

Intersecting branes and 7-manifolds with G_2 holonomy¹

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Abstract: In this talk I discuss intersecting brane configurations coming from explicit metrics with G_2 holonomy. An example of a 7-manifold which representing a \mathbb{R}^3 bundle over a self-dual Einstein space is described and the potential appearing after compactification over the 6-d twistor space is derived.

1 Introduction

The compactification of heterotic string theory on a complex three-dimensional Calabi-Yau space was for a long time the standard way to derive $N = 1$ Super Yang Mills (SYM) in 4 dimensions (or even better the MSSM) from string- or M-theory. In this approach non-Abelian gauge groups as well as chiral matter appear very natural. On the other hand in the dual type II or M-theory picture non-Abelian gauge groups and chiral matter come from singularities or appear on the world volume of D-branes. Concrete $N = 1$ models could be constructed using intersecting branes at angles, where supersymmetry requires that the branes have to intersect at specific angles [1]. Examples are discussed in [2] and non-supersymmetric brane world models can be found in [3], with massless matter living on the common intersection of the branes. One can use different branes to build the brane world models, but especially interesting are the 6-branes which uplift to pure geometry in the 11-d M-theory picture. Assuming that the resulting 4-d external space is given by the Poincare invariant flat Minkowski space, the 11-d geometry becomes: $M_{11} = M_4 \times M_7$. Now, the amount of 4-d supersymmetry is directly related to the number (covariantly constant) Killing spinors on M_7 , which in turn reduces the holonomy of this space. The holonomy of a generic (orientable) 7-d space is given by $SO(7)$ and reduces to G_2 if the space allows for exactly one covariantly constant spinor. If the space allows for 2, 4 or even 8 Killing spinors the holonomy is further reduced to $SU(3)$, $SU(2)$ or becomes trivial, i.e. consist only of the identity. We will be interested in the case with only one Killing spinor and thus having a G_2 manifold we want to address the question of how one can obtain branes and the corresponding gauge group upon dimensional reduction, see also [4, 5].

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2 Branes from geometry

Dp-branes are extended BPS objects that are charged under a (p+1)-form potential. From the Kaluza-Klein reduction one can obtain only a 1-form gauge potential supporting either a particle (0-brane) or the Hodge-dual a co-dimension 3 brane. In 10 dimensions this is the D6-brane which is obtained by the dimensional reduction from $M_7 = R_3 \times X_{TN}$, where X_{TN} is the 4-d Taub-NUT space with the metric

$$ds^2 = \frac{1}{V}(d\chi + n \cos \theta d\varphi)^2 + V(dr^2 + r^2 d\Omega_2) , \quad V = 1 + \frac{n}{r} . \quad (1)$$

From this metric the 6-brane with charge n is obtained by dimensional reduction along the Killing vector $k = \partial_\chi$. Equivalently, a charge- n -6-brane can be seen as n 6-branes put on top of each other and hence there is a $U(n)$ world-volume gauge group. Keeping the 4π -periodicity of χ as for the single brane, this results in a conical singularity, which is nothing but the well known \mathbb{Z}_n orbifold singularity. For this simple case it is straightforward to extract the location and charge of the brane. In fact, often one defines the space in terms of orbifold actions, which gives a clear brane content (as orbifold fixed points). Sometimes however, one wants to read-off the location and charge of the 6-branes directly from a given 7-d metric where the explicit orbifold action is unclear. Of course, branes appear only after dimensional reduction along a given Killing vector k and one finds the identification

$$\begin{aligned} \text{location of 6-branes} & \quad \text{---} \quad \text{fixed point set } L \text{ (i.e. } |k|^2 = 0 \text{) of codimension four} \\ \text{charge of 6-branes} & \quad \text{---} \quad \text{(inverse) surface gravity } \kappa \text{ of } L \text{ (} \kappa^2 = |\nabla k|^2 \text{) .} \end{aligned}$$

For the case discussed above one finds $|k|^2 = \frac{1}{V}$ and $|\nabla k|^2 = \frac{1}{n} + \mathcal{O}(r)$ and the relations are identical fulfilled. It is a well-known fact from black hole solutions that a non-vanishing surface gravity (=non-vanishing Hawking temperature) corresponds to case where the Killing vector is compact (periodic Euclidean time) and the fixed point set is at finite geodesic distance. In the extreme limit the surface gravity vanishes, the Killing vector becomes non-compact or a translational isometry and the fixed point moves to infinity. For the simple 6-brane configuration from above, there is no regular extreme limit, but note, that if $\kappa = 1$ there is no conical singularity and the geometry is locally equivalent to the flat space. Since n is an integer, one can resolve the conical singularity by going to the n -center solution and because $g_{\chi\chi}$ vanishes at each center, between each two centers appear a non-trivial 2-cycle and the resolution of the singularity can also be understood as the blowing up of these 2-cycles. This is of course well-known for 6-branes living in flat space, where the corresponding multicenter solution exists, but it may be not possible for a more general manifold with G_2 holonomy (due to the lack of the corresponding moduli).

Let us also mention, that in addition to co-dimension four singularities, which are interpreted as 6-branes, there can also appear singularities of co-dimension two and six. In general their interpretation is less clear. For the example that we will discuss below, co-dimension two singularities do not appear and the co-dimension 6 objects are T-dual to NS5-branes.

We can also extract the 10-d quantities in detail. Having a Killing vector $k = \partial_\chi$ we write the 11-d metric as usual

$$ds^2 = e^{\frac{4}{3}\phi}(d\chi + C_\mu dx^\mu)^2 + e^{-\frac{2}{3}\phi} ds_{10}^2 \quad (2)$$

and the 10-d fields can be expressed only by the Killing vector as

$$e^{\frac{4}{3}\phi} = |k|^2 , \quad C_\mu = \frac{k_\mu}{|k|^2} , \quad F_{\mu\nu} = \frac{6k^\alpha k_{[\alpha} \partial_\mu k_{\nu]}}{|k|^2} . \quad (3)$$

For the simple example above we find $\phi \sim \log |k|^2 = -\log V$ and the RR-1-form becomes $C = n \cos \theta d\varphi$ so that n is in fact related to the 6-brane charge.

3 Seven-manifolds with G_2 holonomy

Recall, 7-manifolds with G_2 holonomy allow for exactly one covariantly constant Killing spinor which can be used to build covariantly constant differential forms as $F_{n_1 \dots n_p} = \bar{\epsilon} \gamma_{n_1 \dots n_p} \epsilon$. If ϵ is a commuting pseudo Majorana spinor (with a symmetric charge conjugation matrix) on the 7-d space these p -forms are non-vanishing for $p = 0, 3, 4, 7$ [6]. The 0- and its dual 7-forms are trivial from the geometry point of view, the only non-trivial case is given by the covariantly constant 3-form (and its dual 4-form). In a proper parameterization, this form is given by

$$\begin{aligned} \Phi &= \frac{1}{3!} \phi_{abc} e^a \wedge e^b \wedge e^c = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^3 \wedge e^5 + e^5 \wedge e^1 \wedge e^6 + e^6 \wedge e^2 \wedge e^4 + \\ &\quad e^4 \wedge e^7 \wedge e^1 + e^5 \wedge e^7 \wedge e^2 + e^6 \wedge e^7 \wedge e^3, \\ &= e^1 \wedge e^2 \wedge e^3 + e^i \wedge e^m J_{mn}^i \wedge e^n \end{aligned} \tag{4}$$

where e^a ($a = 1 \dots 7$) are the vielbeine of the 7-manifold and J_{mn}^i is the triplet of anti-selfdual complex structures satisfying the quaternionic algebra

$$J^i \cdot J^j = -\mathbb{I} \delta^{ij} + \epsilon^{ijk} J^k. \tag{5}$$

The appearance of a triplet of complex structures suggests an embedding of a hyper Kaehler or quaternionic space in the 7-manifold. In fact, two major classes of G_2 manifolds are [7]: (i) an \mathbb{R}^3 bundle over a quaternionic space Q or (ii) an \mathbb{R}^4 bundle over \mathbb{S}^3 (where \mathbb{R}^4 can be replaced by a more general hyper Kaehler space [8]). For both cases exist a limit in which the space becomes a cone over a 6-d space Y_6 so that the metric can be written as

$$ds_7^2 = d\rho^2 + \rho^2 ds_Y^2. \tag{6}$$

In case (i) Y_6 is the twistor space related to the quaternionic base space (e.g. $Y_6 = \mathbb{C}\mathbb{P}^3$ for $Q = \mathbb{S}^4$ or $Y_6 = U(3)/U(1)^3$ for $Q = \mathbb{C}\mathbb{P}_2$) and case (ii) gives $Y_6 = \mathbb{S}^3 \times \mathbb{S}^3$. Obviously, $\rho = 0$ is a co-dimension 7 singularity, where all branes will meet and this is exactly the point where chiral matter is located [5, 9]. Viewed from the 10-d perspective this matter corresponds to open strings stretched between different branes, which become massless at the common intersection. In 11 dimensions these open strings become membranes wrapping a 2-cycle which terminates at the fixed point of the Killing vector. The kind of matter depends crucially on the type of the singularity [5, 9], usually it transforms in the bi-fundamental representation of the gauge group, but also matter transforming in a tensor as well as tri-fundamental representation has been discussed [10]. In any case, the number of chiral multiplets is related to the second Betti number b_2 and for case (ii) with $Y_6 = \mathbb{S}^3 \times \mathbb{S}^3$ we will get no chiral multiplets (there can be chiral matter if \mathbb{R}^4 is replaced by a more general hyper Kaehler space) whereas for case (i) with $Q = \mathbb{S}^4$ there is one chiral multiplet and for $Q = \mathbb{C}\mathbb{P}_2$ gives two chiral multiplets; more general quaternionic spaces (see e.g. [11, 12, 13, 14]) will of course yield more chiral multiplet. The case with a single chiral multiplet (i.e. with $Q = \mathbb{S}^4$) gives in 10 dimensions the situation with two 6-branes, which are connected by the non-trivial 2-cycle. The case yielding two chiral multiplets describes in 10 dimensions three 6-branes, which exhibits a triality symmetry corresponding to the exchange of the three 6-branes and thus all 6-branes have

the same gauge group on their world volume. More interesting is of course the situation with different gauge groups where each singularity corresponds to a different number of 6-branes. This brings us to the proposal to use a G_2 manifold given as an \mathbb{R}^3 bundle over $\mathbb{WCP}_{n_1 n_2 n_3}$, which has again $b_2 = 2$ and hence describes three intersecting 6-branes with gauge groups related to the weights n_1 , n_2 and n_3 [9]. This space however is known to have further singularities, related to co-dimension 6 fixed points.

4 The explicit example

The mathematical literature [14] provides strong evidence that the metric of the metric of the quaternionic space $\mathbb{WCP}_{n_1 n_2 n_2}$ is basically given by a fourth order polynomial where the roots sum to zero, i.e.

$$F(x) = \kappa(x - r_1)(x - r_2)(x - r_3)(x - r_4) \quad , \quad r_1 + r_2 + r_3 + r_4 = 0 \quad (7)$$

and can be written as

$$ds_4^2 = \frac{q^2 - p^2}{F(p)} dp^2 + \frac{p^2 - q^2}{F(q)} dq^2 + \frac{F(p)}{q^2 - p^2} (d\tau + q^2 d\sigma)^2 + \frac{F(q)}{p^2 - q^2} (d\tau + p^2 d\sigma)^2 \quad (8)$$

and it solves the equations $R_{mn} = 3\kappa g_{mn}$. Since the Weyl tensor satisfies $W + *W = 0$ these spaces are known in the mathematical literature as (anti) self-dual Einstein spaces. We consider the compact case and write for the curvature parameter $\kappa = 1$, but one can of course consider any value. Due to a scaling symmetry, this solution depends on three continuous and one discrete parameter. In the physics literature this metric is known in Minkowski signature ($q \rightarrow iq$ and $r_m \rightarrow ir_m$) as E(A)dS-Kerr-Taub-NUT space [15], where the four parameters have the interpretation as cosmological constant, the rotational parameter and the mass equals the NUT parameter (due to the anti-self-duality constraint). The remaining discrete parameter defines the slicing of the 4-d space by a 2-space of constant positive, negative or vanishing curvature.

The curvature square of this space is $R_{mnr s} R^{mnr s} = 24\kappa^2 + 96n^2/(p+q)^6$, where $p+q=0$ is a non-removable singularity and $n = \partial_x F|_{x=0}$ is the mass (=NUT parameter) for Minkowskean solution. By a proper choice of parameter this singularity can however be put into an unphysical coordinate region where the metric has the signature (2,2). Note, the space has Euclidean signature only if the two relations hold: $F(p) \geq q^2 - p^2$ and $F(q) \geq p^2 - q^2$. With this 4-d base space the 7-d metric reads

$$ds^2 = \frac{dr^2}{\left(1 - \frac{4u_0}{r^4}\right)} + \frac{r^2}{4} \left(1 - \frac{4u_0}{r^4}\right) h_{ab} (dx^a + \xi_i^a A^i) (dx^b + \xi_j^b A^j) + \frac{r^2}{2} ds_4^2 \quad , \quad (9)$$

where $A^i = \frac{1}{2} \omega^{mn} J_{mn}^i$ is the anti-self-dual part of the spin connections ω^{mn} of the 4-d base space and ξ_i^a are three Killing vectors of the S^2 -metric h_{ab} ; for more details see [13, 12]. In the limit where the parameter u_0 vanishes, this metric describes in fact a cone over a 6-d space Y_6 , which is topologically an S^2 fibered over the 4 dimensional base given by the metric (8).

This space has two commuting, tri-holomorphic² Killing vectors ($k_1 = \partial_\sigma$ and $k_2 = \partial_\tau$) and taking a general linear combination: $k = \beta_1 \partial_\tau - \beta_2 \partial_\sigma$, 6-branes are related to co-dimension four fixed point sets. Recall, the fixed point set given by $|k|^2 = 0$ can have different co-dimensions and one finds

²Which leaves the triplet of $SU(2)$ connections invariant.

- co-dimension 7: if $u_0 = 0$ at $r = 0$,
co-dimension 6: at $|\xi|^2 = 0$ and $F(p) = F(q) = 0$,
co-dimension 4: if (i) $|\xi|^2 = 0$ and $q^2 = \frac{\beta_1}{\beta_2} = r_m$ (or $p^2 = \frac{\beta_1}{\beta_2}$) or
if (ii) $F(q) = F(p) = 0$ and $\beta_1 A_\tau^2 - \beta_2 A_\sigma^2 = 0$.

Explicit calculations show no co-dimension 2 fixed points (or better they are at infinite geodesic distance). The co-dimension 4 singularity has the interpretation as 6-branes and the co-dimension 7 case is related to the appearance of chiral fermions. The co-dimension 6 singularity is related to a fixed point of both Killing vectors and hence they correspond to additional NS5-branes in the dual type IIB picture. If one is interested in a “clear” picture of only 6-branes living in a topologically flat space, one has to “turn off” the co-dimension 6 singularities and in this case the number of components of the fixed point set gives the number number of 6-branes [16]. This can be done by equalizing two of the roots of the 4th order polynomial, e.g. $r_1 = r_2$ and $k = r_3^2 \partial_\tau - \partial_\sigma$, see [13].

In concluding let us stress, because the 7-space is non-compact we cannot simply reduce the model to 4 dimensions, but we can of course reduce it over the compact 6-d space and obtain a domain wall solution in 5 dimensions. So, writing the 7-metric as

$$ds^2 = e^{2\varphi_1(r)} dr^2 + e^{2\varphi_2(r)} h_{ab}(dx^a + \xi_i^a A^i)(dx^b + \xi_j^b A^j) + e^{2\varphi_3(r)} ds_4^2 . \quad (10)$$

The Kaluza-Klein scalars $X_{2,3}$ parameterize the volumes of the \mathbb{S}^2 and the quaternionic space and using well-known reduction formulae [17] it is straightforward to derive their 5-d potential. It is obtained from the Ricci tensor by setting $r = \text{constant}$ and reads

$$V = 2 e^{-\frac{4}{3}(\varphi_2+2\varphi_3)} \left(e^{-2\varphi_2} + 6 \kappa e^{-2\varphi_3} + 1 \right) \quad (11)$$

where the first dilatonic term is due to the conformal rescaling necessary to obtain the Einstein frame and we used that the 2- and 4-dimensional spaces satisfy $R_{ab}^{(2)} = h_{ab}$ and $R^{(4)} = 3\kappa g_{mn}$, where we set $\kappa = 1$ to have a compact space. Like the potentials coming from flux compactifications [18] this potential has no fixed points and hence the scalars cannot be stabilized yielding a singular domain wall. An extrema for the potential would of course imply that the solution would contain either an AdS space or becomes flat, which is not the case for the explicitly solution given in eqs. (9). Similarly to the approach discussed in [19] this model should be embeddable into $N=2, D=5$ gauged supergravity.

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