

# Spin-2 fields on Minkowski space near space-like and null infinity.

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## Abstract

We show that the spin-2 equations on Minkowski space in the gauge of the ‘regular finite initial value problem at space-like infinity’ imply estimates which, together with the transport equations on the cylinder at space-like infinity, allow us to obtain for a certain class of initial data information on the behaviour of the solution near space-like and null infinity of any desired precision.

# 1 Introduction

In a recent article ([1]) Chruściel and Delay have shown the existence of non-trivial solutions to Einstein's vacuum field equations which satisfy Penrose's condition of asymptotic simplicity ([8], [9]) with a prescribed smoothness of the asymptotic structure. The basic step in [1] consists in the construction of a specific class of asymptotically flat Cauchy data of prescribed smoothness on  $\mathbb{R}^3$  which coincide with Schwarzschild data outside a fixed radius and which can be chosen with this property arbitrarily close to Minkowski data. Since the evolution of the data is Schwarzschild near space-like infinity, one has perfect control on the asymptotic structure near space-like infinity and one can construct hyperboloidal data arbitrarily close to Minkowskian hyperboloidal data. A general result of [3] on the hyperboloidal initial value problem then implies the existence of the desired space-times.

While this result finally settles a question which raised some controversy and remained open for forty years, the technique used in [1] precludes the possibility to resolve the main open question concerning asymptotic simplicity: how 'large' is the class of asymptotically simple solutions, or, more precisely, how can this class be characterized in terms of Cauchy data? The answers to this question requires a general and very detailed analysis of the behaviour of the solutions in the domain where 'null infinity touches space-like infinity' and the data constructed in [1] are designed precisely to circumvent this task.

A basic step towards answering this question has been taken in [5], using the general conformal representation of Einstein's vacuum field equations introduced in [4]. This allows us to employ a gauge based on conformally invariant structures, which simplifies the analysis of the equations and of the underlying conformal geometry. It allows us in particular to control, in the given gauge, the location of null infinity in terms of the initial data. Under suitable assumptions on the initial data it has been shown that the standard Cauchy problem can be reformulated to obtain a 'regular finite initial value problem near space-like infinity'. Since this requires the complete information on the asymptotic structure of the initial data near space-like infinity, the data have been assumed in [5] to be time-symmetric. Recent results on non-time-symmetric data [2] will make it possible to extend the analysis of [5] to more general space-times.

In the regular finite initial value problem near space-like infinity, the initial hypersurface  $S$  is a three-manifold diffeomorphic to a closed ball in  $\mathbb{R}^3$  (possibly with its center removed). Its spherical boundary, denoted by  $I^0$ , represents space-like infinity for the initial data on  $S$ . With respect to the solution space-time space-like infinity is represented by a cylinder  $I$  diffeomorphic to  $] -1, 1[ \times I^0$ , which intersects  $S$  at  $I^0$ .

There are coordinates  $\rho \geq 0$  with  $\rho = 0$  on  $I$  and  $\tau$  with  $\tau = 0$  on  $S$ . The range of  $\tau$  is limited by continuous functions  $\tau_{\pm}(p)$ ,  $p \in S$ , with  $\tau_- \leq -1$ ,  $\tau_+ \geq 1$  and  $\tau_{\pm}(p) \rightarrow \pm 1$  as  $p \rightarrow I^0$ , such that the sets  $\mathcal{J}^{\pm} = \{\tau = \tau_{\pm}, \rho > 0\}$  represent future and past null infinity. These sets 'touch' the cylinder  $I$  at the two components  $I^{\pm} = \{\tau = \pm 1, \rho = 0\}$  of its boundary, which are diffeomorphic to  $I^0$ .

While the reduced equations are symmetric hyperbolic on the ‘physical’ part of the underlying manifold, where  $\rho > 0$ , and extend with this property to the cylinder  $I$ , they develop a degeneracy at the spherical sets  $I^\pm$ . It turns out that this degeneracy is not a deficiency of our formulation. As argued in the following it rather indicates in a precise way certain innate features of the generic asymptotic structure. Understanding the consequences of this degeneracy is the main open problem of the subject of asymptotics. Its resolution will provide a host of detailed information on asymptotically flat space-times, new insight into the field equations, and it will open the door for various theoretical and practical applications.

The cylinder  $I$  represents a boundary of the physical manifold. However, it is not a boundary for the conformal field equations in the sense that one could prescribe boundary data on  $I$ . The evolution of the unknown  $u$  in the conformal field equations, which comprises frame and connection coefficients and the components of the non-physical Ricci tensor and the rescaled conformal Weyl tensor, is governed on  $I$  by an intrinsic system of *transport equations* induced on  $I$  by the conformal field equations. In fact, for suitably chosen smooth initial data, the complete system of derivatives  $u^p = \partial_\rho^p u|_I$ ,  $p = 0, 1, 2, \dots$ , is determined by intrinsic symmetric hyperbolic transport equations on  $I$  and corresponding initial data on  $I^0$ .

It has been shown in [5] that the transport equations can be solved explicitly to arbitrary order, provided certain algebraic complexities can be handled (e.g. with an algebraic computer program). In [7] it has been demonstrated that our analysis can be related to earlier analyses of null infinity and that the functions  $u^p$  supply interesting information on the space-time near space-like and null infinity. We expect that under suitable assumptions on the initial data the setting developed in [5] will allow us to obtain perfect control on the asymptotic structure of space-time.

The explicit calculation of certain components of the functions  $u^p$  on  $I$  shows that, in general, logarithmic singularities of the form  $(1 - \tau)^k \log^j(1 - \tau)$  will occur in the  $u^p$  near  $I^+$ , where  $\tau \rightarrow 1$ . These singularities can even be observed for data which are analytic at space-like infinity. It turns out, however, that they vanish for data which satisfy a certain set of ‘regularity conditions’ (cf. [5] for details and [6] for a discussion of the related conjecture concerning the solution space-time and an associated subconjecture concerning the smoothness of the functions  $u^p$  on  $\bar{I} = I \cup I^+ \cup I^-$ ).

Due to the hyperbolicity of the reduced equations, the tendency of the quantities  $u^p$  to become singular at  $I^\pm$  is likely to spread along characteristics sufficiently close to null infinity, thus destroying the possibility to define an asymptotic structure at null infinity of the desired smoothness if the regularity conditions are not satisfied to a sufficiently high order. But even if the latter are satisfied at all orders and if it can be shown that they imply the smoothness of the functions  $u^p$  on  $\bar{I}$ , the decision about the existence of a smooth extension of the solution to null infinity is still complicated because the degeneracy of the equations at  $I^\pm$  interferes with the known techniques to deduce estimates for solutions of symmetric hyperbolic systems.

This requires us to go beyond the general theory and to look for specific features of the conformal field equations which might allow us to overcome this difficulty.

The degeneracy occurs in the part of the reduced equations which is deduced from the Bianchi equation  $\nabla_\mu d^\mu{}_{\nu\lambda\rho} = 0$  for the rescaled conformal Weyl tensor  $d^\mu{}_{\nu\lambda\rho} = \Theta^{-1} C^\mu{}_{\nu\lambda\rho}$ . All the other equations remain regular near  $I^\pm$ . The type of degeneracy remains if the equations are linearized at Minkowski space in the specific gauge near space-like infinity introduced in [5]. We are thus led to analyse the spin-2 equation on Minkowski space in this particular gauge. It is the purpose of the present article to discuss this situation. We shall show that in the given setting the spin-2 equations imply certain estimates, which, combined with the transport equations on the cylinder at space-like infinity, yield any desired information on the asymptotic behaviour of the solution (cf. the relations (23), (24)). This result on the solution will not be stated as a theorem (cf. also the remarks following (24)), because our main interest here lies in the nature of argument and in the underlying particularities of the setting and the equations. We expect that arguments of this type will also help us tackle the quasi-linear problem.

## 2 Minkowski space near space-like and null infinity

Let  $y^\mu$  be coordinates on Minkowski space in which the metric takes the standard form  $\tilde{g} = \eta_{\mu\nu} dy^\mu dy^\nu$  with  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . We want to prescribe initial data for the spin-2 equation on the hypersurface  $\tilde{S} = \{y^0 = 0\}$  and study the behaviour of the solution on the conformal extension of Minkowski space in a neighbourhood of space-like infinity which includes a part of null infinity.

To discuss the field on such a neighbourhood, we perform on  $N = \{y_\mu y^\mu < 0\}$  the coordinate transformation  $y^\mu \rightarrow x^\mu = -\frac{y^\mu}{y_\nu y^\nu}$  and formally extend the domain of validity of the coordinates  $x^\mu$  to include the part  $\mathcal{J}'$  of the set  $\{x_\mu x^\mu = 0\}$  adjacent to  $N$ . In the new coordinates, or in the associated spatial polar coordinates with radial coordinate  $\rho = \sqrt{\sum_{\mu=1}^3 (x^\mu)^2}$ , the metric takes the forms

$$\tilde{g} = \frac{1}{(x_\lambda x^\lambda)^2} \eta_{\mu\nu} dx^\mu dx^\nu = \frac{1}{(\rho^2 - (x^0)^2)^2} ((dx^0)^2 - d\rho^2 - \rho^2 d\sigma^2),$$

where  $d\sigma^2$  denotes the standard line element on the 2-sphere. Introducing the conformal factor  $\Omega = -x_\lambda x^\lambda$  on  $N$  we find that  $\Omega$  and  $g' \equiv \Omega^2 \tilde{g}$  extend smoothly to the set  $\mathcal{J}'$ . The conformal factor  $\Omega$  vanishes there but  $g'$  remains regular. The point  $i^0 = \{x^\mu = 0\}$  then represents spacelike infinity and the sets  $\mathcal{J}^\pm = \mathcal{J}' \cap \{x_\mu x^\mu = 0, \pm x^0 > 0\}$  represent parts of future resp. past null infinity of Minkowski space.

We can reconstruct this representation of Minkowski space by solving the conformal field equations with data on the set  $S' = \{x^0 = 0\}$  which are given

by the intrinsic 3-metric  $h = -(d\rho^2 + \rho^2 d\sigma^2)$  and the second fundamental form  $\chi = 0$  induced by  $g'$ , by the conformal factor  $\Omega = \rho^2$ , and by certain fields derived from  $h$  and  $\Omega$ . If we slightly perturb  $h$  and  $\chi$  now (keeping the conformal constraints satisfied) to obtain more general solutions, the rescaled conformal Weyl tensor will develop a singularity at the point  $i = \{x^\mu = 0\}$  in  $S'$  as soon as the data acquire a non-vanishing ADM-mass. The following gauge arose from the desire to analyse this situation (cf. [5]).

To define a different scaling of the metric and a new coordinate  $\tau$  we choose a function  $\kappa = \rho\mu$ , where  $\mu$  is a smooth positive function of  $\rho \in \mathbb{R}$  satisfying  $\mu(0) = 1$ , and set  $x^0 = \tau\kappa$ . With the conformal factor

$$\Theta = \frac{\rho}{\mu} (1 - \tau^2 \mu^2) = \frac{1}{\kappa} \Omega, \quad (1)$$

we then find

$$g \equiv \Theta^2 \tilde{g} = \frac{1}{\kappa^2} \left( \kappa^2 d\tau^2 + 2\tau\kappa\kappa' d\tau d\rho - (1 - \tau^2 \kappa'^2) d\rho^2 - \rho^2 d\sigma^2 \right). \quad (2)$$

With  $\rho$  and  $\tau$  and suitable spherical coordinates we have  $N = \{\rho > 0, -\frac{1}{\mu(\rho)} < \tau < \frac{1}{\mu(\rho)}\}$  and  $\mathcal{J}^\pm = \{\rho > 0, \tau = \pm \frac{1}{\mu(\rho)}\}$ . We set  $I = \{\rho = 0, |\tau| < 1\}$ ,  $I^0 = \{\rho = 0, \tau = 0\}$ , and denote by  $I^\pm = \{\rho = 0, \tau = \pm 1\}$  the sets where  $I$  ‘touches’  $\mathcal{J}^\pm$ . It is understood here that  $I$  is diffeomorphic to  $[-1, 1] \times S^2$ ,  $I^0$  and  $I^\pm$  are diffeomorphic to  $S^2$  and that the spherical coordinates extend as smooth coordinates to  $I$  and  $I^\pm$ . Finally, we set  $S = \tilde{S} \cup I^0$ ,  $\bar{N} = N \cup \mathcal{J}^+ \cup \mathcal{J}^- \cup \bar{I}$  with  $\bar{I} = I \cup I^+ \cup I^-$  and consider  $\rho, \tau$  and the spherical coordinates as coordinates on  $\bar{N} \simeq [-1, 1] \times [0, \infty[ \times S^2$ . In these coordinates the expressions above for  $\Theta$  and  $\Omega = \kappa\Theta$  extend smoothly to  $\bar{N}$  and the coordinate expression for  $g$  extends smoothly and without degeneracy to  $N \cup \mathcal{J}^+ \cup \mathcal{J}^-$  while it becomes singular on  $\bar{I}$ . In this new representation the set  $I^0$  corresponds to  $i$  and with respect to  $N$  space-like infinity is represented now by the cylinder  $I$ , which can be regarded as a kind of ‘blow-up’ of the point  $i^0$ . Note that the differential structure defined here near  $I$  is completely different from the differential structure defined by the coordinates  $x^\mu$  near  $i^0$ .

To write out the spin-2 equation  $\nabla^f_{a'} \phi_{abcf} = 0$  we introduce a pseudo-orthonormal frame  $c_{aa'}$  satisfying  $g(c_{aa'}, c_{bb'}) = \epsilon_{ab} \epsilon_{a'b'}$  and  $\bar{c}_{aa'} = c_{aa'}$ . As real null vector fields we choose

$$c_{00'} = \frac{1}{\sqrt{2}} \{(1 - \kappa' \tau) \partial_\tau + \kappa \partial_\rho\}, \quad c_{11'} = \frac{1}{\sqrt{2}} \{(1 + \kappa' \tau) \partial_\tau - \kappa \partial_\rho\},$$

on  $\bar{N} \setminus \bar{I}$ . We note that with these conventions the (linearized) radiation field on  $\mathcal{J}^+$  and the null data on the outgoing null hypersurfaces tangent to  $c_{11'}$  correspond to the components  $\phi_0 \equiv \phi_{0000}$  and  $\phi_4 \equiv \phi_{1111}$  respectively.

The vectors  $c_{01'}$ ,  $c_{01'}$  are then necessarily tangent to the spheres  $\tau, \rho = \text{const.}$  and can not define smooth global vector fields. Since this leads to various awkward expressions, all fields  $c_{01'}$ ,  $c_{01'}$  satisfying the normalization condition above will be considered. We thus define a 5-dimensional subbundle of the

bundle of frames with structure group  $U(1)$  which projects onto  $\bar{N} \setminus \bar{I}$ , the projection corresponding to the standard Hopf map  $SU(2) \rightarrow SU(2)/U(1) \simeq S^2$ . We lift all our structures on  $\bar{N} \setminus \bar{I}$  to this subbundle and keep the notation used before.

Allowing  $\rho$  to take the value 0, we extend everything, including the projection. We consider thus  $\tau, \rho$ , and  $s \in SU(2)$  as ‘coordinates’ on the extended subbundle, which we denote again by  $\bar{N}$ . The lifted conformal factor and the lifted metric are again given by (1) and (2), with  $d\sigma^2$  denoting the pull back of the line element on  $S^2$  to  $SU(2)$ . Thus  $\bar{N}$  is diffeomorphic to  $] -1, 1[ \times ] 0, \infty[ \times SU(2)$  now and the sets  $I, I^\pm, \mathcal{J}^\pm$  etc. are now considered as subsets of  $\bar{N}$  which are defined by the same conditions on  $\tau$  and  $\rho$  as before.

We consider the basis

$$u_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (3)$$

of the Lie algebra of  $SU(2)$  with commutation relations  $[u_i, u_j] = \epsilon_{ijk} u_k$  and denote by  $Z_i, i = 1, 2, 3$ , the (real) left invariant vector field generated by  $u_i$  on the (real) Lie group  $SU(2)$ . We consider  $Z_3$  as the vertical vector field which generates the group  $U(1)$  acting on the fibres of  $\bar{N}$  and set  $X = -2iZ_3$ . Defining the complex conjugate vector fields  $X_\pm = -(Z_2 \pm iZ_1)$  and setting

$$c_{01'} = -\frac{1}{\sqrt{2}} \mu X_+, \quad c_{01''} = -\frac{1}{\sqrt{2}} \mu X_-,$$

we obtain smooth vector fields  $c_{aa'}, Z_3$  on  $\bar{N} \setminus \bar{I}$ , which extend smoothly to  $\bar{I}$  and satisfy  $g(Z_3, \cdot) = 0$ , and  $g(c_{aa'}, c_{bb'}) = \epsilon_{ab} \epsilon_{a'b'}$  on  $\bar{N} \setminus \bar{I}$ .

The connection form induced on  $\bar{N} \setminus \bar{I}$  defines connection coefficients with respect to  $c_{aa'}$ , which take the values

$$\Gamma_{00'01} = \Gamma_{11'01} = -\frac{1}{2\sqrt{2}}(\mu + \rho\mu'), \quad \Gamma_{01'11} = \Gamma_{10'00} = \frac{1}{\sqrt{2}}\rho\mu',$$

$$\Gamma_{00'00} = \Gamma_{00'11} = \Gamma_{01'00} = \Gamma_{01'01} = \Gamma_{10'01} = \Gamma_{10'11} = \Gamma_{11'00} = \Gamma_{11'11} = 0,$$

and extend smoothly to  $\bar{I}$ .

Making use of the conformal covariance of the spin-2 equations, we write them in the new conformal gauge in terms of the frame and the connection coefficients above and obtain

$$0 = (1 + (\mu + \rho\mu')\tau) \partial_\tau \phi_k - \rho \mu \partial_\rho \phi_k + \mu X_+ \phi_{k+1} + ((2-k)\mu + 3\rho\mu') \phi_k, \quad (4)$$

$$0 = (1 - (\mu + \rho\mu')\tau) \partial_\tau \phi_{k+1} + \rho \mu \partial_\rho \phi_{k+1} + \mu X_- \phi_k + ((1-k)\mu - 3\rho\mu') \phi_{k+1}, \quad (5)$$

for  $k = 0, 1, 2, 3$ , where we set as usual  $\phi_0 = \phi_{0000}, \phi_1 = \phi_{1000}$ , etc. Notice that the coefficients of these equations are defined and smooth for all values of  $\tau, \rho$  and  $s$  and the equations thus make sense and are regular on  $\bar{N}$ .

These equations can be obtained immediately from equations given in [5] by linearizing the latter, in the given gauge, at Minkowski space. The underlying construction, which may appear rather arbitrary here, has a geometrical background for which we refer the reader to [5]. In particular, the curves on which  $\rho$  and  $s$  are constant are conformal geodesics with parameter  $\tau$ , the set  $I$  is attained as a limit set of these curves, and the field  $c_{aa'}$  are parallelly propagated along these curves with respect to a certain Weyl connection for  $g$ .

### 3 The spin-2 equations near space-like and null infinity

Equations (4), (5) imply the equivalent system of evolution equations

$$0 = (1 + \kappa' \tau) \partial_\tau \phi_0 - \kappa \partial_\rho \phi_0 + \mu X_+ \phi_1 + (2\mu + 3\rho\mu') \phi_0,$$

$$0 = 2 \partial_\tau \phi_1 + \mu X_+ \phi_2 + \mu X_- \phi_0 + 2\mu \phi_1,$$

$$0 = 2 \partial_\tau \phi_2 + \mu X_+ \phi_3 + \mu X_- \phi_1$$

$$0 = 2 \partial_\tau \phi_3 + \mu X_+ \phi_4 + \mu X_- \phi_2 - 2\mu \phi_3,$$

$$0 = (1 - \kappa' \tau) \partial_\tau \phi_4 + \kappa \partial_\rho \phi_4 + \mu X_- \phi_3 - (2\mu + 3\rho\mu') \phi_4,$$

and the constraints

$$0 = \kappa' \tau \partial_\tau \phi_1 - \kappa \partial_\rho \phi_1 + \frac{\mu}{2} (X_+ \phi_2 - X_- \phi_0) + 3\rho\mu' \phi_1,$$

$$0 = \kappa' \tau \partial_\tau \phi_2 - \kappa \partial_\rho \phi_2 + \frac{\mu}{2} (X_+ \phi_3 - X_- \phi_1) + 3\rho\mu' \phi_2,$$

$$0 = \kappa' \tau \partial_\tau \phi_3 - \kappa \partial_\rho \phi_3 + \frac{\mu}{2} (X_+ \phi_4 - X_- \phi_2) + 3\rho\mu' \phi_3.$$

The latter reduce to interior equations and thus imply conditions on the data on  $S = \{\tau = 0\} \subset \bar{N}$ . We shall not study them in any detail here.

For the following discussion it is convenient to assume that  $\mu' < 0$  for  $\rho > 0$ . The evolution equations are then symmetric hyperbolic on the set  $\bar{N} \setminus (I^+ \cup I^-)$ . Writing them in the form  $A^\mu \partial_\mu \phi = B \phi$ , where  $\phi$  denotes a column vector with entries  $\phi_k$ , we get  $A^\tau = \text{diag}(1 + \kappa' \tau, 2, 2, 2, 1 - \kappa' \tau)$ . Thus, while being positive definite on  $\bar{N} \setminus (I^+ \cup I^-)$  and ensuring by this the hyperbolicity of the evolution equations, the matrix  $A^\tau$  loses this property on  $I^\pm$  and it does so for any choice of  $\mu$  as above. It will be seen below that this renders the standard

energy estimates useless at  $I^\pm$ . This degeneracy represents the central problem of our discussion.

Data for the linear spin-2 equations which are smooth on the hypersurface  $\{y^0 = 0\}$  of Minkowski space are known to develop into a smooth solution on Minkowski space and thus imply smooth fields  $\phi_k$  on  $N$  in the gauge above. Whether these fields extend smoothly to  $I$  and  $\mathcal{J}^\pm$  clearly depends on the behaviour of the data near space-like infinity. Linearizing the data considered in [5] at Minkowski space, we obtain a class of non-trivial data  $\phi_k$  which extend in our gauge smoothly to all of  $S$ . The data in [5] were chosen to be time-symmetric. Linearizing similarly the data for the non-linear equations obtained from those discussed in [2] provides a large class of non-time-symmetric data for the linear spin-2 equation which also extend smoothly to  $I^0$ .

The equations above extend smoothly across  $I$  into a domain where  $\rho < 0$  and remain symmetric hyperbolic there. Extending the data on  $S$  smoothly into a region of  $\{\tau = 0\}$  in which  $\rho < 0$ , not necessarily observing the constraints there, and evolving the data with the extended equations, we find that the solution on  $N$  extends smoothly to  $I$ . Because of the degeneracy of  $A^\tau$  the standard theory for symmetric hyperbolic systems does not allow us to derive statements about the behaviour of the solutions at  $I^\pm$  and, as a consequence, also not on  $\mathcal{J}^\pm$ .

A detailed inspection of the equations and the data shows, that the degeneracy at  $I^\pm$  can have consequences for the asymptotic smoothness of the solutions. It is an important feature of the evolution equations above that the matrix  $A^\rho$  vanishes on  $I$  and the system thus implies an intrinsic system of transport equations on  $I$  which determines the solution on  $I$  in terms of the data implied by  $\phi$  on  $I^0$ . In fact, applying formally the operators  $\partial_\rho^p$  to the equations and restricting to the set  $I$  we find that the functions  $\phi^p = \partial_\rho^p \phi|_I$  satisfy a system of linear symmetric hyperbolic transport equations for  $p = 0, 1, 2, \dots$ . By expanding the functions  $\phi^p$  in a suitable function system on  $SU(2)$  the equations can be reduced to systems of ODE's which can be solved explicitly. The solutions can be read off directly from the results in [5]. The details of this will not be important for the following discussion. What is important, though, is the fact that one obtains solutions which are regular (in fact, polynomial) in  $\tau$  but besides those for  $p \geq 2$  also solutions which are singular at  $I^\pm$ . The latter are of the form

$$\left(\frac{1-\tau}{2}\right)^{p-k+2} \left(\frac{1+\tau}{2}\right)^{p+k-2} I_k^p(\tau)$$

where

$$I_k^p(\tau) \equiv \int_0^\tau \frac{d\sigma}{(1-\sigma)^{p-k+3}(1+\sigma)^{p+k-1}}$$

with constants of integration  $e_k, f_k$  which are determined by the data on  $S$ . Expanding the integral one finds that these solutions behave like

$$(1-\tau)^{p-k+2} (1+\tau)^{p+k-2} \log(1-\tau) + \textit{analytic}$$

as  $\tau \rightarrow 1$ . Note that with increasing  $p$  the singularity gets less severe. The solutions thus develop in general logarithmic singularities at  $I^\pm$ . These singularities can be expected to spread along the characteristics  $\mathcal{J}^\pm$  of the evolution equations.

Obviously, the occurrence of these singularities depends on the constants  $e_k, f_k$ . In [5] certain ‘regularity conditions’ on the time symmetric data have been derived. In terms of the free data given there, i.e. a conformal metric  $h$  on  $\tilde{S} = \mathbb{R}^3$  which is assumed to extend smoothly if the latter is suitably compactified in the form  $\tilde{S} \rightarrow S' = \tilde{S} \cup \{i\} \simeq S^3$  by adding a point  $i$  at space-like infinity, these conditions read

$$D_{(a_q b_q} \dots D_{a_1 b_1} b_{abcd})(i) = 0, \quad q = 0, 1, 2, \dots, q_*, \quad (6)$$

where we employ the space spinor notation and  $b_{abcd}$  denotes the Cotton spinor of  $h$ . We expect that for given  $p_*$  condition (6) ensures the non-existence of logarithmic singularities in the  $u^p$  for  $p = 0, 1, \dots, p_*$  if it is satisfied for sufficiently large  $q_*$ . From the calculation in [5] it follows that this statement is true in the linearized setting considered above, if (6) is replaced by its linearization at the Minkowski data  $h = -(d\rho^2 + \rho^2 d\sigma^2)$ ,  $\chi = 0$ . In this case the  $\phi^p$  extend smoothly, in fact as polynomials in  $\tau$ , to  $I^\pm$ .

However, the degeneracy of  $A^\tau$  there still does not allow us to draw conclusions about the smoothness of the solutions on  $\mathcal{J}^\pm$  by applying standard techniques for symmetric hyperbolic systems.

For the following discussion it will be convenient to use a more specific gauge. We set  $\mu \equiv 1$  such that  $\mathcal{J}^\pm = \{\tau = \pm 1, \rho > 0\}$ . The evolution equations then take the form

$$0 = E_0 \equiv (1 + \tau) \partial_\tau \phi_0 - \rho \partial_\rho \phi_0 + X_+ \phi_1 + 2 \phi_0, \quad (7)$$

$$0 = E_1 \equiv 2 \partial_\tau \phi_1 + X_- \phi_0 + X_+ \phi_2 + 2 \phi_1, \quad (8)$$

$$0 = E_2 \equiv 2 \partial_\tau \phi_2 + X_- \phi_1 + X_+ \phi_3, \quad (9)$$

$$0 = E_3 \equiv 2 \partial_\tau \phi_3 + X_- \phi_2 + X_+ \phi_4 - 2 \phi_3, \quad (10)$$

$$0 = E_4 \equiv (1 - \tau) \partial_\tau \phi_4 + \rho \partial_\rho \phi_4 + X_- \phi_3 - 2 \phi_4, \quad (11)$$

and the constraints read

$$0 = C_1 \equiv \tau \partial_\tau \phi_1 - \rho \partial_\rho \phi_1 - \frac{1}{2} X_- \phi_0 + \frac{1}{2} X_+ \phi_2, \quad (12)$$

$$0 = C_2 \equiv \tau \partial_\tau \phi_2 - \rho \partial_\rho \phi_2 - \frac{1}{2} X_- \phi_1 + \frac{1}{2} X_+ \phi_3, \quad (13)$$

$$0 = C_2 \equiv \tau \partial_\tau \phi_3 - \rho \partial_\rho \phi_3 - \frac{1}{2} X_- \phi_2 + \frac{1}{2} X_+ \phi_4. \quad (14)$$

Equations (7), (11) now degenerate everywhere on the sets  $\{\tau = \pm 1, \rho \geq 0\}$ . However, for  $\rho > 0$  this happens only because we have chosen the coordinate  $\tau$  to be constant on the characteristics  $\mathcal{J}^\pm$ .

Two other classes of characteristics which will be important for us. They are given by

$$C_-^{\rho_*} = \{(1 + \tau) \rho = \rho_*, \rho > 0, |\tau| < 1\}, \quad C_+^{\rho_*} = \{(1 - \tau) \rho = \rho_*, \rho > 0, |\tau| < 1\},$$

where  $\rho_*$  is given positive number. These correspond to spherically symmetric outgoing and ingoing null hypersurfaces in Minkowski space. As  $\rho_* \rightarrow 0$ , the sets  $C_\pm^{\rho_*}$  approach the sets  $\bar{I} \cup \mathcal{J}^\pm$  respectively in a limit which is non-uniform near the sets  $I^\pm$ .

If data for (7) to (11) are prescribed on the subset  $S_{\rho_1, \rho_2} = \{\rho_1 < \rho < \rho_2\}$  of  $S$ , with  $0 < \rho_1 < \rho_2$ , the solution will in the domain where  $\tau > 0$  be determined uniquely on the open set  $D_{\rho_1, \rho_2}^+ \subset N$  which is bounded below by  $S$ , on the left hand side by  $C_+^{\rho_1}$ , and on the right hand side by  $C_-^{\rho_2}$ . It follows that solutions of the extended equations for data on  $S$  which are smoothly extended through  $I^0$  depend on  $\bar{N}$  only on the data on  $S$ .

Let  $\phi_k$  be a solution of (7) to (11) on  $D_{\rho_1, \rho_2}^+$ . Considering  $E_k = E_k[\phi_j]$  as operators acting on the  $\phi_j$ , a direct calculation using the commutation relations of the fields  $Z_i$  shows that  $E_j[X\phi_k - 2(2-k)\phi_k] = 0$  on  $D_{\rho_1, \rho_2}^+$ , for  $j, k = 0, \dots, 4$ . It follows that  $X\phi_k = 2(2-k)\phi_k$  hold on  $D_{\rho_1, \rho_2}^+$  if these equations are satisfied on  $S_{\rho_1, \rho_2}$ , i.e. the evolution equations preserve the spin weights. Under the same assumptions a further direct calculation gives on  $D_{\rho_1, \rho_2}^+$  the equations

$$\begin{aligned} \partial_\tau C_1 + \frac{1}{2} X_+ C_2 + C_1 &= 0, \\ \partial_\tau C_2 + \frac{1}{2} X_+ C_3 + \frac{1}{2} X_- C_1 &= 0, \\ \partial_\tau C_3 + \frac{1}{2} X_- C_2 - C_3 &= 0, \end{aligned}$$

in which the operator  $\partial_\rho$  does not occur. This system is symmetric hyperbolic and implies that the quantities  $C_1, C_2, C_3$  vanish on  $D_{\rho_1, \rho_2}^+$  if this is the case on  $S_{\rho_1, \rho_2}$ , i.e. the evolution equations preserve the constraints.

## 4 Structure of solutions near space-like and null infinity

We shall assume in the following that the fields  $\phi_k$  represent a smooth solution of (7) to (11) in  $N$  which arises from smooth data on  $\tilde{S}$  which satisfy there the

constraints and have the correct spin weights. Assuming furthermore that the data extend smoothly to  $I^0$ , we can assume by the arguments above that the  $\phi_k$  provide in fact a smooth solution of (7) to (14) on the set  $\{|\tau| < 1, \rho \geq 0\} \subset \bar{N}$ . The standard argument to derive energy estimates for (7) to (11) proceeds as follows. A direct calculation gives

$$0 = \sum_{k=0}^4 (\bar{\phi}_k E_k + \phi_k \bar{E}_k) \quad (15)$$

$$\begin{aligned} &= \partial_\tau ((1 + \tau) |\phi_0|^2 + 2 |\phi_1|^2 + 2 |\phi_2|^2 + 2 |\phi_3|^2 + (1 - \tau) |\phi_4|^2) + \partial_\rho (-\rho |\phi_0|^2 + \rho |\phi_4|^2) \\ &\quad + X_- (\phi_0 \bar{\phi}_1 + \phi_1 \bar{\phi}_2 + \phi_2 \bar{\phi}_3 + \phi_3 \bar{\phi}_4) + X_+ (\phi_1 \bar{\phi}_0 + \phi_2 \bar{\phi}_1 + \phi_3 \bar{\phi}_2 + \phi_4 \bar{\phi}_3) \\ &\quad + 4 (|\phi_0|^2 + |\phi_1|^2 - |\phi_3|^2 - |\phi_4|^2). \end{aligned}$$

For  $t \in [0, 1]$  and  $\rho_* > 0$  we set

$$N_t = \{0 \leq \tau \leq t, 0 \leq \rho \leq \frac{\rho_*}{1 + \tau}, s \in SU(2)\},$$

$$S_t = \{\tau = t, 0 \leq \rho \leq \frac{\rho_*}{1 + t}, s \in SU(2)\},$$

$$B_t = \{0 \leq \tau \leq t, \rho = \frac{\rho_*}{1 + \tau}, s \in SU(2)\}, \quad I_t = \{0 \leq \tau \leq t, 0 = \rho, s \in SU(2)\}.$$

Choose now  $t$  with  $0 \leq t < 1$ . If (15) is integrated over  $N_t$  with respect to  $d\tau d\rho d\mu$ , where  $d\mu$  denotes the normalized Haar measure on  $SU(2)$ , the terms involving the left invariant operators  $X_\pm$  give no contribution (cf. the proof of lemma 5.1) and an application of Gauss' law gives

$$\begin{aligned} 0 &= \int_{S_t} ((1 + t) |\phi_0|^2 + 2 |\phi_1|^2 + 2 |\phi_2|^2 + 2 |\phi_3|^2 + (1 - t) |\phi_4|^2) d\rho d\mu \\ &\quad - \int_{S_0} (|\phi_0|^2 + 2 |\phi_1|^2 + 2 |\phi_2|^2 + 2 |\phi_3|^2 + |\phi_4|^2) d\rho d\mu \\ &\quad + \int_{I_t} (\rho |\phi_0|^2 - \rho |\phi_4|^2) d\tau d\mu \\ &\quad + \int_{B_t} \{(n_\tau ((1 + \tau) |\phi_0|^2 + 2 |\phi_1|^2 + 2 |\phi_2|^2 + 2 |\phi_3|^2 + (1 - \tau) |\phi_4|^2) \end{aligned}$$

$$\begin{aligned}
& +n_\rho (-\rho |\phi_0|^2 + \rho |\phi_4|^2) \} dv d\mu \\
& +4 \int_{N_t} (|\phi_0|^2 + |\phi_1|^2 - |\phi_3|^2 - |\phi_4|^2) d\tau d\rho d\mu,
\end{aligned}$$

where  $n_\tau = \nu \rho$  and  $n_\rho = \nu (1 + \tau)$ , with a suitable positive normalizing factor  $\nu$ , denote the components of the conormal to  $B_t$  and  $dv d\mu$  is the volume element induced on  $B_t$ . Since the  $\phi_k$  are smooth on  $N_t$  the integral over  $I_t$  vanishes, the integral over  $B_t$  is non-negative and we get

$$\begin{aligned}
(1-t) \int_{S_t} (|\phi_0|^2 + 2|\phi_1|^2 + 2|\phi_2|^2 + 2|\phi_3|^2 + |\phi_4|^2) d\rho d\mu & \quad (16) \\
\leq \int_{S_0} (|\phi_0|^2 + 2|\phi_1|^2 + 2|\phi_2|^2 + 2|\phi_3|^2 + |\phi_4|^2) d\rho d\mu \\
4 \int_{N_t} (|\phi_0|^2 + 2|\phi_1|^2 + 2|\phi_2|^2 + 2|\phi_3|^2 + |\phi_4|^2) d\tau d\rho d\mu,
\end{aligned}$$

which implies by the Gronwall argument

$$\begin{aligned}
& \int_{S_t} (|\phi_0|^2 + 2|\phi_1|^2 + 2|\phi_2|^2 + 2|\phi_3|^2 + |\phi_4|^2) d\rho d\mu \\
& \leq (1-t)^{-5} \int_{S_0} (|\phi_0|^2 + 2|\phi_1|^2 + 2|\phi_2|^2 + 2|\phi_3|^2 + |\phi_4|^2) d\rho d\mu.
\end{aligned}$$

Estimates obtained along these lines are good enough to show the existence of solutions to (7) to (11) for  $\tau$  in a given range  $0 \leq \tau < t_*$  with a fixed  $t_* < 1$ , but they give little information on the behaviour of the solutions near  $\tau = 1$ . The estimate (16) could perhaps be somewhat refined, however, this would not remove the basic difficulty of this type of estimate.

To derive sharp results about the smoothness of the solution near  $\mathcal{J}^\pm \cup I^\pm$  we shall make use of the specific properties of the spin-2 equations in our setting such as their overdeterminedness, the specific structure of the coefficients of the equations, and the existence of transport equations on  $I$ .

The spin-2 equations (4), (5) take in the present gauge the form

$$0 = A_k \equiv (1 + \tau) \partial_\tau \phi_k - \rho \partial_\rho \phi_k + X_+ \phi_{k+1} + (2 - k) \phi_k, \quad (17)$$

$$0 = B_k \equiv (1 - \tau) \partial_\tau \phi_{k+1} + \rho \partial_\rho \phi_{k+1} + X_- \phi_k + (1 - k) \phi_{k+1}, \quad (18)$$

where  $k = 0, \dots, 3$ .

With the operators  $D^{q,p,\alpha} = \partial_\tau^q \partial_\rho^p Z^\alpha$ , where  $Z^\alpha$  denote the operators introduced in lemma (5.1), the equation

$$\begin{aligned} & \overline{D^{q,p,\alpha} \phi_k} D^{q,p,\alpha} A_k + D^{q,p,\alpha} \phi_k \overline{D^{q,p,\alpha} A_k} \\ & + \overline{D^{q,p,\alpha} \phi_{k+1}} D^{q,p,\alpha} B_k + D^{q,p,\alpha} \phi_{k+1} \overline{D^{q,p,\alpha} B_k} = 0, \end{aligned}$$

can be written

$$\begin{aligned} & \partial_\tau((1+\tau)|D^{q,p,\alpha}\phi_k|^2) + \partial_\tau((1-\tau)|D^{q,p,\alpha}\phi_{k+1}|^2) \tag{19} \\ & - \partial_\rho(\rho|D^{q,p,\alpha}\phi_k|^2) + \partial_\rho(\rho|D^{q,p,\alpha}\phi_{k+1}|^2) \\ & + Z^\alpha X_+(\partial_\tau^q \partial_\rho^p \bar{\phi}_k) Z^\alpha (\partial_\tau^q \partial_\rho^p \phi_{k+1}) + Z^\alpha (\partial_\tau^q \partial_\rho^p \bar{\phi}_k) Z^\alpha X_+(\partial_\tau^q \partial_\rho^p \phi_{k+1}) \\ & + Z^\alpha X_-(\partial_\tau^q \partial_\rho^p \phi_k) Z^\alpha (\partial_\tau^q \partial_\rho^p \bar{\phi}_{k+1}) + Z^\alpha (\partial_\tau^q \partial_\rho^p \phi_k) Z^\alpha X_-(\partial_\tau^q \partial_\rho^p \bar{\phi}_{k+1}) \\ & - 2(p-q+k-2)|D^{q,p,\alpha}\phi_k|^2 + 2(p-q-k+1)|D^{q,p,\alpha}\phi_{k+1}|^2 = 0. \end{aligned}$$

The numerical factors in the last two terms, which result from the specific structure of the differential operators in (17), (18), will play a crucial role in the following. Their signs can be suitably adjusted by the choices of  $p$  and  $q$ .

Integration of (19) over  $N_t$  with respect to  $d\tau d\rho d\mu$  gives with Gauss' law

$$\begin{aligned} & \int_{S_t} ((1+t)|D^{q,p,\alpha}\phi_k|^2 + (1-t)|D^{q,p,\alpha}\phi_{k+1}|^2) d\rho d\mu \\ & - \int_{S_0} (|D^{q,p,\alpha}\phi_k|^2 + |D^{q,p,\alpha}\phi_{k+1}|^2) d\rho d\mu \\ & + \int_{I_t} \rho (|D^{q,p,\alpha}\phi_k|^2 - |D^{q,p,\alpha}\phi_{k+1}|^2) d\tau d\mu \\ & + \int_{B_t} \{ (n_\tau(1+\tau) - n_\rho \rho) |D^{q,p,\alpha}\phi_k|^2 + (n_\tau(1-\tau) + n_\rho \rho) |D^{q,p,\alpha}\phi_{k+1}|^2 \} d v d\mu \\ & + \int_{N_t} \{ Z^\alpha X_+(\partial_\tau^q \partial_\rho^p \bar{\phi}_k) Z^\alpha (\partial_\tau^q \partial_\rho^p \phi_{k+1}) + Z^\alpha (\partial_\tau^q \partial_\rho^p \bar{\phi}_k) Z^\alpha X_+(\partial_\tau^q \partial_\rho^p \phi_{k+1}) \\ & + Z^\alpha X_-(\partial_\tau^q \partial_\rho^p \phi_k) Z^\alpha (\partial_\tau^q \partial_\rho^p \bar{\phi}_{k+1}) + Z^\alpha (\partial_\tau^q \partial_\rho^p \phi_k) Z^\alpha X_-(\partial_\tau^q \partial_\rho^p \bar{\phi}_{k+1}) \} d\tau d\rho d\mu \end{aligned}$$

$$\begin{aligned}
& -2(p - q + k - 2) \int_{N_t} |D^{q,p,\alpha} \phi_k|^2 d\tau d\rho d\mu \\
& + 2(p - q - k + 1) \int_{N_t} |D^{q,p,\alpha} \phi_{k+1}|^2 d\tau d\rho d\mu = 0.
\end{aligned}$$

It follows again that the integral over  $B_t$  is non-negative and the integral over  $I_t$  vanishes. For given non-negative integers  $m$  and  $p$  summation now yields in view of lemma (5.1)

$$\begin{aligned}
& (1+t) \sum_{q'+p'+|\alpha|\leq m} \int_{S_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \\
& + (1-t) \sum_{q'+p'+|\alpha|\leq m} \int_{S_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\rho d\mu \\
& + 2 \sum_{q'+p'+|\alpha|\leq m} (p' + p - q' - k + 1) \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \sum_{q'+p'+|\alpha|\leq m} (p' + p - q' + k - 2) \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu.
\end{aligned}$$

With  $p > m + 2$  it follows for  $k = 0, \dots, 3$  that

$$\begin{aligned}
& (1+t) \sum_{q'+p'+|\alpha|\leq m} \int_{S_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \\
& + (1-t) \sum_{q'+p'+|\alpha|\leq m} \int_{S_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\rho d\mu \\
& + 2(p - m - 2) \sum_{q'+p'+|\alpha|\leq m} \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu
\end{aligned}$$

$$+2(p+m+1) \sum_{q'+p'+|\alpha|\leq m} \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu,$$

and thus in particular, for  $k = 0, \dots, 3$ ,

$$\begin{aligned} & \int_{S_t} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu \quad (20) \\ & \leq \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\ & +2(p+m+1) \int_{\tau=0}^t \left( \int_{S_\tau} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu \right) d\tau, \end{aligned}$$

and also

$$\begin{aligned} & 2(p-m-2) \sum_{q'+p'+|\alpha|\leq m} \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_4)|^2 d\tau d\rho d\mu \quad (21) \\ & \leq \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_3)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_4)|^2) d\rho d\mu \\ & +2(p+m+1) \sum_{q'+p'+|\alpha|\leq m} \int_{N_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_3)|^2 d\tau d\rho d\mu. \end{aligned}$$

Inequality (20) implies for  $k = 0, \dots, 3$

$$\begin{aligned} & \int_{N_t} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu, \\ & \leq \frac{e^{2(p+m+1)t} - 1}{2(p+m+1)} \left\{ \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \right\}, \end{aligned}$$

which gives with (21) the estimate

$$\begin{aligned} & \int_{N_t} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_4)|^2 \right) d\tau d\rho d\mu \\ & \leq \frac{e^{2(p+m+1)t}}{2(p-m-2)} \left\{ \sum_{q'+p'+|\alpha|\leq m} \int_{S_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_3)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_4)|^2) d\rho d\mu \right\}, \end{aligned}$$

and thus finally, for  $k = 0, \dots, 4$  and  $p > m + 2$ ,

$$\begin{aligned} & \int_{N_t} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu \quad (22) \\ & \leq C \sum_{k=0}^4 \int_{S_0} \left( \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu. \end{aligned}$$

The constant  $C$  here depends on  $p$  and  $m$  but not on  $t \in [0, 1[$ . Since then the right hand side does not depend on  $t$  we find that the norms on the left hand side are uniformly bounded as  $t \rightarrow 1$ . Note that by using the evolution equations (7) to (11) we can express the Sobolev norm on the right hand side in terms of the initial data and their spatial derivatives in  $S$ .

Observing the nature of the boundaries of the sets  $N_t$  and the Sobolev embedding theorems, we have for  $0 < t \leq 1$  and  $j = 0, 1, \dots$ , a continuous embedding

$$H^{j+3}(\text{int}(N_t)) \rightarrow C^{j,\lambda}(N_t),$$

where  $H$  denotes a standard  $L^2$ -type Sobolev space and  $\lambda$  indicates a local Hölder condition of exponent  $\lambda$  with  $0 < \lambda \leq 1/2$ . The space  $C^{j,\lambda}(N_t)$  consists of functions in  $C^j(\text{int}(N_t))$  which, together with their derivatives of order  $\leq j$ , are locally Hölder continuous, bounded and uniformly continuous on the interior  $\text{int}(N_t)$  of the closed set  $N_t$  and thus extend together with the derivatives of order  $\leq j$  to continuous functions on  $N_t$ . Observing that our 5-dimensional setting is obtained by lifting a 4-dimensional setting we find that the condition on  $\lambda$  can be relaxed to  $0 < \lambda < 1$ .

By the estimate (22) it follows then that for given non-negative integer  $j$  we have

$$\partial_\rho^p \phi_k \in C^{j,\lambda}(N_1) \quad \text{for } p \geq j + 6,$$

which allows us to get by integration the representation

$$\phi_k = \sum_{p'=0}^{p-1} \frac{1}{p'!} \phi_k^{p'} \rho^{p'} + J^p(\partial_\rho^p \phi_k) \quad \text{on } N_1 \quad \text{for } p \geq j + 6, \quad (23)$$

where  $J$  denotes the operator  $f \rightarrow J(f) = \int_0^\rho f(\tau, r, s) dr$  and the functions  $\phi_k^{p'}(\tau, s) = \partial_\rho^{p'} \phi_k|_I$ , which are obtained by integrating the transport equations on  $I$ , are considered as being extended to  $N_1$  as  $\rho$ -independent functions.

Since then

$$\phi_k - \sum_{p'=0}^{p-1} \frac{1}{p'!} \phi_k^{p'} \rho^{p'} \in C^{j,\lambda}(N_1) \quad \text{for } p \geq j + 6, \quad (24)$$

for given  $j$ , we can control the behaviour of the solution near  $\mathcal{J}^\pm \cup I^\pm$  with arbitrary precision. In particular, if the linearization of (6) is satisfied at all orders, the functions  $\phi_k^{p'}$  are smooth on  $\bar{I}$  for all  $p' = 0, 1, 2, \dots$  and the  $\phi_k$  have a smooth extension to  $\mathcal{J}^\pm \cup I^\pm$ .

The situation cannot be expected to improve in the quasi-linear problem. Thus the expansion above suggests that logarithmic singularities will occur in general also in that case. This then says that we cannot find a finite representation of the Cauchy problem near space-like infinity which is more regular than the one obtained in [5].

After introducing suitable function spaces, the representation (23) can be used to derive estimates for the  $\phi_k$  also if logarithmic singularities are present.

We emphasize that our conclusion refers to a gauge where  $\mathcal{J}^+ = \{\tau = 1\}$ . If we had chosen  $\kappa = \rho \mu$  with  $\mu(\rho) = 1$  for  $0 \leq \rho < \rho_{**}$  but  $\mu' < 0$  for  $\rho > \rho_{**}$ , the simple representation (23) would not be valid on the slice  $\{\tau = 1\}$  for  $\rho > \rho_{**}$ . Where the coefficients in the equations will begin to differ from those of (17), (18) the whole string of quantities  $\partial_\rho^{p'} \phi_k$ ,  $p' < p$ , may enter the estimates and the argument needs to be modified.

In the present linear case the conclusions of (i) follows also by a closer inspection of the quantities  $\phi_k^p$  and the observation that due to the specific nature of the equations the sum in (24) does already define a solution (cf. [10] for more details). Our point here is that we found a type of argument which is based on features of the linearized equations which can also be identified in the non-linear setting of [5].

## 5 Appendix

The purpose of this appendix is to introduce a family of left invariant operators on  $SU(2)$  and to prove lemma (5.1). This implies a considerable simplification of our estimates.

Consider the (real) left invariant vector fields  $Z_i$  on  $SU(2)$ . For given multi-index  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  with non-negative integers  $\alpha_i$  we set  $\hat{Z}^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$ , with the understanding that  $\hat{Z}^\alpha = 1$  if  $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3 = 0$ , and consider it as a left invariant operator on the set of smooth function on  $SU(2)$ .

If we identify the infinitesimal algebra over  $\mathbb{R}$  generated by the operators  $Z_i$  with the universal enveloping algebra of  $su(2)$ , the operators  $\hat{Z}^\alpha$  are known to provide a basis of this algebra. For our purposes a different basis leads to a considerable simplification of our estimates. Writing  $\hat{Z}^\alpha$  in the form

$$Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3} = Z_{i_1} \dots Z_{i_{\alpha_1}} Z_{i_{\alpha_1+1}} \dots Z_{i_{\alpha_1+\alpha_2}} Z_{i_{\alpha_1+\alpha_2+1}} \dots Z_{i_{|\alpha|}},$$

we get from it by symmetrization and normalization the operator

$$Z'^\alpha = \frac{1}{\alpha_1! \alpha_2! \alpha_3!} \sum_{\pi \in S_m} Z_{\pi(i_1)} \dots Z_{\pi(i_{|\alpha|})} = \sum_{A=1}^{\frac{m!}{\alpha_1! \alpha_2! \alpha_3!}} \sigma_A,$$

where  $S_m$  denotes the symmetric group and the terms  $\sigma_A$  realize the  $\frac{m!}{\alpha_1! \alpha_2! \alpha_3!}$  possibilities to form out of  $\alpha_i$  (indistinguishable) operators  $Z_i$ ,  $i = 1, 2, 3$ , products of  $|\alpha|$  operators. For  $|\alpha| \leq 1$  we have  $Z'^\alpha = \hat{Z}^\alpha$ . Then equations of the form

$$\frac{m!}{\alpha_1! \alpha_2! \alpha_3!} \hat{Z}^\alpha = Z'^\alpha + \sum_{|\beta| < |\alpha|} a_\beta \hat{Z}^\beta = Z'^\alpha + \sum_{|\beta| < |\alpha|} c_\beta Z'^\beta,$$

hold with constant coefficients  $a_\beta$  and  $c_\beta$ . The first equation is obtained by commuting operators and observing the commutation relations, the second equation is obtained by symmetrizing and normalizing the lower order operators  $\hat{Z}^\beta$ . The operators  $Z'^\alpha$  thus also form a basis of the enveloping algebra of  $su(2)$ .

**Lemma 5.1** *With the normalizing factors  $f(\alpha) = c \sqrt{\frac{\alpha_1! \alpha_2! \alpha_3!}{|\alpha|!}}$ , where  $c$  is a fixed positive constant, the operators  $Z^\alpha = f(\alpha) Z'^\alpha$  satisfy for any smooth complex-valued functions  $f, g$  on  $SU(2)$  for  $k = 1, 2, 3$  the equation*

$$\sum_{|\alpha|=m} (Z^\alpha Z_k f Z^\alpha g + Z^\alpha f Z^\alpha Z_k g) = Z_k \left( \sum_{|\alpha|=m} Z^\alpha f Z^\alpha g \right). \quad (25)$$

*In particular, if  $d\mu$  denotes the normalized Haar measure on  $SU(2)$ ,*

$$\sum_{|\alpha|=m} \int_{SU(2)} (Z^\alpha X_\pm f Z^\alpha g + Z^\alpha f Z^\alpha X_\pm g) d\mu = 0. \quad (26)$$

**Proof:** We consider  $Z'^\alpha Z_1 = (\sum \sigma_A) Z_1$  and study what happens if we commute  $Z_1$  successively with the factors generating the  $\sigma_A$ 's to obtain  $Z_1 Z'^\alpha$ . Each commutation of  $Z_1$  with a factor  $Z_2$  in one of the  $\sigma_A$ 's generates a transition  $Z_2 \rightarrow [Z_2, Z_1] = -Z_3$  and thus a term  $-\sigma'_B$  of  $-Z'^{(\alpha_1, \alpha_2-1, \alpha_3+1)}$ . The number of terms in  $Z'^{(\alpha_1, \alpha_2-1, \alpha_3+1)}$  created by the complete commutation process is then  $\frac{\alpha_2 |\alpha|!}{\alpha_1! \alpha_2! \alpha_3!}$ . Conversely, each term  $-\sigma'_B$  of  $-Z'^{(\alpha_1, \alpha_2-1, \alpha_3+1)}$  can be generated by the commutation process from precisely  $\alpha_3 + 1$  different terms in  $Z'^\alpha$  (there are  $\alpha_3 + 1$  possibilities to replace in  $\sigma'_B$  one of the  $Z_3$ 's by a  $Z_2$ ). The number  $\frac{(\alpha_3+1) |\alpha|!}{\alpha_1! (\alpha_2-1)! (\alpha_3+1)!}$  of terms  $-\sigma'_B$  thus obtained agrees with the number above. It follows that each term of  $-Z'^{(\alpha_1, \alpha_2-1, \alpha_3+1)}$  is generated precisely  $\alpha_3 + 1$  times. An analogous consideration concerning the commutations of  $Z_1$  with factors  $Z_3$ , which generate transitions  $Z_3 \rightarrow [Z_3, Z_1] = Z_2$  and thus terms of  $Z'^{(\alpha_1, \alpha_2+1, \alpha_3-1)}$ , shows that each term of  $Z'^{(\alpha_1, \alpha_2+1, \alpha_3-1)}$  is generated precisely  $\alpha_2 + 1$  times. Similar results are obtained for the commutators of  $Z'^\alpha$  with  $Z_2$  and  $Z_3$ .

Setting now  $Z^\alpha = g(\alpha) Z'^\alpha$  with an as yet undetermined normalizing factor  $g$ , we thus get the relations

$$Z^\alpha Z_1 - Z_1 Z^\alpha$$

$$\begin{aligned}
&= -\frac{(\alpha_3 + 1)g(\alpha)}{g(\alpha_1, \alpha_2 - 1, \alpha_3 + 1)}Z^{(\alpha_1, \alpha_2 - 1, \alpha_3 + 1)} + \frac{(\alpha_2 + 1)g(\alpha)}{g(\alpha_1, \alpha_2 + 1, \alpha_3 - 1)}Z^{(\alpha_1, \alpha_2 + 1, \alpha_3 - 1)}, \\
&\quad Z^\alpha Z_2 - Z_2 Z^\alpha \\
&= -\frac{(\alpha_1 + 1)g(\alpha)}{g(\alpha_1 + 1, \alpha_2, \alpha_3 - 1)}Z^{(\alpha_1 + 1, \alpha_2, \alpha_3 - 1)} + \frac{(\alpha_3 + 1)g(\alpha)}{g(\alpha_1 - 1, \alpha_2, \alpha_3 + 1)}Z^{(\alpha_1 - 1, \alpha_2, \alpha_3 + 1)}, \\
&\quad Z^\alpha Z_3 - Z_3 Z^\alpha \\
&= -\frac{(\alpha_2 + 1)g(\alpha)}{g(\alpha_1 - 1, \alpha_2 + 1, \alpha_3)}Z^{(\alpha_1 - 1, \alpha_2 + 1, \alpha_3)} + \frac{(\alpha_1 + 1)g(\alpha)}{g(\alpha_1 + 1, \alpha_2 - 1, \alpha_3)}Z^{(\alpha_1 + 1, \alpha_2 - 1, \alpha_3)}.
\end{aligned}$$

With these relations it follows by a direct calculation that the  $Z^\alpha$  satisfy equations (25) if

$$g(\alpha_1, \alpha_2 - 1, \alpha_3 + 1) = \sqrt{\frac{\alpha_3 + 1}{\alpha_2}} g(\alpha) \quad \alpha_2 \geq 1,$$

$$g(\alpha_1 + 1, \alpha_2, \alpha_3 - 1) = \sqrt{\frac{\alpha_1 + 1}{\alpha_3}} g(\alpha) \quad \alpha_3 \geq 1,$$

$$g(\alpha_1 - 1, \alpha_2 + 1, \alpha_3) = \sqrt{\frac{\alpha_2 + 1}{\alpha_1}} g(\alpha) \quad \alpha_1 \geq 1.$$

The normalizing factors  $f$  given in the lemma obey these rules.

If  $u$  is the generator of a left invariant vector field  $Z$  on  $SU(2)$ , then

$$\int_{SU(2)} Z f(s) d\mu(s) = \int_{SU(2)} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (f(s) - f(s \exp(\lambda u))) d\mu(s) = 0,$$

for any  $C^1$  function  $f$  on  $SU(2)$  because the measure is right invariant and we may commute the integration with taking the limit. Therefore (26) follows immediately from (25).

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