alpha} \delta_8^{\beta} \right) f_{\alpha\beta\mathcal{B}}^{\mathcal{C}} \\
q_1^{\mathcal{C}\mathcal{B}} &= \frac{1}{4} \left( \delta_1^i \delta_7^j + \delta_2^i \delta_8^j - \delta_3^i \delta_6^j + \delta_4^i \delta_5^j \right) \left( \delta_i^I \delta_j^J - \delta_i^J \delta_j^I + \delta_i^{I-8} \delta_j^{J-8} - \delta_i^{J-8} \delta_j^{I-8} \right) f_{[IJ]\mathcal{B}}^{\mathcal{C}} \\
q_2^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \left( \delta_7^{\alpha} \delta_7^{\beta} - \delta_8^{\alpha} \delta_8^{\beta} \right) f_{\alpha\beta\mathcal{B}}^{\mathcal{C}} & q_3^{\mathcal{C}\mathcal{B}} &= \frac{1}{2} \left( \delta_7^{\alpha} \delta_8^{\beta} + \delta_8^{\alpha} \delta_7^{\beta} \right) f_{\alpha\beta\mathcal{B}}^{\mathcal{C}}.
\end{aligned} \tag{3.8}$$

The  $p$ -, resp.  $q$ -generators stand in one-to-one correspondence to the following  $SL(3)$  resp.  $SL(2)$  generator matrices in the defining representations that satisfy the same commutation relations (e.g.  $[p_8, p_6] = 3p_3$  and  $[\tilde{p}_8, \tilde{p}_6] = 3\tilde{p}_3$ ) and hence are much better suited to read off such commutation relations than above  $E_8$  generator definitions.

$$\begin{aligned}
\tilde{p}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & \tilde{p}_2 &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \tilde{p}_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\tilde{p}_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \tilde{p}_5 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \tilde{p}_6 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
\tilde{p}_7 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \tilde{p}_8 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & & \\
\tilde{q}_1 &= \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} & \tilde{q}_2 &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} & \tilde{q}_3 &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}
\end{aligned} \tag{3.9}$$

The noncompact directions of  $SL(3)$  are hence equivalent to the symmetric traceless  $3 \times 3$  matrices, and since the axes of any tensor ellipsoid can be aligned with the coordinate axes by a suitable rotation, we parametrize the scalars from  $SL(3)$  by applying all  $SO(3)$  rotations (which we parametrize by  $yzx$  Euler angles, since such a system is better behaved in terms of coordinate singularities than a  $zxx$  system<sup>9</sup>) to all linear combinations of two linear independent commuting diagonal matrices. Thus a convenient coordinate parametrization of our seven-dimensional singlet space is given by

$$\begin{aligned}
\mathcal{V} &= \exp(r_1 p_1) \exp(r_2 p_3) \exp(r_3 p_2) \\
&\quad \exp(z p_8 - s p_7) \exp(-r_3 p_2) \\
&\quad \exp(-r_2 p_3) \exp(-r_1 p_1) \\
&\quad \exp(r_5 q_1) \exp(v q_2) \exp(-r_5 q_1)
\end{aligned} \tag{3.10}$$

and the potential we obtain reads

$$\begin{aligned}
-8g^{-2}V &= \\
&27 + 3 \cosh(4z) + 3 \cosh(4z) \cos(2r_2) - 3 \cosh(4z) \cos(2r_1) \\
&- 3 \cosh(4z) \cos(2r_1) \cos(2r_2) + \frac{1}{4} \cosh(4s) \\
&+ \frac{1}{4} \cosh(4s) \cos(2r_2) - \frac{1}{4} \cosh(4s) \cos(2r_1)
\end{aligned}$$

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<sup>9</sup>Hence, the yaw, pitch, roll system used in aviation is also of  $zxy$  type.

$$\begin{aligned}
& -\frac{1}{4} \cosh(4s) \cos(2r_1) \cos(2r_2) + 9 \cosh(2s) \cosh(2z) \\
& + \frac{3}{4} \cosh(2s) \cosh(6z) - 3 \cos(2r_3) \sinh(2z) \sinh(2s) \\
& + \frac{1}{4} \cos(2r_3) \sinh(6z) \sinh(2s) \\
& - 3 \cosh(2s) \cosh(2z) \cos(2r_2) - \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_2) \\
& - 3 \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& + \frac{1}{4} \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + 3 \cosh(2s) \cosh(2z) \cos(2r_1) + \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_1) \\
& - 9 \cos(2r_1) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& + \frac{3}{4} \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + 3 \cosh(2s) \cosh(2z) \cos(2r_1) \cos(2r_2) \\
& + \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\
& - 12 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2z) \sinh(2s) \\
& + \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& + 3 \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{1}{4} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + \cosh(2v) + 9 \cosh(v) \cosh(4z) \\
& - 3 \cosh(v) \cosh(4z) \cos(2r_2) \\
& + 3 \cosh(v) \cosh(4z) \cos(2r_1) \\
& + 3 \cosh(v) \cosh(4z) \cos(2r_1) \cos(2r_2) \\
& - \frac{1}{4} \cosh(2v) \cosh(4s) - \frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_2) \\
& + \frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_1) \\
& + \frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \\
& + 15 \cosh(v) \cosh(2s) \cosh(2z) - \frac{3}{4} \cosh(2v) \cosh(2s) \cosh(6z) \\
& + 3 \cosh(v) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{1}{4} \cosh(2v) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + 3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_2) \\
& + \frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_2) \\
& + 3 \cosh(v) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{1}{4} \cosh(2v) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& - 3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_1) \\
& - \frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \\
& + 9 \cosh(v) \cos(2r_1) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{3}{4} \cosh(2v) \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& - 3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_1) \cos(2r_2) \\
& - \frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\
& + 12 \cosh(v) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2z) \sinh(2s) \\
& - \cosh(2v) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& - 3 \cosh(v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& + \frac{1}{4} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s).
\end{aligned}$$

### 3.3.4 Residual symmetry of $SO(5)_{\text{diag}}$

For an embedding of  $SO(5)_{\text{diag}}$  analogous to the  $SO(6)$  embedding studied in the previous section (i.e.  $SO(5)$  leaves fixed the spinor indices 6, 7, 8), the manifold of invariant scalars is 14-dimensional, and therefore too big to justify its use for this approach. However, a particularly interesting subgroup to consider is an analogously formed  $(SO(5) \times SO(3))_{\text{diag}}$ , since there is a known de Sitter vacuum of  $SO(5, 3)$  gauged  $N = 8$   $D = 4$  supergravity. Employing the techniques of the last section, it is very tempting to go even a bit further and delete the 6, 8 and 7, 8-rotations from  $SO(3)$  to break to  $SO(5) \times SO(2)$ , since the invariant manifold then is the six-dimensional space  $\mathbb{R} \times SL(3)/SO(3)$ .

We first want to consider the simpler  $(SO(5) \times SO(3))_{\text{diag}}$  case whose three singlets are the two noncompact directions of the  $SL(2)$  commuting with  $E_{7(7)}$  which we again parametrize as in 3.4, plus an extra singlet given by

$$M^C_{\mathcal{B}} = \frac{1}{4} \left( 3 \sum_{j=1}^5 \delta_{\alpha}^j \delta_{\beta}^j - 5 \sum_{j=6}^8 \delta_{\alpha}^j \delta_{\beta}^j \right) f_{\alpha\beta\mathcal{B}}^C, \quad (3.11)$$

and thus the total parametrization is

$$\mathcal{V} = \exp(sM) \exp(wW) \exp(zZ) \exp(-wW) \quad (3.12)$$

from which we obtain the potential

$$\begin{aligned} -8g^{-2}V &= 25 + 15 \cosh(4s) + 15 \cosh(s) \cosh(z) + 15/2 \cosh(3s) \cosh(z) \\ &+ 3/2 \cosh(5s) \cosh(z) - 15 \cos(w) \sinh(z) \sinh(s) \\ &+ 15/2 \cos(w) \sinh(z) \sinh(3s) - 3/2 \cos(w) \sinh(z) \sinh(5s). \end{aligned} \quad (3.13)$$

The eight generators of  $SL(3)$  commuting with  $SO(5) \times SO(2)$  are

$$\begin{aligned} h_{\alpha\beta} &= \delta_{\alpha}^5 \delta_{\beta}^6 - \delta_{\alpha}^6 \delta_{\beta}^5 \\ p_1^C_{\mathcal{B}} &= \frac{1}{2} f_{[IJ]\mathcal{B}}^C H_{IJ}^{[IJ]} H_i^I H_j^J \delta^{i\bar{j}} \\ p_2^C_{\mathcal{B}} &= -\frac{1}{8} f_{[IJ]\mathcal{B}}^C \left( H_i^I H_j^J \gamma_{\alpha\beta}^{ij} - H_i^I H_j^J \gamma_{\alpha\beta}^{\bar{i}\bar{j}} \right) h_{\alpha\beta} \\ p_3^C_{\mathcal{B}} &= -\frac{1}{4} f_{[IJ]\mathcal{B}}^C H_i^I H_j^J \gamma_{\alpha\beta}^{i\bar{j}} h_{\alpha\beta} \\ p_4^C_{\mathcal{B}} &= \frac{1}{2} f_{\dot{\alpha}\dot{\beta}\mathcal{B}}^C \delta^{\dot{\alpha}\dot{\beta}} \\ p_5^C_{\mathcal{B}} &= -f_{\alpha\beta\mathcal{B}}^C h^{\alpha\beta} \\ p_6^C_{\mathcal{B}} &= f_{\dot{\alpha}\dot{\beta}\mathcal{B}}^C \delta_{\dot{\gamma}\dot{\delta}}^{\dot{\alpha}\dot{\beta}} \left( \delta_1^{\dot{\gamma}} \delta_4^{\dot{\delta}} + \delta_3^{\dot{\gamma}} \delta_2^{\dot{\delta}} + \delta_5^{\dot{\gamma}} \delta_8^{\dot{\delta}} + \delta_6^{\dot{\gamma}} \delta_7^{\dot{\delta}} \right) \\ p_7^C_{\mathcal{B}} &= f_{\alpha\beta\mathcal{B}}^C \left( \delta_6^{\alpha} \delta_6^{\beta} + \delta_7^{\alpha} \delta_7^{\beta} \right) \\ p_8^C_{\mathcal{B}} &= f_{\alpha\beta\mathcal{B}}^C \delta^{\alpha\beta} - p_7^C_{\mathcal{B}} \end{aligned} \quad (3.14)$$

which again are given in the proper order to share the commutation relations of the  $\tilde{p}$  from (3.9), hence we can again use the parametrization

$$\begin{aligned} \mathcal{V} = & \exp(r_1 p_1) \exp(r_2 p_3) \exp(r_3 p_2) \\ & \exp(z p_8 - s p_7) \exp(-r_3 p_2) \\ & \exp(-r_2 p_3) \exp(-r_1 p_1) \end{aligned} \quad (3.15)$$

and obtain the potential given in the appendix.

### 3.3.5 Residual symmetry of $(SO(4) \times SO(4))_{\text{diag}}$

Of the many different ways to embed  $SO(4) \times SO(4)$  into  $SO(8) \times SO(8)$ , we only consider the one analogous to the previous  $SO(5)_{\text{diag}}$  and  $SO(6)_{\text{diag}}$  constructions: if we label our gauge group as  $SO(8)_L \times SO(8)_R$  and perform the split  $\mathbf{8}_{L/R,s} \rightarrow (\mathbf{4}_{L/R,1}, \mathbf{4}_{L/R,2})$ , then we consider the diagonal  $SO(4)_{\text{diag}(L1,R1)} \times SO(4)_{\text{diag}(L2,R2)}$ . There are four singlets under this  $SO(4) \times SO(4)$ , parametrizing the space  $(SL(2)/U(1))^2$ , five singlets under an analogous  $SO(4) \times SO(3)$ , eleven singlets under  $SO(4) \times SO(2)$ , and ten singlets under an  $SO(3) \times SO(3)$ .

If we parametrize the  $SO(4) \times SO(3)$  singlet space via

$$\begin{aligned} S_{1B}^C &= \frac{1}{4} \left( \sum_{j=1}^4 \delta_\alpha^j \delta_\beta^j - \sum_{j=5}^8 \delta_\alpha^j \delta_\beta^j \right) f_{\alpha\beta B}^C \\ S_{2B}^C &= \frac{1}{4} \left( \sum_{j=1}^4 \delta_\alpha^j \delta_\beta^j - \sum_{j=5}^8 \delta_\alpha^j \delta_\beta^j \right) f_{\alpha\dot{\beta}B}^C \\ X_B^C &= \left( \sum_{j=5}^7 \delta_\alpha^j \delta_\beta^j - 3\delta_\alpha^8 \delta_\beta^8 \right) f_{\alpha\beta B}^C \\ \mathcal{V} &= \exp(wW) \exp(zZ) \exp(-wW) \\ & \exp(v[S_1, S_2]) \exp(S_1) \exp(-v[S_1, S_2]) \exp(xX), \end{aligned} \quad (3.16)$$

we obtain the potential

$$\begin{aligned} -8g^{-2}V = & 21 + 3 \cosh(8x) + 12 \cosh(z) \cosh(2x) + 4 \cosh(z) \cosh(6x) \\ & + 12 \cos(w) \sinh(2x) \sinh(z) - 4 \cos(w) \sinh(6x) \sinh(z) \\ & + 12 \cosh(s) \cosh(2x) + 4 \cosh(s) \cosh(6x) + 4 \cosh(s) \cosh(z) \\ & + \frac{9}{2} \cosh(s) \cosh(z) \cosh(4x) - \frac{1}{2} \cosh(s) \cosh(z) \cosh(12x) \\ & - \frac{3}{2} \cosh(s) \cos(w) \sinh(4x) \sinh(z) \\ & + \frac{1}{2} \cosh(s) \cos(w) \sinh(12x) \sinh(z) - 12 \cos(v) \sinh(2x) \sinh(s) \\ & + 4 \cos(v) \sinh(6x) \sinh(s) + \frac{3}{2} \cos(v) \cosh(z) \sinh(4x) \sinh(s) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \cos(v) \cosh(z) \sinh(12x) \sinh(s) \\
& +4 \cos(v) \cos(w) \sinh(z) \sinh(s) \\
& -\frac{9}{2} \cos(v) \cos(w) \cosh(4x) \sinh(z) \sinh(s) \\
& +\frac{1}{2} \cos(v) \cos(w) \cosh(12x) \sinh(z) \sinh(s)
\end{aligned}$$

which reduces to the potential on the  $SO(4) \times SO(4)$  singlet space

$$-8g^{-2}V = 24 + 16 \cosh(z) + 16 \cosh(s) + 8 \cosh(s) \cosh(z) \quad (3.17)$$

by setting  $x = 0$ .

### 3.4 Vacua

For many submanifolds for which these restricted potentials can be analytically calculated, they turn out to be too complicated for a fully analytic determination of their extremal structure (using presently available technology); one way to proceed from here is to make educated guesses and further restrict the analysis to directions in the potential with special properties by letting the angular parameters run through a set of special values. While this technique was used in [21] to identify analoga of all stationary points of the  $SO(8) \times SO(8)$  potential that correspond to the known extrema of  $SO(8)$  gauged  $N = 8$   $D = 4$  supergravity with at least remaining  $SU(3)$  symmetry (except the vacuum with  $G_2$  symmetry which was identified in [19], a more promising approach seems to be to use these analytic expressions<sup>10</sup> in a numerical search for vacua<sup>11</sup> whose results then are subjected to educated inspection to give conjectures about exact locations of stationary points. Typically, numerical search will produce values for angular coordinates very close to rational multiples of  $\pi$ , or simple relations between some of the hyperbolic angular coordinates (for example, one being close to the negative value of another one). By substituting these conjectured properties back into the analytic potential, the problem typically is simplified far enough to produce a complete set of coordinates which then are subjected to (3.2) to filter out 'fake' solutions this process may have generated.<sup>12</sup>

For the more accessible vacua, we collect eigenvalues of the vector and scalar mass matrices as well as the  $A_1$  and  $A_3$  fermion and gravitino mass

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<sup>10</sup>whose evaluation is much faster than a full numerical exponentiation of  $E_8$  generator matrices – which nevertheless is also available and useful to cross-check results

<sup>11</sup>Since stationary points of this potential usually are saddle points, one has to minimize the length of the gradient.

<sup>12</sup>This procedure is documented in full detail in one of the examples provided in the `LambdaTensor` package.

tensors in tables like the following corresponding to the trivial vacuum at the origin  $\mathcal{V}_B^C = \delta_B^C$  with  $(n_L, n_R) = (8, 8)$  supersymmetry:

$\Lambda/(2g^2)$	-16	(3.18)
$\mathcal{M}/g^2$	$-12_{(\times 128)}$	
$\mathcal{M}^{\text{vec}}/g$	$0_{(\times 128)}$	
$A_1$	$2_{(\times 8)}, -2_{(\times 8)}$	
$A_3$	$0_{(\times 128)}$	

Multiplicities are given as subscripts in these tables; the Goldstone modes are contained in the  $m^2 = 0$   $\mathcal{M}$ -eigenvalues and goldstino modes (identified by projection with  $A_2$ ) will be marked with an asterisk.

### 3.4.1 The $G_2 \times G_2$ vacuum

Finding vacua is a highly nontrivial task; checking the existence of a vacuum (and its properties) once it has been identified is – at least in principle – amenable to a manual calculation. We demonstrate this by presenting explicit intermediate quantities for the particular case of the vacuum with unbroken gauge symmetry  $G_2 \times G_2$  and  $(n_L, n_R) = (1, 1)$  supersymmetry.

This vacuum is located at

$$\begin{aligned}
M^{\mathcal{A}} &:= \sqrt{2} \left( H_{\alpha\beta}^{\mathcal{A}} \delta_8^\alpha \delta_8^\beta - H_{\dot{\alpha}\dot{\beta}}^{\mathcal{A}} \delta_8^{\dot{\alpha}} \delta_8^{\dot{\beta}} \right) \\
m &:= \frac{1}{2} \log \left( \frac{7}{3} + \frac{2}{3} \sqrt{10} \right) \\
\mathcal{V}_B^C &= \exp \left( m M^{\mathcal{A}} f_{\mathcal{A}B}{}^C \right)
\end{aligned} \tag{3.19}$$

Even and odd powers of the generator  $N_B^C := M^{\mathcal{A}} f_{\mathcal{A}B}{}^C$  (from which one can directly read off its exponential in terms of sin and cos contributions) are given by

$$\begin{aligned}
P_{ij}^{(7v)} &:= \delta_{ij} - \delta_i^8 \delta_j^8, & P_{\alpha\beta}^{(7s)} &:= \delta_{\alpha\beta} - \delta_\alpha^8 \delta_\beta^8, & P_{\dot{\alpha}\dot{\beta}}^{(7c)} &:= \delta_{\dot{\alpha}\dot{\beta}} - \delta_{\dot{\alpha}}^8 \delta_{\dot{\beta}}^8 \\
D_{ij}^{(7v)} &:= \delta_{ij} - 8 \delta_i^8 \delta_j^8, & D_{\alpha\beta}^{(7s)} &:= \delta_{\alpha\beta} - 8 \delta_\alpha^8 \delta_\beta^8, & D_{\dot{\alpha}\dot{\beta}}^{(7c)} &:= \delta_{\dot{\alpha}\dot{\beta}} - 8 \delta_{\dot{\alpha}}^8 \delta_{\dot{\beta}}^8 \\
E_{\alpha\dot{\beta}}^{(7)} &:= \gamma_{\alpha\dot{\beta}}^i \delta_8^i - 8 \delta_\alpha^8 \delta_{\dot{\beta}}^8 \\
A_{ijkl} &:= \delta_{kl}^{mn} P_{im}^{(7v)} P_{jn}^{(7v)} \\
B_{ijkl} &:= \delta_{kl}^{ij} - A_{ijkl} \\
Q_{ij\dot{\alpha}\dot{\beta}}^{(7c)} &:= \delta_\beta^8 \delta_\gamma^8 P_{\dot{\delta}\dot{\alpha}}^{(7c)} P_{ik}^{(7v)} P_{jl}^{(7v)} \gamma_{\dot{\delta}\dot{\gamma}}^{kl} \\
Q_{ij\dot{\alpha}\dot{\beta}}^{(c)} &:= \delta_\beta^8 \delta_\gamma^8 \gamma_{\dot{\alpha}\dot{\gamma}}^{ij} - Q_{ij\dot{\alpha}\dot{\beta}}^{(7c)} \\
Q_{ij\alpha\beta}^{(7s)} &:= \delta_\alpha^8 \delta_\gamma^8 P_{\delta\beta}^{(7s)} P_{ik}^{(7v)} P_{jl}^{(7v)} \gamma_{\beta\gamma}^{kl}
\end{aligned}$$



$$\begin{aligned}
Q_{ij\alpha\beta}^{(s)} &:= \delta_\alpha^8 \delta_\gamma^8 \gamma_{\beta\gamma}^{ij} - Q_{ij\alpha\beta}^{(7s)} \\
R_{ij\dot{\alpha}\dot{\beta}}^{(7c)} &:= \delta_\alpha^8 \delta_i^8 \delta_\beta^8 \gamma_{\alpha\dot{\alpha}}^j + \delta_\beta^8 \delta_j^8 \delta_\alpha^8 \gamma_{\beta\dot{\beta}}^i - 2\delta_8^i \delta_8^j \delta_8^\alpha \delta_8^\beta \\
R_{ij\dot{\alpha}\dot{\beta}}^{(c)} &:= \gamma_{\beta\dot{\beta}}^i \gamma_{\alpha\dot{\alpha}}^j \delta_\gamma^8 \delta_\alpha^8 - R_{ij\dot{\alpha}\dot{\beta}}^{(7c)} \\
R_{ij\alpha\beta}^{(7s)} &:= \delta_\beta^8 \delta_i^8 \delta_\alpha^8 \gamma_{\alpha\dot{\alpha}}^j + \delta_\alpha^8 \delta_j^8 \delta_\beta^8 \gamma_{\beta\dot{\beta}}^i - 2\delta_8^i \delta_8^j \delta_8^\alpha \delta_8^\beta \\
R_{ij\alpha\beta}^{(s)} &:= \gamma_{\beta\dot{\beta}}^i \gamma_{\alpha\dot{\alpha}}^j \delta_\gamma^8 \delta_\alpha^8 - R_{ij\alpha\beta}^{(7s)} \\
(N^C_{\mathcal{B}})^{2n+1}/\sqrt{2} &= -\frac{1}{2} H_{IJ}^C H_B^{\dot{\alpha}\dot{\beta}} H_i^I H_j^J \left( Q_{ij\dot{\beta}\dot{\alpha}}^{(c)} + 2^{2n} Q_{ij\dot{\beta}\dot{\alpha}}^{(7c)} \right) \\
&\quad + \frac{1}{2} H_{IJ}^C H_B^{\alpha\beta} H_i^I H_j^J \left( Q_{ij\alpha\beta}^{(s)} + 2^{2n} Q_{ij\alpha\beta}^{(7s)} \right) \\
&\quad + \frac{1}{2} H_{IJ}^C H_B^{\alpha\beta} H_i^I H_j^J \left( Q_{ij\beta\alpha}^{(s)} + 2^{2n} Q_{ij\beta\alpha}^{(7s)} \right) \\
&\quad - \frac{1}{2} H_{IJ}^C H_B^{\dot{\alpha}\dot{\beta}} H_i^I H_j^J \left( Q_{ij\dot{\alpha}\dot{\beta}}^{(c)} + 2^{2n} Q_{ij\dot{\alpha}\dot{\beta}}^{(7c)} \right) \\
&\quad - \frac{1}{2} H_{\dot{\alpha}\dot{\beta}}^C H_B^{IJ} H_i^I H_j^J \left( Q_{ij\dot{\beta}\dot{\alpha}}^{(c)} + 2^{2n} Q_{ij\dot{\beta}\dot{\alpha}}^{(7c)} \right) \\
&\quad + \frac{1}{2} H_{\alpha\beta}^C H_B^{IJ} H_i^I H_j^J \left( Q_{ij\alpha\beta}^{(s)} + 2^{2n} Q_{ij\alpha\beta}^{(7s)} \right) \\
&\quad + \frac{1}{2} H_{\alpha\beta}^C H_B^{IJ} H_i^I H_j^J \left( Q_{ij\beta\alpha}^{(s)} + 2^{2n} Q_{ij\beta\alpha}^{(7s)} \right) \\
&\quad - \frac{1}{2} H_{\dot{\alpha}\dot{\beta}}^C H_B^{IJ} H_i^I H_j^J \left( Q_{ij\dot{\alpha}\dot{\beta}}^{(s)} + 2^{2n} Q_{ij\dot{\alpha}\dot{\beta}}^{(7s)} \right) \\
&\quad + H_{IJ}^C H_B^{\alpha\beta} H_i^I H_j^J \left( R_{ij\alpha\beta}^{(s)} + 2^{2n} R_{ij\alpha\beta}^{(7s)} \right) \\
&\quad + H_{IJ}^C H_B^{\dot{\alpha}\dot{\beta}} H_i^I H_j^J \left( R_{ij\dot{\alpha}\dot{\beta}}^{(c)} + 2^{2n} R_{ij\dot{\alpha}\dot{\beta}}^{(7c)} \right) \\
&\quad + H_{\alpha\beta}^C H_B^{IJ} H_i^I H_j^J \left( R_{ij\alpha\beta}^{(s)} + 2^{2n} R_{ij\alpha\beta}^{(7s)} \right) \\
&\quad + H_{\dot{\alpha}\dot{\beta}}^C H_B^{IJ} H_i^I H_j^J \left( R_{ij\dot{\alpha}\dot{\beta}}^{(c)} + 2^{2n} R_{ij\dot{\alpha}\dot{\beta}}^{(7c)} \right) \\
(N^C_{\mathcal{B}})^{2n} &= H_{\dot{\alpha}\dot{\beta}}^C H_B^{\dot{\gamma}\dot{\delta}} \left( \frac{1}{2} \delta_{\dot{\alpha}\dot{\gamma}} \delta_{\dot{\beta}\dot{\delta}} + 2^{2n-1} \delta_{\dot{\alpha}\dot{\gamma}}^8 \delta_{\dot{\beta}\dot{\delta}}^8 \right) \\
&\quad + 2^{2n-1} \delta_{\dot{\beta}\dot{\delta}} \delta_{\dot{\alpha}}^8 \delta_{\dot{\gamma}}^8 - 2^{2n} \delta_{\dot{\alpha}}^8 \delta_{\dot{\beta}}^8 \delta_{\dot{\gamma}}^8 \delta_{\dot{\delta}}^8 \\
&\quad + H_{\alpha\beta}^C H_B^{\gamma\delta} \left( \frac{1}{2} \delta_{\alpha\gamma} \delta_{\beta\delta} + 2^{2n-1} \delta_{\alpha\gamma}^8 \delta_{\beta\delta}^8 \right) \\
&\quad + 2^{2n-1} \delta_{\beta\delta} \delta_{\alpha}^8 \delta_{\gamma}^8 - 2^{2n} \delta_{\alpha}^8 \delta_{\beta}^8 \delta_{\gamma}^8 \delta_{\delta}^8 \\
&\quad + H_{\alpha\beta}^C H_B^{\dot{\gamma}\dot{\delta}} \left( \frac{1}{2} \gamma_{\alpha\dot{\gamma}}^i \gamma_{\beta\dot{\delta}}^j \delta_i^8 \delta_j^8 - 2^{2n-1} \gamma_{\alpha\dot{\gamma}}^i \delta_i^8 \delta_{\beta\dot{\delta}}^8 \delta_{\dot{\delta}}^8 \right) \\
&\quad - 2^{2n-1} \gamma_{\beta\dot{\delta}}^i \delta_i^8 \delta_{\alpha}^8 \delta_{\dot{\gamma}}^8 + 2^{2n} \delta_{\alpha}^8 \delta_{\beta}^8 \delta_{\dot{\gamma}}^8 \delta_{\dot{\delta}}^8 \\
&\quad + H_{\dot{\alpha}\dot{\beta}}^C H_B^{\gamma\delta} \left( \frac{1}{2} \gamma_{\dot{\gamma}\dot{\alpha}}^i \gamma_{\delta\dot{\beta}}^j \delta_i^8 \delta_j^8 - 2^{2n} \gamma_{\dot{\gamma}\dot{\alpha}}^i \delta_i^8 \delta_{\beta}^8 \delta_{\dot{\delta}}^8 \right)
\end{aligned}$$

$$\begin{aligned}
& -2^{2n} \gamma_{\beta\delta}^i \delta_\alpha^8 \delta_\gamma^8 \delta_\gamma^8 + 2^{2n+1} \delta_\alpha^8 \delta_\beta^8 \delta_\gamma^8 \delta_\delta^8 \\
& + 2^{2n-6} H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \delta_i^8 E_{\alpha\gamma}^{(7)} \gamma_{\alpha\beta}^{jkl} \\
& + 2^{2n-6} H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \delta_l^8 E_{\alpha\gamma}^{(7)} \gamma_{\alpha\beta}^{ijk} \\
& + 2^{2n-6} H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \delta_j^8 E_{\alpha\gamma}^{(7)} \gamma_{\alpha\beta}^{ikl} \\
& + 2^{2n-6} H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \delta_k^8 E_{\alpha\gamma}^{(7)} \gamma_{\alpha\beta}^{ijl} \\
& H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \left( 2^{2n-3} A_{ijkl} - \frac{1}{16} (1 + 7 \cdot 2^{2n-2}) B_{ijkl} \right. \\
& \left. + 2^{2n-5} \gamma_{\alpha\beta}^{ijkl} P_{\alpha\beta}^{(7s)} + 2^{2n-5} \gamma_{\dot{\alpha}\dot{\beta}}^{ijkl} P_{\dot{\alpha}\dot{\beta}}^{(7c)} \right) \\
& H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \left( 2^{2n-3} A_{i\bar{j}\bar{k}\bar{l}} - \frac{1}{16} (1 + 7 \cdot 2^{2n-2}) B_{i\bar{j}\bar{k}\bar{l}} \right. \\
& \left. + 2^{2n-5} \gamma_{\alpha\beta}^{i\bar{j}\bar{k}\bar{l}} P_{\alpha\beta}^{(7s)} + 2^{2n-5} \gamma_{\dot{\alpha}\dot{\beta}}^{i\bar{j}\bar{k}\bar{l}} P_{\dot{\alpha}\dot{\beta}}^{(7c)} \right) \\
& H_{IJ}^C H_B^{KL} H_i^I H_j^J H_k^K H_l^L \left( \delta_{ik} \delta_{\bar{j}\bar{l}} + (2^{2n-2} - 1) \delta_{ik} \delta_j^8 \delta_l^8 \right. \\
& \left. + (2^{2n-2} - 1) \delta_{\bar{j}\bar{l}} \delta_i^8 \delta_k^8 - (2^{2n-1} - 2) \delta_i^8 \delta_j^8 \delta_k^8 \delta_l^8 \right) \quad (3.20)
\end{aligned}$$

The  $T$ -tensor for this vacuum is then given by

$$\begin{aligned}
K_{\dot{\alpha}\dot{\beta}kl}^{(c)} & := \frac{2}{3} \sqrt{5} \left( 2 \gamma_{\dot{\alpha}\dot{\gamma}}^{kl} \delta_\gamma^8 \delta_\beta^8 - \gamma_{\dot{\alpha}\dot{\gamma}}^{il} \delta_k^8 \delta_\gamma^8 \delta_i^8 \delta_\beta^8 - \gamma_{\dot{\alpha}\dot{\gamma}}^{ki} \delta_l^8 \delta_\gamma^8 \delta_i^8 \delta_\beta^8 \right) \\
K_{\alpha\beta kl}^{(s)} & := \frac{2}{3} \sqrt{5} \left( 2 \gamma_{\alpha\gamma}^{kl} \delta_\gamma^8 \delta_\beta^8 - \gamma_{\alpha\gamma}^{il} \delta_k^8 \delta_\gamma^8 \delta_i^8 \delta_\beta^8 - \gamma_{\alpha\gamma}^{ki} \delta_l^8 \delta_\gamma^8 \delta_i^8 \delta_\beta^8 \right) \\
K_{ijkl}^{(v)} & := -\frac{47}{6} \delta_{kl}^{ij} - \frac{1}{6} \delta_{kl}^{mn} D_{im}^{(7v)} D_{jn}^{(7v)} - \frac{1}{8} D_{\alpha\beta}^{(7s)} \gamma_{\alpha\beta}^{ijkl} - \frac{1}{8} D_{\dot{\alpha}\dot{\beta}}^{(7c)} \gamma_{\dot{\alpha}\dot{\beta}}^{ijkl} \\
T_{AB} & = \frac{1}{2} \left( H_A^{\dot{\alpha}\dot{\beta}} H_B^{IJ} + H_B^{\dot{\alpha}\dot{\beta}} H_A^{IJ} \right) H_I^{\bar{i}} H_J^{\bar{j}} K_{\dot{\alpha}\dot{\beta}\bar{i}\bar{j}}^{(c)} \\
& + \frac{1}{2} \left( H_A^{\alpha\beta} H_B^{IJ} + H_B^{\alpha\beta} H_A^{IJ} \right) H_I^{\bar{i}} H_J^{\bar{j}} K_{\beta\alpha\bar{i}\bar{j}}^{(s)} \\
& - \frac{2}{3} \sqrt{5} \left( H_A^{\alpha\beta} H_B^{IJ} + H_B^{\alpha\beta} H_A^{IJ} \right) H_I^{\bar{i}} H_J^{\bar{j}} \left( \delta_\beta^8 \delta_i^8 \delta_\gamma^8 \gamma_{\alpha\gamma}^{\bar{j}} - \delta_\alpha^8 \delta_j^8 \delta_\gamma^8 \gamma_{\beta\gamma}^{\bar{i}} \right) \\
& - \frac{2}{3} \sqrt{5} \left( H_A^{\dot{\alpha}\dot{\beta}} H_B^{IJ} + H_B^{\dot{\alpha}\dot{\beta}} H_A^{IJ} \right) H_I^{\bar{i}} H_J^{\bar{j}} \left( \delta_\beta^8 \delta_i^8 \delta_\gamma^8 \gamma_{\gamma\alpha}^{\bar{j}} - \delta_\alpha^8 \delta_j^8 \delta_\gamma^8 \gamma_{\gamma\beta}^{\bar{i}} \right) \\
& + \frac{8}{3} \sqrt{5} \left( H_A^{\alpha\beta} H_B^{\gamma\delta} + H_B^{\alpha\beta} H_A^{\gamma\delta} \right) \left( \delta_i^8 \delta_\beta^8 \delta_\delta^8 \gamma_{\alpha\gamma}^i - \delta_i^8 \delta_\alpha^8 \delta_\gamma^8 \gamma_{\beta\delta}^i \right) \\
& - \frac{1}{12} \left( H_A^{IJ} H_B^{KL} + H_B^{IJ} H_A^{KL} \right) H_I^{\bar{i}} H_J^{\bar{j}} H_K^k H_L^l \delta_j^8 \delta_m^8 \gamma_{\alpha\beta}^{mikl} D_{\alpha\beta}^{(7s)} \\
& + \frac{1}{12} \left( H_A^{IJ} H_B^{KL} + H_B^{IJ} H_A^{KL} \right) H_I^{\bar{i}} H_J^{\bar{j}} H_K^k H_L^l \delta_k^8 \delta_m^8 \gamma_{\alpha\beta}^{m\bar{i}\bar{k}\bar{l}} D_{\alpha\beta}^{(7s)}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left( H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{IJ} + H_{\mathcal{B}}^{\alpha\beta} H_{\mathcal{A}}^{IJ} \right) H_{\mathcal{I}}^{\bar{i}} H_{\mathcal{J}}^{\bar{j}} K_{\alpha\beta\bar{i}\bar{j}}^{(s)} \\
& -4H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} \left( \delta_{\beta}^{\delta} \delta_{\delta}^{\alpha} \delta_{\alpha\gamma} - \delta_{\alpha}^{\delta} \delta_{\gamma}^{\delta} \delta_{\beta\delta} \right) \\
& -4H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \left( \delta_{\dot{\beta}}^{\dot{\delta}} \delta_{\dot{\delta}}^{\dot{\alpha}} \delta_{\dot{\alpha}\dot{\gamma}} - \delta_{\dot{\alpha}}^{\dot{\delta}} \delta_{\dot{\gamma}}^{\dot{\delta}} \delta_{\dot{\beta}\dot{\delta}} \right) \\
& -\frac{1}{2} \left( H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{IJ} + H_{\mathcal{B}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{A}}^{IJ} \right) H_{\mathcal{I}}^i H_{\mathcal{J}}^j K_{\dot{\beta}\dot{\alpha}ij}^{(c)} \\
& +\frac{1}{4} H_{\mathcal{A}}^{IJ} H_{\mathcal{B}}^{KL} H_{\mathcal{I}}^{\bar{i}} H_{\mathcal{J}}^{\bar{j}} H_{\mathcal{K}}^{\bar{k}} H_{\mathcal{L}}^{\bar{l}} K_{\bar{i}\bar{j}\bar{k}\bar{l}}^{(v)} \\
& -\frac{1}{4} H_{\mathcal{A}}^{IJ} H_{\mathcal{B}}^{KL} H_{\mathcal{I}}^i H_{\mathcal{J}}^j H_{\mathcal{K}}^k H_{\mathcal{L}}^l K_{ijkl}^{(v)} \\
& +\frac{2}{3} H_{\mathcal{A}}^{IJ} H_{\mathcal{B}}^{KL} H_{\mathcal{I}}^i H_{\mathcal{J}}^{\bar{j}} H_{\mathcal{K}}^k H_{\mathcal{L}}^{\bar{l}} \left( \delta_{\bar{j}}^{\delta} \delta_{\bar{l}}^{\delta} P_{ik}^{(7)} - \delta_{\bar{i}}^{\delta} \delta_{\bar{k}}^{\delta} P_{\bar{j}\bar{l}}^{(7)} \right)
\end{aligned} \tag{3.21}$$

and the  $A_{1,2,3}$  tensors are

$$\begin{aligned}
A_1^{IJ} &= H_{\mathcal{I}}^I H_{\mathcal{J}}^J \left( 4P_{ij}^{(7v)} + \frac{8}{3} \delta_i^{\delta} \delta_j^{\delta} \right) - H_{\mathcal{I}}^I H_{\mathcal{J}}^J \left( 4P_{\bar{i}\bar{j}}^{(7v)} + \frac{8}{3} \delta_{\bar{i}}^{\delta} \delta_{\bar{j}}^{\delta} \right) \\
A_2^{IA} &= \frac{4}{3} \sqrt{5} \left( H_{\mathcal{I}}^I H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} \delta_{\dot{\alpha}}^{\delta} \delta_{\dot{\beta}}^{\delta} P_{ij}^{(7v)} \gamma_{\gamma\dot{\beta}}^j - H_{\mathcal{I}}^I H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} \delta_{\dot{\beta}}^{\delta} \delta_{\dot{\alpha}}^{\delta} P_{\bar{i}\bar{j}}^{(7v)} \gamma_{\alpha\dot{\gamma}}^{\bar{j}} \right. \\
& \quad \left. - H_{\mathcal{I}}^I H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} \delta_{\dot{\alpha}}^{\delta} \delta_{\dot{\beta}}^{\delta} P_{ij}^{(7v)} \gamma_{\beta\dot{\gamma}}^j - H_{\mathcal{I}}^I H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} \delta_{\dot{\beta}}^{\delta} \delta_{\dot{\alpha}}^{\delta} P_{\bar{i}\bar{j}}^{(7v)} \gamma_{\gamma\dot{\alpha}}^{\bar{j}} \right) \\
A_3^{\dot{A}\dot{B}} &= -8 H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \left( P_{\alpha\gamma}^{(7s)} \delta_{\beta}^{\delta} \delta_{\delta}^{\delta} - P_{\beta\delta}^{(7c)} \delta_{\alpha}^{\delta} \delta_{\gamma}^{\delta} \right) \\
& -8 H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \left( P_{\dot{\alpha}\dot{\gamma}}^{(7c)} \delta_{\beta}^{\delta} \delta_{\delta}^{\delta} - P_{\beta\delta}^{(7s)} \delta_{\dot{\alpha}}^{\delta} \delta_{\dot{\gamma}}^{\delta} \right) \\
& -\frac{4}{3} \left( H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} + H_{\mathcal{A}}^{\dot{\beta}\dot{\alpha}} H_{\mathcal{B}}^{\dot{\delta}\dot{\gamma}} \right) \delta_i^{\delta} \times \\
& \quad \times \left( \delta_j^{\delta} \gamma_{\gamma\dot{\alpha}}^i \gamma_{\beta\dot{\delta}}^j - 4 \gamma_{\gamma\dot{\alpha}}^i \delta_{\beta}^{\delta} \delta_{\delta}^{\delta} - 4 \gamma_{\beta\dot{\delta}}^i \delta_{\alpha}^{\delta} \delta_{\gamma}^{\delta} \right)
\end{aligned} \tag{3.22}$$

And one sees that indeed, the stationarity condition  $3A_1^{IJ}A_2^{JA} = A_2^{IB}A_3^{\dot{B}\dot{A}}$  is satisfied.

Thus, we obtain:

$\Lambda/(2g^2)$	$-256/9$
$\mathcal{M}/g^2$	$1040/9_{(\times 1)}, 16_{(\times 1)}, 0_{(\times 28)}, -112/9_{(\times 49)}, -80/3_{(\times 49)}$
$\mathcal{M}^{\text{vec}}/g$	$20/3_{(\times 7)}, 4/3_{(\times 7)}, 0_{(\times 100)}, -4/3_{(\times 7)}, -20/3_{(\times 7)}$
$A_1$	$4_{(\times 7)}, 8/3_{(\times 1)}, -8/3_{(\times 1)}, -4_{(\times 7)}$
$A_3$	$12_{(\times 7)*}, 28/3_{(\times 1)}, 4_{(\times 7)}, 4/3_{(\times 49)},$ $-4/3_{(\times 49)}, -4_{(\times 7)}, -28/3_{(\times 1)}, -12_{(\times 7)*}$

### 3.4.2 The other vacua

For the other vacua, we do not present the full calculation but just list their locations and properties.

The vacuum with largest unbroken symmetry has a remaining symmetry of  $SO(7)^\pm \times SO(7)^\pm$  (both are equivalent here), and is located at

$$\mathcal{V}^c_{\mathcal{B}} = \exp \left( (\text{Arcosh } 2) \delta_8^\alpha \delta_8^\beta H_{\alpha\beta}^A f_{AB}^c \right) \quad (3.23)$$

– all supersymmetry is broken here, but it is nevertheless stable:

$\Lambda/(2g^2)$	$-25$	
$\mathcal{M}/g^2$	$96_{(\times 1)}, 0_{(\times 14)}, -9_{(\times 64)}, -24_{(\times 49)}$	
$\mathcal{M}^{\text{vec}}/g$	$6_{(\times 7)}, 0_{(\times 114)}, -6_{(\times 7)}$	(3.24)
$A_1$	$7/2_{(\times 8)}, -7/2_{(\times 8)}$	
$A_3$	$21/2_{(\times 8)^*}, 3/2_{(\times 56)}, -3/2_{(\times 56)}, -21/2_{(\times 8)^*}$	

There is a further vacuum with  $(n, n_R) = (2, 2)$  supersymmetry and a remaining symmetry of  $SU(3) \times U(1) \times SU(3) \times U(1)$  at

$$\mathcal{V}^c_{\mathcal{B}} = \exp \left( \frac{1}{2} (\text{Arcosh } 3) f_{AB}^c \left( H_{\alpha\beta}^A \left( \delta_3^\alpha \delta_3^\beta + \delta_5^\alpha \delta_5^\beta \right) + H_{\dot{\alpha}\dot{\beta}}^A \left( \delta_4^{\dot{\alpha}} \delta_6^{\dot{\beta}} + \delta_6^{\dot{\alpha}} \delta_4^{\dot{\beta}} \right) \right) \right) \quad (3.25)$$

with mass spectrum

$\Lambda/(2g^2)$	$-36$	
$\mathcal{M}/g^2$	$160_{(\times 1)}, 28_{(\times 4)}, 0_{(\times 38)}, -20_{(\times 36)}, -32_{(\times 49)}$	
$\mathcal{M}^{\text{vec}}/g$	$8_{(\times 7)}, 2_{(\times 12)}, 0_{(\times 90)}, -2_{(\times 12)}, -8_{(\times 7)}$	(3.26)
$A_1$	$5_{(\times 6)}, 3_{(\times 2)}, -3_{(\times 2)}, -5_{(\times 6)}$	
$A_3$	$15_{(\times 6)^*}, 11_{(\times 2)}, 5_{(\times 14)}, 1_{(\times 42)},$ $-1_{(\times 42)}, -5_{(\times 14)}, -11_{(\times 2)}, -15_{(\times 6)^*}$	

Another AdS vacuum without any supersymmetry that breaks  $SO(8) \times SO(8)$  down to  $SU(4)$  and has  $\Lambda/(2g^2) = -52$  is located at

$$\begin{aligned} a &:= \frac{1}{4} \log \left( \frac{5}{2} + \frac{1}{2} \sqrt{21} \right) \\ b &:= \text{Arcosh } 2 \\ \mathcal{V}^c_{\mathcal{B}} &= \exp \left( f_{AB}^c \left( a H_{\alpha\beta}^A \delta^{\alpha\beta} + b H_{\dot{\alpha}\dot{\beta}}^A \left( \delta_3^{\dot{\alpha}} \delta_5^{\dot{\beta}} + \delta_5^{\dot{\alpha}} \delta_3^{\dot{\beta}} \right) \right) \right) \end{aligned} \quad (3.27)$$

The mass spectrum for this vacuum is slightly more difficult to calculate than for the preceding cases and has not been determined yet.

A further AdS vacuum that breaks all supersymmetry and has a remaining symmetry of  $SU(3) \times U(1) \times U(1)$  is given by

$$\begin{aligned}
A_{\alpha\beta} &:= \delta_{\alpha\beta} - \delta_\alpha^3 \delta_\beta^3 - \delta_\alpha^5 \delta_\beta^5 \\
k &:= \sqrt{78 + 14\sqrt{33}} \\
m &= \sqrt{6 + 6\sqrt{33}} \\
p &:= \log\left(\frac{1}{6}\sqrt{18 + 6\sqrt{33} + 6m}\right) \approx 0.4616649 \\
q &:= -\frac{1}{2}\log\left(\frac{7}{2} + \frac{1}{2}\sqrt{33} - \frac{1}{2}k\right) \approx 1.2694452 \\
\mathcal{V}_B^C &= \exp\left(f_{AB}^C \left(pH_{\alpha\beta}^A A^{\alpha\beta} + qH_{\dot{\alpha}\dot{\beta}}^A \left(\delta_3^{\dot{\alpha}} \delta_5^{\dot{\beta}} + \delta_5^{\dot{\alpha}} \delta_3^{\dot{\beta}}\right)\right)\right) \quad (3.28) \\
\Lambda/(2g^2) &= -3564 \left(1453 + 253\sqrt{33} + 116k + 20\sqrt{33}k\right) \times \\
&\quad \left(3 + \sqrt{33} + m\right)^{-2} \left(7 + \sqrt{33} + k\right)^{-2} \times \\
&\quad \left(24 + 6\sqrt{33} - 3m - \sqrt{33}m\right)^{-1} \approx -49.82132
\end{aligned}$$

Again, the mass spectrum has not been determined analytically yet.

Furthermore, there is strong numerical evidence for a further vacuum with remaining symmetry of only  $SU(3)$  at

$$\begin{aligned}
p &\approx 0.43045295, \quad q \approx 0.03708009, \quad r \approx 1.16386200 \\
A_{\alpha\beta}^{(s)} &= \sum_{n \in \{3,5\}} -q \delta_\alpha^n \delta_\beta^n + \sum_{n \in \{1,2,4,6,7,8\}} -p \delta_\alpha^n \delta_\beta^n - r \delta_\alpha^3 \delta_\beta^5 - r \delta_\alpha^5 \delta_\beta^3 \\
A_{\alpha\beta}^{(c)} &= \sum_{n \in \{3,5\}} q \delta_\alpha^n \delta_\beta^n + \sum_{n \in \{1,2,4,6,7,8\}} p \delta_\alpha^n \delta_\beta^n + r \delta_\alpha^3 \delta_\beta^5 + r \delta_\alpha^5 \delta_\beta^3 \\
\mathcal{V}_B^C &= \exp\left(f_{AB}^C \left(pH_{\alpha\beta}^A A_{\alpha\beta}^{(s)} + qH_{\dot{\alpha}\dot{\beta}}^A A_{\dot{\alpha}\dot{\beta}}^{(c)}\right)\right) \quad (3.29) \\
\Lambda/(2g^2) &\approx 105.527621
\end{aligned}$$

So far, no analytic expression has been found for this vacuum candidate. The corresponding parameters for (3.4) are  $\phi = \pi/4$ ,  $a = 15\pi/4$ ,  $w = \pi/4$ ,  $\lambda_1 \approx -0.5563132$ ,  $\lambda_2 \approx -1.6459494$ ,  $z \approx -1.8786964$ .



# Chapter 4

## Noncompact gauge groups

### 4.1 On noncompact gauge groups in supergravity

Standard quantum field theory textbook lore [52] tells that gauge groups have to be compact to avoid the appearance of negative norm states [28]. Nevertheless, there are some cases of theories which contain scalar fields where a positive definite metric can be constructed from these scalars that allows an implementation of noncompact gauge invariance. (See [14] and, in particular, [33] for an explanation of the construction of  $SO(7, 1)$  gauged  $N = 8$  supergravity.)

Generally, a necessary condition for a gauge group candidate is that a maximal subset of the vectors transform in the adjoint representation. When it is not possible to absorb the extra charged vectors that transform non-trivially under the gauge group (for example, in  $D = 5$  the vectors can be dualized into massive self-dual 2-forms [45, 30]), as is the case in four dimensions, then the whole set of vectors must be used. This gives quite severe restrictions on possible gaugings in higher dimensions; in particular, for  $D = 4$   $N = 8$  one basically has just the  $SO(p, 8 - p)$  noncompact versions of  $SO(8)$  and their contractions [15], while for  $N = 16$ ,  $D = 3$  a much bigger freedom exists due to the duality between vectors and scalars. Of the maximal semisimple subgroups of  $E_{8(+8)}$ , the analogous noncompact versions of  $SO(8) \times SO(8)$  of the form  $SO(p, 8 - p) \times SO(p, 8 - p)$ , which are contained in  $E_{8(+8)}$  via their embedding in  $SO(8, 8)$ , as well as the exceptional cases  $E_{8(+8)}$ ,  $E_{7(+7)} \times SL(2)$ ,  $E_{7(-5)} \times SU(2)$ ,  $E_{6(+6)} \times SL(3)$ ,  $E_{6(+2)} \times SU(2, 1)$ ,  $E_{6(-14)} \times SU(3)$ ,  $G_2 \times F_{4(-20)}$ , and  $G_{2(+2)} \times F_{4(+4)}$  are possible.

## 4.2 $SO(p, 8 - p)^2$

Going back to the  $E_8 \rightarrow SO(8)_L \times SO(8)_R$  decomposition  $\mathbf{248} = (\mathbf{1}, \mathbf{28}) + (\mathbf{28}, \mathbf{1}) + (\mathbf{8}_v, \mathbf{8}_v) + (\mathbf{8}_s, \mathbf{8}_s) + (\mathbf{8}_c, \mathbf{8}_c)$ , we obtain  $SO(8, 8)$  from  $SO(16)$  by using a triality rotation to exchange the role of vectors and spinors. Then, the compact subgroup of the first factor of  $SO(p, 8 - p)_1 \times SO(p, 8 - p)_2$  is obtained by splitting each  $(\mathbf{8}_s) \rightarrow (\mathbf{p}, \mathbf{8} - \mathbf{p})$  and combining the  $SO(p)_L$  with the  $SO(8 - p)_R$  subgroup. (And vice versa for the second factor.) As before, the  $P^{27000}\Theta = 0$  projection condition requires a ratio of gauge coupling constants of  $-1$ . Hence, the embedding tensor of  $SO(p, 8 - p)^2$  is given by

$$\begin{aligned}
P_{\alpha\beta} &:= \sum_{j=1}^p \delta_\alpha^p \delta_\beta^p, & Q_{\alpha\beta} &:= \delta_{\alpha\beta} - P_{\alpha\beta} \\
H_{\mathcal{A}}^{ij} &:= H_{\mathcal{A}}^{[IJ]} H_{[IJ]}^{IJ} H_{\mathcal{I}}^i H_{\mathcal{J}}^j, & H_{\mathcal{A}}^{\bar{i}\bar{j}} &:= H_{\mathcal{A}}^{[IJ]} H_{[IJ]}^{IJ} H_{\mathcal{I}}^{\bar{i}} H_{\mathcal{J}}^{\bar{j}} \\
\Theta_{\mathcal{AB}}^{SO(p,8-p)^2} &= \frac{1}{16} H_{\mathcal{A}}^{ij} H_{\mathcal{B}}^{kl} \gamma_{\alpha\beta}^{ij} \gamma_{\gamma\delta}^{kl} (P^{\alpha\gamma} P^{\beta\delta} - Q^{\alpha\gamma} Q^{\beta\delta}) \\
&\quad + \frac{1}{16} H_{\mathcal{A}}^{\bar{i}\bar{j}} H_{\mathcal{B}}^{\bar{k}\bar{l}} \gamma_{\alpha\beta}^{\bar{i}\bar{j}} \gamma_{\gamma\delta}^{\bar{k}\bar{l}} (Q^{\alpha\gamma} Q^{\beta\delta} - P^{\alpha\gamma} P^{\beta\delta}) \\
&\quad + 2H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} P_{\alpha\gamma} Q_{\beta\delta} - 2H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} Q_{\alpha\gamma} P_{\beta\delta}. \tag{4.1}
\end{aligned}$$

The subgroups presented in the previous chapter have been chosen in such a way to maximize the amount of information that can be obtained on various gaugings of the theory by simply redoing the calculations using the invariant scalar submanifold parametrizations given there with other (compatible) embedding tensors.<sup>1</sup> It may happen that in some cases, some of the singlets under the subgroup in question of the compact subgroup of the gauge group are part of the noncompact gauge group, and hence correspond to trivial flat directions of the potential, but this does not spoil the analysis.

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<sup>1</sup>Note that the double ratio  $\frac{\text{Effort}(\text{Submanifold M, hand calculation Nr. 2}) / \text{Effort}(\text{Submanifold M, machine calculation Nr. 2})}{\text{Effort}(\text{Submanifold M, hand calculation Nr. 1}) / \text{Effort}(\text{Submanifold M, machine calculation Nr. 1})}$  hardly could be larger, since all we have to do in the machine approach is to change the value  $p$  and go for a cup of coffee!



Using the  $G_{2\text{diag}}$  singlet parametrization (3.5) for  $SO(7, 1)^2$  gives

$$\begin{aligned}
-8g^{-2}V = & \frac{909}{32} - \frac{7}{8} \cosh(2s) - \frac{49}{32} \cosh(4s) + \frac{6461}{512} \cosh(s) \cosh(z) \\
& - \frac{1001}{512} \cosh(3s) \cosh(z) - \frac{203}{512} \cosh(5s) \cosh(z) - \frac{137}{512} \cosh(7s) \cosh(z) \\
& + \frac{21}{8} \cos(2v) + \frac{63}{32} \cos(4v) + \frac{7}{2} \cos(2v) \cosh(2s) \\
& - \frac{49}{8} \cos(2v) \cosh(4s) - \frac{21}{8} \cos(4v) \cosh(2s) + \frac{21}{32} \cos(4v) \cosh(4s) \\
& + \frac{5145}{1024} \cos(2v) \cosh(s) \cosh(z) - \frac{5229}{1024} \cos(2v) \cosh(3s) \cosh(z) \\
& + \frac{217}{1024} \cos(2v) \cosh(5s) \cosh(z) - \frac{133}{1024} \cos(2v) \cosh(7s) \cosh(z) \\
& - \frac{21}{512} \cos(4v) \cosh(s) \cosh(z) - \frac{63}{512} \cos(4v) \cosh(3s) \cosh(z) \\
& + \frac{147}{512} \cos(4v) \cosh(5s) \cosh(z) - \frac{63}{512} \cos(4v) \cosh(7s) \cosh(z) \\
& - \frac{105}{1024} \cos(6v) \cosh(s) \cosh(z) + \frac{189}{1024} \cos(6v) \cosh(3s) \cosh(z) \\
& - \frac{105}{1024} \cos(6v) \cosh(5s) \cosh(z) + \frac{21}{1024} \cos(6v) \cosh(7s) \cosh(z) \\
& - \frac{28987}{2048} \cos(v-w) \sinh(z) \sinh(s) - \frac{651}{512} \cos(v+w) \sinh(z) \sinh(s) \\
& - \frac{2835}{2048} \cos(v-w) \sinh(z) \sinh(3s) - \frac{3255}{512} \cos(v+w) \sinh(z) \sinh(3s) \\
& - \frac{623}{2048} \cos(v-w) \sinh(z) \sinh(5s) + \frac{21}{512} \cos(v+w) \sinh(z) \sinh(5s) \\
& - \frac{343}{2048} \cos(v-w) \sinh(z) \sinh(7s) - \frac{63}{512} \cos(v+w) \sinh(z) \sinh(7s) \\
& + \frac{1323}{1024} \cos(3v-w) \sinh(z) \sinh(s) - \frac{7875}{2048} \cos(3v+w) \sinh(z) \sinh(s) \\
& - \frac{385}{1024} \cos(3v-w) \sinh(z) \sinh(3s) + \frac{2541}{2048} \cos(3v+w) \sinh(z) \sinh(3s) \\
& + \frac{35}{1024} \cos(3v-w) \sinh(z) \sinh(5s) + \frac{609}{2048} \cos(3v+w) \sinh(z) \sinh(5s) \\
& - \frac{49}{1024} \cos(3v-w) \sinh(z) \sinh(7s) - \frac{399}{2048} \cos(3v+w) \sinh(z) \sinh(7s) \\
& + \frac{35}{2048} \cos(5v-w) \sinh(z) \sinh(s) + \frac{315}{1024} \cos(5v+w) \sinh(z) \sinh(s) \\
& - \frac{189}{2048} \cos(5v-w) \sinh(z) \sinh(3s) + \frac{63}{1024} \cos(5v+w) \sinh(z) \sinh(3s) \\
& + \frac{175}{2048} \cos(5v-w) \sinh(z) \sinh(5s) - \frac{189}{1024} \cos(5v+w) \sinh(z) \sinh(5s) \\
& - \frac{49}{2048} \cos(5v-w) \sinh(z) \sinh(7s) + \frac{63}{1024} \cos(5v+w) \sinh(z) \sinh(7s) \\
& + \frac{315}{2048} \cos(7v+w) \sinh(z) \sinh(s) - \frac{189}{2048} \cos(7v+w) \sinh(z) \sinh(3s) \\
& + \frac{63}{2048} \cos(7v+w) \sinh(z) \sinh(5s) - \frac{9}{2048} \cos(7v+w) \sinh(z) \sinh(7s)
\end{aligned} \tag{4.2}$$

and by setting  $v = 0$ , we again obtain the potential on the manifold of  $SO(7)$  singlets

$$\begin{aligned}
-8g^{-2}V = & 33 - 7 \cosh(4s) + \frac{35}{2} \cosh(s) \cosh(z) - 7 \cosh(3s) \cosh(z) \\
& - \frac{1}{2} \cosh(7s) \cosh(z) - \frac{35}{2} \cos(w) \sinh(z) \sinh(s) \\
& - 7 \cos(w) \sinh(z) \sinh(3s) - \frac{1}{2} \cos(w) \sinh(z) \sinh(7s).
\end{aligned} \tag{4.3}$$

The potentials obtained by using the  $SO(6)_{\text{diag}}$  singlet parametrization (3.9) for the embedding tensors of  $SO(7, 1)^2$  and  $SO(6, 2)^2$  as well as the  $(SO(5) \times SO(2))_{\text{diag}}$  potential for  $(SO(5) \times SO(3))^2$  are given in appendix A. The potential of the  $(SO(5) \times SO(3))^2$  gauged theory on the manifold of  $(SO(5) \times SO(3))_{\text{diag}}$  singlets parametrized by (3.11) reads

$$\begin{aligned}
-8g^{-2}V = & 25 - 15 \cosh(4s) - 15 \cosh(s) \cosh(z) + \frac{15}{2} \cosh(3s) \cosh(z) \\
& + \frac{3}{2} \cosh(5s) \cosh(z) + 15 \cos(w) \sinh(z) \sinh(s) \\
& + \frac{15}{2} \cos(w) \sinh(z) \sinh(3s) - \frac{3}{2} \cos(w) \sinh(z) \sinh(5s).
\end{aligned} \tag{4.4}$$

For the gauge group  $SO(4, 4)^2$ , the potential on the manifold of  $(SO(4) \times SO(4))_{\text{diag}}$  singlets parametrized by (3.16) with  $x = 0$  is

$$-8g^{-2}V = 24 - 16 \cosh(z) - 16 \cosh(s) + 8 \cosh(s) \cosh(z), \quad (4.5)$$

and including the extra  $(SO(4) \times SO(3))_{\text{diag}}$  singlet, this reads

$$\begin{aligned} -8g^{-2}V &= 21 + 3 \cosh(8x) - 12 \cosh(z) \cosh(2x) - 4 \cosh(z) \cosh(6x) \\ &\quad - 12 \cos(w) \sinh(2x) \sinh(z) + 4 \cos(w) \sinh(6x) \sinh(z) \\ &\quad - 12 \cosh(s) \cosh(2x) - 4 \cosh(s) \cosh(6x) + 4 \cosh(s) \cosh(z) \\ &\quad + \frac{9}{2} \cosh(s) \cosh(z) \cosh(4x) \\ &\quad - \frac{1}{2} \cosh(s) \cosh(z) \cosh(12x) - \frac{3}{2} \cosh(s) \cos(w) \sinh(4x) \sinh(z) \\ &\quad + \frac{1}{2} \cosh(s) \cos(w) \sinh(12x) \sinh(z) + 12 \cos(v) \sinh(2x) \sinh(s) \\ &\quad - 4 \cos(v) \sinh(6x) \sinh(s) + \frac{3}{2} \cos(v) \cosh(z) \sinh(4x) \sinh(s) \\ &\quad - \frac{1}{2} \cos(v) \cosh(z) \sinh(12x) \sinh(s) + 4 \cos(v) \cos(w) \sinh(z) \sinh(s) \\ &\quad - \frac{9}{2} \cos(v) \cos(w) \cosh(4x) \sinh(z) \sinh(s) \\ &\quad + \frac{1}{2} \cos(v) \cos(w) \cosh(12x) \sinh(z) \sinh(s). \end{aligned} \quad (4.6)$$

If we do the same calculation with the embedding tensor of  $SO(7, 1)^2$ , we obtain

$$\begin{aligned} -8g^{-2}V &= 21 - 3 \cosh(8x) + 12 \cosh(z) \cosh(2x) - 4 \cosh(z) \cosh(6x) \\ &\quad + 12 \cos(w) \sinh(2x) \sinh(z) + 4 \cos(w) \sinh(6x) \sinh(z) \\ &\quad + 12 \cosh(s) \cosh(2x) - 4 \cosh(s) \cosh(6x) + 4 \cosh(s) \cosh(z) \\ &\quad - \frac{3}{2} \cosh(s) \cosh(z) \cosh(4x) \\ &\quad - \frac{1}{2} \cosh(s) \cosh(z) \cosh(12x) + \frac{9}{2} \cosh(s) \cos(w) \sinh(4x) \sinh(z) \\ &\quad + \frac{1}{2} \cosh(s) \cos(w) \sinh(12x) \sinh(z) - 12 \cos(v) \sinh(2x) \sinh(s) \\ &\quad - 4 \cos(v) \sinh(6x) \sinh(s) - \frac{9}{2} \cos(v) \cosh(z) \sinh(4x) \sinh(s) \\ &\quad - \frac{1}{2} \cos(v) \cosh(z) \sinh(12x) \sinh(s) + 4 \cos(v) \cos(w) \sinh(z) \sinh(s) \\ &\quad + \frac{3}{2} \cos(v) \cos(w) \cosh(4x) \sinh(z) \sinh(s) \\ &\quad + \frac{1}{2} \cos(v) \cos(w) \cosh(12x) \sinh(z) \sinh(s) \end{aligned} \quad (4.7)$$

## 4.3 Exceptional gauge groups

### 4.3.1 $E_{7(+7)} \times SL(2)$

Since for the gauge group  $E_{8(+8)}$ , the potential reduces to just a cosmological constant, the nontrivial case with the largest number of noncompact directions in the gauge group is that of  $E_{7(+7)} \times SL(2)$ .

Here, the embedding tensor is given by

$$\begin{aligned}
\Theta_{AB}^{E_7(+7) \times SL(2)} &= -\frac{1}{2} H_A^{[IJ]} H_B^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} \\
&\quad \left( H_I^i H_J^j H_K^k H_L^l \delta_{kl}^{ij} + H_I^{\bar{i}} H_J^{\bar{j}} H_K^{\bar{k}} H_L^{\bar{l}} \delta_{\bar{k}\bar{l}}^{\bar{i}\bar{j}} \right. \\
&\quad + H_I^i H_J^j H_K^{\bar{k}} H_L^{\bar{l}} \delta_{\bar{k}\bar{l}}^{ij} + H_I^{\bar{i}} H_J^{\bar{j}} H_K^k H_L^l \delta_{kl}^{\bar{i}\bar{j}} \\
&\quad \left. + 2 H_I^i H_J^{\bar{j}} H_K^k H_L^{\bar{l}} (\delta_{ik} \delta_{\bar{j}\bar{l}} + \delta_{\bar{j}k} \delta_{i\bar{l}} - \delta_{i\bar{j}} \delta_{k\bar{l}}) \right) \\
&\quad + H_A^{\alpha\beta} H_B^{\gamma\delta} (\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\beta\gamma} \delta_{\alpha\delta} - \delta_{\alpha\beta} \delta_{\gamma\delta}) \\
&\quad + H_A^{\dot{\alpha}\dot{\beta}} H_B^{\dot{\gamma}\dot{\delta}} (\delta_{\dot{\alpha}\dot{\gamma}} \delta_{\dot{\beta}\dot{\delta}} + \delta_{\dot{\beta}\dot{\gamma}} \delta_{\dot{\alpha}\dot{\delta}} - \delta_{\dot{\alpha}\dot{\beta}} \delta_{\dot{\gamma}\dot{\delta}})
\end{aligned} \tag{4.8}$$

Using the tools we developed in previous sections, it is natural to consider the scalar potential on the manifold of  $SO(6) \subset SO(8)$  singlets by using the parametrization (3.10). Since the  $SL(2)$  parametrized by  $v, r_5$  is part of the gauge group, and hence corresponds to flat directions in the potential, these two parameters drop out. Furthermore, three of the five noncompact directions of the  $SO(6)$ -invariant  $SL(3)$  singlets lie in the gauge group, and the smallest group containing the remaining two orthogonal directions is  $SL(2)$ , which we can parametrize by<sup>2</sup>

$$\begin{aligned}
S^c_{\mathcal{B}} &= \left( \delta_7^\alpha \delta_8^\beta - \delta_8^\alpha \delta_7^\beta \right) H_{\alpha\beta}^A f_{AB}^C \\
V^c_{\mathcal{B}} &= \frac{1}{2} \delta^{i\bar{j}} H_i^I H_{\bar{j}}^J H_{IJ}^A f_{AB}^C \\
\mathcal{V} &= \exp(vV) \exp(sS) \exp(-vV)
\end{aligned} \tag{4.9}$$

and obtain the potential

$$-8g^{-2}V = 22 - 6 \cosh(4s), \tag{4.10}$$

which obviously does not have any nontrivial stationary points.

### 4.3.2 $E_{7(-5)} \times SU(2)$

An embedding of this gauge group into  $E_{8(+8)}$  can be obtained by deleting the  $+2-3$  root from the extended Dynkin diagram 1.2. The ladder operators corresponding to the remaining roots will generate the noncompact real form  $E_{7(-5)} \times SU(2)$ . In order to define the embedding tensor, we have to introduce a further  $\mathbf{8} \rightarrow \mathbf{4} + \mathbf{4}$  index split for the vectors of the left  $SO(8)$  of  $SO(8) \times SO(8)$ ; using superscripts  $A, B$  to discern both  $\mathbf{4}$ , we have  $I \rightarrow (i, \bar{j}) \rightarrow (i^A, i^B, \bar{j})$ .

<sup>2</sup>Note that  $S^c_{\mathcal{B}}$  was incorrectly translated from machine form – where index counting starts at zero – to conventional notation in [19].

The embedding tensor is then given by

$$\begin{aligned}
\Theta_{\mathcal{AB}}^{E_{7(-5)} \times SU(2)} = & \frac{1}{2} H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} \delta_{\alpha\gamma} \left( \delta_{\beta\delta} + \gamma_{\beta\delta}^{ijkl} \delta_i^1 \delta_j^2 \delta_k^3 \delta_l^4 \right) \\
& + \frac{1}{2} H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \delta_{\dot{\alpha}\dot{\gamma}} \left( \delta_{\dot{\beta}\dot{\delta}} + \gamma_{\dot{\beta}\dot{\delta}}^{ijkl} \delta_i^1 \delta_j^2 \delta_k^3 \delta_l^4 \right) \\
& + \frac{1}{2} H_{\mathcal{A}}^{[IJ]} H_{\mathcal{B}}^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} H_I^i H_J^j H_K^k H_L^l \\
& \left( H_i^{iA} H_j^{jA} H_k^{kA} H_l^{lA} \delta_{k^A l^A}^{i^A j^A} - H_i^{iB} H_j^{jB} H_k^{kB} H_l^{lB} \delta_{k^B l^B}^{i^B j^B} \right. \\
& \left. + 24 \delta_{ijkl}^{1234} \right) \\
& - \frac{1}{2} H_{\mathcal{A}}^{[IJ]} H_{\mathcal{B}}^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} H_I^i H_J^j H_K^k H_L^l \delta_{\bar{k}\bar{l}}^{\bar{i}\bar{j}} \\
& - H_{\mathcal{A}}^{[IJ]} H_{\mathcal{B}}^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} H_I^i H_J^j H_K^k H_L^l \delta_{\bar{j}\bar{l}}^{\bar{i}\bar{A}} H_i^A H_k^A \delta_{i^A k^A}
\end{aligned} \tag{4.11}$$

Shooting into the blue sky, we take the positive roots + + + + + + + + and + + + + - - - - and parametrize the four noncompact directions of the corresponding  $SL(2) \times SL(2)$  via

$$\begin{aligned}
A^{ij} & := \delta_i^1 \delta_j^2 + \delta_i^3 \delta_j^4 + \delta_i^5 \delta_j^6 + \delta_i^7 \delta_j^8 \\
V_{\mathcal{B}}^C & = \frac{1}{2} \delta^{ij} H_i^I H_j^J H_{IJ}^A f_{\mathcal{AB}}^C \\
S_{\mathcal{B}}^C & = \left( -\delta_3^\alpha \delta_3^\beta + \delta_5^\alpha \delta_5^\beta \right) H_{\alpha\beta}^A f_{\mathcal{AB}}^C \\
W_{\mathcal{B}}^C & = \frac{1}{2} \left( H_i^I H_j^J A^{ij} - H_i^I H_j^J A^{\bar{i}\bar{j}} \right) H_{IJ}^A f_{\mathcal{AB}}^C \\
Z_{\mathcal{B}}^C & = \left( +\delta_3^\alpha \delta_3^\beta + \delta_5^\alpha \delta_5^\beta \right) H_{\alpha\beta}^A f_{\mathcal{AB}}^C
\end{aligned}$$

$$\mathcal{V} = \exp(vV) \exp(sS) \exp(-vV) \exp(wW) \exp(zZ) \exp(-wW) \tag{4.12}$$

and obtain the potential

$$\begin{aligned}
-8g^{-2}V = & \frac{39}{4} + \frac{1}{4} \cosh(4z) + \frac{1}{4} \cosh(4s) \\
& + 6 \cosh(2s) \cosh(2z) - \frac{1}{4} \cosh(4s) \cosh(4z)
\end{aligned} \tag{4.13}$$

which indeed does have a stationary point corresponding to a true vacuum at  $s = z = \frac{1}{2} \text{Arcosh } 2$ . The corresponding singlet generator  $\delta_5^\alpha \delta_5^\beta H_{\alpha\beta}^A f_{\mathcal{AB}}^C \text{Arcosh } 2$  breaks the  $(SO(12) \times SO(3))_{E_7} \times SU(2)_{SU(2)}$  compact subgroup of the  $E_{7(-5)} \times SU(2)$  gauge group in quite an interesting way: the  $SU(2)$  gauge group factor with positive root +1 + 2 forms a diagonal  $SO(3)$  with an  $SO(3)$  factor within the  $SO(12)$  of the  $SO(12) \times SO(3)$  maximal compact subgroup of  $E_{7(-5)}$ ; the positive root of this latter  $SO(3)$  is +3 + 4. The  $SO(3)$  factor of forementioned  $SO(12) \times SO(3)$  (with positive root +1 - 2) remains unbroken, and a third  $SO(3)$  factor with positive root +3 - 4 appears. The rest of  $SO(12)$  then forms a  $SO(7)$  with an  $SU(4)$  with simple

roots  $+5 - 6, +6 - 7, +7 - 8$  as simply laced subgroup. Hence, the remaining gauge symmetry is  $SO(7) \times SO(3) \times SO(3) \times SO(3)_{\text{diag}}$ , and this vacuum furthermore features  $N = 4$  supersymmetry.

### 4.3.3 $E_{6(-14)} \times SU(3)$

For this gauge group, no attempt has been made so far to find a nontrivial vacuum; here, we only give the embedding tensor. The subset  $+4 - 5, +5 - 6, +6 - 7, +7 - 8, +7 + 8, + - - - - - +$  of simple roots from (1.11) with respect to the compact Cartan subalgebra corresponds to an embedding of  $E_{6(-14)}$  into  $E_{8(+8)}$ .  $SU(3)$  is likewise obtained from the roots  $-1 - 2, +2 - 3$ . This is sufficient information to complete the construction. It gives:

$$\begin{aligned}
D_{\alpha\beta}^{(46s)} &:= \delta_{\alpha}^4 \delta_{\beta}^4 + \delta_{\alpha}^6 \delta_{\beta}^6, & D_{\dot{\alpha}\dot{\beta}}^{(46c)} &:= \delta_{\dot{\alpha}}^4 \delta_{\dot{\beta}}^4 + \delta_{\dot{\alpha}}^6 \delta_{\dot{\beta}}^6 \\
E_{ij} &:= \delta_i^7 \delta_j^7 + \delta_i^8 \delta_j^8, & F_{ij} &:= \delta_{ij} - E_{ij} \\
\Theta_{AB}^{E_{6(-14)} \times SU(3)} &= H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} \delta^{\alpha\gamma} D_{\beta\delta}^{(46s)} + H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \delta^{\dot{\alpha}\dot{\gamma}} D_{\dot{\beta}\dot{\delta}}^{(46c)} \\
&\quad + H_{\mathcal{A}}^{[IJ]} H_{\mathcal{A}}^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} \\
&\quad \left( -H_I^i H_J^{\bar{j}} H_K^k H_L^{\bar{l}} \delta_{\bar{j}\bar{l}} E_{ik} - \frac{1}{2} H_I^{\bar{i}} H_J^{\bar{j}} H_K^{\bar{k}} H_L^{\bar{l}} \delta_{\bar{k}\bar{l}}^{\bar{i}\bar{j}} \right. \\
&\quad \left. + \frac{1}{2} H_I^i H_J^j H_K^k H_L^l (\delta_{kl}^{mn} F_{im} F_{jn} - \delta_{kl}^{mn} E_{im} E_{jn} \right. \\
&\quad \left. - \frac{1}{32} \gamma_{\alpha\beta}^{ijkl} (\delta_{\alpha\beta} - 4 D_{\alpha\beta}^{(46s)}) - \frac{1}{32} \gamma_{\dot{\alpha}\dot{\beta}}^{ijkl} (\delta_{\dot{\alpha}\dot{\beta}} - 4 D_{\dot{\alpha}\dot{\beta}}^{(46c)}) \right).
\end{aligned} \tag{4.14}$$

### 4.3.4 $E_{6(+2)} \times SU(2, 1)$

Applying a  $+++++$  Weyl reflection to the previously given set of simple  $E_6$  roots yields the roots  $+1 + 8, +4 - 5, +5 - 6, +6 - 7, +7 - 8$ , and  $- - - - - + +$ , which correspond to an embedding of  $E_{6(+2)}$  into  $E_{8(+8)}$ . In the same way, we obtain the corresponding  $SU(2, 1)$  from  $- - + + + + +$  and  $+2 - 3$ . Unfortunately, the corresponding embedding tensor is quite unwieldy, so we do not give it here, as it perhaps makes sense to try to find a construction which is more convenient for further calculations.

### 4.3.5 $E_{6(+6)} \times SL(3)$

One way to obtain the embedding tensor for this group is to take as  $SL(3)$  the subgroup of  $E_{8(+8)}$  that commutes with the compact  $(SO(6) \times SO(2))_{\text{diag}}$  subgroup of the diagonal of the gauge group  $SO(6, 2)^2$  constructed in sec. 3.3.3; the corresponding generators were explicitly given by  $p_{1\dots 8}$  in (3.8). The group commuting with this  $SL(3)$  is just  $E_{6(+6)}$ . With the intention to embed

$F_{4(+4)}$  into  $E_{6(+6)}$  in mind, a perhaps somewhat more useful construction is given by starting again from the extended Dynkin diagram, but this time with roots corresponding to the noncompact Cartan subalgebra, that is, (1.15).

The embedding tensor thus obtained reads

$$\begin{aligned}
D_{\alpha\beta}^{(3)} &:= \sum_{n=1}^3 \delta_\alpha^n \delta_\beta^n \\
D_{\alpha\beta}^{(5)} &:= \sum_{n=4}^8 \delta_\alpha^n \delta_\beta^n \\
E_{\alpha\beta\gamma\delta} &:= D_{\alpha\gamma}^{(5)} D_{\beta\delta}^{(5)} - D_{\alpha\gamma}^{(3)} D_{\beta\delta}^{(3)} \\
F_{\alpha\beta\gamma\delta} &:= \delta_\alpha^1 \delta_\beta^2 \delta_\gamma^1 \delta_\delta^2 + \delta_\alpha^1 \delta_\beta^3 \delta_\gamma^1 \delta_\delta^3 - \delta_\alpha^2 \delta_\beta^3 \delta_\gamma^2 \delta_\delta^3 \\
G_{\alpha\beta\gamma\delta} &:= \delta_\alpha^2 \delta_\beta^2 \delta_\gamma^3 \delta_\delta^3 + \delta_\alpha^3 \delta_\beta^3 \delta_\gamma^2 \delta_\delta^2 - \delta_\alpha^2 \delta_\beta^3 \delta_\gamma^3 \delta_\delta^2 - \delta_\alpha^3 \delta_\beta^2 \delta_\gamma^2 \delta_\delta^3 \\
\Theta_{AB}^{E_{6(+6)} \times SL(3)} &= H_A^{[IJ]} H_B^{[KL]} H_{[IJ]}^{IJ} H_{[KL]}^{KL} \\
&\quad \left( -\frac{1}{32} H_I^i H_J^j H_K^k H_L^l \gamma_{\alpha\beta}^{ij} \gamma_{\gamma\delta}^{kl} E_{\alpha\beta\gamma\delta} \right. \\
&\quad -\frac{1}{32} H_I^{\bar{i}} H_J^{\bar{j}} H_K^{\bar{k}} H_L^{\bar{l}} \gamma_{\alpha\beta}^{\bar{i}\bar{j}} \gamma_{\gamma\delta}^{\bar{k}\bar{l}} E_{\alpha\beta\gamma\delta} \\
&\quad -\frac{1}{16} H_I^{\bar{i}} H_J^{\bar{j}} H_K^k H_L^l \gamma_{\alpha\beta}^{\bar{i}\bar{j}} \gamma_{\gamma\delta}^{kl} F_{\alpha\beta\gamma\delta} \\
&\quad -\frac{1}{16} H_I^i H_J^j H_K^{\bar{k}} H_L^{\bar{l}} \gamma_{\alpha\beta}^{ij} \gamma_{\gamma\delta}^{\bar{k}\bar{l}} F_{\alpha\beta\gamma\delta} \\
&\quad \left. -\frac{1}{2} H_I^i H_J^{\bar{j}} H_K^k H_L^{\bar{l}} \left( \frac{1}{4} \gamma_{\alpha\beta}^{i\bar{j}} \gamma_{\gamma\delta}^{k\bar{l}} \left( \delta_1^\alpha \delta_1^\gamma D_{\beta\delta}^{(5)} + \delta_2^\alpha \delta_3^\beta \delta_2^\gamma \delta_3^\delta \right) \right. \right. \\
&\quad \left. \left. + \frac{1}{24} \gamma_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{i\bar{j}} \gamma_{\beta_1 \beta_2 \beta_3 \beta_4}^{k\bar{l}} \delta_1^{\alpha_1} \delta_1^{\beta_1} D_{\alpha_2 \beta_2}^{(5)} D_{\alpha_3 \beta_3}^{(5)} D_{\alpha_4 \beta_4}^{(5)} \right) \right) \\
&\quad + H_A^{\alpha\beta} H_B^{\gamma\delta} \left( D_{\alpha\gamma}^{(5)} D_{\beta\delta}^{(5)} - D_{\alpha\gamma}^{(3)} D_{\beta\delta}^{(3)} + D_{\alpha\delta}^{(3)} D_{\beta\gamma}^{(3)} - D_{\alpha\beta}^{(3)} D_{\gamma\delta}^{(3)} + 2G_{\alpha\beta\gamma\delta} \right) \\
&\quad + \frac{1}{2} H_A^{\dot{\alpha}\dot{\beta}} H_B^{\dot{\gamma}\dot{\delta}} \left( \frac{1}{4} \gamma_{\alpha\beta}^{\dot{\alpha}\dot{\beta}} \gamma_{\gamma\delta}^{\dot{\gamma}\dot{\delta}} \left( \delta_1^\alpha \delta_1^\gamma D_{\beta\delta}^{(5)} + \delta_2^\alpha \delta_3^\beta \delta_2^\gamma \delta_3^\delta \right) \right) \\
&\quad \left. + \frac{1}{24} \gamma_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\dot{\alpha}\dot{\beta}} \gamma_{\beta_1 \beta_2 \beta_3 \beta_4}^{\dot{\gamma}\dot{\delta}} \delta_1^{\alpha_1} \delta_1^{\beta_1} D_{\alpha_2 \beta_2}^{(5)} D_{\alpha_3 \beta_3}^{(5)} D_{\alpha_4 \beta_4}^{(5)} \right)
\end{aligned} \tag{4.15}$$

### 4.3.6 $F_{4(+4)} \times G_{2(+2)}$

The  $F_{4(+4)}$  subgroup can be obtained from  $E_{6(+6)}$  above;  $G_{2(+2)}$  is then the space of all  $F_{4(+4)}$  singlets. In particular, the  $F_{4(+4)}$  Lie algebra is generated by  $(T_{+7+8} + T_{-7-8})$ ,  $(T_{+7+8} - T_{-7-8})$ ,  $(T_{+6-7} + T_{-6+7})$ ,  $(T_{+6-7} - T_{-6+7})$ ,  $(T_{+-----+} + T_{-+++++--+} + T_{+4-5} + T_{-4+5})$ ,  $(T_{+-----+} - T_{-+++++--+} + T_{+4-5} - T_{-4+5})$ ,  $(T_{+5-6} + T_{-6+5} + T_{+7-8} + T_{-7+8})$ ,  $(T_{+5-6} - T_{-6+5} + T_{+7-8} - T_{-7+8})$ . Unfortunately, this construction mutilates the  $SO(8) \times SO(8)$  components of  $E_{8(+8)}$  so badly that no reasonably simple expression for the embedding tensor could be found in that language.

### 4.3.7 $G_{2(-14)} \times F_{4(-20)}$

The subgroup of  $SO(8)_L$  that leaves the last vector, spinor, and co-spinor index fixed is  $G_2$ ; all generators of  $E_8$  that commute with this  $G_2$  form a  $F_{4(-20)}$

whose compact subgroup is  $SO(9)$ . The linear combination of embedding tensors of these two groups with a relative gauge coupling  $g_{G_2}/g_{F_4} = 3/2$  satisfies the  $P^{27000}$  projection condition and thus corresponds to a possible gauging. It is given by

$$\begin{aligned}
p_{ij} &:= \delta_{ij} - 2\delta_i^8\delta_j^8 \\
d_{ijkl} &:= \delta_{k'l'}^i p_{ii'} p_{jj'} p_{kk'} p_{ll'} \\
\Theta_{\mathcal{AB}} &= H_{\mathcal{A}}^{\alpha\beta} H_{\mathcal{B}}^{\gamma\delta} \delta_{\alpha\gamma} \delta_{\beta\delta}^8 \delta_{\delta}^8 \\
&\quad + H_{\mathcal{A}}^{\dot{\alpha}\dot{\beta}} H_{\mathcal{B}}^{\dot{\gamma}\dot{\delta}} \delta_{\dot{\alpha}\dot{\gamma}} \delta_{\dot{\beta}\dot{\delta}}^8 \delta_{\dot{\delta}}^8 \\
&\quad - H_{\mathcal{A}}^{\underline{[IJ]}} H_{\mathcal{B}}^{\underline{[K L]}} H_{\underline{[IJ]}}^{\underline{IJ}} H_{\underline{[K L]}}^{\underline{KL}} H_I^i H_J^{\bar{j}} H_K^k H_L^{\bar{l}} \delta_{\bar{j}\bar{l}} \delta_i^8 \delta_k^8 \\
&\quad - \frac{1}{2} H_{\mathcal{A}}^{\underline{[IJ]}} H_{\mathcal{B}}^{\underline{[K L]}} H_{\underline{[IJ]}}^{\underline{IJ}} H_{\underline{[K L]}}^{\underline{KL}} H_I^{\bar{i}} H_J^{\bar{j}} H_K^{\bar{k}} H_L^{\bar{l}} \delta_{\bar{i}\bar{j}}^{\bar{k}\bar{l}} \\
&\quad - \frac{1}{16} H_{\mathcal{A}}^{\underline{[IJ]}} H_{\mathcal{B}}^{\underline{[K L]}} H_{\underline{[IJ]}}^{\underline{IJ}} H_{\underline{[K L]}}^{\underline{KL}} H_I^i H_J^j H_K^k H_L^l \\
&\quad \left( 8 d_{ijkl} - \frac{1}{7} \gamma_{\alpha\beta}^{ijkl} (\delta_{\alpha\beta} - 7\delta_{\alpha}^8 \delta_{\beta}^8) - \frac{1}{7} \gamma_{\dot{\alpha}\dot{\beta}}^{ijkl} (\delta_{\dot{\alpha}\dot{\beta}} - 7\delta_{\dot{\alpha}}^8 \delta_{\dot{\beta}}^8) \right).
\end{aligned} \tag{4.16}$$

The main problem in this case is to find a suitable invariance subgroup of the gauge group which is small enough to show nontrivial structure, but does not give too many singlets. Here, we choose that particular subgroup  $SU(3) \times SU(3)$  of the group  $SO(8)_L \times SO(8)_R$  which stabilizes the vectors  $v_1^i = \delta^{i7}$ ,  $v_2^i = \delta^{i8}$ ,  $v_3^{\bar{i}} = \delta^{\bar{i}7}$ ,  $v_4^{\bar{i}} = \delta^{\bar{i}8}$  as well as the spinors  $\psi^{\alpha L} = \delta^{\alpha L 8}$ ,  $\psi^{\alpha R} = \delta^{\alpha R 8}$  (and which therefore is also a subgroup of  $G_2 \times F_{4(-20)}$ ).

This group is stabilized by a subgroup  $SU(2,1) \times SU(2,1)$  of  $E_{8(8)}$ , hence we have to deal with an eight-dimensional submanifold of the supergravity scalars here. The intersection of this eight-dimensional manifold with the gauge group is four-dimensional, but unfortunately, unlike for the parametrization considered in the  $E_{7(7)} \times SL(2)$  case, the smallest group containing the four directions orthogonal to the gauge group is the full  $SU(2,1) \times SU(2,1)$ , hence we parametrize the full eight-dimensional manifold. Using the generators  $X_{(A,B)}$  of both  $SO(3)$  subalgebras as well as those

of two noncompact directions  $Y_{(A,B)}$

$$\begin{aligned}
Y_{(A)\mathcal{B}}^C &= -\frac{1}{2} \left( \delta_2^\alpha \delta_8^\beta - \delta_8^\alpha \delta_2^\beta \right) f_{\alpha\beta\mathcal{B}}^C \\
X_{(A)1\mathcal{B}}^C &= 2 \left( \delta_7^i \delta_8^{\bar{j}} - \delta_8^i \delta_7^{\bar{j}} \right) f_{[i\bar{j}]\mathcal{B}}^C \\
X_{(A)2\mathcal{B}}^C &= 2 \left( \delta_7^i \delta_7^{\bar{j}} + \delta_8^i \delta_8^{\bar{j}} \right) f_{[i\bar{j}]\mathcal{B}}^C \\
X_{(A)3\mathcal{B}}^C &= 2 \left( \delta_7^i \delta_8^{\bar{j}} f_{[i\bar{j}]\mathcal{B}}^C - \delta_7^{\bar{i}} \delta_8^j f_{[\bar{i}j]\mathcal{B}}^C \right) \\
Y_{B\mathcal{B}}^C &= -\frac{1}{2} \left( \delta_2^\alpha \delta_8^\beta + \delta_8^\alpha \delta_2^\beta \right) f_{\alpha\beta\mathcal{B}}^C \\
X_{(B)1\mathcal{B}}^C &= -2 \left( \delta_7^i \delta_8^{\bar{j}} + \delta_8^i \delta_7^{\bar{j}} \right) f_{[i\bar{j}]\mathcal{B}}^C \\
X_{(B)2\mathcal{B}}^C &= -2 \left( \delta_7^i \delta_7^{\bar{j}} - \delta_8^i \delta_8^{\bar{j}} \right) f_{[i\bar{j}]\mathcal{B}}^C \\
X_{(B)3\mathcal{B}}^C &= -2 \left( \delta_7^i \delta_8^{\bar{j}} f_{[i\bar{j}]\mathcal{B}}^C + \delta_7^{\bar{i}} \delta_8^j f_{[\bar{i}j]\mathcal{B}}^C \right),
\end{aligned} \tag{4.17}$$

we parametrize the eight-dimensional singlet manifold as

$$\begin{aligned}
\mathcal{V} &= \exp(r_1 X_{(A)1}) \exp(r_2 X_{(A)2}) \exp(r_3 X_{(A)3}) \\
&\exp(r_4 X_{(B)1}) \exp(r_5 X_{(B)2}) \exp(r_6 X_{(B)3}) \\
&\exp(s Y_{(A)}) \exp(z Y_{(B)}) \\
&\exp(-r_6 X_{(B)3}) \exp(-r_5 X_{(B)2}) \exp(-r_4 X_{(B)1}) \\
&\exp(-r_3 X_{(A)3}) \exp(-r_2 X_{(A)2}) \exp(-r_1 X_{(A)1})
\end{aligned} \tag{4.18}$$

and obtain the potential given (A.6).

## 4.4 Vacua

For every model with noncompact gauge group for which we find a nontrivial vacuum, we also give data for the trivial  $\mathcal{V}_{\mathcal{B}}^C = \delta_{\mathcal{B}}^C$  vacuum; of relevance is for example the square root of the ratio of cosmological constants, since this should correspond to the ratio of central charges of the boundary CFT.

For the gauge group  $SO(7, 1) \times SO(7, 1)$ , the vacuum at the origin has:

$\Lambda/2g^2$	-9	
$\mathcal{M}/g^2$	$16_{(\times 1)}, 0_{(\times 14)}, -5_{(\times 64)}, -8_{(\times 49)}$	
$\mathcal{M}^{\text{vec}}/g$	$2_{(\times 7)}, 0_{(\times 114)}, -2_{(\times 7)}$	
$A_1$	$3/2_{(\times 8)}, -3/2_{(\times 8)}$	
$A_3$	$7/2_{(\times 8)}, 1/2_{(\times 56)}, -1/2_{(\times 56)}, -7/2_{(\times 8)}$	

(4.19)

There is a further stable AdS vacuum with remaining symmetry  $G_2 \times G_2$  and no supersymmetry at

$$\mathcal{V}_{\mathcal{B}}^C = \exp \left( f_{AB}^C H_{\alpha\beta}^A \delta_8^\alpha \delta_8^\beta \cdot \text{Arcosh } 2 \right) \tag{4.20}$$



with mass spectrum

$\Lambda/(2g^2)$	$-211/16$	
$\mathcal{M}/g^2$	$195/4_{(\times 1)}, 45/2_{(\times 1)}, 0_{(\times 28)}, -9/2_{(\times 49)}, -33/4_{(\times 49)}$	
$\mathcal{M}^{\text{vec}}/g$	$9/2_{(\times 7)}, 3_{(\times 7)}, 0_{(\times 100)}, -3_{(\times 7)}, -9/2_{(\times 7)}$	(4.21)
$A_1$	$35/8_{(\times 1)}, 19/8_{(\times 7)}, -19/8_{(\times 7)}, -35/8_{(\times 1)}$	
$A_3$	$105/8_{(\times 1)*}, 57/8_{(\times 7)*}, 33/8_{(\times 7)}, 15/8_{(\times 49)},$ $-15/8_{(\times 49)}, -33/8_{(\times 7)}, -57/8_{(\times 7)*}, -105/8_{(\times 1)*}$	

Furthermore, there is strong numerical evidence for another vacuum with  $\Lambda/(2g^2) = -13$  and remaining symmetry  $SO(6) \times SO(6)$  whose location is approximately given by

$$\mathcal{V}^c_B \approx \exp \left( f_{AB}{}^C H_{\alpha\beta}^A \left( -1.1311837 \delta_7^\alpha \delta_7^\beta + 0.3317440 \delta_7^\alpha \delta_8^\beta \right. \right. \\ \left. \left. + 0.1023203 \delta_8^\alpha \delta_7^\beta - 0.0084405 \delta_8^\alpha \delta_8^\beta \right) \right) \quad (4.22)$$

or, in parameters for (3.10),  $z \approx -s \approx 0.2906220$ ,  $v \approx -1.2037291$ ,  $r_1 \approx 3\pi$ ,  $r_3 \approx 5\pi/2$ ,  $r_2 \approx 6.3825153$ ,  $r_5 \approx 0.3689107$ . So far, no corresponding analytic expression has been obtained for this vacuum.

For the  $SO(5, 3) \times SO(5, 3)$  gauged model, the origin is an AdS vacuum:

$\Lambda/2g^2$	$-1$	
$\mathcal{M}/g^2$	$8_{(\times 9)}, 3_{(\times 64)}, 0_{(\times 55)}$	
$\mathcal{M}^{\text{vec}}/g$	$2_{(\times 15)}, 0_{(\times 98)}, -2_{(\times 15)}$	(4.23)
$A_1$	$1/2_{(\times 8)}, -1/2_{(\times 8)}$	
$A_3$	$5/2_{(\times 24)}, 3/2_{(\times 40)}, -3/2_{(\times 40)}, -5/2_{(\times 24)}$	

Furthermore, we find a deSitter vacuum with remaining symmetry of  $SO(5) \times SO(5) \times SO(3)_{\text{diag}}$  at

$$\mathcal{V}^c_B = \exp \left( \frac{1}{2} \text{Arcosh } 5 f_{AB}{}^C H_{\alpha\beta}^A \left( \delta_6^\alpha \delta_6^\beta + \delta_7^\alpha \delta_7^\beta + \delta_8^\alpha \delta_8^\beta \right) \right) \quad (4.24)$$

with mass spectrum

$\Lambda/(2g^2)$	$11$	
$\mathcal{M}/g^2$	$96_{(\times 5)}, 45_{(\times 48)}, 24_{(\times 25)}, 0_{(\times 33)}, -3_{(\times 16)}, -48_{(\times 1)}$	
$\mathcal{M}^{\text{vec}}/g$	$6_{(\times 15)}, 0_{(\times 98)}, -6_{(\times 15)}$	(4.25)
$A_1$	$5/2_{(\times 8)}, -5/2_{(\times 8)}$	
$A_3$	$15/2_{(\times 8^*, \times 16)}, 9/2_{(\times 40)}, -9/2_{(\times 40)}, -15/2_{(\times 8^*, \times 16)}$	

The vacuum at the origin of the  $SO(4, 4) \times SO(4, 4)$  gauged model is a Minkowski vacuum:

$\Lambda/(2g^2)$	0	
$\mathcal{M}/g^2$	$4_{(\times 96)}, 0_{(\times 32)}$	
$\mathcal{M}^{\text{vec}}/g$	$2_{(\times 16)}, 0_{(\times 96)}, -2_{(\times 16)}$	(4.26)
$A_1$	$0_{(\times 16)}$	
$A_3$	$2_{(\times 64)}, -2_{(\times 64)}$	

Here we find a further deSitter vacuum with remaining symmetry  $SO(4) \times SO(4) \times SO(4)_{\text{diag}}$  at

$$\mathcal{V}_{\mathcal{B}}^c = \exp\left(\frac{1}{2}\text{Arcosh } 2 f_{AB}{}^C H_{\alpha\beta}^A \sum_n = 1^4 \delta_n^\alpha \delta_n^\beta\right) \quad (4.27)$$

with mass spectrum

$\Lambda/(2g^2)$	2	
$\mathcal{M}/g^2$	$12_{(\times 49)}, 9_{(\times 32)}, 0_{(\times 46)}, -12_{(\times 1)}$	
$\mathcal{M}^{\text{vec}}/g$	$3_{(\times 16)}, 0_{(\times 96)}, -3_{(\times 16)}$	(4.28)
$A_1$	$1_{(\times 8)}, -1_{(\times 8)}$	
$A_3$	$3_{(\times 8^*, \times 56)}, -3_{(\times 8^*, \times 56)}$	

For the  $G_{2(-14)} \times F_{4(-20)}$  gauged model, the origin has  $N = (7, 9)$  supersymmetry, and the gauge group is broken down to its maximal compact subgroup  $G_2 \times SO(9)$ :

$\Lambda/(2g^2)$	-4	
$\mathcal{M}/g^2$	$0_{(\times 16)}, -3_{(\times 112)}$	
$\mathcal{M}^{\text{vec}}/g$	$1_{(\times 16)}, 0_{(\times 112)}$	(4.29)
$A_1$	$1_{(\times 7)}, -1_{(\times 9)}$	
$A_3$	$2_{(\times 16)}, 0_{(\times 112)}$	

There is a further AdS vacuum with  $N = (0, 1)$  supersymmetry which breaks gauge group to  $SU(3) \times SO(7)$  at

$$\mathcal{V}_{\mathcal{B}}^c = \exp\left(\frac{1}{2}\text{Arcosh } 7 f_{AB}{}^C H_{\dot{\alpha}\dot{\beta}}^A \delta_8^{\dot{\alpha}} \delta_2^{\dot{\beta}}\right) \quad (4.30)$$

with mass spectrum

$\Lambda/(2g^2)$	$-25/4$	
$\mathcal{M}/g^2$	$24_{(\times 1)}, 0_{(\times 37)}, -9/4_{(\times 48)}, -6_{(\times 42)}$	
$\mathcal{M}^{\text{vec}}/g$	$4_{(\times 1)}, 3_{(\times 6)}, 3/2_{(\times 8)}, 1_{(\times 7)}, 0_{(\times 91)}, -1/2_{(\times 8)}, -3_{(\times 7)}$	(4.31)
$A_1$	$11/4_{(\times 1)}, 7/4_{(\times 6)}, -5/4_{(\times 1)}, -7/4_{(\times 8)}$	
$A_3$	$33/4_{(\times 1)^*}, 21/4_{(\times 6)^*}, 17/4_{(\times 1)}, 11/4_{(\times 8)}, 9/4_{(\times 7)}, 3/4_{(\times 48)}, -3/4_{(\times 42)}, -7/4_{(\times 7)}, -21/4_{(\times 8)^*}$	

For the  $E_{7(-5)} \times SU(2)$  exceptional gauging, the vacuum at the origin breaks the gauge group down to  $SO(12) \times SO(3) \times SU(2)$ , with  $N = (4, 12)$  supersymmetry:

$\Lambda/(2g^2)$	$-4$	(4.32)
$\mathcal{M}/g^2$	$0_{(\times 64)}, -3_{(\times 64)}$	
$\mathcal{M}^{\text{vec}}/g$	$1_{(\times 64)}, 0_{(\times 64)}$	
$A_1$	$1_{(\times 4)}, -1_{(\times 12)}$	
$A_3$	$2_{(\times 64)}, 0_{(\times 64)}$	

As explained before in sec. 4.3.2, there is a further nontrivial vacuum at

$$\mathcal{V}_B^c = \exp\left(\delta_5^\alpha \delta_5^\beta H_{\alpha\beta}^A f_{AB}{}^C \text{Arcosh } 2\right) \quad (4.33)$$

with remaining symmetry  $SO(7) \times SO(3) \times SO(3) \times SO(3)_{\text{diag}}$ ,  $N = (0, 4)$ :

$\Lambda/(2g^2)$	$-25/4$	(4.34)
$\mathcal{M}/g^2$	$24_{(\times 1)}, 0_{(\times 106)}, -6_{(\times 21)}$	
$\mathcal{M}^{\text{vec}}/g$	$4_{(\times 4)}, 3_{(\times 3)}, 3/2_{(\times 32)}, 1_{(\times 28)},$ $0_{(\times 22)}, -1/2_{(\times 32)}, -3_{(\times 7)}$	
$A_1$	$11/4_{(\times 4)}, -5/4_{(\times 4)}, -7/4_{(\times 8)}$	
$A_3$	$33/4_{(\times 4)^*}, 17/4_{(\times 4)}, 11/4_{(\times 32)},$ $9/4_{(\times 28)}, 3/4_{(\times 24)}, -7/4_{(\times 28)}, -21/4_{(\times 8)^*}$	



# Chapter 5

## A four-dimensional example

For maximal ( $N = 8$ ) gauged supergravity in four dimensions with compact gauge group  $SO(8)$ , the known nontrivial vacua are one with a remaining symmetry of  $SO(3) \times SO(3)$  and cosmological constant  $-14g^2$  [50] as well as five further ones for which it has been proven by exhaustive analysis of the potential restricted to a submanifold of six scalars that they are the only vacua with a remaining symmetry of at least  $SU(3)$ . It is tempting to try to apply the tools presented here to go even further and break  $SO(8)$  down to a smaller subgroup. In particular, among the different embeddings of  $SO(3)$  into  $SO(8)$ , there is a very simple one for which the  $SO(3)$ -invariant submanifold of the 70-dimensional space  $E_{7(7)}/SU(8)$  is ten-dimensional. This is given by the following construction: under its maximal compact subgroup,  $SU(8)$ ,  $E_{7(7)}$  decomposes into the adjoint representation as well as the self-dual and anti-self-dual 4-forms  $\mathbf{133} \rightarrow \mathbf{63} + \mathbf{35}_{\text{sd}} + \mathbf{35}_{\text{asd}}$ . With respect to the gauge group  $SO(8)$ , the  $\mathbf{35}_{\text{sd}}$  and  $\mathbf{35}_{\text{asd}}$  are just the symmetric traceless matrices over the spinors and co-spinors, while  $\mathbf{63}$  decomposes into the adjoint  $\mathbf{28}$  and a further  $\mathbf{35}$  given by the symmetric traceless matrices over the vectors. Hence we split  $E_{7(7)}$  adjoint indices via  $\mathbf{A} \rightarrow ((\underline{\alpha\beta}), (\underline{\dot{\alpha}\dot{\beta}}), (\underline{ab}), [\underline{ij}])$ . If we furthermore split the fundamental representation via  $\mathbf{56} \rightarrow \mathbf{28} + \mathbf{28}$ ,  $\mathbf{P} \rightarrow ([\underline{ij}]^a, [\underline{ij}]^b)$  and introduce the auxiliary tensors

$$\Psi_{\underline{(\alpha\beta)}}^{ijkl} := \begin{cases} \gamma_{\alpha\beta}^{ijkl} \left( \delta_n^\alpha \delta_n^\beta - \delta_{n+1}^\alpha \delta_{n+1}^\beta \right) & \text{for } \underline{(\alpha\beta)} = n \in \{1 \dots 7\} \\ \gamma_{\alpha\beta}^{ijkl} \left( \delta_p^\alpha \delta_q^\beta + \delta_q^\alpha \delta_p^\beta \right) & \text{for } \begin{cases} \underline{(\alpha\beta)} = n \in \{8 \dots 35\}, \\ \underline{[pq]} = n - 8, \\ p, q \text{ such that } H_{\underline{[pq]}}^{pq} = 1 \end{cases} \end{cases} \quad (5.1)$$

$$\dot{\Psi}_{\underline{(\dot{\alpha}\dot{\beta})}}^{ijkl} := \text{ditto, with } \gamma_{\dot{\alpha}\dot{\beta}}^{ijkl} \text{ in place of } \gamma_{\alpha\beta}^{ijkl} \quad (5.2)$$

$$\Omega_{\underline{(ij)b}^c} = \begin{cases} i(\delta_n^b \delta_n^c - \delta_{n+1}^b \delta_{n+1}^c) & \text{for } \underline{(ij)} = n \in \{1..7\} \\ i(\delta_p^b \delta_q^c + \delta_q^b \delta_p^c) & \text{for } \begin{cases} \underline{(ij)} = n \in \{8..35\}, \\ \underline{[pq]} = n - 8, \\ p, q \text{ such that } H_{\underline{[pq]}}^{pq} = 1 \end{cases} \end{cases} \quad (5.3)$$

then generators in the  $E_{7(7)}$  fundamental representation can be obtained from<sup>1</sup> (here, capital letters  $I, J, \dots$  also designate  $SO(8)$  indices)

$$\begin{aligned} t_{\text{AP}}^{\text{Q}} &= \frac{1}{8} H_{\underline{\text{A}}}^{(\alpha\beta)} H_{\underline{[ij]}}^{ij} H_{\underline{\text{Q}}}^{[ij]^a} H_{\underline{KL}}^{[KL]} H_{\underline{[KL]b}}^{\text{P}} \Psi_{\underline{(\alpha\beta)}}^{ijKL} \\ &+ \frac{1}{8} H_{\underline{\text{A}}}^{(\alpha\beta)} H_{\underline{[kl]}}^{kl} H_{\underline{\text{Q}}}^{[kl]^b} H_{\underline{IJ}}^{[IJ]} H_{\underline{[IJ]^a}}^{\text{P}} \Psi_{\underline{(\alpha\beta)}}^{klIJ} \\ &+ \frac{i}{8} H_{\underline{\text{A}}}^{(\dot{\alpha}\dot{\beta})} H_{\underline{[ij]}}^{ij} H_{\underline{\text{Q}}}^{[ij]^a} H_{\underline{KL}}^{[KL]} H_{\underline{[KL]b}}^{\text{P}} \dot{\Psi}_{\underline{(\dot{\alpha}\dot{\beta})}}^{ijKL} \\ &- \frac{i}{8} H_{\underline{\text{A}}}^{(\dot{\alpha}\dot{\beta})} H_{\underline{[kl]}}^{kl} H_{\underline{\text{Q}}}^{[kl]^b} H_{\underline{IJ}}^{[IJ]} H_{\underline{[IJ]^a}}^{\text{P}} \dot{\Psi}_{\underline{(\dot{\alpha}\dot{\beta})}}^{klIJ} \\ &+ 2H_{\underline{\text{A}}}^{(ab)} H_{\underline{[ij]^a}}^{\text{Q}} H_{\underline{\text{P}}}^{[IJ]^a} \delta_{j'}^{J'} \delta_{i'j'}^{ij} \delta_{IJ}^{I'J'} \Omega_{\underline{(ab)}i'}^{I'} \\ &+ 2H_{\underline{\text{A}}}^{[ab]} H_{\underline{[ij]^a}}^{\text{Q}} H_{\underline{\text{P}}}^{[IJ]^a} \delta_{j'}^{J'} \delta_{i'j'}^{ij} \delta_{IJ}^{I'J'} H_{\underline{[ab]}}^{ab} \delta_a^{i'} \delta_b^{J'} \\ &- 2H_{\underline{\text{A}}}^{(ab)} H_{\underline{[kl]b}}^{\text{Q}} H_{\underline{\text{P}}}^{[KL]^b} \delta_{l'}^{L'} \delta_{k'l'}^{kl} \delta_{KL}^{K'L'} \Omega_{\underline{(ab)k}'}^{K'} \\ &+ 2H_{\underline{\text{A}}}^{[ab]} H_{\underline{[kl]b}}^{\text{Q}} H_{\underline{\text{P}}}^{[KL]^b} \delta_{l'}^{L'} \delta_{k'l'}^{kl} \delta_{KL}^{K'L'} H_{\underline{[ab]}}^{ab} \delta_a^{k'} \delta_b^{K'}. \end{aligned} \quad (5.4)$$

Structure constants of  $E_{7(7)}$  then satisfy  $t_{\text{AP}}^{\text{Q}} t_{\text{bQ}}^{\text{R}} - t_{\text{BP}}^{\text{Q}} t_{\text{AQ}}^{\text{R}} = f_{\text{AB}}^{\text{C}} t_{\text{CP}}^{\text{S}}$ . With the normalization

$$\eta_{\text{AB}} := \frac{1}{288} f_{\text{AC}}^{\text{D}} f_{\text{BD}}^{\text{C}}, \quad (5.5)$$

metric tensor entries are  $\pm 1$  on the diagonal and  $\pm 1/2$  for neighbouring entries on diagonals on a **35**.

The potential for this model is given as follows: for an element of  $E_{7(7)}$  given as  $56 \times 56$  matrix  $\mathcal{V}_{\text{P}}^{\text{Q}}$ , one first forms the tensor  $T_i^{kij}$  via

$$u_{ij}^{IJ} := \mathcal{V}_{\underline{[IJ]}}^{[ij]} H_{\underline{[ij]^a}}^{ij} H_{\underline{IJ}}^{[IJ]^a}$$

<sup>1</sup>What happens here is quite obvious, but considerably obfuscated by the necessity of successive index mappings to get rid of all factor-two ambiguities: in Matrix notation, the generators of  $E_7$  in the fundamental representation decompose into  $28 \times 28$  blocks; the right-top and left-bottom blocks are formed from the self-dual and anti-self-dual  $SU(8)$  4-forms, which are equivalent to the  $\mathbf{35}_s$  and  $\mathbf{35}_c$  from  $SO(8)$ , while the blocks on the major diagonal are essentially given by  $SU(8)$  generators and their conjugates; cf. (3.10) in [50]. (Note that the generators in the right-bottom corner should be the complex conjugated ones there!)

$$\begin{aligned}
u^{kl}{}_{KL} &:= \mathcal{V} \frac{[kl]}{[KL]} H_{[kl]^b}^{kl} H_{KL}^{[KL]^b} \\
v_{ij}{}_{KL} &:= \mathcal{V} \frac{[ij]}{[KL]} H_{[ij]^a}^{ij} H_{KL}^{[KL]^b} \\
v^{klIJ} &:= \mathcal{V} \frac{[kl]}{[IJ]} H_{[kl]^b}^{kl} H_{IJ}^{[IJ]^a} \\
T_l{}^{kij} &:= \frac{1}{8} (u^{ij}{}_{IJ} + v^{ijIJ}) (u_{lm}{}^{JK} u^{km}{}_{KI} - v_{lmJK} v^{kmKI}).
\end{aligned} \tag{5.6}$$

Then, the  $A_1$  and  $A_2$  tensors are given by

$$A_1^{ij} = -\frac{4}{21} T_m{}^{ijm}, \quad A_2^{ijk} = -\frac{4}{3} T_l{}^{i'j'k'} \delta_{i'j'k'} \tag{5.7}$$

and the potential is

$$V(\mathcal{V}) = g^2 \left( \frac{1}{24} A_{2i}^{jkl} A_{2i}^{*jkl} - \frac{3}{4} A_1^{ij} A_1^{*ij} \right). \tag{5.8}$$

If we consider the  $SO(3)$  subgroup leaving invariant the vector coordinates 4...8, then the  $SO(3)$  invariant scalars are

$$\begin{aligned}
S_n{}^{\mathbf{Q}}{}_{\mathbf{P}} &= \frac{1}{16} \Psi_{(\alpha\beta)}^{123n} H_{\mathbf{B}}^{(\alpha\beta)} \eta^{\mathbf{BA}} t_{\mathbf{AP}}{}^{\mathbf{Q}} \\
C_n{}^{\mathbf{Q}}{}_{\mathbf{P}} &= \frac{1}{16} \dot{\Psi}_{(\dot{\alpha}\dot{\beta})}^{123n} H_{\mathbf{B}}^{(\dot{\alpha}\dot{\beta})} \eta^{\mathbf{BA}} t_{\mathbf{AP}}{}^{\mathbf{Q}}
\end{aligned} \tag{5.9}$$

where  $n \in \{4, 5, 6, 7, 8\}$

and the task is now to evaluate (5.8) for an arbitrary linear combination

$$V \left( \exp \left( \sum_{n=4}^8 (s_n S_n + c_n C_n) \right) \right)$$

with ten parameters  $s_n, c_n$ . Although it is easily possible to analytically exponentiate every single one of the (noncommuting) generators  $S_n, C_n$ , this is not the case for an arbitrary linear combination, hence it is necessary to find a different parametrization of this space more amenable to an analytic treatment. It is mostly the complexity of these other parametrizations which allow analytic treatment that make the resulting formulae so complicated, thus it is natural to ask the question whether there might be an alternative approach to this class of problems that avoids this step entirely.

Here, we will make use of  $\mathcal{V}(SX) = U(S)\mathcal{V}(X)U(S^{-1})$  with  $S \in su(8)$  and start with  $\exp(\rho S_4)$  to which we apply an alternating sequence of  $SU(8)$  rotations with real and imaginary generators that turn part of the  $4_s$ -component

into the  $4_c$  component, then part of the  $4_s$  component into the  $5_s$  component, then  $5_s \rightarrow 5_c$ ,  $4_s \rightarrow 6_s$ , etc. In particular, if we define

$$\begin{aligned} w_n &:= s_n + ic_n \\ R^{SO(8)a}{}_b(j, k, \alpha) &:= \delta_{ab} + \delta_{aj}\delta_{kb} \sin \alpha - \delta_{aj}\delta_{kb} \sin \alpha \\ &\quad + (\delta_{aj}\delta_{bj} + \delta_{ak}\delta_{bk}) (\cos \alpha - 1) \\ R^{SU(8)a}{}_b(j, k, \alpha) &:= \delta_{ab} + \delta_{aj}\delta_{bj} (e^{i\alpha} - 1) + \delta_{ak}\delta_{bk} (e^{-i\alpha} - 1) \end{aligned} \quad (5.10)$$

and promote  $SU(8)$  rotations to  $E_7$  via

$$\begin{aligned} R^{\mathbb{P}_q} &= \frac{1}{2} \left( R^k{}_i R^l{}_j H_{[ij]}^{ij} H_{[kl]}^{[kl]} H_{[kl]^a}^{\mathbb{P}} H_{[ij]^a}^{\mathbb{Q}} \right. \\ &\quad \left. + \bar{R}^k{}_i \bar{R}^l{}_j H_{[ij]}^{ij} H_{[kl]}^{[kl]} H_{[kl]^b}^{\mathbb{P}} H_{[ij]^b}^{\mathbb{Q}} \right) \end{aligned} \quad (5.11)$$

then we have

$$V \left( \exp \left( \sum_{n=4}^8 (s_n S_n + c_n C_n) \right) \right) = V (RA \exp(\rho S_4) A^{-1} R^{-1}) \quad (5.12)$$

where

$$\begin{aligned} \tilde{\phi}_{45} &:= \frac{1}{2} \text{Arg} \frac{w_5}{w_4} & \tilde{\omega}_{45} &:= \text{atan} \frac{w_5 e^{-i\tilde{\phi}_{45}}}{w_4 e^{i\tilde{\phi}_{45}}} & w_{45} &:= \frac{w_4 e^{i\tilde{\phi}_{45}}}{\cos(\tilde{\omega}_{45})} \\ \tilde{\phi}_{46} &:= \frac{1}{2} \text{Arg} \frac{w_6}{w_{45}} & \tilde{\omega}_{46} &:= \text{atan} \frac{w_6 e^{-i\tilde{\phi}_{46}}}{w_{45} e^{i\tilde{\phi}_{46}}} & w_{46} &:= \frac{w_{45} e^{i\tilde{\phi}_{46}}}{\cos(\tilde{\omega}_{46})} \\ \tilde{\phi}_{47} &:= \frac{1}{2} \text{Arg} \frac{w_7}{w_{46}} & \tilde{\omega}_{47} &:= \text{atan} \frac{w_7 e^{-i\tilde{\phi}_{47}}}{w_{46} e^{i\tilde{\phi}_{47}}} & w_{47} &:= \frac{w_{46} e^{i\tilde{\phi}_{47}}}{\cos(\tilde{\omega}_{47})} \\ \tilde{\phi}_{48} &:= \frac{1}{2} \text{Arg} \frac{w_8}{w_{47}} & \tilde{\omega}_{48} &:= \text{atan} \frac{w_8 e^{-i\tilde{\phi}_{48}}}{w_{47} e^{i\tilde{\phi}_{48}}} & w_{48} &:= \frac{w_{47} e^{i\tilde{\phi}_{48}}}{\cos(\tilde{\omega}_{48})} \\ \alpha &:= \text{Arg} w_{48} & \rho &:= |w_{48}| \\ \phi_{\square} &:= -\tilde{\phi}_{\square} & \omega_{\square} &:= -\tilde{\omega}_{\square} \end{aligned}$$

$$\begin{aligned} R &:= R^{SU(8)}(4, 5, \phi_{45}) R^{SO(8)}(4, 5, \omega_{45}) R^{SU(8)}(4, 6, \phi_{46}) \times \\ &\quad \times R^{SO(8)}(4, 6, \omega_{46}) R^{SU(8)}(4, 7, \phi_{47}) R^{SO(8)}(4, 7, \omega_{47}) \times \\ &\quad \times R^{SU(8)}(4, 8, \phi_{48}) R^{SO(8)}(4, 8, \omega_{48}) \end{aligned}$$

$$\begin{aligned} W &:= 2 [S_4, C_4] \\ A^{\mathbb{P}_q} &:= \delta^{\mathbb{P}_q} - W^{\mathbb{P}_q} \sin(\alpha/2) + W^{\mathbb{P}_R} W^{\mathbb{R}_q} (\cos(\alpha/2) - 1). \end{aligned} \quad (5.13)$$

With this parametrization, the calculation of the potential is straightforward, but still a solid computational challenge, even if we make use of the observation that it is independent of  $\alpha$ . By trial and error, the author found out that forming the tensors  $A_1$  and  $A_2$  is unproblematic, but even after



reducing the maximal number of variables which the `LambdaTensor` package can handle from 28 to 14 (in order to halve memory requirements to store terms), the calculation of their abs-squares, in particular the component  $A_1^{44}$ , exceeds memory limitations of 32-bit computer architectures. This step was performed by splitting the corresponding difficult terms into four pieces each, doing multiplication component-wise and writing out intermediate quantities to disk.<sup>2</sup> Even then, peak memory requirements well exceed 1 GB of RAM. The result is quite long, but numerical checks indicate its correctness. To conserve it in printed form, it is given in appendix A.4, where a special notation is also introduced to present it. Furthermore, it has been made available in electronic form [23].

One may well consider this potential as defining the present upper limit of what may be done with the `LambdaTensor` package with reasonable effort. Since memory requirements are much more a problem than run time here, it would be possible to add code to the package which makes use of the experience gained in this calculation to automatically handle such situations. Corresponding functionality will be included in a new version of the `LambdaTensor` package as soon as some related nontrivial design decisions have been resolved. Remarkably, despite its complexity, it is possible to translate this formula to a machine code version which can be evaluated in a sufficiently short time<sup>3</sup> and with sufficient accuracy to make a numerical search for vacua feasible. So far, there is only numerical evidence for one further nontrivial stationary point of this potential with a cosmological constant close to  $-14g^2$  which also shows up when setting  $\phi_{47} = \omega_{47} = \phi_{48} = \omega_{48} = 0$  and thereby restricting this potential further to the submanifold of  $SO(3) \times SO(2)$ -invariant scalars. Since there is a known vacuum with  $SO(3) \times SO(3)$  symmetry and cosmological constant  $-14g^2$ , no attempt was made so analyze this candidate for a vacuum any further.

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<sup>2</sup>The advantages to have direct access to low-level details of the implementation of symbolic algebra should be obvious.

<sup>3</sup>roughly 1.2 ms on a 1.8 GHz Pentium-IV



# Chapter 6

## Specialized High-Performance Symbolic Algebra

### 6.1 The LambdaTensor package

Due to considerable complexity of the group-theoretic calculations involved, as well as due to the large number of different individual possible gaugings to be considered, there is a strong incentive to employ computer aid in the study of supergravity scalar potentials. Since none of the readily available packages for symbolic algebra turned out to be powerful enough to perform computations on the level of complexity required here, a new tool had to be developed for interactive work with large Lie algebras and Lie groups on the symbolic as well as numeric level. This approach eventually led to the implementation of the `LambdaTensor` package, designed for efficient symbolic and numeric calculations on sparse and nonsparse higher-rank tensors, which finally was released as a library under a free software license (version 2.1 of the GNU Lesser General Public License), since the expectation is that other areas of research may as well benefit from an efficient implementation of this functionality.

This chapter is intended to explain not only why `LambdaTensor` does exist at all, but also give reasons underlying the design decisions that give it its present form, hence providing important background information for users of `LambdaTensor`.

Concerning the original formulation of the problem, there are two complementary ways how such a package could be designed: either as a tool to perform symbolic manipulations on the level of tensor equations themselves, or as a tool to operate on explicit coordinate instantiations of tensor equations. For `LambdaTensor`, the latter approach was chosen for a variety

of reasons: first, as desirable as a toolkit for computer-controlled interactive tensor equation manipulation may be, one important drawback when it comes to the application of such a framework to the problem at hand is that it would *a priori* not be possible to switch over to numerical counter-checks of results obtained on the purely symbolic level, or to do a fully numeric search for stationary points in the high-dimensional potentials of gauged extended supergravity theories which would not be accessible at the symbolic level. Second, since the most promising method available at present to systematically obtain information about vacua of these theories is to consider the potential on a subspace of the full scalar manifold which is invariant under a subgroup of the gauge group, and since one is in particular interested in making this investigation as exhaustive as possible by making the invariant subspace as large, and hence the subgroup of the gauge group as small as is feasible with available resources, the number of terms in such a purely analytic approach is bound to explode, removing some of the aesthetic as well as computational advantages of this route.

Furthermore, the change in perspective induced by the choice to focus on explicit coordinate representations of generators of exceptional groups turned out to be very fruitful for the solution of the mundane sub-problem of finding exact analytic expressions for Lie group elements obtained by exponentiation of certain generators of special relevance, practically rendering this important step trivial. For investigations of the potential of  $N = 8$ ,  $D = 4$  supergravity, this step is greatly simplified by the peculiar  $SU(8)$  sub-structure of the fundamental  $\mathbf{56}$  of  $E_{7(7)}$  available there, while for  $E_{8(8)}$ , no corresponding construction seems to exist. Since no attempt was made to understand how the underlying structure of the exceptional Lie algebras involved simplifies exponentiation of generators given explicitly in coordinate representation, this is (for this work) perhaps to be considered a lucky coincidence.

## 6.2 Design and implementation

Judging from the design of similarly specialized packages for computer-aided calculations, such as GAP, LiE, Form, R, SPSS, and the like, one very common approach is to implement a standalone executable program which employs an interpreter for a simplistic programming language (frequently also usable interactively), providing commands that wrap internal functionality implementing the relevant algorithms. For `LambdaTensor`, an approach fundamentally different from this conventional one was chosen, which therefore should be justified.

The task to implement an interactively usable symbolic algebra package

asks for a run-time system which takes care of dynamic memory management, provides an interactive top-level, and allows to define new values and functions at run-time. It turns out that these requirements which one would like to impose on a run-time system are common enough to be relevant for a much wider class of highly dynamic programs, and therefore the question arises whether it is feasible to do an efficient and aggressively optimized implementation of a bare core of such a programmable run-time system providing a high-quality implementation of widely used functionality, with the idea in mind that having such a standardized system available should take away the burden of writers of dynamic programs to build one themselves as the foundation for their work. Instead, the idea is to evolve an existing core system by supplementing it with additional definitions. Although it is therefore of vital importance that this core subsystem is extensible and programmable at the level of its implementation itself (in contrast to systems that provide an extension language built on top of them), it should be seen as an advantage if it provided metasyntactic capabilities but hardly any syntax by itself; any existing syntax imposes restrictions on the language the application writer is going to build on top of it, and it can not be anticipated (and should not be tried to) what form the final application language is going to have.

Such a programmable efficiently implemented minimal core system is just what a COMMON LISP system is intended to provide; the fact that LISP syntax looks so different from almost anything else is readily explained in this context: LISP is not intended to have any syntax; instead, a LISP program essentially is the syntax tree which in other languages is generated from code during the first compilation step.

A decision was made not to supplement `LambdaTensor` with an own package-specific language, as for example MAXIMA – which is also LISP-based – does, since this would only have meant restricting functionality available to the end user to a subset of LISP, while adding the burden for the user to learn yet another language<sup>1</sup> which he eventually is bound to drop again, since advanced users will most likely sooner or later want to work with `LambdaTensor` at the level it was written itself. Hence, since there is no reason to assume LISP to be worse than any proprietary limited language specific to only a single application, the application and extension language for `LambdaTensor` is also LISP.

The immediate price that has to be paid by implementing a system like

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<sup>1</sup>not to speak of the author to design and implement and document one – which certainly is quite far away from the original research topic of investigating the extremal structure of gauged extended supergravity!

`LambdaTensor` by extending a COMMON LISP system, in particular CMU Common LISP (CMUCL), is that due to CMUCL containing an optimizing machine code compiler with debugger as well as a base set of further LISP- and Unix-related definitions of considerable size, a running `LambdaTensor` process will typically occupy more than 40 MB of RAM. (This, however, has to be seen in relation to the size of typical intermediate values that appear in calculations one uses this package for, usually some 100 MB.) Furthermore, and perhaps even much more important, CMUCL is (with a few exceptions) only well supported on x86-based free unix platforms (Linux and BSD), hence in particular not available for some workstation or supercomputer architectures one would like to run `LambdaTensor` on. This is of importance, since x86 is a 32-bit architecture, meaning that one has to deal with an address space limit of 4 GB. As there is no realistic problem-induced limit to which supergravity potential calculations could be taken using the present approach, technical limitations how far one can go will be either induced by a memory or computation time barrier. Tests have shown that for these calculations, memory limitations are much more severe than calculation time limitations.

There is a variant of CMU CL, called SBCL, which (with minor adjustments) can be used to run `LambdaTensor` on non-x86 Unix workstations, like PPC- or Alpha-based architectures, which might at a first glance seem especially attractive, since the 64-bit Alpha processor offers a 43-bit virtual address space. Unfortunately, SBCL only runs as 32-bit application on all 64-bit architectures it has been ported to, so nothing is gained here. In fact, to the present author's best knowledge, there does not exist a freely available sufficiently evolved COMMON LISP implementation using 64-bit addressing at the time of this writing. For the LISP derivative Scheme, such solutions do exist, e.g. Marc Feely's Gambit[26]. Since the `LambdaTensor` code employs only a comparatively small subset of the COMMON LISP language, which was deliberately chosen to simplify semi-automatic porting to other LISP-like systems, migration to Scheme may become an interesting option in the future.

### 6.3 Central Algorithms

The `LambdaTensor` package is by orders of magnitude more efficient than general-purpose symbolic algebra packages would be for the specialized task it was designed for.<sup>2</sup> Due to its novel ways to handle symbolic algebra, we will give an overview over the central algorithms that make it fast. This section

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<sup>2</sup>Therefore, it was nominated as one of the best three contributions to the *Heinz Billing Award for the Advancement of Scientific Computation* in 2002[4].

should also help to clarify questions on which optimizations are performed automatically by `LambdaTensor` and which are not. For more details, the reader should also consult [18], the `LambdaTensor` documentation, and the LISP source.

As explained in [18], one fundamentally important idea is to utilize the observation that tensors showing up in group theory calculations frequently are very sparsely occupied – for example, in the conventions used here, structure constants of the largest (248-dimensional) exceptional Lie group  $E_8 f_{AB}^C$  contain only 49 440 out of  $248^3 = 15\,252\,992$  nonzero entries – and hence, we can make good use of efficient implementations of abstract algorithms that can handle sparsely occupied higher-rank tensors. Efficient code working on sparse matrices is widely used and readily available; the appropriate algorithms for handling higher-rank tensors are also quite well-known, albeit in a very different context: relational databases.

In particular, at the level of explicit tensor entries, forming a quantity like

$$M_{abc} = N_{gha} P^{gh}_{bc} \quad (6.1)$$

translates as follows into the language of relational databases (SQL syntax used here):

```
SELECT t1.index3 as index1,
       t2.index3 as index2,
       t2.index4 as index3,
       SUM(t1.val*t2.val)
FROM tensor1 t1, tensor2 t2
WHERE t1.index1=t2.index1 AND t1.index2=t2.index2
GROUP BY t1.index3, t2.index3, t2.index4;
```

Unfortunately, it is not feasible to just connect to an existing SQL database system (like PostgreSQL), create relations for tensors, and use existing implementations of these algorithms by doing all the calculations in the database, for various reasons. Besides considerations concerning the efficiency of communication, and considerable additional computational overhead due to databases having different aims, one major problem is that extending the database system to abstract from the implementation of sum and product here, as is necessary as soon as we want to work with data types not natively supported by the database (which are frequently limited to integers and floatingpoint numbers) would bring along too many technical problems. Hence, what is required is a re-implementation of the underlying database algorithms with numerical as well as symbolic tensor computations as applications in mind. Furthermore, this implementation has to be abstract enough to allow all relevant arithmetic operations to be provided as

parameters, so that one may switch between approximate numerics, exact (i.e. rational number) numerics, and symbolic calculations.<sup>3</sup>

The underlying data structure chosen for the implementation of sparse tensors is that of a multidimensional hash. Just as an array can be regarded as a dictionary mapping every index from a given range of natural numbers to a corresponding value with fast (i.e. independent of the size of the array) lookups, a hash is a generalization of this concept where the set of keys of the dictionary is not restricted to a range of natural numbers, but may be an arbitrary set of values which can effectively be compared for equality. (In our case, a vector of indices.) In brief, the idea behind implementations of hashing algorithms<sup>4</sup> in its simplest (one-dimensional) form is to make a naive implementation of a dictionary which does lookups by traversing a list of key-value pairs (and hence has  $O(N)$  time complexity for lookups in a dictionary of  $N$  elements) fast by splitting this key-value list into an array of such key-value lists of limited average length, where the first lookup step consists of finding the right array entry (called a *hash bucket*) holding the key-value list in which a given key, if present in the hash, is bound to lie. This is done by the help of a hash function which maps keys to bucket numbers. Such hash functions should be easy to calculate and distribute keys evenly between buckets. Whenever a new entry is made into a hash, its key is also first mapped to a bucket number, and then, the corresponding key-value-pair is either appended to the key-value-list in that bucket, or merged with a previously existing entry for this key. Should the average occupation of a hash grow beyond a given limit (like, two key-value pairs per bucket), then a rehash is initiated, which means that the underlying array is deleted and all the data it contained is transferred to another array larger than the previous one by a given factor. This process is invisible to the user of the hash data structure. One-dimensional hashing is easily generalized to the multi-dimensional case (using a higher-rank array of buckets), which is not explained in detail here. (Readers wanting to know more may find it instructive to use the LISP inspector<sup>5</sup> to get an impression how this is done in the underlying implementation. At least, it should be noted here that although LISP has built-in support for higher-rank arrays, it would do

---

<sup>3</sup>The ability to implement and use arbitrary arithmetics on tensor entries has proven to be of great value during the debugging phase of the symbolic algebras provided within this package. For example, it is easy to (even automatically) lift an existing implementation of arithmetic operations on symbolic terms to an implementation working on pairs of terms and numerical values of these terms for a given occupation of variables that signals an error whenever a discrepancy between these values shows up.

<sup>4</sup>see e.g. [36]

<sup>5</sup>(`inspect` `<TENSOR>`)



considerable harm to efficiency to actually use these instead of implementing higher-rank arrays on flattened rank-1 arrays, despite recommendations in [46]. The reason is that LISP's method of array access for higher-rank arrays whose rank is not known at compile time (as in our case) via `(apply #'aref ...)`, resp. `(setf (apply #'aref ...) ...)` would cause an intolerable amount of unnecessary consing.)

While for usual applications, the effectivity of hashing will increase with increasing number of buckets (and hence, decreasing average occupation), one frequent operation in database-related applications (like tensor multiplication in the example above) is to iterate over all entries of a hash. This, of course, works best if the number of unoccupied hash buckets is small, so creating overly sparse hashes also hinders performance. Note that version 1.0 of `LambdaTensor` does not shrink hashes automatically for which occupation density falls below a given threshold, so e.g. calculations with an empty (i.e. all entries zero) tensor which was created from a sparsely occupied one by setting entries to zero will at the average not be much faster than calculations with the original sparsely occupied tensor. (In most cases, this is not an issue.) If more than a certain share of all possible entries of a sparse tensor are set to nonzero values (currently, about one-quarter), `LambdaTensor` will internally rehash the tensor to a conventional nonsparse representation.

The tensor index hash function must map a dimension- $N$  index from the range  $0 \dots (N - 1)$  to the range of hash buckets.<sup>6</sup>

For `LambdaTensor`, the hash function currently employed is simple integer modulus. At a first glance, this may seem to be a very dangerous design decision, since a good hash function should try to distribute entries evenly among buckets, and therefore apply some perturbing operations on the hash key before taking the modulus. Here, the idea is that in most cases, group theory itself will take care of the task of even distribution of elements among buckets (since there usually are no preferred coordinates, or obvious special relations causing hash bucket clashes).<sup>7</sup> In fact, looking in detail at a typical higher-rank tensor in the current implementation, like the 49 440 nonzero entries of the structure constants of  $E_{8(8)}$  `e8-fabc`,<sup>8</sup> one discovers that these are stored in a sparse  $41 \times 41 \times 27$  array with hash bucket occupation statis-

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<sup>6</sup>In LISP, it is conventional to start counting of indices at zero, just as in C. One important advantage of such a convention is that linear indices corresponding to flattened versions of higher-rank arrays are more naturally expressed in the original indices than with the FORTRAN convention to start array index counting at 1.

<sup>7</sup>Note that simple integer modulus is also the hash function for integer numbers in CMU CL and GCL, though not in CLISP.

<sup>8</sup>Using package-private internal functions, this information can be extracted via  
`(array-dimensions (lambdatensor::sp-array-data e8-fabc))`  
and  
`(sort (seq-statistics (map 'simple-array #'length`

tics  $0_{\times 15948}$   $1_{\times 15740}$   $2_{\times 9143}$   $3_{\times 3230}$   $4_{\times 1019}$   $5_{\times 222}$   $6_{\times 63}$   $7_{\times 16}$   $8_{\times 6}$ , which is reasonably close to a typical random distribution on available hash buckets.<sup>9</sup>

Since the sparse array handling part of `LambdaTensor` has to deal with sparse and nonsparse arrays without exposing this to the user, in particular provide tensor arithmetic operations for the cases nonsparse-nonsparse, nonsparse-sparse and sparse-sparse, and furthermore supports specialized efficient implementations for certain types of sparse and nonsparse tensors which employ machine arithmetics where available (i.e. for `double-float` and `complex double-float` tensors), this module contains a considerable amount of code to cover all the possible cases resulting from the product of these distinctions.

One further notable feature of `LambdaTensor` is that in products of multiple tensors, the order in which multiplications are executed is scheduled in such a way as to minimize the total number of multiplications to be performed. Let us briefly illustrate this in a very simplistic matrix example: if  $A$  is a nonsparse  $10 \times 2$  matrix,  $B$  a nonsparse  $2 \times 4$  matrix, and  $C$  a nonsparse  $4 \times 8$  matrix, then the  $10 \times 8$  matrix  $ABC$  can be calculated by performing the first or second product first. Calculating  $(AB)C$  will require  $10 \cdot 2 \cdot 4$  multiplications for the calculation of the  $10 \times 4$  matrix  $AB$  and  $10 \cdot 4 \cdot 8$  multiplications to form the other matrix product, for a total of 400 multiplications. Likewise, calculation of  $A(BC)$  only requires  $2 \cdot 4 \cdot 8 + 10 \cdot 2 \cdot 8 = 224$  multiplications. `LambdaTensor` is aware of such optimizations even for products of multiple tensors with arbitrary contractions between tensors and will automatically try to choose that particular multiplication order which results in the smallest number of arithmetic operations. For sparse tensors, this is done heuristically by assuming independent equidistribution of tensor entries. (Should this heuristic approach fail due to non-independence of the distribution of entries and produce overly expensive calculations, one may always resort to determining multiplication order by splitting such products manually.)

Even with efficient tensor arithmetics available, one further problem is the symbolic complexity of analytic expressions involved in supergravity potential calculations. The widespread approach to first introduce coordinates on certain special submanifolds of the symmetric space  $E_{8(8)}/SO(16)$  by a procedure reminiscent of Euler angle parametrizations of  $SO(3)$  inevitably

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`(lambdatensor::sp-array-linear-data e8-fabc))) #'< :key #'car)`

<sup>9</sup>Things look worse for tensors like `so8-sigma-ijkl-ab` or `epsilon8`, but this may in part be traced back to the small index range in every coordinate, which is supposed to cause major distortions for every hash function mapping coordinates to a smaller range. Still, the quality of this naive and easy to calculate hash function turns out to be sufficient even in these cases, although there is clearly room for improvement.

generates very complicated analytic expressions for the  $\mathcal{V}$ -matrix and all further intermediate quantities derived from it whose complexity will explode with the number of (compact or noncompact) angular coordinates. Therefore, it is also of vital importance to have an efficient machine representation of such terms in order to make such calculations feasible on manifolds with enough dimensions to produce interesting results (i.e. new vacua).

The conventional way to represent an analytic expression, in this particular example Maple's internal representation of the term

$$2 \sinh \lambda \sinh 7\mu \cos 4\alpha + 28 \cosh \lambda \cosh 3\mu, \quad (6.2)$$

is shown in figure<sup>10</sup> (6.1).<sup>11</sup> The general underlying idea here is to use a representation general enough to handle arbitrary terms, but to try to save space by not duplicating subterms.<sup>12</sup> The major drawback of such general representations of terms is that for specialized applications, in particular for the problem at hand, they tend to conceal some possible simplifications or reductions which take a more natural form in a term representation that fits the application. In this particular case, all that a general-purpose symbolic algebra package like Maple can do is to factor out common subterms or perform partial reductions by applying simple trigonometric identities.<sup>13</sup>

Concerning trigonometric manipulations, the most useful term representation for doing calculations would be

$$\langle \text{Term} \rangle = \sum_j k_j \exp \left( \sum_{j_k} c_{j_k} v_{j_k} \right) \quad (6.3)$$

where the  $k_j$  and  $c_{j_k}$  are (complex) rational numbers and  $v_{j_k}$  variables. Although calculations involving only terms of this form are conceptually simple, the big drawback of this representation is the large number of individual summands; even an expression as simple as (6.2) would consist of 12 different summands if written in this form, which come in two groups of almost identical summands that differ only in signs of coefficients. Hence, the idea suggests itself to use this explicit exponential form but store terms in a packed format

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<sup>10</sup>Diagrams have been generated with **graphviz** (Trademark by AT&T).

<sup>11</sup>Cf. [32] for details concerning Maple's term representations.

<sup>12</sup>One way to implement this is to use a hash of weak pointers on all currently known terms to map a newly generated expression to a pre-existing memory representation of that expression.

<sup>13</sup>Indeed, only a very small subset of all possible reductions is found and applied, as can be seen by transforming a potential like (3.5) to expanded exponential form via `simplify(normal(expand(convert(Phi,exp))))` and then trying to reduce this to an expression of comparable length using Maple's builtin simplification functions.



where every such group of summands that are identical up to sign flips is represented by the first term in a suitable lexicographical ordering and information about additional symmetrizations and anti-symmetrizations. (As a further optimization, one observes that coefficients are either real and rational, or purely imaginary and rational, so it suffices to store only a real number and one bit of information whether the corresponding coefficient is real or imaginary.) Such a term representation turns out to be of remarkably compact form, yet contain enough information to do efficient calculations that also yield terms of such compact form – which conventional symbolic algebra cannot do with similar efficiency. The internal representation of the term (6.2) in this format is shown in figure (6.2). In both memory representation diagrams (on 32-bit architectures), one rectangular box corresponds to one 32-bit cell.<sup>14</sup>

Addition of such terms is straightforward to implement. Multiplication is much more involved, since product forming of summands with overlapping symmetrizers involves partial unpacking of one factor. The most difficult step for multiplication is the combination of resulting contributions into packed terms. Indeed, there are some situations involving symmetrizers that cover multiple variables at once in which possible symmetrizers overlap in such an unfortunate way that the optimal form can not be found. Experiments have shown that this rarely happens for the calculation of potentials of four variables, but may become more common the larger the number of variables. Nevertheless, degradation of this method due to this effect is a minor issue in most applications.

Further attempts to use memoization or subterm identification as performed by Maple with this new scheme to implement symbolic algebra have not resulted in noticeable improvements in performance or memory requirements so far.

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<sup>14</sup>This will also be changed in subsequent versions, which will only allow a maximum of 14 different angular variables, but considerably reduce memory requirements.

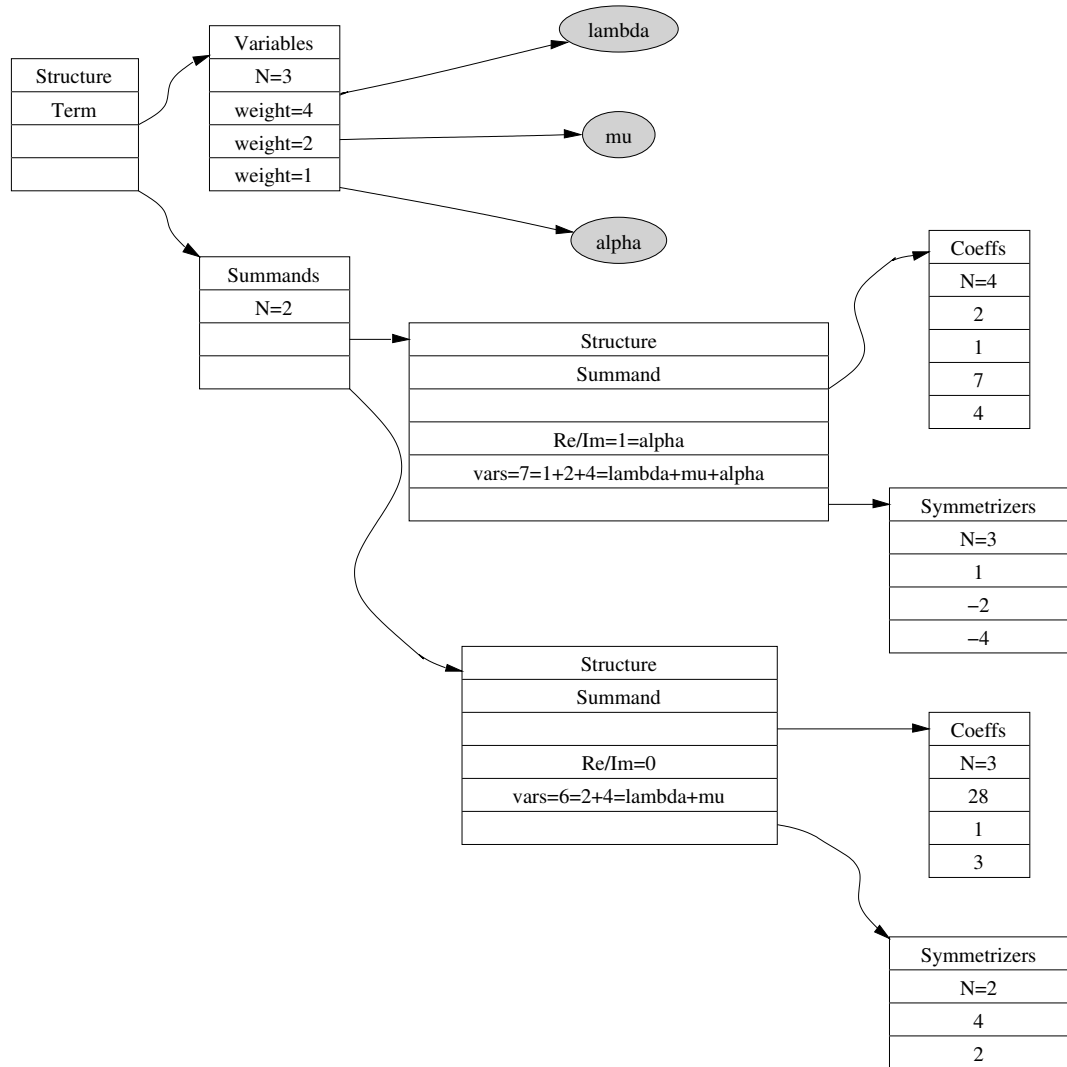


Figure 6.2: Specialized trigonometric term representation (example)

## 6.4 Provided functionality and the user interface

The main user interface to multilinear algebra, given by the function `sp-x`, evolved into its present form through a process of experimentation with a considerable number of different approaches. The basic idea is to subsume the elementary operations of index re-ordering, tensor contraction and tensor product forming in one function that resembles conventional formalism as closely as possible. Since the underlying tensor operations themselves typically require a large number of machine operations to be executed, it is perhaps affordable here to add an overhead of a few extra operations to provide a decent interface. This approach turned out to be much more convenient to use than the one implemented by Maple, which only provides bare operations like tensor contraction and product forming of two (not multiple) tensors and in which indices have to be labeled by position, not by symbolic names. In contrast, `LambdaTensor` index names can be arbitrary LISP values which are compared for equality by `EQUALP`.<sup>15</sup>

Although it is not difficult to implement the index matching code in such a way that upper and lower indices are discerned and summation can only take place between an upper and a lower index, a deliberate choice was made not to do so, in part due to the prominent role of the special orthogonal groups in supergravity, for which there is no need to discern between upper and lower indices (and hence, doing so would only introduce superfluous  $\delta^{ij}$ -metric tensors both in formulae and in code), and in part since it is more advantageous in this case to build code whose main purpose is to catch user errors on top of a library providing bare functionality than into it. (Making it mandatory would only limit the potential given to users that know how and when to deliberately break the rules.)

It would clearly be very desirable to have a close one-to-one correspondence between formulae in a machine representation suited for calculations and conventional notation in physics (especially since manual conversion of a large number of formulae can be a considerable source of errors), and one is tempted to try to use LISP's forementioned metasyntactic capabilities to implement such a 'language' that resembles conventional formalism as close as possible, but despite considerable effort, any previous attempts in that direction have not produced satisfactory results so far. The underlying problem seems to be that casting the somewhat casual practice of omitting simple embedding tensors by the introduction of (sequences of) index splitting con-

---

<sup>15</sup>Index names of the form `(cons :fix <number>)` play a special role and denote indices fixed to a number.

ventions into the framework of a set of strict rules suited for a computer so far only produced very elaborate rulesets with many special cases that turned out to be difficult to handle and by far not as convenient as usual physical notation.

## 6.5 Future developments

Since the first public release of this package [18], this package has considerably evolved. Besides a few bug fixes to the documentation and functions of minor importance that did not endanger the validity of any of the results obtained with this package, as well as necessary simple improvements in a few places like the serializer, a considerable amount of new functionality has been implemented that will be part of the next release. One important improvement is the extension of LISP number arithmetics to finite-dimensional field extensions (or even nondivision algebras) over the rational numbers. It is a happy coincidence that a lot of data can be extracted from the  $D = 3$  maximal gauged supergravity potentials using almost exclusively rational arithmetics; there nevertheless are also quite some cases where rational arithmetics is not sufficient, and having access to a direct implementation of finite-dimensional algebras over  $\mathbb{Q}$  that are specified via a multiplication table can for some tasks circumvent common problems of conventional symbolic algebra.<sup>16</sup>

In a certain sense complementary to the large sparse tensor functionality provided by the initial release is the group-theoretic approach based on roots and weight vectors that is implemented by LiE [47]; Since it is occasionally very useful to have this functionality directly available when working with explicit tensors, and not only via a detour through another program, the next version will also implement more abstract group theory.<sup>17</sup>

As was already mentioned, the next release will change the internal representation of trigonometric term arithmetics to a much denser format, limiting the number of different names for angular variables in a calculation to 14; this limit is perhaps not too unreasonable. (It can be changed back if necessary.)

On the more experimental side, the next release will also contain first

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<sup>16</sup>For example, MapleVR5.1 does not automatically simplify  $\frac{97+56\sqrt{3}}{(2+\sqrt{3})^4}$  to 1. Even making the denominator a rational number only produces  $-(97 + 56\sqrt{3})(-97 + 56\sqrt{3})$ . While one may get rid of such artefacts by applying `x→expand(rationalize(x))` or `radnormal` from the library with the same name to end results, this default behaviour certainly does not help to reduce expression swell in intermediate quantities.

<sup>17</sup>A strong incentive to implement some important group-theoretic algorithms was given by research that culminated in a paper with H. Nicolai on the structure of  $E_{10}$  and  $E_{11}$  [44].



rudimentary support for distributed computing where asynchronous I/O is used to broadcast and receive calculation requests and results via a TCP stream socket.

Finally, the installation procedure for two major commercial Linux distributions which do not use the Debian package format will be simplified.



# Chapter 7

## Conclusion and Outlook

As the analysis of a few example cases with interesting structure has shown, the tools presented in this thesis greatly simplify the task of the determination of nontrivial stationary points of gauged extended supergravity theories. The underlying mathematical ideas of the approach taken here (i.e. restriction of the potential to maximal submanifolds of the whole nonlinear space of scalars which are invariant under some given subgroup of the gauge group) have been in use for many years and still are considered the most promising ansatz to at least obtain information about vacua with a certain amount of unbroken gauge symmetry. By employing and inventing new methods in symbolic algebra, it was possible to take this approach to previously unreached heights, concerning both the study of a large variety of different related models in a reasonable amount of time and the level of detail to which these investigations can be carried.

On the computational side of this work, three major advances conspire to make this technological jump possible: first, a reasonably efficient implementation of multilinear algebra on sparsely occupied tensors of higher rank and large dimension, necessary to effectively handle explicit realisations of exceptional Lie groups (not algebras). This part is quite straightforward, since the underlying algorithms by now have been known for decades in the context of relational databases, only that an implementation of these algorithms with group theory as an application in mind has to the author's best knowledge not been available so far. What should be considered as a new approach, however, is to use  $\lambda$  abstraction to fully parametrize the database part by the implementation of arithmetics on tensor entries, allowing maximal flexibility in the application of this framework. Second, a highly memory efficient problem specific encoding of symbolic expressions that appear in these calculations as tensor entries. This is noteworthy since here it is possible for terms of sufficient complexity to outperform conventional methods for the memory

representation of symbolic expressions by more than one order of magnitude. Third, related to the novel problem-specific tight encoding of information carried by symbolic expressions, term algebra can be implemented in such a way that far more simplifying trigonometric identities are discovered than with a conventional approach. This greatly helps in reducing intermediate expression swell.<sup>1</sup>

The algorithmic tools developed for this work are probably at present the most efficient ones available for working with explicit coordinate representations of large Lie groups; since they might be useful in a much broader context, they have been made publicly available in source form under a free software license [18].

As is exemplified by a dozen new possible vacua that have been found in this work, the extremal structure of gauged three-dimensional supergravities is far richer than that of any higher-dimensional supergravity model. It should be emphasized that these models are likely to still have many more vacua than the ones given here. In particular, we mostly have been concerned with vacua on submanifolds that are invariant under subgroups which are embedded into  $E_{8(+8)}$  in a very simple fashion. In particular, we did not consider a single case here where the gauge group  $SO(8) \times SO(8)$  is broken down to a diagonal subgroup whose definition makes use of an extra triality rotation. Also for the four- and five-dimensional case, a much broader investigation would be possible. Concerning especially the four-dimensional case, there is an embedding of  $SO(3)$  into  $SO(8)$  which gives rise to a 20-dimensional invariant submanifold of the scalar manifold - certainly too large for an analytic treatment. Nevertheless, a numerical search for vacua which employ direct numerical exponentiation of  $56 \times 56$  matrices should be feasible in a highly distributed environment (maybe using computer power from volunteers on the internet). Another question which has not been answered so far that might be solved by a computer-aided exhaustive search is whether there are any semisimple nonmaximal subgroups of  $E_{8(+8)}$  that can be promoted

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<sup>1</sup>To give an explicit example, the calculations presented in [21] originally were performed by employing a term representation comparable to that used by conventional symbolic algebra packages, but with additional provisions to recognize some types of non-local reductions of summands typically not performed by usual symbolic algebra systems. The original form of the five-parameter potential as it dropped out of the tensor calculation filled 475 pages, but could then in a separate step be reduced down to just one single page[22]. Due to exponential swell in expression complexity, calculations with eight- or even nine-parameter potentials, as presented here, would have been entirely impossible with such an ‘usual’ implementation of term algebra. On the other hand, it is hard to imagine how far out of reach of conventional symbolic algebra a potential like (5.8) which we obtained by these novel methods and which would fill well over 200 pages in a less compact notation is.

to gauge groups.

One of the big pending problems is to clarify the relation between the  $N = 16$   $D = 3$  gauged supergravity models and higher-dimensional supergravity; at present one can only speculate whether this might lead to a theory beyond eleven-dimensional supergravity. The  $N = 16$  models may play a role in the supergravity description of matrix string theories [35, 40]; this has to be clarified in the future. In particular, not much is known about the (super)conformal theories corresponding to the AdS solutions of gauged  $N = 16$  supergravity.



# Appendix A

## Potentials too lengthy to be given in the main text

### A.1 The $SO(6)_{\text{diag}}$ invariant submanifold

For the gauge group  $SO(6, 2) \times SO(6, 2)$ , the potential corresponding to the parametrization (3.8) reads:

$$\begin{aligned} -8g^{-2}V = & 27 + 3 \cosh(4z) + 3 \cosh(4z) \cos(2r_2) - 3 \cosh(4z) \cos(2r_1) \\ & - 3 \cosh(4z) \cos(2r_1) \cos(2r_2) + \frac{1}{4} \cosh(4s) \\ & + \frac{1}{4} \cosh(4s) \cos(2r_2) - \frac{1}{4} \cosh(4s) \cos(2r_1) \\ & - \frac{1}{4} \cosh(4s) \cos(2r_1) \cos(2r_2) + 9 \cosh(2s) \cosh(2z) \\ & + \frac{3}{4} \cosh(2s) \cosh(6z) - 3 \cos(2r_3) \sinh(2z) \sinh(2s) \\ & + \frac{1}{4} \cos(2r_3) \sinh(6z) \sinh(2s) \\ & - 3 \cosh(2s) \cosh(2z) \cos(2r_2) - \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_2) \\ & - 3 \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\ & + \frac{1}{4} \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\ & + 3 \cosh(2s) \cosh(2z) \cos(2r_1) + \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_1) \\ & - 9 \cos(2r_1) \cos(2r_3) \sinh(2z) \sinh(2s) \\ & + \frac{3}{4} \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\ & + 3 \cosh(2s) \cosh(2z) \cos(2r_1) \cos(2r_2) \\ & + \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\ & - 12 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2z) \sinh(2s) \\ & + \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\ & + 3 \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\ & - \frac{1}{4} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) + \cosh(2v) \\ & - 9 \cosh(v) \cosh(4z) + 3 \cosh(v) \cosh(4z) \cos(2r_2) \\ & - 3 \cosh(v) \cosh(4z) \cos(2r_1) - 3 \cosh(v) \cosh(4z) \cos(2r_1) \cos(2r_2) \\ & - \frac{1}{4} \cosh(2v) \cosh(4s) - \frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_2) \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_1) \\
& +\frac{1}{4} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \\
& -15 \cosh(v) \cosh(2s) \cosh(2z) - \frac{3}{4} \cosh(2v) \cosh(2s) \cosh(6z) \\
& -3 \cosh(v) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& -\frac{1}{4} \cosh(2v) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& -3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_2) \\
& +\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_2) \\
& -3 \cosh(v) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& -\frac{1}{4} \cosh(2v) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& +3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_1) \\
& -\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \\
& -9 \cosh(v) \cos(2r_1) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& -\frac{3}{4} \cosh(2v) \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& +3 \cosh(v) \cosh(2s) \cosh(2z) \cos(2r_1) \cos(2r_2) \\
& -\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\
& -12 \cosh(v) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2z) \sinh(2s) \\
& -\cosh(2v) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +3 \cosh(v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& +\frac{1}{4} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s).
\end{aligned}$$

If we use the same parametrization for the  $SO(7,1) \times SO(7,1)$  gauged theory, we get:

$$\begin{aligned}
-8g^{-2}V & = \\
& \frac{1663}{64} + \frac{43}{64} \cos(2r_5) + \frac{9}{64} \cos(4r_3) \\
& -\frac{3}{64} \cos(4r_3) \cos(2r_5) - \frac{3}{16} \cos(2r_2) \\
& +\frac{21}{64} \cos(4r_2) + \frac{1}{16} \cos(2r_2) \cos(2r_5) \\
& -\frac{7}{64} \cos(4r_2) \cos(2r_5) + \frac{3}{16} \cos(2r_2) \cos(4r_3) \\
& +\frac{3}{64} \cos(4r_2) \cos(4r_3) - \frac{1}{16} \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{64} \cos(4r_2) \cos(4r_3) \cos(2r_5) - \frac{9}{64} \cos(2r_1) \\
& +\frac{3}{64} \cos(2r_1) \cos(2r_5) - \frac{15}{64} \cos(2r_1) \cos(4r_3) \\
& +\frac{5}{64} \cos(2r_1) \cos(4r_3) \cos(2r_5) + \frac{3}{16} \cos(2r_1) \cos(2r_2) \\
& +\frac{21}{64} \cos(2r_1) \cos(4r_2) - \frac{1}{16} \cos(2r_1) \cos(2r_2) \cos(2r_5) \\
& -\frac{7}{64} \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& -\frac{3}{16} \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& -\frac{3}{16} \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& -\frac{3}{16} \cos(2r_1) \cos(2r_2) \cos(4r_3) \\
& +\frac{3}{64} \cos(2r_1) \cos(4r_2) \cos(4r_3) \\
& +\frac{1}{16} \cos(2r_5) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& +\frac{1}{16} \cos(2r_5) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& +\frac{1}{16} \cos(2r_1) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{64} \cos(2r_1) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& +3 \cosh(4z) + 3 \cosh(4z) \cos(2r_2) - 3 \cosh(4z) \cos(2r_1) \\
& -3 \cosh(4z) \cos(2r_1) \cos(2r_2) - \frac{15}{64} \cosh(4s)
\end{aligned}$$



$$\begin{aligned}
& + \frac{5}{64} \cosh(4s) \cos(2r_5) - \frac{9}{64} \cosh(4s) \cos(4r_3) \\
& + \frac{3}{64} \cosh(4s) \cos(4r_3) \cos(2r_5) - \frac{3}{16} \cosh(4s) \cos(2r_2) \\
& + \frac{3}{64} \cosh(4s) \cos(4r_2) + \frac{1}{16} \cosh(4s) \cos(2r_2) \cos(2r_5) \\
& - \frac{1}{64} \cosh(4s) \cos(4r_2) \cos(2r_5) \\
& - \frac{3}{16} \cosh(4s) \cos(2r_2) \cos(4r_3) \\
& - \frac{3}{64} \cosh(4s) \cos(4r_2) \cos(4r_3) \\
& + \frac{1}{16} \cosh(4s) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& + \frac{1}{64} \cosh(4s) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& + \frac{9}{64} \cosh(4s) \cos(2r_1) - \frac{3}{64} \cosh(4s) \cos(2r_1) \cos(2r_5) \\
& + \frac{15}{64} \cosh(4s) \cos(2r_1) \cos(4r_3) \\
& - \frac{5}{64} \cosh(4s) \cos(2r_1) \cos(4r_3) \cos(2r_5) \\
& + \frac{3}{16} \cosh(4s) \cos(2r_1) \cos(2r_2) \\
& + \frac{3}{64} \cosh(4s) \cos(2r_1) \cos(4r_2) \\
& - \frac{1}{16} \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(2r_5) \\
& - \frac{1}{64} \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& + \frac{3}{16} \cosh(4s) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& + \frac{3}{16} \cosh(4s) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& + \frac{3}{16} \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(4r_3) \\
& - \frac{3}{64} \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(4r_3) \\
& - \frac{1}{16} \cosh(4s) \cos(2r_5) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& - \frac{1}{16} \cosh(4s) \cos(2r_5) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& - \frac{1}{16} \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& + \frac{1}{64} \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& + 9 \cosh(2s) \cosh(2z) - \frac{3}{4} \cosh(2s) \cosh(6z) \\
& + \frac{1}{4} \cosh(2s) \cosh(6z) \cos(2r_5) \\
& - 3 \cos(2r_3) \sinh(2z) \sinh(2s) - 3 \cosh(2s) \cosh(2z) \cos(2r_2) \\
& + \frac{3}{8} \cosh(2s) \cosh(6z) \cos(2r_2) \\
& - \frac{3}{8} \cosh(2s) \cosh(6z) \cos(4r_2) \\
& - \frac{1}{8} \cosh(2s) \cosh(6z) \cos(2r_2) \cos(2r_5) \\
& + \frac{1}{8} \cosh(2s) \cosh(6z) \cos(4r_2) \cos(2r_5) \\
& - 3 \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{1}{3} \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& - \frac{1}{3} \cos(4r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + \frac{1}{8} \cos(2r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& + \frac{1}{8} \cos(4r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& + 3 \cosh(2s) \cosh(2z) \cos(2r_1) \\
& - 9 \cos(2r_1) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& - \frac{3}{4} \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& + \frac{1}{4} \cos(2r_1) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& + 3 \cosh(2s) \cosh(2z) \cos(2r_1) \cos(2r_2) \\
& - \frac{3}{8} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\
& - \frac{3}{8} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(4r_2) \\
& + \frac{1}{8} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \cos(2r_5)
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{8} \cosh(2s) \cosh(6z) \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& -12 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2z) \sinh(2s) \\
& -\frac{3}{4} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +\frac{3}{4} \sin(2r_3) \sin(3r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +3 \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2z) \sinh(2s) \\
& +3 \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& -\frac{3}{8} \cos(2r_1) \cos(4r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& +\frac{1}{4} \cos(2r_5) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& -\frac{1}{4} \cos(2r_5) \sin(2r_3) \sin(3r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& +\frac{1}{8} \cos(2r_1) \cos(4r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& -\frac{43}{64} \cosh(2v) - \frac{43}{64} \cosh(2v) \cos(2r_5) \\
& +\frac{3}{64} \cosh(2v) \cos(4r_3) \\
& +\frac{3}{64} \cosh(2v) \cos(4r_3) \cos(2r_5) - \frac{1}{16} \cosh(2v) \cos(2r_2) \\
& +\frac{7}{64} \cosh(2v) \cos(4r_2) - \frac{1}{16} \cosh(2v) \cos(2r_2) \cos(2r_5) \\
& +\frac{7}{64} \cosh(2v) \cos(4r_2) \cos(2r_5) \\
& +\frac{1}{16} \cosh(2v) \cos(2r_2) \cos(4r_3) \\
& +\frac{1}{64} \cosh(2v) \cos(4r_2) \cos(4r_3) \\
& +\frac{1}{16} \cosh(2v) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& +\frac{1}{64} \cosh(2v) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{3}{64} \cosh(2v) \cos(2r_1) - \frac{3}{64} \cosh(2v) \cos(2r_1) \cos(2r_5) \\
& -\frac{5}{64} \cosh(2v) \cos(2r_1) \cos(4r_3) \\
& -\frac{5}{64} \cosh(2v) \cos(2r_1) \cos(4r_3) \cos(2r_5) \\
& +\frac{1}{16} \cosh(2v) \cos(2r_1) \cos(2r_2) \\
& +\frac{7}{64} \cosh(2v) \cos(2r_1) \cos(4r_2) \\
& +\frac{1}{16} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(2r_5) \\
& +\frac{7}{64} \cosh(2v) \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& -\frac{1}{16} \cosh(2v) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& -\frac{1}{16} \cosh(2v) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& -\frac{1}{16} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(4r_3) \\
& +\frac{1}{64} \cosh(2v) \cos(2r_1) \cos(4r_2) \cos(4r_3) \\
& -\frac{1}{16} \cosh(2v) \cos(2r_5) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& -\frac{1}{16} \cosh(2v) \cos(2r_5) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& -\frac{1}{16} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& +\frac{1}{64} \cosh(2v) \cos(2r_1) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& +3 \cos(r_5) \sinh(4z) \sinh(v) - 9 \cos(2r_2) \cos(r_5) \sinh(4z) \sinh(v) \\
& -3 \cos(2r_1) \cos(r_5) \sinh(4z) \sinh(v) \\
& -3 \cos(2r_1) \cos(2r_2) \cos(r_5) \sinh(4z) \sinh(v) \\
& -\frac{5}{64} \cosh(2v) \cosh(4s) - \frac{5}{64} \cosh(2v) \cosh(4s) \cos(2r_5) \\
& -\frac{3}{64} \cosh(2v) \cosh(4s) \cos(4r_3) \\
& -\frac{3}{64} \cosh(2v) \cosh(4s) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_2) \\
& +\frac{1}{64} \cosh(2v) \cosh(4s) \cos(4r_2)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_2) \cos(2r_5) \\
& +\frac{1}{64} \cosh(2v) \cosh(4s) \cos(4r_2) \cos(2r_5) \\
& -\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_2) \cos(4r_3) \\
& -\frac{1}{64} \cosh(2v) \cosh(4s) \cos(4r_2) \cos(4r_3) \\
& -\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{64} \cosh(2v) \cosh(4s) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& +\frac{3}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \\
& +\frac{3}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_5) \\
& +\frac{5}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_3) \\
& +\frac{5}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_3) \cos(2r_5) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \\
& +\frac{1}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_2) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(2r_5) \\
& +\frac{1}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(4r_3) \\
& -\frac{1}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(4r_3) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_5) \sin(4r_3) \sin(r_2) \sin(2r_1) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_5) \sin(4r_3) \sin(3r_2) \sin(2r_1) \\
& +\frac{1}{16} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(2r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{64} \cosh(2v) \cosh(4s) \cos(2r_1) \cos(4r_2) \cos(4r_3) \cos(2r_5) \\
& -\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \\
& +3 \cosh(2s) \cos(r_5) \sinh(2z) \sinh(v) \\
& -\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_5) \\
& -9 \cosh(2z) \cos(2r_3) \cos(r_5) \sinh(2s) \sinh(v) \\
& +\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_2) \\
& -\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(4r_2) \\
& -9 \cosh(2s) \cos(2r_2) \cos(r_5) \sinh(2z) \sinh(v) \\
& +\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_2) \cos(2r_5) \\
& -\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(4r_2) \cos(2r_5) \\
& -\frac{1}{8} \cosh(2v) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cosh(2v) \cos(4r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& -9 \cosh(2z) \cos(2r_2) \cos(2r_3) \cos(r_5) \sinh(2s) \sinh(v) \\
& -\frac{1}{8} \cosh(2v) \cos(2r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cosh(2v) \cos(4r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& -3 \cosh(2s) \cos(2r_1) \cos(r_5) \sinh(2z) \sinh(v) \\
& -\frac{1}{4} \cosh(2v) \cos(2r_1) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& +9 \cosh(2z) \cos(2r_1) \cos(2r_3) \cos(r_5) \sinh(2s) \sinh(v) \\
& -\frac{1}{4} \cosh(2v) \cos(2r_1) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \\
& -\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(4r_2) \\
& -3 \cosh(2s) \cos(2r_1) \cos(2r_2) \cos(r_5) \sinh(2z) \sinh(v)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(2r_2) \cos(2r_5) \\
& -\frac{1}{4} \cosh(2v) \cosh(2s) \cosh(6z) \cos(2r_1) \cos(4r_2) \cos(2r_5) \\
& -\frac{1}{4} \cosh(2v) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +\frac{1}{4} \cosh(2v) \sin(2r_3) \sin(3r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +\frac{1}{8} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cosh(2v) \cos(2r_1) \cos(4r_2) \cos(2r_3) \sinh(6z) \sinh(2s) \\
& +12 \cosh(2z) \cos(r_5) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(v) \\
& -3 \cosh(2z) \cos(2r_1) \cos(2r_2) \cos(2r_3) \cos(r_5) \sinh(2s) \sinh(v) \\
& -\frac{1}{4} \cosh(2v) \cos(2r_5) \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +\frac{1}{4} \cosh(2v) \cos(2r_5) \sin(2r_3) \sin(3r_2) \sin(2r_1) \sinh(6z) \sinh(2s) \\
& +\frac{1}{8} \cosh(2v) \cos(2r_1) \cos(2r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s) \\
& -\frac{1}{8} \cosh(2v) \cos(2r_1) \cos(4r_2) \cos(2r_3) \cos(2r_5) \sinh(6z) \sinh(2s).
\end{aligned}$$

## A.2 The $(SO(5) \times SO(2))_{\text{diag}}$ invariant submanifold

For the gauge group  $SO(5, 3) \times SO(5, 3)$ , the potential corresponding to the parametrization (3.15) reads:

$$\begin{aligned}
& -8g^{-2}V = \\
& 23 - 5 \cosh\left(\frac{12}{5}v\right) - \frac{15}{2} \cosh\left(\frac{2}{5}v + 4z\right) + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \\
& -\frac{1}{8} \cosh(2v + 4z) + \frac{3}{2} \cosh(2v - 4z) - \frac{1}{8} \cosh(4v + 4z) \\
& + \cosh\left(\frac{4}{5}v + 4z\right) \cos(2r_2) + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_2) \\
& - \cosh\left(\frac{4}{5}v + 4z\right) \cos(2r_2) - \frac{1}{2} \cosh(2v - 4z) \cos(2r_2) \\
& - \cosh(4v + 4z) \cos(2r_2) + \frac{5}{2} \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \\
& + \frac{1}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) - \frac{5}{4} \cosh\left(\frac{8}{5}v + 4z\right) \cos(2r_1) \\
& - \cosh(2v - 4z) \cos(2r_1) - \frac{1}{8} \cosh(4v + 4z) \cos(2r_1) \\
& + \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) \\
& + \frac{1}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) \cos(2r_2) \\
& - \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) \\
& - \cosh(2v - 4z) \cos(2r_1) \cos(2r_2) - \frac{1}{8} \cosh(4v + 4z) \cos(2r_1) \cos(2r_2) \\
& - \frac{25}{2} \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) + \frac{45}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \\
& - \frac{15}{4} \cosh\left(\frac{4}{5}v - 2z\right) \cosh(2s) + \frac{5}{2} \cosh(2v + 2z) \cosh(2s) \\
& - \cosh(4v - 2z) \cosh(2s) + \frac{5}{2} \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& - \frac{15}{8} \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& + \cos(2r_3) \sinh(2s) \sinh(2v + 2z) - \frac{1}{8} \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& - \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_2) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) \\
& + \cosh(2v + 2z) \cosh(2s) \cos(2r_2) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_2) \\
& + \frac{1}{2} \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{15}{8} \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& -\cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& +\cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& -\cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& -\cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& -\frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \\
& +\cosh\left(\frac{8}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& +\cosh(2v + 2z) \cosh(2s) \cos(2r_1) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_1) \\
& -\frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& +\frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& +\frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& -\cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& +\cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& -\cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& -\frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& +\cosh\left(\frac{8}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& +\cosh(2v + 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& +\cosh(4v - 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& -10 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& +\frac{15}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& +5 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& -2 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(2v + 2z) \\
& +\frac{1}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(4v - 2z) \\
& +\frac{3}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& -\frac{15}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& -\frac{15}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& +\frac{1}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& -\frac{1}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v - 2z).
\end{aligned}$$

We skip the case of  $SO(6, 2) \times SO(6, 2)$  gauging, since with this parametrization,  $SO(6)$  would be implemented on the spinor indices 1, 2, 3, 4, 5, 8, and this relabelling is a bit inconvenient. If we use this parametrization with the  $SO(7, 1) \times SO(7, 1)$  gauging, we get:

$$\begin{aligned}
& -8g^{-2}V = \\
& 23 - 5 \cosh\left(\frac{12}{5}v\right) + \frac{15}{2} \cosh\left(\frac{2}{5}v + 4z\right) \\
& +\frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) - \frac{5}{4} \cosh\left(\frac{8}{5}v + 4z\right) \\
& -\cosh(2v - 4z) - \frac{1}{8} \cosh(4v + 4z) \\
& -\cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_2) \\
& +\frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_2) \\
& -\cosh\left(\frac{8}{5}v + 4z\right) \cos(2r_2) + \frac{1}{2} \cosh(2v - 4z) \cos(2r_2) \\
& -\frac{1}{8} \cosh(4v + 4z) \cos(2r_2) - \frac{5}{2} \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \\
& +\frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) - \frac{5}{4} \cosh\left(\frac{8}{5}v + 4z\right) \cos(2r_1) \\
& +\frac{1}{2} \cosh(2v - 4z) \cos(2r_1) - \frac{1}{8} \cosh(4v + 4z) \cos(2r_1)
\end{aligned}$$

$$\begin{aligned}
& -\frac{15}{2} \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) \\
& + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) \cos(2r_2) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) \\
& + \frac{1}{8} \cosh(2v - 4z) \cos(2r_1) \cos(2r_2) - \frac{1}{8} \cosh(4v + 4z) \cos(2r_1) \cos(2r_2) \\
& + \frac{25}{2} \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) + \frac{45}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \\
& - \frac{15}{4} \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) - \frac{5}{2} \cosh(2v + 2z) \cosh(2s) \\
& - \cosh(4v - 2z) \cosh(2s) - \frac{5}{2} \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& - \frac{15}{8} \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& - \cos(2r_3) \sinh(2s) \sinh(2v + 2z) - \frac{1}{8} \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_2) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) \\
& - \cosh(2v + 2z) \cosh(2s) \cos(2r_2) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_2) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& - \frac{15}{8} \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& - \frac{1}{8} \cosh(2v + 2z) \cosh(2s) \cos(2r_1) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_1) \\
& + \frac{15}{2} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& + \frac{15}{4} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& + \frac{15}{4} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& + \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& + \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + \frac{15}{8} \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& - \frac{1}{2} \cosh(2v + 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + 10 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& + \frac{15}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& + 5 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& + 2 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(2v + 2z) \\
& + \frac{1}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(4v - 2z) \\
& - \frac{1}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& - \frac{15}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - \frac{15}{4} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& - \frac{1}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& - \frac{1}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v - 2z).
\end{aligned}$$

Finally, for the compact gauge group  $SO(8) \times SO(8)$ , the potential reads:

$$\begin{aligned}
& -8g^{-2}V = \\
& 23 + 5 \cosh\left(\frac{12}{5}v\right) + \frac{15}{2} \cosh\left(\frac{2}{5}v + 4z\right) \\
& + \frac{15}{2} \cosh\left(\frac{4}{5}v - 4z\right) + \frac{5}{4} \cosh\left(\frac{8}{5}v + 4z\right) + \frac{3}{2} \cosh(2v - 4z) - \frac{1}{8} \cosh(4v + 4z) \\
& - \frac{15}{8} \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_2) + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_2) \\
& + \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_2) - \frac{1}{2} \cosh(2v - 4z) \cos(2r_2) \\
& - \cosh(4v + 4z) \cos(2r_2) - \frac{5}{2} \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \\
& + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) + \frac{5}{4} \cosh\left(\frac{8}{5}v + 4z\right) \cos(2r_1) \\
& - \frac{15}{8} \cosh(2v - 4z) \cos(2r_1) - \frac{1}{8} \cosh(4v + 4z) \cos(2r_1) \\
& - \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) + \frac{15}{8} \cosh\left(\frac{4}{5}v - 4z\right) \cos(2r_1) \cos(2r_2) \\
& + \cosh\left(\frac{2}{5}v + 4z\right) \cos(2r_1) \cos(2r_2) - \frac{1}{2} \cosh(2v - 4z) \cos(2r_1) \cos(2r_2) \\
& - \cosh(4v + 4z) \cos(2r_1) \cos(2r_2) + \frac{25}{2} \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \\
& + \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) + \frac{15}{4} \cosh\left(\frac{8}{5}v - 2z\right) \cosh(2s) \\
& + \cosh(2v + 2z) \cosh(2s) - \frac{3}{8} \cosh(4v - 2z) \cosh(2s) \\
& - \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) - \frac{15}{8} \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& + \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& + \cos(2r_3) \sinh(2s) \sinh(2v + 2z) - \frac{1}{8} \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_2) \\
& - \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_2) \\
& + \cosh(2v + 2z) \cosh(2s) \cos(2r_2) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_2) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& - \frac{15}{8} \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& + \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& + \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& - \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \\
& - \cosh\left(\frac{8}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \\
& + \cosh(2v + 2z) \cosh(2s) \cos(2r_1) + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_1) \\
& + \frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& + \frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - \frac{15}{8} \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& - \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(2v + 2z) \\
& + \cos(2r_1) \cos(2r_3) \sinh(2s) \sinh(4v - 2z) \\
& + \cosh\left(\frac{2}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& - \frac{15}{8} \cosh\left(\frac{4}{5}v + 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& - \cosh\left(\frac{8}{5}v - 2z\right) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + \cosh(2v + 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + \frac{1}{8} \cosh(4v - 2z) \cosh(2s) \cos(2r_1) \cos(2r_2) \\
& + 10 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{2}{5}v - 2z\right) \\
& + \frac{15}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{4}{5}v + 2z\right) \\
& - 5 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh\left(\frac{8}{5}v - 2z\right) \\
& - 2 \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(2v + 2z)
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{2} \sin(2r_3) \sin(r_2) \sin(2r_1) \sinh(2s) \sinh(4v-2z) \\
& -\frac{5}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(\frac{2}{5}v-2z) \\
& -\frac{15}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(\frac{4}{5}v+2z) \\
& +\frac{5}{4} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(\frac{8}{5}v-2z) \\
& +\frac{1}{2} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(2v+2z) \\
& -\frac{1}{8} \cos(2r_1) \cos(2r_2) \cos(2r_3) \sinh(2s) \sinh(4v-2z)
\end{aligned}$$

### A.3 The $SU(3) \times SU(3)$ invariant submanifold

The parametrization (4.18) yields for the  $G_{2(-14)} \times F_{4(-20)}$  gauged model the following potential:

$$\begin{aligned}
-8g^{-2}V = & \frac{24125}{2048} + \frac{9}{2048} \cos(8r_5) + \frac{9}{2048} \cos(8r_2) + \frac{115}{128} \sin(4r_5) \sin(4r_2) \\
& -\frac{27}{2048} \cos(8r_2) \cos(8r_5) - \frac{1}{64} \cos(4r_2) \cos(4r_3-4r_6) \cos(4r_5) \\
& -\frac{9}{2048} \cos(8r_1-8r_4) - \frac{9}{2048} \cos(8r_1-8r_4) \cos(8r_5) \\
& -\frac{1}{64} \cos(4r_1-4r_4) \cos(4r_3-4r_6) + \frac{1}{64} \sin(4r_5) \sin(4r_3-4r_6) \sin(4r_1-4r_4) \\
& -\frac{9}{2048} \cos(8r_1-8r_4) \cos(8r_2) - \frac{9}{512} \cos(4r_1-4r_4) \sin(8r_5) \sin(8r_2) \\
& +\frac{115}{128} \cos(4r_1-4r_4) \cos(4r_2) \cos(4r_5) - \frac{9}{2048} \cos(8r_1-8r_4) \cos(8r_2) \cos(8r_5) \\
& +\frac{1}{64} \sin(4r_3-4r_6) \sin(4r_2) \sin(4r_1-4r_4) \\
& -\frac{1}{64} \cos(4r_1-4r_4) \cos(4r_3-4r_6) \sin(4r_5) \sin(4r_2) + \frac{449}{512} \cosh(z) \\
& -\frac{65}{2048} \cosh(2z) - \frac{3}{512} \cosh(z) \cos(8r_5) + \frac{3}{2048} \cosh(2z) \cos(8r_5) \\
& -\frac{3}{512} \cosh(z) \cos(8r_2) + \frac{3}{2048} \cosh(2z) \cos(8r_2) \\
& -\frac{29}{32} \cosh(z) \sin(4r_5) \sin(4r_2) + \frac{1}{128} \cosh(2z) \sin(4r_5) \sin(4r_2) \\
& +\frac{9}{512} \cosh(z) \cos(8r_2) \cos(8r_5) - \frac{9}{2048} \cosh(2z) \cos(8r_2) \cos(8r_5) \\
& +\frac{1}{64} \cosh(2z) \cos(4r_2) \cos(4r_3-4r_6) \cos(4r_5) + \frac{3}{512} \cosh(z) \cos(8r_1-8r_4) \\
& -\frac{3}{2048} \cosh(2z) \cos(8r_1-8r_4) + \frac{3}{512} \cosh(z) \cos(8r_1-8r_4) \cos(8r_5) \\
& -\frac{3}{2048} \cosh(2z) \cos(8r_1-8r_4) \cos(8r_5) + \frac{1}{64} \cosh(2z) \cos(4r_1-4r_4) \cos(4r_3-4r_6) \\
& -\frac{1}{64} \cosh(2z) \sin(4r_5) \sin(4r_3-4r_6) \sin(4r_1-4r_4) \\
& +\frac{3}{512} \cosh(z) \cos(8r_1-8r_4) \cos(8r_2) - \frac{3}{2048} \cosh(2z) \cos(8r_1-8r_4) \cos(8r_2) \\
& +\frac{3}{128} \cosh(z) \cos(4r_1-4r_4) \sin(8r_5) \sin(8r_2) \\
& -\frac{3}{512} \cosh(2z) \cos(4r_1-4r_4) \sin(8r_5) \sin(8r_2) \\
& -\frac{29}{32} \cosh(z) \cos(4r_1-4r_4) \cos(4r_2) \cos(4r_5) \\
& +\frac{3}{512} \cosh(z) \cos(8r_1-8r_4) \cos(8r_2) \cos(8r_5) \\
& +\frac{1}{128} \cosh(2z) \cos(4r_1-4r_4) \cos(4r_2) \cos(4r_5) \\
& -\frac{3}{2048} \cosh(2z) \cos(8r_1-8r_4) \cos(8r_2) \cos(8r_5) \\
& -\frac{1}{64} \cosh(2z) \sin(4r_3-4r_6) \sin(4r_2) \sin(4r_1-4r_4) \\
& +\frac{1}{64} \cosh(2z) \cos(4r_1-4r_4) \cos(4r_3-4r_6) \sin(4r_5) \sin(4r_2) \\
& +\frac{449}{512} \cosh(s) - \frac{65}{2048} \cosh(2s) - \frac{3}{512} \cosh(s) \cos(8r_5) \\
& +\frac{3}{2048} \cosh(2s) \cos(8r_5) - \frac{3}{512} \cosh(s) \cos(8r_2) \\
& +\frac{3}{2048} \cosh(2s) \cos(8r_2) - \frac{29}{32} \cosh(s) \sin(4r_5) \sin(4r_2) \\
& +\frac{1}{128} \cosh(2s) \sin(4r_5) \sin(4r_2) + \frac{9}{512} \cosh(s) \cos(8r_2) \cos(8r_5) \\
& -\frac{9}{2048} \cosh(2s) \cos(8r_2) \cos(8r_5) + \frac{1}{64} \cosh(2s) \cos(4r_2) \cos(4r_3-4r_6) \cos(4r_5)
\end{aligned}$$



$$\begin{aligned}
& + \frac{3}{512} \cosh(s) \cos(8r_1 - 8r_4) - \frac{3}{2048} \cosh(2s) \cos(8r_1 - 8r_4) \\
& + \frac{3}{512} \cosh(s) \cos(8r_1 - 8r_4) \cos(8r_5) - \frac{3}{2048} \cosh(2s) \cos(8r_1 - 8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cos(4r_1 - 4r_4) \cos(4r_3 - 4r_6) \\
& - \frac{1}{64} \cosh(2s) \sin(4r_5) \sin(4r_3 - 4r_6) \sin(4r_1 - 4r_4) \\
& + \frac{3}{512} \cosh(s) \cos(8r_1 - 8r_4) \cos(8r_2) - \frac{3}{2048} \cosh(2s) \cos(8r_1 - 8r_4) \cos(8r_2) \\
& + \frac{3}{128} \cosh(s) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& - \frac{3}{512} \cosh(2s) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& - \frac{29}{32} \cosh(s) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& + \frac{3}{512} \cosh(s) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5) \\
& + \frac{1}{128} \cosh(2s) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& - \frac{3}{2048} \cosh(2s) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5) \\
& - \frac{1}{64} \cosh(2s) \sin(4r_3 - 4r_6) \sin(4r_2) \sin(4r_1 - 4r_4) \\
& + \frac{1}{64} \cosh(2s) \cos(4r_1 - 4r_4) \cos(4r_3 - 4r_6) \sin(4r_5) \sin(4r_2) \\
& + \frac{341}{128} \cosh(s) \cosh(z) - \frac{21}{512} \cosh(s) \cosh(2z) \\
& - \frac{21}{512} \cosh(2s) \cosh(z) - \frac{107}{2048} \cosh(2s) \cosh(2z) \\
& + \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_5) - \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_5) \\
& - \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_5) + \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_5) \\
& + \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_2) - \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_2) \\
& - \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_2) + \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_2) \\
& + \frac{1}{8} \cosh(s) \cosh(z) \sin(4r_5) \sin(4r_2) + \frac{1}{32} \cosh(s) \cosh(2z) \sin(4r_5) \sin(4r_2) \\
& + \frac{1}{32} \cosh(2s) \cosh(z) \sin(4r_5) \sin(4r_2) \\
& - \frac{5}{128} \cosh(2s) \cosh(2z) \sin(4r_5) \sin(4r_2) - \frac{3}{128} \cosh(s) \cosh(z) \cos(8r_2) \cos(8r_5) \\
& + \frac{3}{512} \cosh(s) \cosh(2z) \cos(8r_2) \cos(8r_5) + \frac{3}{512} \cosh(2s) \cosh(z) \cos(8r_2) \cos(8r_5) \\
& - \frac{3}{2048} \cosh(2s) \cosh(2z) \cos(8r_2) \cos(8r_5) \\
& - \frac{1}{64} \cosh(2s) \cosh(2z) \cos(4r_2) \cos(4r_3 - 4r_6) \cos(4r_5) \\
& - \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_1 - 8r_4) + \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_1 - 8r_4) \\
& + \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_1 - 8r_4) - \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_1 - 8r_4) \\
& - \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_5) \\
& + \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_5) \\
& + \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_5) \\
& - \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_5) \\
& - \frac{1}{64} \cosh(2s) \cosh(2z) \cos(4r_1 - 4r_4) \cos(4r_3 - 4r_6) \\
& + \frac{1}{64} \cosh(2s) \cosh(2z) \sin(4r_5) \sin(4r_3 - 4r_6) \sin(4r_1 - 4r_4) \\
& - \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_2) \\
& + \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_2) \\
& + \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_2) \\
& - \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_2) \\
& - \frac{1}{32} \cosh(s) \cosh(z) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& + \frac{1}{128} \cosh(s) \cosh(2z) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& + \frac{1}{128} \cosh(2s) \cosh(z) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& - \frac{1}{512} \cosh(2s) \cosh(2z) \cos(4r_1 - 4r_4) \sin(8r_5) \sin(8r_2) \\
& + \frac{7}{8} \cosh(s) \cosh(z) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& - \frac{1}{128} \cosh(s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{32} \cosh(s) \cosh(2z) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& + \frac{1}{512} \cosh(s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5) \\
& + \frac{1}{32} \cosh(2s) \cosh(z) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& + \frac{1}{512} \cosh(2s) \cosh(z) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5) \\
& - \frac{5}{128} \cosh(2s) \cosh(2z) \cos(4r_1 - 4r_4) \cos(4r_2) \cos(4r_5) \\
& - \frac{1}{2048} \cosh(2s) \cosh(2z) \cos(8r_1 - 8r_4) \cos(8r_2) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(2z) \sin(4r_3 - 4r_6) \sin(4r_2) \sin(4r_1 - 4r_4) \\
& + \frac{1}{64} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(s) \\
& + \frac{229}{64} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(s) \\
& - \frac{1}{64} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(s) \\
& + \frac{3}{64} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(s) \\
& + \frac{1}{64} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(s) \\
& + \frac{1}{64} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(s) \\
& - \frac{1}{64} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(s) \\
& - \frac{1}{64} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(s) \\
& - \frac{5}{128} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(s) \\
& - \frac{3}{128} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(2s) \\
& - \frac{5}{128} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(2s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(2s) \\
& - \frac{3}{128} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(2r_1 - 2r_4) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(2s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(2s) \\
& + \frac{1}{128} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(6r_1 - 6r_4) \sinh(z) \sinh(2s) \\
& + \frac{1}{256} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(2s) \\
& - \frac{27}{256} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(2s) \\
& - \frac{1}{256} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(2s) \\
& + \frac{3}{256} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(2r_1 - 2r_4) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \sin(2r_3 - 2r_6) \sin(2r_2 - 6r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \sin(2r_3 - 2r_6) \sin(2r_2 + 2r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(2s) \\
& - \frac{1}{256} \sin(2r_3 - 2r_6) \sin(6r_2 - 2r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(2s) \\
& - \frac{1}{256} \sin(2r_3 - 2r_6) \sin(6r_2 + 6r_5) \sin(6r_1 - 6r_4) \sinh(2z) \sinh(2s) \\
& - \frac{239}{64} \cos(2r_1 - 2r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& - \frac{1}{64} \cos(2r_1 - 2r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& + \frac{3}{64} \cos(2r_1 - 2r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& - \frac{1}{64} \cos(2r_1 - 2r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{64} \cos(6r_1 - 6r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& + \frac{1}{64} \cos(6r_1 - 6r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& + \frac{1}{64} \cos(6r_1 - 6r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& + \frac{1}{64} \cos(6r_1 - 6r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(s) \\
& + \frac{5}{128} \cos(2r_1 - 2r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& + \frac{1}{128} \cos(2r_1 - 2r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& - \frac{3}{128} \cos(2r_1 - 2r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& + \frac{1}{128} \cos(2r_1 - 2r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(s) \\
& + \frac{5}{128} \cos(2r_1 - 2r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& + \frac{1}{128} \cos(2r_1 - 2r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& - \frac{3}{128} \cos(2r_1 - 2r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& + \frac{1}{128} \cos(2r_1 - 2r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& - \frac{1}{128} \cos(6r_1 - 6r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(z) \sinh(2s) \\
& + \frac{27}{256} \cos(2r_1 - 2r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& - \frac{1}{256} \cos(2r_1 - 2r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& + \frac{3}{256} \cos(2r_1 - 2r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& - \frac{1}{256} \cos(2r_1 - 2r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \cos(6r_1 - 6r_4) \cos(2r_2 - 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \cos(6r_1 - 6r_4) \cos(2r_2 + 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \cos(6r_1 - 6r_4) \cos(6r_2 - 6r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& + \frac{1}{256} \cos(6r_1 - 6r_4) \cos(6r_2 + 2r_5) \cos(2r_3 - 2r_6) \sinh(2z) \sinh(2s) \\
& - \frac{1}{64} \cosh(2s) \cosh(2z) \cos(4r_1 - 4r_4) \cos(4r_3 - 4r_6) \sin(4r_5) \sin(4r_2)
\end{aligned}$$

Using this parametrization with the embedding tensor of  $SO(8) \times SO(8)$  gives:

$$\begin{aligned}
-8g^{-2}V = & \frac{8391}{256} + \frac{39}{256} \cos(8r_5) + \frac{39}{256} \cos(8r_4) + \frac{39}{256} \cos(8r_4) \cos(8r_5) \\
& + \frac{39}{256} \cos(8r_2) - \frac{9}{256} \cos(8r_2) \cos(8r_5) - \frac{9}{256} \cos(8r_2) \cos(8r_4) \\
& - \frac{9}{256} \cos(8r_2) \cos(8r_4) \cos(8r_5) + \frac{39}{256} \cos(8r_1) \\
& - \frac{9}{256} \cos(8r_1) \cos(8r_5) - \frac{9}{256} \cos(8r_1) \cos(8r_4) \\
& - \frac{9}{256} \cos(8r_1) \cos(8r_4) \cos(8r_5) + \frac{39}{256} \cos(8r_1) \cos(8r_2) \\
& - \frac{9}{256} \cos(8r_1) \cos(8r_2) \cos(8r_5) - \frac{9}{256} \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& + \frac{45}{8} \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& - \frac{9}{256} \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) + \frac{423}{64} \cosh(z) \\
& + \frac{157}{256} \cosh(2z) - \frac{13}{64} \cosh(z) \cos(8r_5) + \frac{13}{256} \cosh(2z) \cos(8r_5) \\
& - \frac{13}{64} \cosh(z) \cos(8r_4) + \frac{13}{256} \cosh(2z) \cos(8r_4)
\end{aligned}$$

$$\begin{aligned}
& -\frac{13}{64} \cosh(z) \cos(8r_4) \cos(8r_5) + \frac{13}{256} \cosh(2z) \cos(8r_4) \cos(8r_5) \\
& -\frac{9}{64} \cosh(z) \cos(8r_2) - \frac{3}{256} \cosh(2z) \cos(8r_2) \\
& + \frac{3}{64} \cosh(z) \cos(8r_2) \cos(8r_5) - \frac{3}{256} \cosh(2z) \cos(8r_2) \cos(8r_5) \\
& + \frac{3}{64} \cosh(z) \cos(8r_2) \cos(8r_4) - \frac{3}{256} \cosh(2z) \cos(8r_2) \cos(8r_4) \\
& + \frac{3}{64} \cosh(z) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& - \frac{9}{256} \cosh(2z) \cos(8r_2) \cos(8r_4) \cos(8r_5) - \frac{9}{64} \cosh(z) \cos(8r_1) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_5) + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_4) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_4) \\
& + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& - \frac{9}{64} \cosh(z) \cos(8r_1) \cos(8r_2) - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_2) \\
& + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& - 6 \cosh(z) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& + \frac{3}{64} \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{3}{8} \cosh(2z) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& - \frac{3}{256} \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{423}{64} \cosh(s) + \frac{157}{256} \cosh(2s) - \frac{9}{64} \cosh(s) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_5) - \frac{9}{64} \cosh(s) \cos(8r_4) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_4) - \frac{9}{64} \cosh(s) \cos(8r_4) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_4) \cos(8r_5) - \frac{13}{64} \cosh(s) \cos(8r_2) \\
& + \frac{13}{256} \cosh(2s) \cos(8r_2) + \frac{3}{64} \cosh(s) \cos(8r_2) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_2) \cos(8r_5) + \frac{3}{64} \cosh(s) \cos(8r_2) \cos(8r_4) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_2) \cos(8r_4) \\
& + \frac{3}{64} \cosh(s) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& - \frac{13}{64} \cosh(s) \cos(8r_1) + \frac{13}{256} \cosh(2s) \cos(8r_1) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_5) - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_5) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_4) - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_4) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& - \frac{13}{64} \cosh(s) \cos(8r_1) \cos(8r_2) + \frac{13}{256} \cosh(2s) \cos(8r_1) \cos(8r_2) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& - 6 \cosh(s) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& + \frac{3}{64} \cosh(s) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{3}{8} \cosh(2s) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& - \frac{3}{256} \cosh(2s) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5)
\end{aligned}$$

$$\begin{aligned}
& + \frac{279}{16} \cosh(s) \cosh(z) - \frac{3}{64} \cosh(s) \cosh(2z) - \frac{3}{64} \cosh(2s) \cosh(z) \\
& - \frac{145}{256} \cosh(2s) \cosh(2z) + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_5) \\
& - \frac{3}{64} \cosh(s) \cosh(2z) \cos(8r_5) + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_5) + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_4) \\
& - \frac{3}{64} \cosh(s) \cosh(2z) \cos(8r_4) + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_4) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_4) \\
& + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_4) \cos(8r_5) \\
& - \frac{3}{64} \cosh(s) \cosh(2z) \cos(8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_4) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_4) \cos(8r_5) \\
& + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_2) + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_2) \\
& - \frac{3}{64} \cosh(2s) \cosh(z) \cos(8r_2) - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_2) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_2) \cos(8r_5) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_2) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_2) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_2) \cos(8r_5) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_2) \cos(8r_4) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_2) \cos(8r_4) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_2) \cos(8r_4) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_2) \cos(8r_4) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_1) + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \\
& - \frac{3}{64} \cosh(2s) \cosh(z) \cos(8r_1) - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_5) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_5) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_4) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_4) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_4) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_4) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& + \frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_4) \cos(8r_5) \\
& + \frac{3}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_2) \\
& + \frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_2) \\
& - \frac{3}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_2) \\
& - \frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_2) \\
& - \frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_5)
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& +\frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& -\frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_5) \\
& -\frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& +\frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& +\frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& -\frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \\
& +6 \cosh(s) \cosh(z) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& -\frac{1}{16} \cosh(s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& +\frac{1}{64} \cosh(s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& +\frac{1}{64} \cosh(2s) \cosh(z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5) \\
& -\frac{3}{8} \cosh(2s) \cosh(2z) \cos(4r_1) \cos(4r_2) \cos(4r_4) \cos(4r_5) \\
& -\frac{1}{256} \cosh(2s) \cosh(2z) \cos(8r_1) \cos(8r_2) \cos(8r_4) \cos(8r_5)
\end{aligned}$$

## A.4 The $D = 4$ potential on a manifold of ten $SO(3)$ singlets

For the potential whose construction was described in ch.5, we introduce the following abbreviations:

$$\begin{aligned}
A & := \cos(2\omega_{48}) & B & := \cos(4\omega_{48}) \\
C & := \cos(6\omega_{48}) & D & := \cos(8\omega_{48}) \\
E & := \cos(2\omega_{47}) & F & := \cos(4\omega_{47}) \\
G & := \cos(6\omega_{47}) & H & := \cos(8\omega_{47}) \\
I & := \cos(2\phi_{47} - 4\phi_{48}) & J & := \cos(2\omega_{46}) \\
K & := \cos(4\omega_{46}) & L & := \cos(6\omega_{46}) \\
M & := \cos(8\omega_{46}) & N & := \cos(2\phi_{46} - 4\phi_{47}) \\
O & := \cos(2\phi_{46} - 2\phi_{47} - 4\phi_{48}) & P & := \cos(4\omega_{45}) \\
Q & := \cos(8\omega_{45}) & R & := \cos(4\phi_{45}) \\
S & := \cos(2\phi_{45}) & T & := \cos(4\phi_{46}) \\
U & := \cos(2\phi_{46} + 4\phi_{47}) & V & := \cos(2\phi_{46} + 2\phi_{47} + 4\phi_{48}) \\
W & := \cos(2\omega_{45}) & X & := \sin(4\phi_{46}) \\
Y & := \sin(2\phi_{45}) & Z & := \cos(6\omega_{45}) \\
a & := \sin(2\phi_{46} + 4\phi_{47}) & b & := \sin(2\phi_{46} + 2\phi_{47} + 4\phi_{48}) \\
c & := \cosh(1/2\rho) & d & := \cosh(\rho) \\
e & := \cosh(3/2\rho) & f & := \cosh(2\rho)
\end{aligned} \tag{A.1}$$

Furthermore, since every denominator in this potential is a power of two, we define  $a\%b := a/2^b$ . Then, the 12 240 terms of the potential are given on the following eight pages.

















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+452626SPULHD -94199SPUKEA -278216SPUREB -38196SPUREK -94236SPUREK
-27920SPUPKFA -94216SPUPKFB +34206SPUPKFC +94236SPUPKFD +94199SPUPKGA +274216SPUPKGB
+34196SPUPKGC +94236SPUPKGD +638226SPUPKEA +638226SPUPKEB +94236SPUPKEC +94236SPUPKED
-214216SPULFA -214216SPULFB -38216SPULFC -38246SPULFD +214216SPULGA
+214216SPULGB +38216SPULGC +38246SPULGD +214216SPULHA +214246SPULHB +38246SPULHC +38246SPULHD
-38216SPULGE +38246SPULGD +214216SPULHA +214246SPULHB +38246SPULHC +38246SPULHD
-103118226WREBY +151418216WJFBY +236298226WJGDY +117818216WKEBY +638206WKFY
-13658216WKGDBY -236298226WLEBY -47258216WLFY -15758226WJGDY -236298226WJGDY
-47258216WLFY -15758226WJGDY -47258216WLFY -47258216WLFY -47258216WLFY
-15758226WJGDY -3158216ZLFY -1088226ZLGDY -1207618236WJFBDY -151418226WJFBDY
+236298226WJGDBY -117818226WKEBY +638216WKFY +13658226WKGDBY +236298226WJGDBY
+47258226WLFY +15758226WJGDBY +236298226WJGDBY +47258226WLFY +15758226WJGDBY
+47258226WLFY +9458216ZLFY +3158226ZLGDY -15758226WJGDBY +3158226ZLFY
+1058234ZLGDY -1478226WREBY +21638216WJFBDY -33758226WJGDBY +16838216WKEBY
-98206WKFY -1958216WKEBY -33758226WLEBY -6758216WLFY -258226WJGDBY
-33758226WLFY -6758216WLFY -2288226ZLGDY -6758216ZLFY -1358206WKEBY
-458216ZKGBY -2288226ZLFY -458216ZLFY -158226ZLGDY +132545825WJFBDY
-2163824WJFBDY +3375825WJGDBY -1683824WKEBY +98236WKFY +195824WKEBY
+3375825WJGDBY +675824WJFBDY +2288226ZLFY +3158226ZLGDY -675824ZLFY
+2288226ZLFY +675824ZLFY +1358236ZLFY +45824ZLFY +2288226ZLFY
+45824ZLFY +15825ZLFY -19388216SPVGB -6518206SPVGF +15758216SPVGG
+6518206SPVGE +2738186SPVFB +3158206SPVFC +15758216SPVFE +3158206SPVFF
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-2738186SPVUL -3158216SPVUK -15758226SPVLE -3158216SPVLF -1058226SPVLG
-27698216SPVJE -938206SPVJF +2258216SPVJG -938206SPVKE +398196SPVKE
+458206SPVKG +2258216SPVLE +458206SPVLF +158216SPVLF +2769824SPVUE
+938216SPVJF +2258216SPVJG +938216SPVKE -398226SPVKE -458216SPVKG -2258246SPVLE
-458216SPVLF -158246SPVLE +151418206WLEBY -151418206WLEBY -151418206WLEBY
+708758226WREBY -3428186WJFBDY -73298186WJFBDY +3428186WJFBDY
+141788216WJFBDY +47258206WJGDBY +47258206WJGDBY +47258206WJGDBY
+47258226WJGDBY -40958196WKEBY -66578196WKEBY +40958196WKEBY
+148478216WKEBY -8198186WKFBDY -5258186WKFBDY +8198186WKFBDY -21638216WKFBDY
-2738186WKEBY +2738186WKEBY +2738186WKEBY -2738186WKEBY -47258206WLEBY
+47258206WLEBY +47258206WLEBY +47258226WLEBY -9458196WLFBDY +9458196WLFBDY
+9458196WLFBDY +9458216WLFBDY -3158206WLEBY -3158206WLEBY +3158206WLEBY
+3158226WLEBY -47258206ZLEBY +47258206ZLEBY +47258206ZLEBY +47258226ZLEBY
-9458196ZLFBDY +9458196ZLFBDY +9458196ZLFBDY +9458196ZLFBDY -3158206ZLFBDY
+3158206ZLFBDY +9458216ZLFBDY +3158226ZLFBDY -9458196ZLFBDY +9458196ZLFBDY
+9458196ZLFBDY +9458216ZLFBDY -1898186ZLFBDY +1898186ZLFBDY +1898186ZLFBDY
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-141788226WJFBDY +47258216WJGDBY -47258216WJGDBY +47258216WJGDBY
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-9458206WLFBDY -9458226WLFBDY +3158216WLFBDY -3158216WLFBDY -3158216WLFBDY
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-3158216ZLFBDY -3158216ZLFBDY -3158236ZLFBDY +9458206ZLFBDY -9458206ZLFBDY
+9458206ZLFBDY -9458226ZLFBDY -1898196ZLFBDY -1898196ZLFBDY -1898196ZLFBDY
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+6758206WJGDBY +6758206WJGDBY +6758206WJGDBY +6758226WJGDBY -5858196WKEBY
-9518196WKEBY +5858196WKEBY +21218216WKEBY -1178186WKEBY -758186WKEBY
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+398216WKEBY -6758206WLEBY +6758206WLEBY +6758206WLEBY +6758226WLEBY
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+5858226WKEBY +9518226WKEBY -5858226WKEBY -21218246WKEBY +1178216WJFBDY
+758216WJFBDY -1178216WJFBDY -309206WKFBDY +398216WKFBDY -398226WKFBDY
-398226WKFBDY -398246WJGDBY +6758236WLEBY -6758236WLEBY -6758236WLEBY

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-6758256WLEBY +1358226WLFBDY -1358226WLFBDY -1358226WLFBDY -1358246WLFBDY
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-6758236ZLFBDY -6758236ZLFBDY -6758236ZLFBDY -6758236ZLFBDY -1358226ZLFBDY
-1358226ZLFBDY -1358246ZLFBDY +458236ZLFBDY +458236ZLFBDY -458236ZLFBDY
-458256ZLFBDY +1358226ZLFBDY -1358226ZLFBDY -1358226ZLFBDY -1358246ZLFBDY
+2738186ZLFBDY -2738186ZLFBDY -2738186ZLFBDY -2738186ZLFBDY +98226ZLFBDY
-98226ZLFBDY -98226ZLFBDY -98226ZLFBDY -98226ZLFBDY +458236ZLFBDY -458236ZLFBDY
-458236ZLFBDY -458256ZLFBDY +98226ZLFBDY -98226ZLFBDY -98226ZLFBDY -98226ZLFBDY
+38236ZLFBDY -38236ZLFBDY -38236ZLFBDY -38236ZLFBDY -6518196SPVUEA
+60278196SPVUEB +6518196SPVUEC -47258216SPVUED -2738186SPVUEE +3998186SPVUEF
-2738186SPVUEG +9458206SPVUEH +3158186SPVUEI -1358196SPVUEJ -3158196SPVUEK
-3158216SPVUEL +2738186SPVUEM +3998186SPVUEN -2738186SPVUEO -9458206SPVUEP
-1058176SPVUEQ -218176SPVUER -1058176SPVUES -1898196SPVUEF +638186SPVUEG
+638186SPVUEH -638186SPVUEI -638206SPVUEJ +1358196SPVUEK -1358196SPVUEL
-1358196SPVUEM -3158216SPVUEO +638186SPVUEP -638186SPVUEQ -638186SPVUER
-638206SPVUEF +218196SPVUEG -218196SPVUEH -218196SPVUEI -218196SPVUEJ
+6518206SPVUEK -60278206SPVUEL -6518206SPVUEM -47258216SPVUEO -2738186SPVUEP
-3998196SPVUEQ -2738196SPVUER +9458216SPVUES -3158206SPVUEF +3158206SPVUEG
+3158206SPVUEH -3158206SPVUEI -2738186SPVUEJ -3998196SPVUEK -2738196SPVUEL
+9458216SPVUEM -1058186SPVUEO +218186SPVUEP +1058186SPVUEQ +1898206SPVUER
-638196SPVUEK +638196SPVUEL +638196SPVUEM +638196SPVUEO +638196SPVUEP
-3158206SPVUEQ +638216SPVUER -21206SPVUEK -3158206SPVUEL -21206SPVUEM
+638196SPVUEO +638216SPVUEK -21206SPVUEL -21206SPVUEM -21206SPVUEO
+398186SPVUEA +578186SPVUEB -398186SPVUEC -1358206SPVUEF +458196SPVUEG
-458196SPVUEH -458196SPVUEI -458196SPVUEJ -398186SPVUEK +578186SPVUEL
-398186SPVUEM -1358206SPVUEO -138176SPVUEP -38176SPVUEQ -138176SPVUER
-273196SPVUEK +98186SPVUEL -98186SPVUEM -98186SPVUEO -98206SPVUEP +458196SPVUEA
-458196SPVUEB -458196SPVUEC -458196SPVUEE -98186SPVUEF -98186SPVUEG -38216SPVUEH
+938226SPVUEA -8618226SPVUEB -938226SPVUEC +6758246SPVUEE -398216SPVUEA
-578216SPVUEB +398216SPVUEC -1358236SPVUEE -458226SPVUEG -458226SPVUEH
+458226SPVUEI +458246SPVUEJ -398216SPVUEK -578216SPVUEL +398216SPVUEM
+1358236SPVUEK -158206SPVUEA +38206SPVUEB +158206SPVUEC +38206SPVUEE
-98216SPVUEG +98216SPVUEH +98216SPVUEI +98216SPVUEJ +98216SPVUEK +98216SPVUEL
+458226SPVUEM +458246SPVUEO -98216SPVUEP -98216SPVUEQ +98216SPVUER +98216SPVUEO
-38226SPVUEG +38226SPVUEH +38226SPVUEI +38246SPVUEO

```

# Appendix B

## Explicit definitions reproducing all major results given in the text

For the sake of reproducibility, `LambdaTensor` definitions that reproduce the main results of this work are given here. This may also serve as a starting point for further investigations of the potentials of gauged maximal supergravities and furthermore gives many examples how to work with the `LambdaTensor` package. In order not to make this appendix overly long, most consistency cross-checks originally present in the code have been omitted here. Furthermore, due to typographical reasons (LISP code easily tends to become quite wide), and since this appendix presupposes quite deep knowledge of LISP, the decision was made to typeset it in a rather small font. Although the `LambdaTensor` package (which itself is much too large to be presented here, consisting of 20 000+ lines of code and documentation) should be considered an integral part of this work, no knowledge of LISP is necessary to use the data on vacua obtained here. However, it should perhaps be noted that, despite considerable diligence on the author's side, there still is some chance that in the process of transliteration from LISP to physical notation, which still is not fully automatized and hence prone to human errors, occasional mistakes occurred, or some ambiguity remained. Thus, for any formula in question, the LISP definition should be considered as known good and machine checked.

### B.1 The partial $SU(3)_{\text{diag}}$ potential

```
(eval-when (compile load eval)
  (progn
```

```

(require :e8-supergravity)
(use-package :lambdatensor)
(use-package :tf-spellbook)))

;; First of all, we have to set up the set of variables
;; we are going to use

(setf *poexp-vars*
      (map 'simple-array #'identity
           '(w z lambda1 lambda2 a phi theta psi)))

(setf (sp-arith-converter *sp-arith-poexp*)
      #'(lambda (x) (poexp-converter x *poexp-vars*)))

;; The embedding tensor

(defvar theta-so8xso8 (e8-theta-so-p 8))

;; E7 is parametrized as in [Phys.\ Lett.\ B {\bf 128} (1983) 169.]
;; These singlets correspond to G1+ and G2+ given there.

(defvar ac-g1+
      (sp-x '(e8-a)
            1/4
            '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
            '(, (sp+
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 1) (:fix 2) (:fix 3)
                                             alpha beta))
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 1) (:fix 4) (:fix 5)
                                             alpha beta))
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 1) (:fix 6) (:fix 7)
                                             alpha beta)))
            alpha beta)))

(defvar ac-g2+
      (sp-x '(e8-a)
            1/4
            '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
            '(, (sp+
                 '(-1 , (sp-x '(alpha beta)
                              '(,so8-sigma-ijkl-ab (:fix 0) (:fix 2) (:fix 4) (:fix 6)
                                                    alpha beta)))
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 2) (:fix 5) (:fix 7)
                                             alpha beta))
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 3) (:fix 4) (:fix 7)
                                             alpha beta))
                 (sp-x '(alpha beta)
                       '(,so8-sigma-ijkl-ab (:fix 0) (:fix 3) (:fix 5) (:fix 6)
                                             alpha beta)))
            alpha beta)))

;; Next, we have to lift anti-hermitian SU(8) generators
;; built out of LISP numbers to the corresponding SO(16) generators

(defun translate-su8-so16 (su8)
  (declare (optimize (speed 3)))
  (let ((so16 (make-sp-array '(16 16))))
    (sp-do
      #'(lambda (pos val)
          (declare (type (simple-array fixnum (2)) pos))
          (let* ((p1 (aref pos 0))
                 (p2 (aref pos 1))
                 (p1+8 (+ p1 8))
                 (p2+8 (+ p2 8))
                 ;; NOTE: if we were using some more generic algebra,
                 ;; we would have to employ (sp-arith-conj arith) here.
                 (vr (realpart val))
                 (vi (imagpart val)))
            (declare (fixnum p1 p2 p1+8 p2+8))
            (sp-set! so16 vr p1 p2)
            (sp-set! so16 vr p1+8 p2+8)
            (sp-set! so16 vi p1+8 p2) ; real p2 turned into imag p1
            (sp-set! so16 (- vi) p1 p2+8) ; imag turned into -real
          ))
      su8)
    (sp-scale so16 1/2)))

```



```

(defun translate-su8-gen-e8 (su8)
  (let ((so16 (translate-su8-so16 su8))
        (sp-x '(e8-M e8-N)
              '(,so16 I J)
              '(,ixmap-e8-so16-i-j e8-IJ I J)
              '(,e8-fabc e8-IJ e8-N e8-M))))

;; With these, we define the E8 analoga of the a,phi,theta,psi
;; rotation generators.

(defvar gen-a
  (let ((su8 (make-sp-array '(8 8)
                            :with-entries
                            '((-3 6 6) (-3 7 7))))
        (i-dotimes (j 6)
                    (setf (sp-ref su8 j j) 1))
        (translate-su8-gen-e8 (sp-scale su8 (complex 0 -1/4)))))

(defvar gen-phi
  (let ((su8 (make-sp-array '(8 8)
                            :with-entries
                            '(1 6 6) (-1 7 7))))
        (translate-su8-gen-e8 (sp-scale su8 (complex 0 1)))))

(defvar gen-theta
  (let ((su8 (make-sp-array '(8 8)
                            :with-entries
                            '((-1 6 7) (1 7 6))))
        (translate-su8-gen-e8 su8)))

(defvar gen-psi gen-theta)

;; Parametrization of SL(2)

(defvar ac-w
  (sp-x '(e8-a)
        '(,ixmap-e8-so16-i-j e8-a j16 k16)
        '(,ixmap-so16-i16-i j16 j)
        '(,ixmap-so16-i16-i* k16 k)
        '(, (sp-scale (sp-id 8) 1/4) j k)))

(defvar ac-z
  (sp-x '(e8-a)
        '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
        '(, (sp-scale (sp-id 8) 1/4) alpha beta)))

;; Note that there are four further singlets from the (56,2)
;; in 248 -> (133,1) + (56,2) + (1,3) which we simply
;; ignore here to reduce complexity. Hence,
;; We will probably not obtain all stationary points
;; that break symmetry down to  $SU(3)_{\text{diag}}$ ,
;; and furthermore, we will probably get other false positives
;; than coordinate system artefacts, which have to be eliminated.

;; This corresponds to the potential (3.9) from hep-th/0207206,
;; but here with the  $SL(2)$  part fully parametrized.

(defvar phi-so8xso8-e7xsl2-and-tensors
  (labels
    ((e8-gen (x)
             (sp-ac-to-generator e8-fabc x))
     (poexp-simplify (x)
                     (sp-map #'(lambda (z) (poexp* z 1)) x)))
    (let* ((rot-w (poexp-make-rot (e8-gen ac-w) 'w))
           (rot-z (poexp-make-rot (e8-gen ac-z) 'z))
           (rot-a (poexp-make-rot gen-a 'a))
           (rot-phi (poexp-make-rot gen-phi 'phi))
           (rot-theta (poexp-make-rot gen-theta 'theta))
           (rot-psi (poexp-make-rot gen-psi 'psi))
           (rot-g1+ (poexp-make-rot (e8-gen ac-g1+) 'lambda1))
           (rot-g2+ (poexp-make-rot (e8-gen ac-g2+) 'lambda2))
           (rot-g12+ (poexp-simplify (sp* rot-g1+ rot-g2+)))
           (rot-compact-e7
            (poexp-simplify (sp* (poexp-simplify (sp* rot-a rot-phi))
                               (poexp-simplify (sp* rot-theta rot-psi)))))
           (rot-sl2 (poexp-simplify (sp* rot-w rot-z)))
           (rot-e7 (poexp-simplify (sp* rot-compact-e7 rot-g12+)))
           (v-matrix (poexp-simplify (sp* rot-sl2 rot-e7)))
           ;; Due to local  $S_0(16)$  invariance of the potential, we can ignore
           ;; the compact rhs factors.

```

```

*e8-last-computed-a1*
*e8-last-computed-a2*
*e8-last-computed-t-tensor*
(declare (special *e8-last-computed-a1*
                  *e8-last-computed-a2*
                  *e8-last-computed-t-tensor*))
(let ((potential (e8-potential-from-v-theta v-matrix theta-so8xso8)))
  (:(potential . ,potential)
   (:a1 . ,*e8-last-computed-a1*)
   (:a2 . ,*e8-last-computed-a2*)
   (:(t-tensor . ,*e8-last-computed-t-tensor*))
  )))

```

## B.2 The $SO(6)_{\text{diag}}$ potentials

```

(eval-when (compile load eval)
  (progn
    (require :e8-supergravity)
    (use-package :lambdatensor)
    (use-package :tf-spellbook)))

;; XXX still relics in the code for forming an embedding tensor!

(defun e8-so-p-8-p-diag-gen-acs (p &key (only-first nil))
  (let ((gen-acs-1 nil)
        (gen-acs-2 nil)
        (dotimes (k p)
          (dotimes (j k)
            (ppush gen-acs-1
                  (cons +1
                        (sp-x '(e8)
                              '(,ixmap-e8-so16-ij e8 ij16)
                              '(,ixmap-so16-ij-i-j ij16 i16 j16)
                              '(,ixmap-so16-I16-i i16 i8*)
                              '(,ixmap-so16-I16-i j16 j8)
                              '(,so8-sigma-ij-ab i8 j8 (:fix ,j) (:fix ,k))
                              )))
            (ppush gen-acs-2
                  (cons -1
                        (sp-x '(e8)
                              '(,ixmap-e8-so16-ij e8 ij16)
                              '(,ixmap-so16-ij-i-j ij16 i16 j16)
                              '(,ixmap-so16-I16-i* i16 i8*)
                              '(,ixmap-so16-I16-i* j16 j8*)
                              '(,so8-sigma-ij-ab i8* j8* (:fix ,j) (:fix ,k))
                              )))))
    (if (not only-first)
        (do ((j p (i1+ j)))
            ((i= j 8)
             (do ((k (i1+ j) (i1+ k)))
                 ((i= k 8)
                  (ppush gen-acs-2
                        (cons -1
                              (sp-x '(e8)
                                    '(,ixmap-e8-so16-ij e8 ij16)
                                    '(,ixmap-so16-ij-i-j ij16 i16 j16)
                                    '(,ixmap-so16-I16-i i16 i8)
                                    '(,ixmap-so16-I16-i j16 j8)
                                    '(,so8-sigma-ij-ab i8 j8 (:fix ,j) (:fix ,k))
                                    )))
                  (ppush gen-acs-1
                        (cons +1
                              (sp-x '(e8)
                                    '(,ixmap-e8-so16-ij e8 ij16)
                                    '(,ixmap-so16-ij-i-j ij16 i16 j16)
                                    '(,ixmap-so16-I16-i* i16 i8*)
                                    '(,ixmap-so16-I16-i* j16 j8*)
                                    '(,so8-sigma-ij-ab i8* j8* (:fix ,j) (:fix ,k))
                                    ))))))
        (map 'simple-array #'(lambda (x y) (sp+ (cdr x) (cdr y))) gen-acs-1 gen-acs-2)))

(defvar diag6 (e8-so-p-8-p-diag-gen-acs 6))

(defvar diag6-singlets (sp-ac-singlets e8-fabc diag6 e8-scalars))
(defvar diag6-singlets+so16 (sp-ac-singlets e8-fabc diag6 e8-scalars+so16))
(defvar diag6a (e8-so-p-8-p-diag-gen-acs 6 :only-first t))
(defvar diag6a-singlets (sp-ac-singlets e8-fabc diag6a e8-scalars)) ; ; seven

```

```

(defvar diag6a-singlets+so16 (sp-ac-singlets e8-fabc diag6a e8-scalars+so16)) ; eleven (SL(3) x SL(2) = 8+3)

(defvar u1-generator (sp+ (aref diag6-singlets+so16 5) (aref diag6-singlets+so16 8)))

(defvar sl2
  (make-array 3 :initial-contents
    (list (sp+ '(1/4 ,u1-generator))
          (sp-x '(e8)
                '(,ixmap-e8-so16-alphabeta e8 alpha beta)
                '(,(make-sp-array '(8 8) :with-entries '((+1/2 6 6) (-1/2 7 7))) alpha beta))
          (sp-x '(e8)
                '(,ixmap-e8-so16-alphabeta e8 alpha beta)
                '(,(make-sp-array '(8 8) :with-entries '((+1/2 6 7) (+1/2 7 6))) alpha beta))))))

(defvar sl3
  (let* ((sl3-v0 (sp-lin-indep-ac-commutators e8-fabc diag6-singlets+so16 diag6-singlets+so16))
        (s (copy-seq sl3-v0)))
    (setf (aref s 6) (sp+ '(-1 ,(aref sl3-v0 6)) '(1/4 ,(aref sl3-v0 7))))
    (setf (aref s 7) (sp+ '(+1 ,(aref sl3-v0 6)) '(1/4 ,(aref sl3-v0 7))))
    (map 'simple-array
        #'(lambda (x scale) (sp-scale x scale)) s '(-1/6 -1/6 -1/2 1/6 1/6 1/2 1/2 1/2))
    ;; The minus signs for the first three generators make the structure constants equal +epsilon3.
    ))

(defvar sl3-g_ab
  (let ((mx (make-sp-array '(8 8))))
    (dotimes (j 8)
      (dotimes (k 8)
        (let ((sprod (* 1/60
                      (sp-ref (sp-x '() '(,e8-g_ab a b) '(,(aref sl3 j) a) '(,(aref sl3 k) b))))))
          (sp-set! mx sprod j k)))
      mx))

(defvar sl3-g^ab (sp-invert sl3-g_ab))

(defun sl3-decompose (e8-ac)
  (let ((v (make-sp-array '(8))))
    (dotimes (c 8)
      (let ((coeff (* (sp-ref sl3-g^ab c c)
                     (sp-ref (sp-x '() '(,e8-eta p q) '(,(aref sl3 c) p) '(,e8-ac q))))))
        (sp-set! v coeff c)))
    v))

(defvar sl3-fabc
  (let ((fabc (make-sp-array '(8 8 8))))
    (dotimes (a 8)
      (dotimes (b 8)
        (let* ((ab (e8-ac-[] (aref sl3 a) (aref sl3 b)))
              (vc (sl3-decompose ab)))
          (sp-do #'(lambda (p v) (sp-set! facb v a b (aref p 0)) vc)))
        facb))

;; (sp-check-jacobi sl3-fabc) ==> T

(defvar data-so62
  (let* ((delta8-ij (sp-generate-indexsplit-tensor 16 '(8)))
        (delta8-ij-8 (sp-generate-indexsplit-tensor 16 '(8) :offset 8))
        (p1 (sp-x '(e8-c e8-b)
                  1/2 1/2
                  '(,(make-sp-array '(8 8) :with-entries '((1 0 6) (1 1 7) (-1 2 5) (1 3 4))) i j)
                  '(,(sp+ '(+1 ,(sp-x '(Ix Jx i j) '(,delta8-ij Ix i) '(,delta8-ij-8 Jx j)))
                          '(-1 ,(sp-x '(Ix Jx i j) '(,delta8-ij-8 Jx i) '(,delta8-ij Ix j)))
                          '(+1 ,(sp-x '(Ix Jx i j) '(,delta8-ij-8 Ix i) '(,delta8-ij Jx j)))
                          '(-1 ,(sp-x '(Ix Jx i j) '(,delta8-ij Jx i) '(,delta8-ij-8 Ix j)))
                          )
                  Ix Jx i j)
                  '(,ixmap-so16-ij-i-j IJ Ix Jx)
                  '(,ixmap-e8-so16-ij e8-a IJ)
                  '(,e8-fabc e8-a e8-b e8-c)))
        (p2 (sp-x '(e8-c e8-b)
                  1/2 1/2
                  '(,(sp-id 8) i j)
                  '(,(sp+ '(+1 ,(sp-x '(Ix Jx i j) '(,delta8-ij-8 Ix i) '(,delta8-ij Jx j)))
                          '(-1 ,(sp-x '(Ix Jx i j) '(,delta8-ij Ix i) '(,delta8-ij-8 Jx j)))) Ix Jx i j)
                  '(,ixmap-e8-so16-ij e8-a ij)
                  '(,ixmap-so16-ij-i-j ij Ix Jx)
                  '(,e8-fabc e8-a e8-b e8-c)))
        (p3 (sp-x '(e8-c e8-b)
                  -1/2 1/2

```

```

      '(, (make-sp-array '(8 8) :with-entries '((1 0 6) (1 1 7) (-1 2 5) (1 3 4))) i j)
      '(, (sp+ '(+1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij Ix i) '(, (delta8-ij Jx j)))
                '(-1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij Jx i) '(, (delta8-ij Ix j)))
                '(-1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij-8 Ix i) '(, (delta8-ij-8 Jx j)))
                '(+1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij-8 Jx i) '(, (delta8-ij-8 Ix j)))
                )
                Ix Jx i j)
      '(, (ixmap-so16-ij-i-j IJ Ix Jx)
      '(, (ixmap-e8-so16-ij e8-a IJ)
      '(, (e8-fabc e8-a e8-b e8-c)))
(p4 (sp-x '(e8-c e8-b)
      1/2
      '(, (make-sp-array '(8 8) :with-entries '((1 0 1) (-1 2 3) (1 4 5) (1 6 7))) c* d*)
      '(, (sp+ '(+1 ,(sp-x '(a* b* c* d*) '(, (sp-id 8) a* c*) '(, (sp-id 8) b* d*))
                '(-1 ,(sp-x '(a* b* c* d*) '(, (sp-id 8) b* c*) '(, (sp-id 8) a* d*))
                a* b* c* d*)
      '(, (ixmap-e8-so16-alpha*beta* e8-a a* b*)
      '(, (e8-fabc e8-a e8-b e8-c)))
(p5 (sp-x '(e8-c e8-b)
      1/2
      '(, (sp-id 8) a* b*)
      '(, (ixmap-e8-so16-alpha*beta* e8-a a* b*)
      '(, (e8-fabc e8-a e8-b e8-c)))
(p6 (sp-x '(e8-c e8-b)
      '(, (make-sp-array '(8 8) :with-entries '((1 6 7) (-1 7 6))) a b)
      '(, (ixmap-e8-so16-alphabeta e8-a a b)
      '(, (e8-fabc e8-a e8-b e8-c)))
(p7 (sp-x '(e8-c e8-b)
      1/2
      '(, (sp-id 8) a b)
      '(, (ixmap-e8-so16-alphabeta e8-a a b)
      '(, (e8-fabc e8-a e8-b e8-c)))
(p8
  (sp+ p7
    '(-2
      ,(sp-x '(e8-c e8-b)
        '(, (make-sp-array '(8 8) :with-entries '((1 6 6) (+1 7 7))) a b)
        '(, (ixmap-e8-so16-alphabeta e8-a a b)
        '(, (e8-fabc e8-a e8-b e8-c))))))
;; =====
(q1 (sp-x '(e8-c e8-b)
      1/4 1/2
      '(, (make-sp-array '(8 8) :with-entries '((1 0 6) (1 1 7) (-1 2 5) (1 3 4))) i j)
      '(, (sp+ '(+1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij Ix i) '(, (delta8-ij Jx j)))
                '(-1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij Jx i) '(, (delta8-ij Ix j)))
                '(+1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij-8 Ix i) '(, (delta8-ij-8 Jx j)))
                '(-1 ,(sp-x '(Ix Jx i j) '(, (delta8-ij-8 Jx i) '(, (delta8-ij-8 Ix j)))
                )
                Ix Jx i j)
      '(, (ixmap-so16-ij-i-j IJ Ix Jx) ; note to include a factor 1/2 since we are mapping I J -> IJ!
      '(, (ixmap-e8-so16-ij e8-a IJ)
      '(, (e8-fabc e8-a e8-b e8-c)))
(q2 (sp-x '(e8-c e8-b)
      1/2
      '(, (make-sp-array '(8 8) :with-entries '((1 6 6) (-1 7 7))) a b)
      '(, (ixmap-e8-so16-alphabeta e8-a a b)
      '(, (e8-fabc e8-a e8-b e8-c)))
(q3 (sp-x '(e8-c e8-b)
      1/2
      '(, (make-sp-array '(8 8) :with-entries '((1 6 7) (1 7 6))) a b)
      '(, (ixmap-e8-so16-alphabeta e8-a a b)
      '(, (e8-fabc e8-a e8-b e8-c)))
'((:sl3 . ,(vector p1 p2 p3 p4 p5 p6 p7 p8))
  (:sl2 . ,(vector q1 q2 q3))))

#|
(dotimes (j 8)
  (format t "~A: ~A ~j"
    (sp-multiple-p (sp-ac-to-generator e8-fabc (aref sl3 0))
      (aref (cav data-so62 :sl3) 0))))
=> 0: 1 1: 1 2: 1 3: 1 4: 1 5: 1 6: 1 7: 1
|#

;; dr is the defining representation.

(defvar sl3-dr
  (map 'simple-array #'(lambda (e) (make-sp-array '(3 3) :with-entries e))
    '(
      ((1 1 2) (-1 2 1))
      ((1 0 1) (-1 1 0))
      ((1 2 0) (-1 0 2))
    ))

```

```

;; ---
((1 1 2) (1 2 1))
((1 0 1) (1 1 0))
((-1 2 0) (-1 0 2))
((-1 0 0) (1 1 1))
((1 0 0) (1 1 1) (-2 2 2))))

(defvar sl2-dr
  (map 'simple-array #'(lambda (e) (make-sp-array '(2 2) :with-entries e))
    '(
      ((1/2 0 1) (-1/2 1 0))
      ((1/2 0 0) (-1/2 1 1))
      ((1/2 0 1) ( 1/2 1 0))
    )))

(defvar sl3-g-ab-dr
  (let ((mx (make-sp-array '(8 8))))
    (dotimes (j 8)
      (dotimes (k 8)
        (let ((sprod (sp-trace (sp* (aref sl3-dr j) (aref sl3-dr k))))
              (sp-set! mx sprod j k))))
      mx))

  (defvar sl3-g^ab-dr (sp-invert sl3-g-ab-dr))

  (defvar sl3-fabc-dr
    (labels
      ((sl3-decompose-dr (gen)
         (let ((v (make-sp-array '(8))))
           (dotimes (c 8)
             (let ((coeff (* (sp-ref sl3-g^ab-dr c c) (sp-trace (sp* gen (aref sl3-dr c)))))
                   (sp-set! v coeff c)))
             v)))
        (let ((fabc (make-sp-array '(8 8 8))))
          (dotimes (a 8)
            (dotimes (b 8)
              (let* ((ab (sp-[] (aref sl3-dr a) (aref sl3-dr b)))
                    (vc (sl3-decompose-dr ab)))
                (sp-do #'(lambda (p v) (sp-set! facb v a b (aref p 0)) vc))))
              facb)))

    ;; (sp+ sl3-fabc '(-1 ,sl3-fabc-dr)) => 0

    ;; The generic SL(3) coset element (mod right SO(16) rotation),
    ;; extended by a SL(2) rotation

    (defvar v-rot-so62
      (let* ((rot-s (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 6) -1)) 's))
             ;; Note: this corresponds to diag(1,-1,0)
             (rot-z (poexp-make-rot (sp-ac-to-generator e8-fabc (aref sl3 7)) 'z))
             (rot-sz (sp* rot-s rot-z))
             (rot-x1 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 0) +1)) 'r1))
             (rot-x2 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 2) +1)) 'r2))
             (rot-x3 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 1) +1)) 'r3))
             (rot-x123-sz (sp* rot-x1 (sp* rot-x2 (sp* rot-x3 rot-sz))))
             (rot-v (poexp-make-rot (sp-ac-to-generator e8-fabc (aref sl2 1)) 'v))
             (rot-x5 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref sl2 0)) 'r5))
             (rot-sl2 (sp* rot-x5 rot-v))
             (rot-sl3xsl2 (sp* rot-x123-sz rot-sl2))
            )
      (sp-map #'(lambda (x) (poexp* 1 x) rot-sl3xsl2)))

    (defvar phi-so6-so62 (poexp* 1 (e8-potential-from-v-theta v-rot-so62 (e8-theta-so-p 6))))
    (defvar phi-so6-so71 (poexp* 1 (e8-potential-from-v-theta v-rot-so62 (e8-theta-so-p 7))))
    (defvar phi-so6-so8 (poexp* 1 (e8-potential-from-v-theta v-rot-so62 (e8-theta-so-p 8))))

```

## B.3 The $SO(5)_{\text{diag}}$ potentials

The code given here is a bit more elaborate than for the other cases to demonstrate the process used to identify the  $SL(3)$  generators.

```

(eval-when (compile load eval)
  (progn

```

```

(require :e8-supergravity)
(use-package :lambdatensor)
(use-package :tf-spellbook)))

(setf *bytes-consed-between-gcs* 40000000 *gc-verbose* nil)

(defun e8-so-p-diag-gen-acs (p &key (start 0))
  (let ((gen-acs-1 nil)
        (gen-acs-2 nil))
    (dotimes (k p)
      (dotimes (j k)
        (ppush gen-acs-1
              (sp-x '(e8)
                    '(,ixmap-e8-so16-ij e8 ij16)
                    '(,ixmap-so16-ij-i-j ij16 i16 j16)
                    '(,ixmap-so16-I16-i i16 i8)
                    '(,ixmap-so16-I16-i j16 j8)
                    '(,so8-sigma-ij-ab i8 j8 (:fix ,(+ j start)) (:fix ,(+ k start))))
              ))
        (ppush gen-acs-2
              (sp-x '(e8)
                    '(,ixmap-e8-so16-ij e8 ij16)
                    '(,ixmap-so16-ij-i-j ij16 i16 j16)
                    '(,ixmap-so16-I16-i* i16 i8*)
                    '(,ixmap-so16-I16-i* j16 j8*)
                    '(,so8-sigma-ij-ab i8* j8* (:fix ,(+ j start)) (:fix ,(+ k start))))
              )))
    (map 'simple-array #'(lambda (x y) (sp+ x y)) gen-acs-1 gen-acs-2)))

(defvar so5_diag-v-ac (e8-so-p-diag-gen-acs 5))
(defvar so3_diag-v-ac (e8-so-p-diag-gen-acs 3 :start 5))
(defvar so2_diag-v-ac (e8-so-p-diag-gen-acs 2 :start 5))

(defvar so5xso3-singlets
  (sp-ac-singlets e8-fabc (concatenate 'list so5_diag-v-ac so3_diag-v-ac) e8-scalars))

(defun v-rot-so5xso3
  (let* ((*HEURISTIC-FACTORIZE-POLYNOME-ALSO-TRY-THESE-ZEROES* '(1/4))
        (ac-W
          (sp-x '(e8-a)
                '(,ixmap-e8-so16-i-j e8-a j16 k16)
                '(,ixmap-so16-i16-i j16 j)
                '(,ixmap-so16-i16-i* k16 k)
                '(,sp-scale (sp-id 8) 1/4) j k)))
        (ac-Z
          (sp-x '(e8-a)
                '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
                '(,sp-scale (sp-id 8) 1/4) alpha beta)))
        (ac-M
          (sp-x '(e8-a)
                '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
                '(,sp-scale
                  (make-sp-array '(8 8) :with-entries
                                '((3 0 0) (3 1 1) (3 2 2) (3 3 3) (3 4 4)
                                  (-5 5 5) (-5 6 6) (-5 7 7)))
                                1/4) alpha beta))))
    (declare (special *HEURISTIC-FACTORIZE-POLYNOME-ALSO-TRY-THESE-ZEROES*))
    (sp* (poexp-make-rot (sp-ac-to-generator e8-fabc ac-M) 's)
         (poexp-make-rot (sp-ac-to-generator e8-fabc ac-W) 'w)
         (poexp-make-rot (sp-ac-to-generator e8-fabc ac-Z) 'z))))))

(defvar phi-so5xso3-so5
  (poexp* 1 (e8-potential-from-v-theta v-rot-so5xso3
                                       (e8-theta-so-p 5))))

(defvar phi-so5xso3-so8
  (poexp* 1 (e8-potential-from-v-theta v-rot-so5xso3
                                       (e8-theta-so-p 8))))

(defvar so5xso2-singlets
  (sp-ac-singlets e8-fabc
                  (concatenate 'list so5_diag-v-ac so2_diag-v-ac)
                  e8-scalars))

(defvar so5xso2-singlets+so16
  (sp-ac-singlets e8-fabc
                  (concatenate 'list so5_diag-v-ac so2_diag-v-ac)
                  e8-scalars+so16))

;; These are ten.

; (length (sp-lin-indep-ac-commutators e8-fabc so5xso2-singlets+so16 so5xso2-singlets+so16))

```

```

;; => 8; 3c+5nc. This must be a SL(3).

;; (length (sp-ac-commutator-closure e8-fabc so5xso2-singlets))
;; => 9

;; (sp-total-dimension (concatenate 'list (sp-lin-indep-ac-commutators e8-fabc so5xso2-singlets+so16 so5xso2-singlets+so16) so5xso2-singlets))
;; => there must be a singlet not contained in SL(3). Hence, 5 Singlets from SL(5) and a single extra one. This is the picture.

(defvar so3-in-sl3
  (sp-lin-indep-ac-commutators e8-fabc so5xso2-singlets so5xso2-singlets))

(defvar non-sl3-singlet
  (aref (sp-ac-singlets e8-fabc so3-in-sl3 so5xso2-singlets) 0)) ; only one

(defvar sl3-noncompact
  (map 'simple-array #'sp-from-array
    (lintrans-core-and-complement
     #'(lambda (v)
         (vector (sp-ref (sp-x '() '(e8-eta a b)
                          '(non-sl3-singlet a)
                          '(sp-from-array v) b))))))
    (map 'simple-array #'sp-to-array so5xso2-singlets))))

(defvar full-sl3 (sp-ac-commutator-closure e8-fabc sl3-noncompact))

;; (length (sp-ac-commutator-closure e8-fabc sl3-noncompact)) => 8
;; Hence, we now really have SL(3).

|#

This shows that our generators are orthonormal.
And all except the first one can easily be normalized to +-2;
for the first one, 6 is natural,
so we may want to identify this with diag(1,1,-2).

(dotimes (j 8)
  (dotimes (k 8)
    (format t "~A ~A: ~A~%" j k
      (sp-ref (sp-x '() '(e8-eta a b)
                  '(aref full-sl3 j) a)
              '(aref full-sl3 k) b))))))

|#

(defvar properly-normalized-sl3
  (map 'simple-array
    #'(lambda (x)
        (let ((sqrt-abs-norm^2/2
              (sqrt
               (* 1/2 (abs
                     (sp-ref (sp-x '()
                                 '(e8-eta a b)
                                 '(x a) '(x b))))))))
            (format t "sqrt-abs-norm^2/2=~A~" sqrt-abs-norm^2/2)
            (if (= (floor sqrt-abs-norm^2/2) sqrt-abs-norm^2/2)
                (sp-scale x (/ (floor sqrt-abs-norm^2/2))
                          x)))
            full-sl3))
    (map 'simple-array
      #'(lambda (x)
          (let ((mx (make-sp-array '(8 8))))
            (dotimes (j 8)
              (dotimes (k 8)
                (let ((sprod (sp-ref (sp-x '() '(e8-eta a b)
                                        '(aref sl3-pn j) a) '(aref sl3-pn k) b))))
                  (sp-set! mx sprod j k))))
            mx)))
      (setf sl3-pn-g^ab (sp-invert sl3-pn-g_ab))

    (defun sl3-pn-decompose (e8-ac)
      (let ((v (make-sp-array '(8))))
        (dotimes (c 8)
          (let ((coeff (* (sp-ref sl3-pn-g^ab c c)
                          (sp-ref (sp-x '() '(e8-eta p q)
                                      '(aref sl3-pn c) p) '(e8-ac q))))))
            (sp-set! v coeff c)))
          v))

    (defvar sl3-pn-fabc
      (let ((fabc (make-sp-array '(8 8 8))))

```

```

(dotimes (a 8)
  (dotimes (b 8)
    (let* ((ab (e8-ac-[] (aref sl3-pn a) (aref sl3-pn b)))
           (vc (sl3-pn-decompose ab)))
      (sp-do #'(lambda (p v) (sp-set! fabc v a b (aref p 0))) vc)))
  fabc))

;; By educated guess and trial and error, the following permutation was found:

(setf sl3-reordered-fabc
  (let ((re-ordering
        (make-sp-array '(8 8)
          :with-entries
          '((1 0 7) (1 7 1) (1 1 4) (1 2 6)
            (1 3 5) (1 4 3) (1 5 2) (1 6 0)))))
    (sp-x '(ar br cr)
          '(,re-ordering a ar) '(,re-ordering b br) '(,re-ordering c cr)
          '(,sl3-pn-fabc a b c))))

;; ...and with this re-ordering, we just have the same sl3 fabc commutation relations as in the S0(6)_diag case.

(defvar sl3
  (map 'simple-array #'(lambda (n) (aref sl3-pn n))
    #(6 7 5 4 1 3 2 0)))

(defvar sl3-g_ab
  (let ((mx (make-sp-array '(8 8))))
    (dotimes (j 8)
      (dotimes (k 8)
        (let ((sprod (sp-ref (sp-x '() '(,e8-eta a b) '(,aref sl3 j) a) '(,aref sl3 k) b))))
          (sp-set! mx sprod j k)))
      mx))

(defvar sl3-g^ab (sp-invert sl3-g_ab))

(defun sl3-decompose (e8-ac)
  (let ((v (make-sp-array '(8))))
    (dotimes (c 8)
      (let ((coeff (* (sp-ref sl3-g^ab c c)
                     (sp-ref (sp-x '()
                                   '(,e8-eta p q)
                                   '(,aref sl3 c) p)
                                   '(,e8-ac q))))))
        (sp-set! v coeff c)))
    v))

(defvar sl3-fabc
  (let ((fabc (make-sp-array '(8 8 8))))
    (dotimes (a 8)
      (dotimes (b 8)
        (let* ((ab (e8-ac-[] (aref sl3 a) (aref sl3 b)))
               (vc (sl3-decompose ab)))
          (sp-do #'(lambda (p v) (sp-set! fabc v a b (aref p 0))) vc)))
      fabc))

;; (sp+ sl3-fabc '(-1 ,sl3-reordered-fabc) => #<empty>
;; Inspection shows that fabc is also the same as for the S0(6) case.

(defvar v-rot-so52
  (let* ((rot-s (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 6) -1)) 's))
         (rot-z (poexp-make-rot (sp-ac-to-generator e8-fabc (aref sl3 7)) 'z))
         (rot-sz (sp* rot-s rot-z))
         (rot-x1 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 0) +1)) 'r1))
         (rot-x2 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 2) +1)) 'r2))
         (rot-x3 (poexp-make-rot (sp-ac-to-generator e8-fabc (sp-scale (aref sl3 1) +1)) 'r3))
         (rot-x123-sz (sp* rot-x1 (sp* rot-x2 (sp* rot-x3 rot-sz))))
         (rot-v (poexp-make-rot (sp-ac-to-generator e8-fabc non-sl3-singlet) 'v))
         (rot-sl3xR (sp* rot-x123-sz rot-v))
         )
    (sp-map #'(lambda (x) (poexp* 1 x)) rot-sl3xR)))

(defvar phi-so5-so8
  (poexp* 1 (e8-potential-from-v-theta v-rot-so52 (e8-theta-so-p 8))))

(defvar phi-so5-so5
  (poexp* 1 (e8-potential-from-v-theta v-rot-so52 (e8-theta-so-p 5))))

```



## B.4 The $SO(4)_{\text{diag}}$ potentials

```

(eval-when (compile load eval)
  (progn
    (require :e8-supergravity)
    (use-package :lambdatensor)
    (use-package :tf-spellbook)))

(eval-when (compile load eval)
  (progn
    (setf lambdatensor::*poexp-vars*
      (coerce '(V S W Z X R0 R1 R2 R3 R4 R5) '(simple-array * (*))))
    (setf (lambdatensor::sp-arith-converter lambdatensor::*sp-arith-poexp*
      #'(lambda (x) (lambdatensor::poexp-converter x lambdatensor::*poexp-vars*)))
      ))

(setf *bytes-consed-between-gcs* 40000000 *gc-verbose* nil)

(defun e8-so-p-diag-gen-acs (p &key (start 0))
  [as in the  $S0(5)$  case]

  (defvar so4a_diag-v-ac (e8-so-p-diag-gen-acs 4))
  (defvar so3a_diag-v-ac (e8-so-p-diag-gen-acs 3))
  (defvar so4b_diag-v-ac (e8-so-p-diag-gen-acs 4 :start 4))
  (defvar so3b_diag-v-ac (e8-so-p-diag-gen-acs 3 :start 4))
  (defvar so2b_diag-v-ac (e8-so-p-diag-gen-acs 2 :start 4))

  (defvar so4xso4-singlets
    (sp-ac-singlets e8-fabc
      (concatenate 'list so4a_diag-v-ac so4b_diag-v-ac)
      e8-scalars))

  (defvar so4xso3-singlets
    (sp-ac-singlets e8-fabc
      (concatenate 'list so4a_diag-v-ac so3b_diag-v-ac)
      e8-scalars))

  (defvar so3xso3-singlets
    (sp-ac-singlets e8-fabc
      (concatenate 'list so3a_diag-v-ac so3b_diag-v-ac)
      e8-scalars))

  ;; ten

  (defvar so4xso2-singlets
    (sp-ac-singlets e8-fabc
      (concatenate 'list so4a_diag-v-ac so2b_diag-v-ac)
      e8-scalars))

  ;; eleven

  (defvar so4xso3-singlets+so16
    (sp-ac-singlets e8-fabc
      (concatenate 'list so4a_diag-v-ac so3b_diag-v-ac)
      e8-scalars+so16))

  ;; seven

  (defvar so3-singlet
    (aref (sp-ac-singlets e8-fabc so4xso4-singlets so4xso3-singlets) 0))

  (defun v-rot-so43 (v w s z x &key (type :symbolic))
    (let* ((ac-W
      (sp-x '(e8-a)
        '(,ixmap-e8-so16-i-j e8-a j16 k16)
        '(,ixmap-so16-i16-i j16 j)
        '(,ixmap-so16-i16-i* k16 k)
        '(,sp-scale (sp-id 8) 1/4) j k)))
      (ac-Z
      (sp-x '(e8-a)
        '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
        '(,sp-scale (sp-id 8) 1/4) alpha beta)))
      (e4 (make-sp-array '(8 8) :with-entries
        '((1 0 0) (1 1 1) (1 2 2) (1 3 3))))
      (f4 (make-sp-array '(8 8) :with-entries
        '((1 4 4) (1 5 5) (1 6 6) (1 7 7))))
      (ac-S1
      (sp-x '(e8-a)
        '(,sp+ '(1/4 ,e4) '(-1/4 ,f4)) a b)
        '(,ixmap-e8-so16-alphabeta e8-a a b)))
      (ac-S2
      (sp-x '(e8-a)
        '(,sp+ '(1/4 ,e4) '(-1/4 ,f4)) a* b*)
        '(,ixmap-e8-so16-alpha*beta* e8-a a* b*)))
      ))

```

```

(ac-V (e8-ac-[] ac-s1 ac-s2))
(ac-X (sp-x '(e8)
            '(,ixmap-e8-so16-alphabeta e8 alpha beta)
            '(,make-sp-array '(8 8)
                              :with-entries '((1 4 4) (1 5 5) (1 6 6) (-3 7 7))
                              alpha beta))))
;; this is just so3-singlet
(if (eq type :symbolic)
    (let* ((rot-wz
            (sp* (poexp-make-rot (sp-ac-to-generator e8-fabc ac-W) w)
                 (poexp-make-rot (sp-ac-to-generator e8-fabc ac-Z) z)))
           (rot-vs
            (sp* (poexp-make-rot (sp-ac-to-generator e8-fabc ac-V) v)
                 (poexp-make-rot (sp-ac-to-generator e8-fabc ac-S1) s)))
           (rot-x (poexp-make-rot (sp-ac-to-generator e8-fabc ac-X) x)))
        (sp* (sp* rot-vs rot-wz) rot-x))
    (labels
      ((gen (g x)
            (sp-exp
             (sp-scale
              (sp-change-arith (sp-ac-to-generator e8-fabc g) *sp-arith-double-float*
                               x))))))
        (let* ((rot-wzw (sp* (gen ac-W w) (gen ac-Z z) (gen ac-W (- w))))
               (rot-vsv (sp* (gen ac-V v) (gen ac-S1 s) (gen ac-V (- v))))
               (rot-x (gen ac-X x)))
              (sp* rot-wzw rot-vsv rot-x))))))

(defvar phi-so43
  (poexp* 1 (e8-potential-from-v-theta
            (v-rot-so43 'v 'w 's 'z 'x :type :symbolic)
            (e8-theta-so-p 8))))

(defvar phi-so43-so4
  (poexp* 1 (e8-potential-from-v-theta
            (v-rot-so43 'v 'w 's 'z 'x :type :symbolic)
            (e8-theta-so-p 4))))

(defvar phi-so43-so7
  (poexp* 1 (e8-potential-from-v-theta
            (v-rot-so43 'v 'w 's 'z 'x :type :symbolic)
            (e8-theta-so-p 7))))

```

## B.5 The $G_{2(-14)} \times F_{4(-20)}$ potential

```

(eval-when (compile load eval)
  (progn
    (setf lambdatensor::*poexp-vars*
          (coerce '(V S W Z X R0 R1 R2 R3 R4 R5 R6) '(simple-array * (*))))
    (setf (lambdatensor::sp-arith-converter lambdatensor::*sp-arith-poexp*)
          #'(lambda (x) (lambdatensor::poexp-converter x lambdatensor::*poexp-vars*)))
    ))

(defvar ac-g2
  (map 'simple-array
    #'(lambda (x)
      (sp-scale
       (sp-x '(e8)
              '(,ixmap-e8-so16-i-j e8 i16 j16)
              '(,ixmap-so16-i16-i i16 i)
              '(,ixmap-so16-i16-i j16 j)
              '(,x i j))
       1/2))
    g2-v8-vs))

(defvar ac-f4 (sp-ac-singlets e8-fabc ac-g2 e8-scalars+so16))

(defvar theta-f4
  (reduce #'(lambda (sf x) (sp+ sf (sp-x '(a b) '(,x a) '(,e8-eta c b) '(,x c)))) ac-f4
         :initial-value (make-sp-array '(248 248))))

;; Now, for g2, we have to trick a bit...

(defvar ac-g2o
  (map 'simple-array #'(lambda (x) (sp-from-vector x '(248)))
    (orthogonal-basis (map 'simple-array #'sp-to-vector ac-g2))))

```

```
;; Note that theta-g2 has an extra factor -1 sneaking in from the division by (negative) generator "length^2".
```

```
(defvar theta-g2
  (reduce #'(lambda (sf x)
    (let ((iproduct (sp-ref (sp-x '()) '(,x a) '(,e8-eta a b) '(,x b))))
      (sp+ sf (sp-scale (sp-x '(a b) '(,x a) '(,e8-eta c b) '(,x c)) (/ 1 iproduct))))
    ac-g2o :initial-value (make-sp-array '(248 248))))

(defvar theta-g2xf4 (sp+ theta-f4 '(+3/2 ,theta-g2)))

;; (e8-p27000 theta-g2xf4) => 0

(setf explicit-theta-g2xf4
  (let*
    ((delta7 (sp-antisymmetrizer 7 2))
     (proj7 (sp+ (sp-id 8) (make-sp-array '(8 8) :with-entries '((-2 7 7))))
     (delta78 (sp-x '(i j k l)
                   '(,delta7 i7 j7 k7 l7)
                   '(,proj7 i7 l) '(,proj7 j7 j)
                   '(,proj7 k7 k) '(,proj7 l7 l)))
     (traceless
      (sp+ (sp-id 8) (make-sp-array '(8 8) :with-entries '((-7 7 7)))))
    (sp+
     (sp-x '(e8-a e8-b)
           '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
           '(,ixmap-e8-so16-alphabeta e8-b gamma delta)
           '(, (sp-id 8) alpha gamma)
           '(, (sp-id 8) beta (:fix 7)) '(, (sp-id 8) delta (:fix 7)))
     (sp-x '(e8-a e8-b)
           '(,ixmap-e8-so16-alpha*beta* e8-a alpha* beta*)
           '(,ixmap-e8-so16-alpha*beta* e8-b gamma* delta*)
           '(, (sp-id 8) alpha* gamma*)
           '(, (sp-id 8) beta* (:fix 7)) '(, (sp-id 8) delta* (:fix 7)))
     (sp-x '(e8-a e8-b)
           -1
           '(,ixmap-e8-so16-ij e8-a ij)
           '(,ixmap-e8-so16-ij e8-b kl)
           '(,ixmap-so16-ij-i-j ij i16 j16)
           '(,ixmap-so16-ij-i-j kl k16 l16)
           '(,ixmap-so16-i16-i i16 i)
           '(,ixmap-so16-i16-i* j16 j)
           '(,ixmap-so16-i16-i k16 k)
           '(,ixmap-so16-i16-i* l16 l)
           '(, (sp-id 8) j l)
           '(, (sp-id 8) i (:fix 7))
           '(, (sp-id 8) k (:fix 7)))
     (sp-x '(e8-a e8-b)
           -1/2
           '(,ixmap-e8-so16-ij e8-a ij)
           '(,ixmap-e8-so16-ij e8-b kl)
           '(,ixmap-so16-ij-i-j ij i16 j16)
           '(,ixmap-so16-ij-i-j kl k16 l16)
           '(,ixmap-so16-i16-i* i16 i)
           '(,ixmap-so16-i16-i* j16 j)
           '(,ixmap-so16-i16-i* k16 k)
           '(,ixmap-so16-i16-i* l16 l)
           '(, (sp-antisymmetrizer 8 2) i j k l))
     (sp-x '(e8-a e8-b)
           1/16
           '(,ixmap-e8-so16-ij e8-a ij)
           '(,ixmap-e8-so16-ij e8-b kl)
           '(,ixmap-so16-ij-i-j ij i16 j16)
           '(,ixmap-so16-ij-i-j kl k16 l16)
           '(,ixmap-so16-i16-i i16 i)
           '(,ixmap-so16-i16-i j16 j)
           '(,ixmap-so16-i16-i k16 k)
           '(,ixmap-so16-i16-i l16 l)
           '(, (sp+ '(8 ,delta78)
                  '(-1/7 ,(sp-x '(i j k l)
                                '(,so8-sigma-ijkl-ab i j k l a b)
                                '(,traceless a b)))
                  '(-1/7 ,(sp-x '(i j k l)
                                '(,so8-sigma-ijkl-a*b* i j k l a b)
                                '(,traceless a b))))
            i j k l))))))

;; (sp+ '(-1 ,explicit-theta-g2xf4) theta-g2xf4)
;; => #<sparse (248 248) array [0/61504 entries, hash space=96, hash density=0.000]>
```

```

;; Checks show that these indeed do form 8-dimensional semisimple groups.

(defvar ac-su3-left
  (map 'simple-array
    #'(lambda (x)
      (sp-scale
        (sp-x '(e8)
          '(,ixmap-e8-so16-i-j e8 i16 j16)
          '(,ixmap-so16-i16-i i16 i)
          '(,ixmap-so16-i16-i j16 j)
          '(,x i j))
        1/2))
    (sp-heuristic-singlets (list (sp-from-vector #(0 0 0 0 0 0 1 0) '(8))
      g2-v8-vs
      :action #'(lambda (vec gen) (sp-x '(a) '(,gen a b) '(,vec b))))))

(defvar ac-su3-right
  (map 'simple-array
    #'(lambda (x)
      (sp-scale
        (sp-x '(e8)
          '(,ixmap-e8-so16-i-j e8 i16 j16)
          '(,ixmap-so16-i16-i* i16 i)
          '(,ixmap-so16-i16-i* j16 j)
          '(,x i j))
        1/2))
    (sp-heuristic-singlets (list (sp-from-vector #(0 0 0 0 0 1 0) '(8))
      g2-v8-vs
      :action #'(lambda (vec gen) (sp-x '(a) '(,gen a b) '(,vec b))))))

;; Note that these are part of G2 and F4:
;;
;; (sp-total-dimension ac-su3-left ac-g2) => 14
;; (sp-total-dimension ac-su3-right ac-f4) => 52

(defvar su3-singlets (sp-ac-singlets e8-fabc (map 'simple-array #'sp+ ac-su3-left ac-su3-right)
  e8-scalars)) ; 12

(defvar su3-singlets+so16 (sp-ac-singlets e8-fabc (map 'simple-array #'sp+ ac-su3-left ac-su3-right)
  e8-scalars+so16)) ; 22

;; Since this is a bit hard to do: alternative suggestion: study breaking to SU(3)xSU(3).

(defvar su33-singlets (sp-ac-singlets e8-fabc (concatenate 'list ac-su3-left ac-su3-right)
  e8-scalars)) ; 8

(defvar su33-singlets+so16 (sp-ac-singlets e8-fabc (concatenate 'list ac-su3-left ac-su3-right)
  e8-scalars+so16)) ; 16

(defvar su33-so16 (sp-ac-singlets e8-fabc (concatenate 'list ac-su3-left ac-su3-right) e8-so16)) ; 8

|#

;; Note: the derivative of the SU(3)^2 singlets 16 is also a 16, hence no U(1) factors.

(length (SP-LIN-INDEP-AC-COMMUTATORS e8-fabc su33-so16 su33-so16)) ; => 6

And these stay 6 under further derivative forming. Hence, SO(3)xSU(3).

|#

(setf a0 (SP-LIN-INDEP-AC-COMMUTATORS e8-fabc su33-so16 su33-so16))
(setf so3-a (list (sp+ (aref a0 0) '(-1 ,(aref a0 5)))
  (sp+ (aref a0 1) '(-1 ,(aref a0 4)))
  (sp+ (aref a0 2) (aref a0 3))))
(setf so3-b (list (sp+ (aref a0 0) (aref a0 5))
  (sp+ (aref a0 1) (aref a0 4))
  (sp+ (aref a0 2) '(-1 ,(aref a0 3)))))

(setf su21-a (sp-ac-commutator-closure e8-fabc
  (concatenate 'list so3-a
    (sp-lin-indep-ac-commutators e8-fabc so3-a su33-singlets))))
(setf su21-b (sp-ac-commutator-closure e8-fabc
  (concatenate 'list so3-b
    (sp-lin-indep-ac-commutators e8-fabc so3-b su33-singlets))))

(defvar v-rot-su33
  (let* ((rot-s (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-a 4) 's))
    (rot-z (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-b 4) 'z))
    (rot-sz (sp* rot-s rot-z)))

```

```

(rot-x1 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-a 0)) 'r1))
(rot-x2 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-a 1)) 'r2))
(rot-x3 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-a 2)) 'r3))
(rot-x4 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-b 0)) 'r4))
(rot-x5 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-b 1)) 'r5))
(rot-x6 (poexp-make-rot (sp-ac-to-generator e8-fabc (aref su21-b 2)) 'r6))
(rot-x123 (sp* rot-x1 (sp* rot-x2 rot-x3)))
(rot-x456 (sp* rot-x4 (sp* rot-x5 rot-x6)))
(rot-xsz (sp* rot-x123 (sp* rot-x456 rot-sz)))
(sp-map #'(lambda (x) (poexp* 1 x)) rot-xsz)))

(map 'simple-array
  #'sp-multiple-p
  (vector (aref su21-a 4) (aref su21-b 4)
    (aref su21-a 0) (aref su21-a 1) (aref su21-a 2)
    (aref su21-b 0) (aref su21-b 1) (aref su21-b 2)
    )
  (vector
    (sp-x '(e8)
      -1/2
      '(,ixmap-e8-so16-alpha*beta* e8 alpha* beta*)
      '(, (sp+ '(+1 , (sp-x '(alpha* beta*) '(, (sp-id 8) alpha* (:fix 1)) '(, (sp-id 8) beta* (:fix 7))))
        '(-1 , (sp-x '(alpha* beta*) '(, (sp-id 8) alpha* (:fix 7)) '(, (sp-id 2) beta* (:fix 1))))))
        alpha* beta*))
    (sp-x '(e8)
      -1/2
      '(,ixmap-e8-so16-alpha*beta* e8 alpha* beta*)
      '(, (sp+ '(+1 , (sp-x '(alpha* beta*) '(, (sp-id 8) alpha* (:fix 1)) '(, (sp-id 8) beta* (:fix 7))))
        '(+1 , (sp-x '(alpha* beta*) '(, (sp-id 8) alpha* (:fix 7)) '(, (sp-id 2) beta* (:fix 1))))))
        alpha* beta*))
    (sp-x '(e8)
      2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)
      '(,ixmap-so16-i16-i i16 i)
      '(,ixmap-so16-i16-i* j16 j)
      '(, (sp+ '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))
        '(-1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 7)) '(, (sp-id 8) j (:fix 6))))))
        i j))
    (sp-x '(e8)
      2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)
      '(,ixmap-so16-i16-i i16 i)
      '(,ixmap-so16-i16-i* j16 j)
      '(, (sp+ '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 6))))
        '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 7)) '(, (sp-id 8) j (:fix 7))))))
        i j))
    (sp-x '(e8)
      2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)
      '(, (sp+ '(+1 , (sp-x '(i16 j16)
        '(,ixmap-so16-i16-i i16 i)
        '(,ixmap-so16-i16-i j16 j)
        '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))
        '(-1 , (sp-x '(i16 j16)
        '(,ixmap-so16-i16-i* i16 i)
        '(,ixmap-so16-i16-i* j16 j)
        '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))))
        i16 j16))
    (sp-x '(e8)
      -2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)
      '(,ixmap-so16-i16-i i16 i)
      '(,ixmap-so16-i16-i* j16 j)
      '(, (sp+ '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))
        '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 7)) '(, (sp-id 8) j (:fix 6))))))
        i j))
    (sp-x '(e8)
      -2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)
      '(,ixmap-so16-i16-i i16 i)
      '(,ixmap-so16-i16-i* j16 j)
      '(, (sp+ '(+1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 6))))
        '(-1 , (sp-x '(i j) '(, (sp-id 8) i (:fix 7)) '(, (sp-id 8) j (:fix 7))))))
        i j))
    (sp-x '(e8)
      -2
      '(,ixmap-e8-so16-ij e8 ij)
      '(,ixmap-so16-ij-i-j ij i16 j16)

```

```

      '(, (sp+ '(+1 ,(sp-x '(i16 j16)
                          '(, (ixmap-so16-i16-i i16 i)
                              '(, (ixmap-so16-i16-i j16 j)
                                  '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))
          '(+1 ,(sp-x '(i16 j16)
                      '(, (ixmap-so16-i16-i* i16 i)
                          '(, (ixmap-so16-i16-i* j16 j)
                              '(, (sp-id 8) i (:fix 6)) '(, (sp-id 8) j (:fix 7))))))
        i16 j16))
    ))
;; => #(1 1 1 1 1 1 1 1)

(defvar phi-g2xf4 (poexp* 1 (e8-potential-from-v-theta v-rot-su33 theta-g2xf4)))
(defvar phi-g2xf4-so8 (poexp* 1 (e8-potential-from-v-theta v-rot-su33 (e8-theta-so-p 8))))
;; (defvar phi-g2xf4-so7 (poexp* 1 (e8-potential-from-v-theta v-rot-su33 (e8-theta-so-p 7))))

```

## B.6 The $E_{7(-5)} \times SU(2)$ potential

```

(eval-when (compile load eval)
  (require :e8-supergravity)
  (require :cartan-dynkin)
  (use-package :tf-spellbook)
  (use-package :lambdatensor))

(setf *bytes-consed-between-gcs* 40000000)

(defvar r+ (sp-find-root-operator e8-fabc e8-v-cartan-ac #(-1/2 -1/2 -1/2 -1/2 -1/2 -1/2 -1/2 -1/2)))
(defvar r- (sp-map #'conjugate r+))

(setf r1 (sp+ '(1/2 ,r+) '(1/2 ,r-)))
(setf r2 (sp+ '(#c(0 1/2) ,r+) '(#c(0 -1/2) ,r-)))
(setf r0 (e8-ac-[] r1 r2))

;; These indeed do close and form a SL(2).

;; Let us add #(-1/2 -1/2 -1/2 -1/2 1/2 1/2 1/2 1/2), which are also not in e7xsu2:

(defvar r2+ (sp-find-root-operator e8-fabc e8-v-cartan-ac #(-1/2 -1/2 -1/2 -1/2 1/2 1/2 1/2 1/2)))
(defvar r2- (sp-map #'conjugate r2+))

(defvar r21 (sp+ '(1/2 ,r2+) '(1/2 ,r2-)))
(defvar r22 (sp+ '(#c(0 1/2) ,r2+) '(#c(0 -1/2) ,r2-)))
(defvar r20 (e8-ac-[] r21 r22))

(defvar v-rot-sl2^2
  (let* ((r0-scaled (sp-scale r0 1/2)) ; then, [r0,r1]=-2r2, [r1,r2]=2r0, [r2,r0]=-2r1
        (r20-scaled (sp-scale r20 1/2))
        (rot-vs
         (sp* (poexp-make-rot (sp-ac-to-generator e8-fabc r0-scaled) 'v)
              (poexp-make-rot (sp-ac-to-generator e8-fabc r1) 's)))
        (rot-wz
         (sp* (poexp-make-rot (sp-ac-to-generator e8-fabc r20-scaled) 'w)
              (poexp-make-rot (sp-ac-to-generator e8-fabc r21) 'z))))
    (sp* rot-vs rot-wz)))

(setf theta-e7xsu2
  (let* ((8-4a (sp-generate-indexsplit-tensor 8 '(4)))
        (8-4b (sp-generate-indexsplit-tensor 8 '(4) :offset 4))
        (part-abcd
         (sp-x '(p a q b)
                1/2
                '(, (sp-id 8) p q)
                '(, (sp-id 8)
                   (sp-x '(a b) '(,so8-sigma-ijkl-ab (:fix 0) (:fix 1) (:fix 2) (:fix 3) a b))) a b)))
        (part-a*b*c*d*
         (sp-x '(p a q b)
                1/2
                '(, (sp-id 8) p q)
                '(, (sp-id 8)
                   (sp-x '(a b) '(,so8-sigma-ijkl-a*b* (:fix 0) (:fix 1) (:fix 2) (:fix 3) a b))) a b)))
        (part-ijkl
         (sp+ '(+1/2 ,(sp-x '(a8 b8 c8 d8)
                            '(, (sp-antisymmetrizer 4 2) a4 b4 c4 d4)
                            '(, (8-4a a8 a4) '(, (8-4a b8 b4) '(, (8-4a c8 c4) '(, (8-4a d8 d4))))
                            '(-1/2 ,(sp-x '(a8 b8 c8 d8)
                                       '(, (sp-antisymmetrizer 4 2) a4 b4 c4 d4)

```

```

      '(,8-4b a8 a4) '(,8-4b b8 b4) '(,8-4b c8 c4) '(,8-4b d8 d4)))
    '(+12 ,(sp-x '(i j k l) '(,(sp-antisymmetrizer 8 4) i j k l (:fix 0) (:fix 1) (:fix 2) (:fix 3))))))
  (part-i*j*k*l* (sp-scale (sp-antisymmetrizer 8 2) -1/2))
  (part-ij*k*l*
   (sp-x '(i j k l)
          -1
          '(,(sp-id 8) j l)
          '(,(sp-id 4) i4 k4)
          '(,8-4b i i4) '(,8-4b k k4))))
  (sp+
   (sp-x '(e8-a e8-b)
          '(,ixmap-e8-so16-alphabeta e8-a alpha beta)
          '(,ixmap-e8-so16-alphabeta e8-b gamma delta)
          '(,part-abcd alpha beta gamma delta))
   (sp-x '(e8-a e8-b)
          '(,ixmap-e8-so16-alpha*beta* e8-a alpha* beta*)
          '(,ixmap-e8-so16-alpha*beta* e8-b gamma* delta*)
          '(,part-a*b*c*d* alpha* beta* gamma* delta*))
   (sp-x '(e8-a e8-b)
          '(,ixmap-e8-so16-ij e8-a ij) '(,ixmap-e8-so16-ij e8-b kl)
          '(,ixmap-so16-ij-i-j ij i16 j16) '(,ixmap-so16-ij-i-j kl k16 l16)
          '(,ixmap-so16-i16-i i16 i) '(,ixmap-so16-i16-i j16 j)
          '(,ixmap-so16-i16-i k16 k) '(,ixmap-so16-i16-i l16 l)
          '(,part-ijkl i j k l))
   (sp-x '(e8-a e8-b)
          '(,ixmap-e8-so16-ij e8-a ij) '(,ixmap-e8-so16-ij e8-b kl)
          '(,ixmap-so16-ij-i-j ij i16 j16) '(,ixmap-so16-ij-i-j kl k16 l16)
          '(,ixmap-so16-i16-i* i16 i) '(,ixmap-so16-i16-i* j16 j)
          '(,ixmap-so16-i16-i* k16 k) '(,ixmap-so16-i16-i* l16 l)
          '(,part-i*j*k*l* i j k l))
   (sp-x '(e8-a e8-b)
          '(,ixmap-e8-so16-ij e8-a ij) '(,ixmap-e8-so16-ij e8-b kl)
          '(,ixmap-so16-ij-i-j ij i16 j16) '(,ixmap-so16-ij-i-j kl k16 l16)
          '(,ixmap-so16-i16-i i16 i) '(,ixmap-so16-i16-i* j16 j)
          '(,ixmap-so16-i16-i k16 k) '(,ixmap-so16-i16-i* l16 l)
          '(,part-ij*k*l* i j k l))))))

(defvar phi-e7xsu2 (e8-potential-from-v-theta v-rot-s12^2 theta-e7xsu2))

#!

* PHI-E7XSU2
#<poexp-term 39/4+1/4 cosh(4 z)+1/4 cosh(4 s)+6 cosh(2 s) cosh(2 z)-1/4 cosh(4 s) cosh(4 z)>

|#

(defvar v-stationary-s12^2
  (let* ((k (* 1/2 (log (+ 2 (sqrt 3.0d0)))))
         (rot (sp-exp
                (sp-change-arith
                 (sp-ac-to-generator e8-fabc (sp+ '(,k ,r1) '(,k ,r21)))
                 *sp-arith-double-float*))))
        (rot))

;; (e8-numeric-potential-all-derivs v-stationary-s12^2 theta-e7xsu2) => truly stationary.
;; A123 condition also shows this.

```

## B.7 The $D = 4$ $N = 8$ potential

Since this problem is so complex that it requires splitting of sub-problems, the code given here is especially instructive when machine limitations have to be overcome.

```

(eval-when (compile load eval)
  (progn
    (require :e7-supergravity)
    (use-package :tf-spellbook)
    (use-package :lamdatensor)))

(eval-when (compile load eval)
  (progn
    (setf lamdatensor::*poexp-vars*

```

```

    (map 'simple-array #'identity '(rho alpha
                                   f34 w34 f35 w35 f36 w36 f37 w37
                                   )))
  (setf (lambdatensor::sp-arith-converter lambdatensor::*sp-arith-poexp*)
        #'(lambda (x) (lambdatensor::poexp-converter x lambdatensor::*poexp-vars*)))
  ))

(setf *bytes-consed-between-gcs* 40000000)

(defvar e7-eta_ab (sp-x '(a b) 1/288 '(,e7-fabc a p q) '(,e7-fabc b q p)))
(defvar e7-eta^ab (sp-invert e7-eta_ab))

(defvar v-e7-so3-singlets-re-ac ; from spinors
  (map 'simple-array * (*))
  #'(lambda (n)
      (let* ((indices (append '(0 1 2) '(,n)))
             (fixation (mapcar #'(lambda (n) '(,n)) indices))
             (psi-s (sp-x '(e7-ac^
                           '(,e7-eta^ab e7-ac^ e7-ac_)
                           '(,ixmap-e7-35s e7-ac_ 35s)
                           '(,e7-35s 35s,@fixation))))
             (psi-s))
          '(3 4 5 6 7)))

(defvar v-e7-so3-singlets-im-ac ; from co-spinors
  (map 'simple-array * (*))
  #'(lambda (n)
      (let* ((indices (append '(0 1 2) '(,n)))
             (fixation (mapcar #'(lambda (n) '(,n)) indices))
             (psi-c (sp-x '(e7-ac^
                           '(,e7-eta^ab e7-ac^ e7-ac_)
                           '(,ixmap-e7-35c e7-ac_ 35c)
                           '(,e7-35c 35c,@fixation))))
             (psi-c))
          '(3 4 5 6 7)))

(defvar so8-T28-8-8 (sp-generate-so-split-tensor 8))

(defvar so3-in-e7-v-ac
  (map 'simple-array * (*))
  #'(lambda (so8-ij)
      (sp-x '(e7-ac
             '(,ixmap-e7-so8
              e7-ac (:fix ,(hv so8-ht-ij-i+j so8-ij))))))
  '((0 . 1) (0 . 2) (1 . 2)))

(defvar so3-in-e7 (map 'simple-array #'(lambda (x) (sp-ac-to-generator e7-Tabc x)) so3-in-e7-v-ac))
(defvar v-e7-so3-singlets-re (map 'simple-array #'(lambda (x) (sp-ac-to-generator e7-Tabc x)) v-e7-so3-singlets-re-ac))
(defvar v-e7-so3-singlets-im (map 'simple-array #'(lambda (x) (sp-ac-to-generator e7-Tabc x)) v-e7-so3-singlets-im-ac))

(defvar v-e7-Tab
  (map 'simple-array #'(lambda (n) (sp-x '(a b) '(,e7-Tabc (:fix ,n) a b))) (v-int-range 133)))

(defun angles-by-coeffs (&rest li-coeffs)
  (let* ((v-coeffs (coerce li-coeffs 'simple-array * (5))) ; error on wrong length.
         (phi3 (aref v-coeffs 0))
         (phi4 (aref v-coeffs 1))
         (phi5 (aref v-coeffs 2))
         (phi6 (aref v-coeffs 3))
         (phi7 (aref v-coeffs 4))
         ;;
         (f34 (* 0.5 (phase (/ phi4 phi3))))
         (ef34+ (cis f34))
         (ef34- (cis (- f34)))
         (phi4f (* ef34- phi4))
         (phi3f (* ef34+ phi3))
         (w34 (atan (/ phi4f phi3f)))
         (phi3f4 (/ phi3f (cos w34)))
         ;;
         (f35 (* 0.5 (phase (/ phi5 phi3f4))))
         (ef35+ (cis f35))
         (ef35- (cis (- f35)))
         (phi5f (* ef35- phi5))
         (phi3f4f (* ef35+ phi3f4))
         (w35 (atan (/ phi5f phi3f4f)))
         (phi3f4f5 (/ phi3f4f (cos w35)))
         ;;
         (f36 (* 0.5 (phase (/ phi6 phi3f4f5))))
         (ef36+ (cis f36))
         (ef36- (cis (- f36)))
         (phi6f (* ef36- phi6))
         (phi3f4f5f (* ef36+ phi3f4f5)))

```



```

(w36 (atan (/ phi6f phi3f4f5f)))
(phi3f4f5f6 (/ phi3f4f5f (cos w36)))
;;
(f37 (* 0.5 (phase (/ phi7 phi3f4f5f6))))
(ef37+ (cis f37))
(ef37- (cis (- f37)))
(phi7f (* ef37- phi7))
(phi3f4f5f6f (* ef37+ phi3f4f5f6))
(w37 (atan (/ phi7f phi3f4f5f6f)))
(phi3f4f5f6f7 (/ phi3f4f5f6f (cos w37)))
;;
)
(vector (- f34) (- w34) (- f35) (- w35) (- f36) (- w36) (- f37) (- w37) phi3f4f5f6f7)))

;; noninfinitesimal su8-so8 rotation

(defun su8-so8-rotation (alpha so8-j so8-k &key (arith *sp-arith-complex-double-float*) (converter (sp-arith-converter arith)))
  (let* ((mx (make-sp-array '(8 8) :arith arith))
        (ca
         (if (eq (sp-arith-name arith) :complex-double-float) ; special case needed, since we don't promote a number.
             (funcall converter (cos alpha))
             (funcall converter '(cos ,alpha))))
        (sa
         (if (eq (sp-arith-name arith) :complex-double-float)
             (funcall converter (sin alpha))
             (funcall converter '(sin ,alpha))))
        (-sa (funcall (sp-arith-sub arith) (funcall converter 0) sa))
        (c1 (funcall converter 1)))
    (dotimes (j 8) (sp-set! mx c1 j j)) ; init diag
    (sp-set! mx ca so8-j so8-j)
    (sp-set! mx ca so8-k so8-k)
    (sp-set! mx sa so8-j so8-k)
    (sp-set! mx -sa so8-k so8-j)
    mx))

;; diag(1,...,[pos-j:=exp(i*phase)],1,...,[pos-k:=exp(-i*phase)],1,...1)

(defun su8-f-rotation (phase j k &key (arith *sp-arith-complex-double-float*) (converter (sp-arith-converter arith)))
  (let* ((mx (make-sp-array '(8 8) :arith arith))
        (c1 (funcall converter 1)))
    (dotimes (n 8) (sp-set! mx c1 n n))
    (cond
     ((eq (sp-arith-name arith) :complex-double-float)
      (sp-set! mx (funcall converter (cis phase)) j j)
      (sp-set! mx (funcall converter (cis (- phase))) k k))
      (t
       (sp-set! mx (funcall converter '(exp (* ,(complex 0 1) ,phase))) j j)
       (sp-set! mx (funcall converter '(exp (* ,(complex 0 -1) ,phase))) k k)))
    mx))

;; combine all the alphas given by angles-by-coeffs into one big SU(8) rotation that does the miracle

(defun full-su8-tf-rotation (vec-alphas &key symbolic (how-many 10))
  (mv-bind (arith converter)
    (if symbolic
        (values *sp-arith-poexp* #'(lambda (term) (fp-naive-to-poexp (convert-to-fp term) *poexp-vars*)))
        (values *sp-arith-complex-double-float* #'(lambda (x) (coerce x '(complex double-float))))))
    (apply #'sp*
      '(
        ,@(if (> how-many 2) (list (su8-f-rotation (aref vec-alphas 0) 3 4 :arith arith :converter converter)) nil)
        ,@(if (> how-many 3) (list (su8-so8-rotation (aref vec-alphas 1) 3 4 :arith arith :converter converter)) nil)
        ,@(if (> how-many 4) (list (su8-f-rotation (aref vec-alphas 2) 3 5 :arith arith :converter converter)) nil)
        ,@(if (> how-many 5) (list (su8-so8-rotation (aref vec-alphas 3) 3 5 :arith arith :converter converter)) nil)
        ,@(if (> how-many 6) (list (su8-f-rotation (aref vec-alphas 4) 3 6 :arith arith :converter converter)) nil)
        ,@(if (> how-many 7) (list (su8-so8-rotation (aref vec-alphas 5) 3 6 :arith arith :converter converter)) nil)
        ,@(if (> how-many 8) (list (su8-f-rotation (aref vec-alphas 6) 3 7 :arith arith :converter converter)) nil)
        ,@(if (> how-many 9) (list (su8-so8-rotation (aref vec-alphas 7) 3 7 :arith arith :converter converter)) nil))))))

;; We must be able to lift SU(8) rotations to E7.
;; This we have to do via a function, not a tensor,
;; since the result is quadratic in the arg.

;; Note: this version is quite slow. Hence, we try a different
;; approach further below, setting tensor indices directly.

(defun e7-su8-rotations (su8)
  (let* ((arith (sp-array-arith su8))
        (mult (sp-arith-mult arith))
        (converter (sp-arith-converter arith))
        (one (funcall converter 1))
        (add (sp-arith-add arith))
        (minus-one (funcall converter -1)))

```

```

(one-half (funcall converter 1/2))
(conj (sp-arith-conj arith))
(su8-conj (sp-map #'(lambda (x) (funcall conj x)) su8))
(anti8 (sp-antisymmetrizer 8 2))
(rotation
  (sp+
    '(+1/2 ,(sp-x '(a56 b56)
      '(,ixmap-56-28a a56 a28)
      '(,ixmap-56-28a b56 b28)
      '(,so8-T28-8-8 a28 a8k a8l)
      '(,so8-T28-8-8 b28 b8i b8j)
      '(,su8 a8k b8i)
      '(,su8 a8l b8j))))
    '(1/2 ,(sp-x '(a56 b56)
      '(,ixmap-56-28b a56 a28)
      '(,ixmap-56-28b b56 b28)
      '(,so8-T28-8-8 a28 a8p a8r)
      '(,so8-T28-8-8 b28 b8q b8s)
      '(,su8-conj a8p b8q)
      '(,su8-conj a8r b8s))))))
(inv-rotation ; adjoint = hermitian conjugate
  (sp+
    '(1/2 ,(sp-x '(a56 b56)
      '(,ixmap-56-28a a56 a28)
      '(,ixmap-56-28a b56 b28)
      '(,so8-T28-8-8 a28 a8k a8l)
      '(,so8-T28-8-8 b28 b8i b8j)
      '(,su8-conj b8i a8k)
      '(,su8-conj b8j a8l))))
    '(1/2 ,(sp-x '(a56 b56)
      '(,ixmap-56-28b a56 a28)
      '(,ixmap-56-28b b56 b28)
      '(,so8-T28-8-8 a28 a8p a8r)
      '(,so8-T28-8-8 b28 b8q b8s)
      '(,su8 b8q a8p)
      '(,su8 b8s a8r))))))
(cons rotation inv-rotation)))

;; As is explicitly demonstrated by test examples,
;; this is equivalent to the function above, but works faster.

(defun e7-su8-rotations-quick (su8)
  (let* ((arith (sp-array-arith su8))
        (mult (sp-arith-mult arith))
        (converter (sp-arith-converter arith))
        (one (funcall converter 1))
        (add (sp-arith-add arith))
        (minus-one (funcall converter -1))
        (one-half (funcall converter 1/2))
        (conj (sp-arith-conj arith))
        )
    (let ((rotation (make-sp-array '(56 56) :arith arith))
          (inv-rotation (make-sp-array '(56 56) :arith arith)))
      (labels
        ((build-rotation (rotation su8)
          (sp-do
            #'(lambda (pos1 val1)
              (sp-do
                #'(lambda (pos2 val2)
                  (if (and (i/= (aref pos1 0) (aref pos2 0))
                        (i/= (aref pos1 1) (aref pos2 1)))
                    (mv-bind (ixpair1 factor1)
                      (if (i< (aref pos1 0) (aref pos2 0))
                          (values (hv so8-ht-ij-i+j (cons (aref pos1 0) (aref pos2 0))) one)
                              (values (hv so8-ht-ij-i+j (cons (aref pos2 0) (aref pos1 0))) minus-one))
                      (mv-bind (ixpair2 factor2)
                        (if (i< (aref pos1 1) (aref pos2 1))
                            (values (hv so8-ht-ij-i+j (cons (aref pos1 1) (aref pos2 1))) one)
                                (values (hv so8-ht-ij-i+j (cons (aref pos2 1) (aref pos1 1))) minus-one))
                          (let* ((contrib (funcall mult one-half factor1 factor2 val1 val2))
                                ; 1/2 is nontrivial multiplicity/symmetrization factor.
                                (conj-contrib (funcall conj contrib))
                                (ixpair1+28 (i+ 28 ixpair1))
                                (ixpair2+28 (i+ 28 ixpair2)))
                            (sp-set! rotation
                              (funcall add contrib
                                (sp-ref rotation ixpair1 ixpair2))
                              ixpair1 ixpair2)
                            (sp-set! rotation
                              (funcall add conj-contrib
                                (sp-ref rotation ixpair1+28 ixpair2+28))
                              ixpair1+28 ixpair2+28))))))))))))))
      su8))

```

```

    su8)))
  (build-rotation rotation su8)
  (build-rotation inv-rotation (sp-map conj (sp-x '(b a) '(,su8 a b))))
  (cons rotation inv-rotation)
  ))))

;; Our "start singlet" is also the first singlet from v-e7-so3-singlets-re-ac:

(defvar phi3s-ac
  (sp-x '(e7-ac^
    '(,e7-eta^ab e7-ac^ e7-ac_)
    '(,ixmap-e7-35s e7-ac_ 35s)
    '(,e7-35s 35s ,@(mapcar #'(lambda (x) '(:fix ,x)) '(0 1 2 3))))))

(defvar phi3c-ac
  (sp-x '(e7-ac^
    '(,e7-eta^ab e7-ac^ e7-ac_)
    '(,ixmap-e7-35c e7-ac_ 35c)
    '(,e7-35c 35c ,@(mapcar #'(lambda (x) '(:fix ,x)) '(0 1 2 3))))))

(defvar phi3s (sp-scale (sp-ac-to-generator e7-Tabc phi3s-ac) 1/16))
(defvar phi3c (sp-scale (sp-ac-to-generator e7-Tabc phi3c-ac) 1/16))
(defvar phi3v (sp-scale (sp-[] phi3s phi3c) 2))
;; without the extra factor 2, all three generators would share the same normalization,
;; but it makes sense for exponentiation.

(defvar phi3s-num (sp-change-arith phi3s *sp-arith-complex-double-float*)
  (defvar phi3c-num (sp-change-arith phi3c *sp-arith-complex-double-float*)
  (defvar phi3v-num (sp-change-arith phi3v *sp-arith-complex-double-float*))

;; Smallest positive angle for rotation=identity for phi3v-num = 2pi

;; Note that due to the occurrence of imaginary entries in these
;; matrices, we have to generate these rotations by hand.
;;
;; In our conventions, phi3v is quite simple: it is a submatrix of id*1/2,
;; but we can only use this isofar as to do simple naive sin/cos exponentiation.

(defvar phi3-sl2-symbolic
  (let* ((phi3v-po (sp-change-arith phi3v *sp-arith-poexp*))
    (sin-alpha-po (fp-naive-to-poexp (convert-to-fp '(sin (* 1/2 alpha))) *poexp-vars*))
    (sin-alpha-po (fp-naive-to-poexp (convert-to-fp '(sin (* -1/2 alpha))) *poexp-vars*))
    (cos-alpha-1-po (poexp* -1 (poexp+ (fp-naive-to-poexp (convert-to-fp '(cos (* 1/2 alpha))) *poexp-vars*) -1)))
    (sin-part (sp-scale phi3v-po sin-alpha-po))
    (sin*-part (sp-scale phi3v-po sin-alpha*-po))
    (cos-1-part (sp-scale (sp* phi3v-po phi3v-po) cos-alpha-1-po))
    (r-alpha (sp+ sin-part cos-1-part (sp-id 56 :arith *sp-arith-poexp*)))
    (r-alpha* (sp+ sin*-part cos-1-part (sp-id 56 :arith *sp-arith-poexp*)))
    (r-rho (poexp-make-rot phi3s 'rho))
    (sp* r-alpha* r-rho r-alpha)
  ))

;; Note that the last two components are arg and abs of the noncompact direction!

(defun e7-10-potential-numeric-angular-direct (v)
  (let* ((rotations (e7-su8-rotations-quick (full-su8-tf-rotation v)))
    (arg-alpha (aref v 8))
    (abs-alpha (aref v 9)) ; XXX N.W. would have an extra 1/sqrt2 factor in there!
    (alpha (* abs-alpha (cis arg-alpha)))
    (rot-phi3s (sp* (car rotations)
      (sp-exp (sp+ '(, (realpart alpha) ,phi3s-num)
        '(, (imagpart alpha) ,phi3c-num)))
      (cdr rotations))))
    (e7-potential-from-v rot-phi3s)))

(defvar e7-10-v56-full
  (let ((rotations (e7-su8-rotations-quick
    (full-su8-tf-rotation #(f34 w34 f35 w35 f36 w36 f37 w37) :symbolic t :how-many 10))))
    (sp* (car rotations)
      phi3-sl2-symbolic
      (cdr rotations))))

;; Make use of the invariance of alpha:

(defvar e7-10-v56
  (let ((rotations (e7-su8-rotations-quick
    (full-su8-tf-rotation #(f34 w34 f35 w35 f36 w36 f37 w37) :symbolic t :how-many 10)))
    (phi3-sl2-noalpha (poexp-make-rot phi3s 'rho))
    (sp* (car rotations)
      phi3-sl2-noalpha

```



```

;; By default, symbols from the LAMBDATENSOR package are not re-serialized. Change that.
(setf (hv *SERIALIZATION-ALLOWED-PACKAGES* "LAMBDATENSOR") t)

;; Another unfortunate thing: we do not have a readdir() in LISP -
;; hence, had to hard-code the following pathnames:

(defvar a1-files
  '(
    "#(0 0)"
    "#(1 1)"
    "#(2 2)"
    "#(3 3)_0-1580630x0-1580630"
    "#(3 3)_0-1580630x158-3160630"
  ...))

(defvar a2-files
  '(
    "#(0 0 3 4)"
    "#(0 0 3 5)"
    "#(0 0 3 6)"
    "#(0 0 3 7)"
  ...))

(defvar a1^2
  (let ((sum 0))
    (dolist (f a1-files)
      (format t "re-serializing ~S%" f)
      (let ((z (re-serialize (qcat (format nil "/tmp/e7-tensor-pieces/a1/~A" f))))))
        (setf sum (poexp+ sum z))))
    (poexp* sum 1)))

(defvar a2^2
  (let ((sum 0))
    (dolist (f a2-files)
      (format t "re-serializing ~S%" f)
      (let ((z (re-serialize (qcat (format nil "/tmp/e7-tensor-pieces/a2/~A" f))))))
        (setf sum (poexp+ sum z))))
    (poexp* sum 1)))

(defvar e7-so3-potential
  (poexp+ (poexp* 1/24 a2^2) (poexp* -3/4 a1^2)))

#|

And indeed:

(poexp-assoc-eval e7-so3-potential '((f34 . 0) (f35 . 0) (f36 . 0) (f37 . 0) (w34 . 0)
                                     (w35 . 0) (w36 . 0) (w37 . 0) (rho . 0)))
=> #C(-6.0d0 0.0d0)

(e7-10-potential-numeric-angular-direct (to-vdf #(1.2 2.1 0.3 0.5 0.7 2.4 1.3 2.2 0.2 1.0)))
=> #C(-7.062007741314236d0 0.0d0)

(poexp-assoc-eval e7-so3-potential '((f34 . 1.2d0) (w34 . 2.1d0) (f35 . 0.3d0) (w35 . 0.5d0)
                                     (f36 . 0.7d0) (w36 . 2.4d0) (f37 . 1.3d0) (w37 . 2.2d0)
                                     (rho . 1.0d0)))
=> #C(-7.0619974280259505d0 0.0d0)

|#

```



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