

On the Superselection Theory of the Weyl Algebra for Diffeomorphism Invariant Quantum Gauge Theories

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Abstract

Much of the work in loop quantum gravity and quantum geometry rests on a mathematically rigorous integration theory on spaces of distributional connections. Most notably, a diffeomorphism invariant representation of the algebra of basic observables of the theory, the Ashtekar-Lewandowski representation, has been constructed. This representation is singled out by its mathematical elegance, and up to now, no other diffeomorphism invariant representation has been constructed. This raises the question whether it is unique in a precise sense.

In the present article we take steps towards answering this question. Our main result is that upon imposing relatively mild additional assumptions, the AL-representation is indeed unique. As an important tool which is also interesting in its own right, we introduce a C^* -algebra which is very similar to the Weyl algebra used in the canonical quantization of free quantum field theories.

1 Introduction

Canonical, background independent quantum field theories of connections [1] play a fundamental role in the program of canonical quantization of general relativity (including all types of matter), sometimes called loop quantum gravity or quantum general relativity. For a review geared to mathematical physicists see [2], for a general overview [3]).

The classical canonical theory can be formulated in terms of smooth connections A on principal G -bundles over a D -dimensional spatial manifold Σ for a compact gauge group G and smooth sections of an associated (under the adjoint representation) vector bundle of $\text{Lie}(G)$ -valued vector densities E of weight one. The pair (A, E) coordinatizes an infinite dimensional symplectic manifold (\mathcal{M}, Σ) whose (strong) symplectic structure s is such that A and E are canonically conjugate.

In order to quantize (\mathcal{M}, s) , it is necessary to smear the fields A, E . This has to be done in such a way, that the smearing interacts well with two fundamental automorphisms of the principal G -bundle, namely the vertical automorphisms formed by G -gauge transformations and the horizontal automorphisms formed by $\text{Diff}(\Sigma)$ diffeomorphisms. These requirements naturally lead to holonomies and electric fluxes, that is,

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exponentiated (path-ordered) smearings of the connection over 1-dimensional submanifolds e of Σ as well as smearings of the electric field over $(D - 1)$ -dimensional submanifolds S

$$h_e[A] = \mathcal{P} \exp \int_e A, \quad E_{S,f}[E] = \int_S *E_I f^I$$

These functions on \mathcal{M} generate a closed Poisson*-algebra \mathcal{P} and separate the points of \mathcal{M} . They do not depend on a choice of coordinates nor on a background metric. Therefore, diffeomorphisms and gauge transformations act on these variables in a remarkably simple way: Let φ be a diffeomorphism of Σ , then

$$\alpha_\varphi(h_e) = h_{\varphi^{-1}e}, \quad \alpha_\varphi(E_{S,f}) = E_{\varphi^{-1}S, \varphi^*f}.$$

Similarly let $g : \Sigma \rightarrow G$ be a gauge transformation, then

$$\alpha_g(h_e) = g(a)h_e g^{-1}(b), \quad \alpha_g(E_{S,f}) = E_{S, g^{-1}fg}$$

where a is the starting point of e and b the endpoint.

Quantization now means to promote \mathcal{P} to an abstract *-algebra \mathfrak{A} and to look for its representations. However, for physical reasons we are not interested in arbitrary representations but those fulfilling the following criteria:

i) *Irreducibility*

The representation space \mathcal{H}_π should contain no proper invariant subspaces, i.e. the span of vectors $\pi(a)v$ should be dense in \mathcal{H}_π for any vector $v \in \mathcal{H}$.

Irreducible representations are the building blocks of the representation theory. If their structure is clarified, more general representations can be constructed from and analyzed in terms of them.

ii) *Diffeomorphism and Gauge Invariance*

Diffeomorphism and gauge transformations are fundamental symmetries of the theory, so if we do not consider a scenario of spontaneous symmetry breaking, they should be symmetries of the ground state of the quantum theory as well.

Thus in our setting we require that there is at least one symmetric state Ω_π in the representation space. More precisely, for the expectation value $\omega_\pi(\cdot) := \langle \Omega_\pi, \cdot \Omega_\pi \rangle_{\mathcal{H}_\pi}$ in that state, we require invariance:

$$\omega_\pi \circ \alpha_\varphi = \omega_\pi, \quad \omega_\pi \circ \alpha_g = \omega_\pi$$

for all diffeomorphisms φ and gauge transformations g .

It is remarkable that so far only one representation has been found which satisfies our assumptions: This is the Ashtekar – Isham – Lewandowski representation π_0 on a Hilbert space $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is the Ashtekar – Isham space of distributional connections (the spectrum of a certain Abelian C*-algebra) and μ_0 is the Ashtekar – Lewandowski measure. Historically, first Ashtekar and Isham [7] were looking for a natural distributional extension $\overline{\mathcal{A}}$ of the space \mathcal{A} of smooth connections, which could serve as the support for gauge invariant measures. Then Ashtekar and Lewandowski found a natural, cylindrical measure [8] which was shown to have a unique σ -additive extension μ_0 by Marolf and Mourão [9]. This measure turned out to be diffeomorphism invariant. More general diffeomorphism invariant measures were found by Baez [10], however, in contrast to μ_0 they are not faithful. That the resulting Hilbert space \mathcal{H}_0 indeed carries a representation of the holonomy – flux algebra was shown only later in [1], essentially that representation π_0 results by having connections and electric fields respectively act as multiplication and functional derivative operators respectively.

The present work was inspired by the question whether the fact that \mathcal{H}_0 is the only representation found so far which satisfies our criteria in fact means that it is the unique representation. In this article we show that upon imposing two additional and rather technical conditions on the representations, the question can

be answered affirmatively: Under these assumptions, the Ashtekar-Lewandowski representation is indeed unique.

Work towards settling this questions has begun in [11], however the results obtained there rest on assumptions that exclude the interesting cases, most notably that of a noncommutative gauge group. However it might still be interesting for the reader to take a look at [11] since the discussion there is much less burdened by the technical subtleties that arise in the general case.

During the completion of this article, a very interesting work has been published by Okolow and Lewandowski [27] that aims at settling the very same question raised in this article. Their method of proof and in part also their assumptions differ from the ones used in the present article, so it is very instructive to compare the two approaches. The hope is that combining methods of the present paper with those of [27] enables one to prove a completely general and satisfactory uniqueness theorem.

Before we conclude this introduction, let us discuss the subtleties that arise due to our general setting as well as the additional assumptions we are going to make.

The first problem that arises comes from the fact that the flux operators are unbounded and so one has to worry about domain problems. In our approach, we will try to circumvent these problems by not working with the fluxes directly but with their exponentiated counterparts. More precisely, we will consider the abstract Weyl algebra formed from holonomies and exponentiated electric fluxes and represent them as bounded operators on a Hilbert space . This algebra can be equipped with a C^* -norm so that \mathfrak{A} turns into a C^* -algebra and we therefore have the powerful representation theory of C^* -algebras at our disposal. However, we will require that the representations under considerations will be weakly continuous for the unitary groups generated by exponentiated fluxes. Therefore their selfadjoint generators, the fluxes themselves, will be well defined operators. In the case of an Abelian gauge group, this approach enables us to completely circumvent any specification of the domains of the fluxes. Due to technical complications for non-Abelian gauge groups, we will however have to make such a specification in that case. This is the first of the two requirements in addition to i) and ii) above that we make in order to prove our uniqueness result.

It is interesting to note that, at least for the case of an Abelian gauge group, our theorem could be compared to von Neumann's theorem [12] (uniqueness of weakly continuous, irreducible representations of the Weyl C^* -algebra of the phase space $(\mathcal{M} = \mathbb{R}^{2N}, \sigma = \sum_{a=1}^N dp_a \wedge dq^a)$ with $N < \infty$ up to unitary equivalence) since it also makes use of irreducibility and continuity. The surprise is that our theorem holds for an infinite number of degrees of freedom and that continuity is required only for one half of the variables (in fact, connections only form an affine space and not a vector space, so continuity of holonomies is even hard to formulate) while in background dependent quantum field theories we are faced with an uncountably infinite number of unitarily inequivalent representations of the canonical commutation relations [13]. There, a unique representation is usually selected by using Lorentz invariance and a specific dynamics, in that sense it is a *dynamical uniqueness*. However, while we use spatial diffeomorphism invariance, in our case we do not make use of any particular dynamics such as the Hamiltonian constraint of quantum general relativity [14] and in that sense it is a *kinematical uniqueness*.

This comparison leads us to the second subtlety and the corresponding additional assumption: The requirements i) and ii) guarantee that the the action of the automorphisms on algebra elements can be unitarily implemented in the representation. However there is a priory little control about the details of the action of these unitary operators in the Hilbert space. We will point out that there is a "natural" way for them to act in the representation Hilbert space, and we will require that this natural action is realized in the representations we consider. This is, however, a priori not the most general possibility. Two scenarios can be envisioned: In the first, one can actually show that the natural action is in fact the only possible one, and then our uniqueness result would be general. In the second, there are actually other viable unitary actions of the diffeomorphisms, and this in turn might lead to a classifications of the representations studied in terms of unitary representations of the diffeomorphism group. The picture then would be very similar to that obtained in the case of free quantum field theories, where Poincare invariant representations can be classified by unitary representations of the Poincare group. Both scenarios would be very interesting in their own ways, and we happily await a future settling of this question.

To summarize, a completely general and satisfactory picture of the diffeomorphism and gauge invariant representations of the algebra of holonomies and fluxes has not yet emerged. However, the results of the present work and that of [27, 11] point to the fact that diffeomorphism invariance is an extremely strong requirement and could mean that in background independent quantum field theories there is much less quantization freedom than in background dependent ones.

To finish, let us give an overview of the structure of the rest of present work:

In section 2 we recall from [1] the essentials of the classical formulation of canonical, background independent theories of connections, that is, the symplectic manifold (\mathcal{M}, σ) and the corresponding classical Poisson*-algebra \mathcal{P} generated by holonomies and electric fluxes.

In section 3 we define the abstract *-algebra \mathfrak{A} and recall from [11] the general representation theory of \mathfrak{A} .

In section 4 we implement irreducibility and spatial diffeomorphism invariance and prove our uniqueness theorem.

Finally, in appendix A we present a simpler proof of our main result using an additional, natural assumption. We also report some additional technical information in appendix B.

2 Preliminaries

Let Σ be an analytic, connected and orientable D -dimensional manifold and G a compact, connected gauge group. A principal G -bundle P over Σ is determined by its local trivializations $\phi_I : U_I \times G \rightarrow P$ subordinate to an atlas $\{U_I\}$ of Σ . These give rise to local, smooth G -valued functions $g_{IJ} : U_I \cap U_J \rightarrow G$ on Σ , called transition function cocycles. A connection over P can be thought of as a collection $\{A_I\}$ of smooth, $\text{Lie}(G)$ -valued one-forms over the respective charts U_I subject to the gauge covariance condition $A_I = -dg_{IJ}g_{IJ}^{-1} + \text{Ad}_{g_{IJ}}(A_J)$ over $U_I \cap U_J$. The space of smooth connections \mathcal{A} over P therefore depends on the bundle P but we will abuse notation in not displaying this dependence.

Similarly, we define a vector bundle E_P associated to P under the adjoint representation whose typical fiber is a $\text{Lie}(G)$ -valued $(D - 1)$ -form on P . An electric field is a local section of E_P which we may think of as a collection $\{E_I\}$ of $\text{Lie}(G)$ -valued $(D - 1)$ -forms on Σ subject to the gauge covariance condition $E_I = \text{Ad}_{g_{IJ}}(E_J)$ over $U_I \cap U_J$. The space of smooth electric fields \mathcal{E} over P depends on P as well but the dependence is also not displayed.

The space \mathcal{A} can be given the structure of a manifold modeled on a Banach space in the usual way (see e.g. [15, 1]). Consider now the cotangent bundle $\mathcal{M} := T^*(\mathcal{A})$. Since \mathcal{A} is a Banach manifold, also \mathcal{M} is and, moreover, we may identify \mathcal{E} with the sections of \mathcal{M} together with the induced topology. The cotangent bundle $\mathcal{M} = \mathcal{A} \times \mathcal{E}$ can be equipped with the following (strong, see e.g. [16]) symplectic structure: Let $\text{Tr} : \text{Lie}(G) \times \text{Lie}(G) \rightarrow \mathbb{C}$ be a natural Ad_G -invariant metric on $\text{Lie}(G)$ then there is a natural pairing $\mathcal{E} \times \mathcal{A} \rightarrow \mathbb{C}$ defined by

$$(F, f) \mapsto F(f) := \int_{\Sigma} \text{Tr}(F \wedge f) \tag{2.1}$$

Since in our case the section of the tangential space of \mathcal{M} can be identified with points in \mathcal{M} , we may define the symplectic structure by

$$s : T(\mathcal{M}) \times T(\mathcal{M}) \rightarrow \mathbb{C}; ((F, f), (F', f')) \mapsto F(f') - F'(f) \tag{2.2}$$

In a concrete gauge field theory the right hand side will be multiplied by a constant which depends on the coupling constant of the theory. In order not to clutter our formulae we will assume that A and E respectively have dimension cm^{-1} and $\text{cm}^{-(D-1)}$ respectively and we set $\hbar = 1$ for simplicity.

The Poisson bracket is uniquely defined by

$$\{.,.\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}); (a, b) \mapsto \chi_a(b) \tag{2.3}$$

where the Hamiltonian vector field χ_a on \mathcal{M} defined by $a \in C^\infty(\mathcal{M})$ is uniquely defined by $i_{\chi_a}s + da = 0$. It is easy to see that the usual Poisson bracket $*$ -algebra $\mathcal{P}' := C^\infty(\mathcal{M})$ is generated from the basic canonical bracket relations

$$\{F(A), F'(A)\} = \{E(f), E(f')\} = 0, \quad \{E(f), F(A)\} = F(f) \quad (2.4)$$

and the reality conditions

$$\overline{E(f)} = E(\bar{f}), \quad \overline{F(A)} = \bar{F}(A) \quad (2.5)$$

The relations (2.4) display A, E respectively as the canonically conjugate configuration and momentum degrees of freedom and therefore \mathcal{A} is called the classical configuration space.

The algebra \mathcal{P}' is, however, not what we are interested in for several reasons:

i) *Gauge Invariance*

The objects $F(A), E(f)$ depend heavily on our choice of trivialization of P . It will be very hard to construct gauge invariant quantities from them, in which we are ultimately interested. In order to do that, we must work with basic functions on \mathcal{M} which are different from the canonical functions $F(A), E(f)$. Of course, these problems could be avoided by fixing a gauge, however, there is no canonical gauge and most gauges are plagued by the Gribov problem.

ii) *Background Independence*

Even when ignoring the just mentioned problems, it is rather hard to construct spatially diffeomorphism invariant (background independent) representations of \mathcal{P}' , in fact, to the best of our knowledge such representations have not been constructed. To see where the problem is, suppose that we want to construct a representation of the form $\mathcal{H} = L_2(\mathcal{S}', d\mu)$ where \mathcal{S}' is the space of tempered distributions on Σ (that is, the topological dual of the space \mathcal{S} of functions of rapid decrease) and μ is a measure thereon. This is the form of the representation for free field theories [17]. Notice that the nuclear topology on \mathcal{S} does not refer to any background structure except for the differentiable structure of Σ , so there is no problem up to this point. The problem arises when we define the measure μ via its generating functional $\mu(F) := \mu(\exp(iF(\cdot)))$. For instance, if μ is a (generalized) free (Gaussian) measure, then $\mu(F) = \exp(-F(C \cdot F)/2)$ where C is a *background metric dependent* appropriate covariance which is needed in order to contract indices in the appropriate way. Interacting measures in more than three spacetime dimensions have not been constructed so far.

A solution to the first problem was suggested for canonical quantum Yang-Mills theories already by Gambini et. al. [18] and for loop quantum gravity by Jacobson, Rovelli and Smolin [19]. The idea is to work with holonomies and electric fluxes. We will explain in detail what we mean by that, because it will be important for what follows. For more details, see [2].

Definition 2.1.

- i) \mathcal{C} is the set of piecewise analytic, continuous, oriented, compactly supported, parameterized curves embedded in Σ . We denote by $b(c), f(c)$ the beginning and final point of c and consider the range $r(c)$ as the image of the compact interval $[0, 1]$ under c .
- ii) If $b(c_2) = f(c_1)$ we define composition $(c_1 \circ c_2)(t) = c_1(2t)$ if $t \in [0, \frac{1}{2}]$ and $(c_1 \circ c_2)(t) = c_2(2t - 1)$ if $t \in [\frac{1}{2}, 1]$. Inversion is defined by $c^{-1}(t) := c(1 - t)$.
- iii) We call $c, c' \in \mathcal{C}$ equivalent, $c \sim c'$, iff c, c' differ by a finite number of reparameterizations and retracings (a segment of a curve of the form $s^{-1} \circ s'$). The set of equivalence classes p in \mathcal{C} is denoted as the set of paths \mathcal{Q} . The functions b, f and the operations $\circ, ^{-1}$ extend from \mathcal{C} to \mathcal{Q} .
- iv) An edge $e \in \mathcal{Q}$ is a path for which an entire analytic representative $c_e \in \mathcal{C}$ exists. For edges the function r extends as $r(e) := r(c_e)$.
- v) An oriented graph γ is determined by a finite number of edges $e \in E(\gamma)$ which intersect at most in their boundaries, called the vertex set $V(\gamma)$.

It is important to realize that in contrast to \mathcal{C} the set \mathcal{Q} is a groupoid with objects the points $x \in \Sigma$ and with the sets of morphisms given by $\text{Mor}(x, y) = \{p \in \mathcal{Q}; b(p) = x, f(p) = y\}$. The notion of paths is motivated by the algebraic properties of the holonomy.

Definition 2.2.

For $A \in \mathcal{A}$ and $p \in \mathcal{Q}$ we define $A(p) := h_{A,p}(1)$ where $h_{A,p} : [0, 1] \rightarrow G$ is uniquely defined by the parallel transport equation

$$\frac{d}{dt}h_{A,p}(t) = h_{A,p}(t)A_a(c_p(t))\dot{c}_p^a(t), \quad h_{A,p}(0) = 1_G \tag{2.6}$$

if p is in the domain of a chart, where $a = 1, \dots, D$ denote tensorial indices and $[c_p] = p$ is a representative.

Due to the covariance condition on connections under change of local trivialization one can show that (2.6) can be extended unambiguously (up to a gauge transformation) to the case that p is not within the domain of a chart. This follows from the fact that it comes from the horizontal lift of c_p which is globally defined. The virtue of definition (2.6) is that it displays $A \in \mathcal{A}$ as an a groupoid morphism $A \in \overline{\mathcal{A}} := \text{Hom}(\mathcal{Q}, G)$. In fact, $A(c) = A([c])$ is reparameterization invariant and $A(p_1 \circ p_2) = A(p_1)A(p_2)$, $A(p^{-1}) = A(p)^{-1}$. Under gauge transformations $g \in \text{Fun}(\Sigma, G)$ we find $A^g(p) = g(b(p))A(p)g(f(p))^{-1}$ which implies that e.g. traces of holonomies along closed paths are gauge invariant. Thus, it is relatively easy to construct gauge invariant functions of the connection from holonomies !

The worry is of course, that $A(p)$ is smeared only in one dimension rather than three such as $F(A)$ was. In order to still obtain a well-defined Poisson algebra, the electric field therefore must be smeared in at least $D - 1$ dimensions. This can be done as follows:

Let S be an open, connected, simply connected, analytic, oriented, compactly supported $(D-1)$ -dimensional submanifold of Σ , called a surface in what follows, let $x_0 \in S$ and for $x \in S$ let $c_{x_0,x} \in \mathcal{C}$ with $b(c_{x_0,x}) = x_0$, $f(c_{x_0,x}) = x$, $r(c_{x_0,x}) \subset S$. Then we define

$$E'(S) := \int_S \text{Ad}_{A(p_{x_0,x})}(E(x)) \tag{2.7}$$

It is easy to see that under gauge transformations $E'^g(S) = \text{Ad}_{g(x_0)}(E'(S))$ so that for instance $\text{Tr}(E'(S)^2)$ is gauge invariant. Notice that the holonomies involved in (2.7) are only necessary if G is non-Abelian. The ugly feature of $E'(S)$ is that it depends not only on S but also on $x_0, p_{x_0,x}$. Consider therefore the non-covariant object

$$E_n(S) := \int_S E^j n_j \tag{2.8}$$

where n_j , $j = 1, \dots, \dim(G)$ is a $\text{Lie}(G)$ -valued scalar. It will be sufficient to normalize the components corresponding to the non Abelian generators by $\delta^{jk}n_j(x)n_k(x) = 1$ because we need it only in order to allow for local gauge transformations which have the effect of simply rotating $n_j(x)$ locally. For the same reason we restrict to $n_j(x) = \text{const.}$ if j corresponds to an Abelian generator. The idea is that while (2.8) does not transform simply, we can still construct gauge invariant functions from it using a limiting procedure that involves making the surfaces smaller and smaller while making their number larger and larger at the same time. Examples are provided by the length, area and volume functionals already mentioned. Thus, the $E_n(S)$ serve as an intermediate objects to build more complicated but gauge invariant composite objects and this is why we want them to be represented as well-defined operators later on, because once they are defined, the composite operators can be defined as well.

For the purposes of this paper we will make also the following additional technical assumption: Notice that if $S = S_1 \cup S_2$ is the disjoint union of surfaces then we have $E_n(S) = E_n(S_1) + E_n(S_2)$. Thus we know the flux $E_n(S)$ if we know it for every connected surface S . If S is a connected surface we can triangulate it into $(D-1)$ -simplices Δ and we have $E_n(S) = \sum_{\Delta} E_n(\Delta)$ even if the different Δ overlap in faces, since they are of measure zero. Now each $(D-1)$ -simplex can be decomposed into D , $(D-1)$ -dimensional, cubes by choosing an interior point of Δ , connecting it with an interior point of each of its boundary $(D-2)$ -simplices,

connecting those points with an interior point of each of its boundary $(D - 3)$ -simplices etc. Thus, we know each $E_n(S)$ if we know it for each $E_n(\square)$ where \square is a $(D - 1)$ -cube. The assumption that we now make is the following: We choose precisely one $(D - 2)$ -face of \square open while all others are closed. In other words, if $\bar{\square}$ denotes the closure of \square and \bar{F} the closure of one of its faces F then $\square = \bar{\square} - \bar{F}$. The classical fluxes satisfy $E_n(\square) = E_n(\bar{\square})$ so this seems to be an innocent assumption. However, it will turn out to be crucial in the quantum theory. From now on we allow only compactly supported, analytical, oriented surfaces S which can be written as a disjoint union $S = \cup_{\square} \square$ of such cubes \square with the specified boundary properties. Since the classical flux $E_n(S)$ through any S can be written as a limit of fluxes through those special S , there is no loss of generality on the classical side. The decisive feature of such a cube \square is that we can choose a closed $(D - 2)$ -surface S such that $\square = \square_1 \cup \square_2$ is a disjoint union with $S \subset \square_1$, $S \cap \square_2 = \emptyset$, $\bar{\square}_1 \cap \bar{\square}_2 = S$ and all three $\square, \square_1, \square_2$ are analytically diffeomorphic. The reason for why this is important will become obvious only in section 4. In order that this works, we must restrict to $D \geq 2$ in what follows. We feel that this assumption is not crucial for our result to hold, however, it avoids tedious case by case considerations of the intersection structure of surfaces. Thus, we ask whether the functions $A(p), E_n(S)$ generate a well-defined Poisson algebra which is induced from (2.4). The answer is as follows [20]:

Definition 2.3.

i) Given a graph γ we define $p_\gamma : \mathcal{A} \rightarrow G^{|E(\gamma)|}$; $A \mapsto \{A(e)\}_{e \in E(\gamma)}$. A function f is said to be cylindrical over γ iff there exists a function $f_\gamma : G^{|E(\gamma)|} \rightarrow \mathbb{C}$ such that $f = f_\gamma \circ p_\gamma$. The functions cylindrical over γ are denoted by Cyl_γ and the $*$ -algebra of cylindrical functions is defined by $Cyl := \cup_{\gamma \in \Gamma} Cyl_\gamma$ where Γ is the set of all compactly supported, oriented, piecewise analytic graphs. Notice that $f \in Cyl_\gamma$ implies $f \in Cyl_{\gamma'}$ for any $\gamma \subset \gamma'$ and we identify the corresponding representatives.

ii) The subalgebras Cyl^n , $n = 0, 1, 2, \dots, \infty$ of Cyl consist of functions of the form $f = f_\gamma \circ p_\gamma$ where $f_\gamma \in C^n(G^{|E(\gamma)|})$.

iii) Vector fields on \mathcal{A} are defined as maps $Y : Cyl^n \rightarrow Cyl^{n-1}$ which satisfy the Leibniz rule and annihilate constants. We will denote them by Vec .

iv) Given an open, compactly supported, connected, simply connected, oriented, analytic surface S and a cylindrical function f we can always find a graph γ over which it is cylindrical and which is adapted to S in the following sense: Any $e \in E(\gamma)$ belongs to precisely one of the following subsets $E_*(\gamma)$ of $E(\gamma)$ where

$$\begin{aligned}
 E_{out}(\gamma) &= \{e \in E(\gamma); e \cap S = \emptyset\}, \\
 E_{in}(\gamma) &= \{e \in E(\gamma); e \cap \bar{S} = e\}, \\
 E_{up}(\gamma) &= \{e \in E(\gamma); e \cap S = b(e), e \text{ points into the direction of } S\} \text{ and} \\
 E_{down}(\gamma) &= \{e \in E(\gamma); e \cap S = b(e), e \text{ points into the opposite direction of } S\}.
 \end{aligned}$$

For $e \in E(\gamma)$ we define $\sigma(S, e) := 0$ if $e \in E_{out}(\gamma) \cup E_{in}(\gamma)$ and we define a) $\sigma(S, e) = 1$ if $e \in E_{up}(\gamma)$ b) $\sigma(S, e) = -1$ if $e \in E_{down}(\gamma)$.

We have supplemented the regularization of the flux vector field, so far only discussed for open surfaces in the literature, to the case that $b(e)$ is a boundary point. Our condition is compatible with the additivity of fluxes. We can now define a real-valued vector field $Y_n(S)$ on Cyl by $(f = p_\gamma^* f_\gamma)$

$$Y_n(S)f := p_\gamma^*(Y_n(S))_\gamma f_\gamma := p_\gamma^*\left(\sum_{e \in E(\gamma)} \sigma(e, S)n_j(b(e))R_e^j f_\gamma\right) \tag{2.9}$$

where $R_e^j = R^j(h_e), [p_\gamma^* h_e](A) = A(e)$ and $R^j = (\frac{d}{dt})_{t=0} L_{\exp(t\tau_j)}$ denotes the generator of left translations on G and $(\tau_j)_{j=1}^{\dim(G)}$ is a basis of $Lie(G)$. One can check that the family of vector fields $\{(Y_n(S))_\gamma\}_{\gamma \in \Gamma}$ is indeed consistent, that is, if $\gamma \subset \gamma'$ and $p_{\gamma'\gamma} := p_\gamma \circ p_{\gamma'}^{-1}$ then for $f \in Cyl_\gamma$ we have $p_{\gamma'\gamma}^*[(Y_S^j)_\gamma f_\gamma] = (Y_S^j)_{\gamma'} p_{\gamma'\gamma}^* f_\gamma$.

v)

Consider the Lie*-algebra $V := \text{Cyl}^\infty \times \text{Vec}$ defined by

$$[(f, Y), (f', Y')] := (Y \cdot f' - Y' \cdot f, [Y, Y']) \quad (2.10)$$

where the *-operation is just complex conjugation and where the second entry on the right hand side of (2.10) denotes the Lie bracket of vector fields. We identify f with $(f, 0)$ and Y with $(0, Y)$. The algebra \mathcal{P} is defined as the free tensor algebra over V modulo the two-sided ideal generated by elements of the form $u \otimes v - v \otimes u - [u, v]$ for any $u, v \in V$ (also called the universal enveloping algebra of V)¹. In what follows we drop the tensor product symbol \otimes as usual.

This definition answers our question in the following sense: Notice that since $A(e), E_n(S)$ are not smeared in D dimensions, the Poisson bracket (2.4) is actually ill-defined. However, one can regularize these functions by fattening out e, S to D -dimensional tubes e_r and disks S_r respectively where r is some regularization parameter, compute the Poisson brackets of the regularized objects and then take the limit $r \rightarrow 0$. The end result is the Lie algebra (2.10) in the sense that $A(e)$ is identified with an element of Cyl^∞ and $E_n(S)$ with the element $Y_n(S) \in \text{Vec}$. Thus, in terms of Poisson brackets we have $\{f, f'\} = 0$, $\{E_n(S), f\} = Y_n(S) \cdot f$, $\{\{E_n(S), E_{n'}(S')\}, f\} = [Y_n(S), Y_{n'}(S')] \cdot f$ for any S, S', n, n' and $f, f' \in \text{Cyl}^\infty$. Notice that this implies that the subalgebra of fluxes becomes non-Abelian in apparent contradiction to the Abelian nature of the D -smeared objects $E(S_r)$. However, as one can show, we have still $\{\{E_n(S_r), E_{n'}(S'_r)\}, f\} = 0$, so the non-Abelianity comes about due to the singular smearing dimension. The reason for not using D -smeared electric fields is that $\{E_n(S_r), f\}$ is no longer an element of Cyl (in the non-Abelian case), it is an integral over elements of Cyl and not a countable linear combination.

The representation theory based on the abstract algebra \mathcal{P} supplemented by appropriate *-relations gets complicated due to the fact that the vector fields will be represented by unbounded operators, so that domain questions will arise. We avoid this by means of passing to the corresponding non-Abelian analogs of the Weyl elements.

Definition 2.4.

For $t \in \mathbb{R}$ define

$$W_t^n(S) := e^{tY_n(S)} = e^{-it[iY_n(S)]} \quad (2.11)$$

where $n^j(x)n^k(x)\delta_{jk} = 1$. The algebra \mathfrak{A} is defined as the free tensor algebra generated by the $(f, W_t^n(S)) \in \text{Cyl}^\infty \times \exp(\text{Vec})$ modulo the two-sided ideal induced by (2.10) and modulo the *-relations

$$f^* = \bar{f}, \quad (W_t^n(S))^* = (W_{-t}^n(S)) = (W_t^n(S))^{-1} \quad (2.12)$$

Due to the non-Abelian nature of the Group G , the relations in \mathfrak{A} induced by (2.10) are somewhat difficult to describe but it is nevertheless explicitly possible. In order to do this, we introduce the following notions.

Definition 2.5.

i)
Let $x \in \Sigma$ be given. The germ $[e]_x$ of an edge e with $b(e) = e(0) = x$ is defined by the infinite number of Taylor coefficients $e^{(n)}(0)$ in some parametrization. Likewise, the germ $[S]_x$ of a surface S with $S(0, \dots, 0) = x$ is defined by the Taylor coefficients $S^{(n_1, \dots, n_{D-1})}(0, \dots, 0)$ in some reparametrization.

ii)
The set of germs $[e]$ ($[S]$) of edges (surfaces) at given $x \in \Sigma$ does not depend on x and will be denoted by $\mathcal{E}(S)$.

iii)
Notice that the germs know about the orientation of e, S and that their knowledge allows us to reconstruct

¹If there are any additional algebraic relations in \mathcal{P} then we enlarge the ideal correspondingly.

$e(t), S(u_1, \dots, u_{D-1})$ up to reparametrization due to analyticity so that they reconstruct e, S . We say that two germs are equal if they reconstruct the same edges and surfaces respectively from a given point x .

iv) Let $x \in \Sigma$, $[e] \in \mathcal{E}$. We define elements $R_{x,[e]}^j \in \text{Vec}$ by (assume w.l.g. that γ is adapted to x in the sense that each edge is either disconnected or outgoing from x)

$$R_{x,[e]}^j p_\gamma^* f_\gamma := p_\gamma^* \sum_{e' \in E(\gamma')} \delta_{x,b(e')} \delta_{[e],[e']} R_{e'}^j f_\gamma \quad (2.13)$$

v) Let $x \in \Sigma$, $[e] \in \mathcal{E}$, $[S] \in \mathcal{S}$. We define

$$\sigma([S], [e]) := \sigma(S', e') \text{ for any } e', S' \text{ s.t. } [e] = [e']_x, [S] = [S']_x \quad (2.14)$$

Lemma 2.1.

i) The vector fields $R_{x,[e]}^j$ satisfy the following commutation relations

$$[R_{x,[e]}^j, R_{x',[e']}^k] = -f^{jk} {}^l \delta_{[e],[e']} \delta_{x,x'} R_{[x],[e]}^l \quad (2.15)$$

where $[\tau_j, \tau_k] = f_{jk} {}^l \tau_l$ defines the structure constants².

ii) The flux vector fields $Y_n(S)$ can be expressed in terms of the $R_{x,[e]}^j$ by the formula

$$Y_n(S) = \sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_j(x) R_{x,[e]}^j \quad (2.16)$$

The proof of lemma 2.1 consists of a straightforward computation in applying the left and right hand sides of (2.15), (2.16) to elements of Cyl^∞ . Formula (2.16) looks cumbersome due to the uncountably infinite sums involved, however, as vector fields on Cyl^∞ they make perfect sense. Notice that (2.16) allows us for the first time to compute the commutator $[Y_S^j, Y_{S'}^k]$ in closed form and one sees immediately that the $Y_n(S)$ do not form a subalgebra in Vec . The advantage of the germ vector fields is that they have an extremely simple and closed algebra among themselves. They are not obviously generated from the $Y_n(S)$ and are thus not of physical interest, however, they are a useful tool in order to perform practical calculations. We stress that only the algebra of vector fields generated from the $Y_n(S)$ are of physical interest, but that algebra is a subalgebra of the bigger algebra generated by the $R_{x,[e]}^j$ and we may exploit that.

We can now compute the commutation relations among the $W_t^n(S) = \exp(tY_n(S))$ and the $f \in \text{Cyl}^\infty$. We have

$$\begin{aligned} W_t^n(S) f(W_t^n(S))^{-1} &= \sum_{m=0}^{\infty} \frac{t^m}{m!} [Y_n(S), f]_{(m)} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{m!} (Y_n(S))^m f = W_t^n(S) \cdot f \end{aligned} \quad (2.17)$$

where the bracket notation denotes the multiple commutator and the last line denotes the application of the exponentiated vector field to a cylindrical function. Let now $f = p_\gamma^* f_\gamma$. Since the R_e^j are mutually commuting we have

$$\begin{aligned} [W_t^n(S) \cdot f](A) &= [p_\gamma^* \prod_{e \in E(\gamma)} e^{t\sigma(S,e)n_j(b(e))R_e^j} f_\gamma](A) \\ &= f_\gamma(\{e^{t\sigma(S,e)n_j(b(e))\tau_j} A(e)\}_{e \in E(\gamma)}) \end{aligned} \quad (2.18)$$

²Since G is a compact, connected Lie group, we have $G/D \cong A \times S$ where D is a central discrete subgroup and A, S are Abelian and semisimple Lie groups respectively. Indices are dragged w.r.t. the Cartan-Killing metric $\text{Tr}(T_j T_k) = -\delta_{jk}$ where $(T_j)_i^k = f_{ij}{}^k$, f_{jkl} totally skew for the semisimple generators.

To see the equality in the last line of (2.18) it is obviously sufficient to show it for one copy of G , that is

$$f_t(h) := [e^{tn_j R^j} f](h) = f(e^{tn_j \tau_j} h) =: f'_t(h) \quad (2.19)$$

for $f \in C^\infty(G)$ and any $t \in \mathbb{R}$. To show this, set $R := n_j R^j$, $\tau := n_j \tau_j$. We clearly have $f_0(h) = f'_0(h) = f(h)$ and

$$\begin{aligned} \left(\frac{d}{dt} f_t\right)(h) &= \left[\frac{d}{ds}\right]_{s=0} (e^{(s+t)R} f)(h) = \left[\frac{d}{ds}\right]_{s=0} (e^{sR} e^{tR} f)(h) = (R e^{tR} f)(h) = (R f_t)(h) \\ \left(\frac{d}{dt} f'_t\right)(h) &= \left[\frac{d}{ds}\right]_{s=0} f(e^{(s+t)\tau} h) = \left[\frac{d}{ds}\right]_{s=0} f(e^{s\tau} e^{t\tau} h) = (R f)(e^{t\tau} h) = (R f'_t)(h) \end{aligned} \quad (2.20)$$

Hence, f_t, f'_t satisfy the same ordinary differential equation and initial conditions and thus (2.19) follows from the uniqueness and existence theorems about ordinary differential equations. Hence, the $W_t^n(S)$ act on cylindrical functions just by left translation in their arguments as was to be expected and the result (2.20) implies that the algebra \mathfrak{A} can be extended to the bounded cylindrical functions Cyl_b on \mathcal{A} (differentiability is no longer necessary) which forms an Abelian subalgebra.

Finally we compute the commutator of Weyl-operators by explicitly using the germ vector fields. We have

$$W_t^n(S) W_{t'}^{n'}(S') (W_t^n(S))^{-1} = \exp(t' n'_j \sum_{x' \in S'} \sum_{[e'] \in \mathcal{E}} \sigma([S']_{x'}, [e']) W_t^n(S) R_{x', [e']}^j W_t^n(S)^{-1}) \quad (2.21)$$

Now

$$W_t^n(S) R_{x', [e']}^j (W_t^n(S))^{-1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} \left[\sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_k(x) R_{x, [e]}^k R_{x', [e']}^j \right]^{(m)} \quad (2.22)$$

and

$$\begin{aligned} & \left[\sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_k(x) R_{x, [e]}^k R_{x', [e']}^{j_1} \right]^{(m)} \\ &= [-f_{k_1 j_1 j_2} \sigma([S]_{x'}, [e']) \chi_S(x') n^{k_1}(x')] \left[\sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_k(x) R_{x, [e]}^k R_{x', [e']}^{j_2} \right]^{(m-1)} \\ &= \sigma([S]_{x'}, [e']) \chi_S(x') n_{j_1 j_2}(x') \left[\sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_k(x) R_{x, [e]}^k R_{x', [e']}^{j_2} \right]^{(m-1)} \\ &= \sigma([S]_{x'}, [e'])^2 \chi_S(x') (n^2)_{j_1 j_2}(x') \left[\sum_{x \in S} \sum_{[e] \in \mathcal{E}} \sigma([S]_x, [e]) n_k(x) R_{x, [e]}^k R_{x', [e']}^{j_2} \right]^{(m-2)} \\ &= \sigma([S]_{x'}, [e'])^m \chi_S(x') (n^m)_{j_1 j_2}(x') R_{x', [e']}^{j_2} \end{aligned} \quad (2.23)$$

where we have defined the matrix n by $n_{jj'} := n^k f_{jkj'}$ and χ_S is the characteristic function of the set S . Hence (2.22) becomes

$$\begin{aligned} & W_t^n(S) n'_j(x') R_{x', [e']}^j (W_t^n(S))^{-1} \\ &= [1 - \chi_S(x')] n'_j(x') R_{x', [e']}^j + \chi_S(x') n'_j(x') \sum_{m=0}^{\infty} \frac{[\sigma([S]_{x'}, [e']) t]^m}{m!} (n^m)_{jk}(x') R_{x', [e']}^k \\ &= [1 - \chi_S(x')] n'_j(x') R_{x', [e']}^j + \chi_S(x') [e^{-t\sigma([S]_{x'}, [e']) \text{ad}_n(x')}]_{jk} R_{x', [e']}^k \end{aligned} \quad (2.24)$$

where for any $\tau \in \text{Lie}(G)$ the matrix ad_τ is defined by $[\text{ad}_\tau]_{jk} \tau_k = [\tau, \tau_j]$. Plugging (2.24) into (2.21) we obtain

$$\begin{aligned} & W_t^n(S) W_{t'}^{n'}(S') (W_t^n(S))^{-1} \\ &= \exp(t' \sum_{[e] \in \mathcal{E}} \sum_{x \in S' - S} n'_j(x) \sigma([S']_x, [e]) \delta_{jk} + \sum_{x \in S \cap S'} n'_j(x) \sigma([S']_x, [e]) [e^{-t\sigma([S]_x, [e]) \text{ad}_n(x)}]_{jk} R_{x, [e]}^k) \end{aligned} \quad (2.25)$$

which is almost of the form of a $W_t^n(S)$ again. That the $W_t^n(S)$ do not close among each other we knew from the associated statement for the Y_S^j , however, we see that the algebra they generate can be computed explicitly.

Finally we equip the algebra \mathfrak{A} with a C^* structure. Since the operator norm in a representation π of \mathfrak{A} on a Hilbert space \mathcal{H} does define a C^* -norm through $\|a\| := \|\pi(a)\|_{\mathcal{H}}$ we just need to find a representation of \mathfrak{A} to close our algebra to a C^* algebra. However, the Ashtekar – Lewandowski Hilbert space \mathcal{H}_0 is a representation space for a representation π_0 , hence such a C^* -norm exists.³

That representation is given by $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is the spectrum of the C^* -subalgebra of \mathfrak{A} given by Cyl and μ_0 is a regular Borel probability measure on $\overline{\mathcal{A}}$ consistently defined by

$$\mu_0(p_\gamma^* f_\gamma) = \int_{G^{|E(\gamma)|}} \prod_{e \in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e \in E(\gamma)}) \quad (2.26)$$

for measurable f_γ and extended by σ -additivity. Then

$$\pi_0(f)\psi = f(A)\psi \text{ and } \pi_0(W_t^n(S))\psi = W_t^n(S)\psi \quad (2.27)$$

so that Cyl_b is represented by bounded multiplication operators while the Weyl elements $W_t^n(S)$ are simply extended from Cyl_b to $L_2(\overline{\mathcal{A}}, d\mu_0)$. Notice that Cyl_b (in particular the continuous functions Cyl^0 on $\overline{\mathcal{A}}$) are dense in \mathcal{H}_0 because \mathcal{H}_0 is the GNS Hilbert space induced by the positive linear functional ω_0 on Cyl_b defined by $\omega_0(f) = \mu_0(f)$. Finally it follows from the left invariance of the Haar measure that $\pi_0(W_t^n(S))$ are unitary operators as they should be. Thus e.g.

$$\begin{aligned} \|f\|_{\mathfrak{A}} &= \|\pi_0(f)\|_{\mathcal{B}(\mathcal{H}_0)} = \sup_{\|\psi\|=1} \|f\psi\|_{\mathcal{H}_0} = \sup_{a \in \mathcal{A}} |f(A)| \\ \|W_t^n(S)\|_{\mathfrak{A}} &= \|\pi_0(W_t^n(S))\|_{\mathcal{B}(\mathcal{H}_0)} = \sup_{\|\psi\|=1} \|W_t^n(S)\psi\|_{\mathcal{H}_0} = 1 \end{aligned} \quad (2.28)$$

where \mathcal{B} denotes the bounded operators on a Hilbert space. The C^* -norm of any other element of \mathfrak{A} can be computed by using the commutation relations and the inner product on \mathcal{H}_0 .

This concludes our exposition about the C^* -algebra \mathfrak{A} .

3 General Representation Theory of \mathfrak{A}

Let us clarify what we mean by a representation of \mathfrak{A} .

Definition 3.1. *By a representation of \mathfrak{A} we mean an $*$ -algebra homomorphism $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ from \mathfrak{A} into the algebra of bounded operators of a Hilbert space \mathcal{H} . Thus $\pi(a + zb) = \pi(a) + z\pi(b)$, $\pi(ab) = \pi(a)\pi(b)$, $\pi(a^*) = [\pi(a)]^\dagger$ for all $a, b \in \mathfrak{A}, z \in \mathbb{C}$.*

The representation theory of \mathfrak{A} is very rich and first steps towards a classification have been made in [11]. An elementary result is the following.

Lemma 3.1.

The representation space \mathcal{H} of \mathfrak{A} is necessarily a direct sum of Hilbert spaces

$$\mathcal{H} = \bigoplus_\nu \mathcal{H}_\nu \quad (3.1)$$

where $\mathcal{H}_\nu = L_2(\overline{\mathcal{A}}, d\mu_\nu)$ is an L_2 space over the spectrum $\overline{\mathcal{A}}$ of Cyl_b , and μ_ν is a probability measure on $\overline{\mathcal{A}}$.

³We note however the possibility that one can find a different C^* closure of the algebra at hand. Since we will not use any specific information about the closure, however, this is of no concern in the present paper.

Proof of lemma 3.1:

Every representation π of \mathfrak{A} on a Hilbert space \mathcal{H} is, in particular, a representation of the Abelian sub * -algebra Cyl_b . Now a general result from C^* -algebra theory [21] says that every non-degenerate representation (that is, $\text{Ker}(\pi) := \{\psi \in \mathcal{H}; \pi(a)\psi = 0 \forall a \in (A)\} = \{0\}$) is a direct sum of cyclic representations π_ν on Hilbert spaces \mathcal{H}_ν . Now since our algebra is unital $1 \in \text{Cyl}_b \subset \mathfrak{A}$ we necessarily have $\pi(1) = \text{id}_{\mathcal{H}}$, hence π is non-degenerate and also its restriction to Cyl_b is. Let Ω_ν be a unit vector in \mathcal{H}_ν which is cyclic for Cyl_b . Define the normalized, positive linear functional on Cyl_b given by

$$\omega_\nu(f) := \langle \Omega_\nu, \pi_\nu(f)\Omega_\nu \rangle_{\mathcal{H}_\nu} \quad (3.2)$$

Since Cyl_b is an Abelian C^* -algebra, by Gel'fands theorem, we may think of it as the algebra of continuous functions $C(\overline{\mathcal{A}})$ on the Gel'fand spectrum $\overline{\mathcal{A}}$ of Cyl_b . Since $\overline{\mathcal{A}}$ is a compact Hausdorff space, by the Riesz representation theorem, the positive linear functional ω_ν uniquely determines a regular Borel probability measure μ_ν on $\overline{\mathcal{A}}$ via

$$\omega_\nu(f) = \int_{\overline{\mathcal{A}}} d\mu_\nu(A) f(A) \quad (3.3)$$

hence we may choose w.l.g. $\Omega_\nu = 1$, $\pi_\nu(f) = f \text{id}_{\mathcal{H}_\nu}$ (multiplication operator) and $\mathcal{H}_\nu = L_2(\overline{\mathcal{A}}, d\mu_\nu)$ (by the GNS construction, any other choice corresponds to a unitary transformation).

□

A generic element $\psi \in \mathcal{H}$ is therefore given by

$$\psi = \oplus_\nu \psi_\nu = \sum_\nu \psi_\nu 1^\nu \quad (3.4)$$

where $\psi_\nu \in \mathcal{H}_\nu$, $1^\nu = \delta_{\nu'}^\nu \cdot 1$ and $\sum_\nu \|\psi_\nu\|_\nu^2 < \infty$. Here we have denoted the norm on \mathcal{H}_ν by $\|\cdot\|_\nu$.

It follows that

$$\pi(f)\psi = \oplus_\nu f\psi_\nu = f \oplus_\nu \psi_\nu = f\psi \quad (3.5)$$

whence

$$\pi(f) = f \text{id}_{\mathcal{H}} \quad (3.6)$$

is simply a multiplication operator on \mathcal{H} .

Notice that while the subalgebra Cyl_b has no off-diagonal entries, that is in general not the case for the $\pi(W_t^n(S))$. Also, while π_ν is a cyclic representation for Cyl_b , π is not necessarily cyclic for Cyl_b , one will generically assume it to be cyclic for the full algebra \mathfrak{A} only (that is, there is a vector $\Omega \in \mathcal{H}$ such that the set of states given by $\pi(a)\Omega$, $a \in \mathfrak{A}$ is dense in \mathcal{H}). In what follows we will only consider representations which are cyclic for \mathfrak{A} (otherwise we can decompose π further into cyclic representations by the above theorem, hence cyclic representations are the basic building blocks).

This all that one can say so far about general representations of \mathfrak{A} without making further assumptions. To get further structural control over the representation theory one must examine restricted situations of physical interest. In the next section we will study the important class of diffeomorphism invariant representations which are those realized in nature (nature *is* diffeomorphism invariant, so there is no need to study other representations at all, at least from a physics point of view).

4 Diffeomorphism Invariant Representations of \mathfrak{A} and a Uniqueness Theorem

The group $\text{Diff}^\omega(\Sigma)$ of analytic diffeomorphisms on Σ has a natural representation as outer automorphisms on \mathfrak{A} defined for any $\varphi \in \text{Diff}^\omega(\Sigma)$ by

$$\begin{aligned} \alpha_\varphi(p_\gamma^* f_\gamma) &= p_{\varphi^{-1}(\gamma)}^* f_\gamma \\ \alpha_\varphi(W_t^n(S)) &= W_t^{n \circ \varphi}(\varphi^{-1}(S)) \end{aligned} \quad (4.1)$$

and extended by the automorphism property $\alpha_\varphi(ab) = \alpha_\varphi(a)\alpha_\varphi(b)$, $\alpha_\varphi(a + zb) = \alpha_\varphi(a) + z\alpha_\varphi(b)$. It is trivial to check that $\alpha_\varphi \circ \alpha_{\varphi'} = \alpha_{\varphi \circ \varphi'}$.

Likewise, the set $\text{Fun}(\Sigma, G)$, which forms a group under pointwise multiplication, has a natural representation as outer automorphisms on \mathfrak{A} defined for any $g \in \text{Fun}(\Sigma, G)$ by

$$\begin{aligned} [\alpha_g(p_\gamma^* f_\gamma)](A) &= f_\gamma(\{g(b(e))A(e)g(f(e))^{-1}\}) \\ \alpha_g(W_t^n(S)) &= W_t^{n^g}(S) \end{aligned} \quad (4.2)$$

where $n^g(x) = \text{Ad}_{g(x)}(n(x))$, $n(x) = n_j \tau_j$. As one can check, with these definitions we have $\alpha_\varphi \circ \alpha_g \circ \alpha_{\varphi^{-1}} = \alpha_{\varphi * g}$ so that the combined kinematical gauge group acquires the structure of a semidirect product $\mathcal{G} = \text{Fun}(\Sigma, G) \triangleright \text{Diff}^\omega(\Sigma)$ if we define $\alpha_{(g,\varphi)} := \alpha_g \circ \alpha_\varphi$ with $\text{Fun}(\Sigma, G)$ as invariant subgroup.

Definition 4.1.

i)
A cyclic representation π of \mathfrak{A} is said to be diffeomorphism invariant provided that there is a unitary representation

$$U_\pi : \text{Diff}^\omega(\Sigma) \rightarrow \mathcal{B}(\mathcal{H}); \varphi \mapsto U_\pi(\varphi) \quad (4.3)$$

of the diffeomorphism group and a cyclic invariant vector $\Omega \in \mathcal{H}$ such that

$$U_\pi(\varphi)\pi(a)U_\pi(\varphi)^{-1} = \pi(\alpha_\varphi(a)) \text{ and } U_\pi(\varphi)\Omega = \Omega \quad (4.4)$$

for all $a \in \mathfrak{A}$, $\varphi \in \text{Diff}^\omega(\Sigma)$. There are similar definitions of gauge invariant or kinematically invariant representations when replacing $\text{Diff}^\omega(\Sigma)$ by $\text{Fun}(\Sigma, G)$ or \mathcal{G} .

ii)

Consider each $\psi \in \mathcal{H}$ as a vector-valued function of $A \in \overline{\mathcal{A}}$ according to

$$\psi(A) := \oplus_\nu \psi_\nu(A) \quad (4.5)$$

The **natural** (pull-back) representation of $\text{Diff}^\omega(\Sigma)$ is defined by

$$U_\pi(\varphi)\psi := \oplus_\nu \alpha_\varphi(\psi_\nu) \quad (4.6)$$

Likewise the **natural** representation of $\text{Fun}(\Sigma, G)$ is defined by

$$U_\pi(g)\psi := \oplus_\nu \alpha_g(\psi_\nu) \quad (4.7)$$

The name *natural* representation is due to the fact that it is the natural lift of the action of diffeomorphisms or gauge transformations on functions of \mathcal{A} , that is $f(\varphi^* A) \cong [\alpha_\varphi(f)](A)$ and $f(A^g A) \cong [\alpha_g(f)](A)$ to functions of $\overline{\mathcal{A}}$. The natural representation has the feature of leaving constant functions invariant, which are therefore natural candidates for cyclic invariant vectors.

A natural starting point for cyclic invariant representations exists, provided one manages to find a positive linear functional ω on \mathfrak{A} with the invariance property

$$\omega(\alpha_\varphi(a)) = \omega(a) \quad (4.8)$$

for all $a \in \mathfrak{A}$, $\varphi \in \text{Diff}^\omega(\Sigma)$. Namely, let $\pi_\omega, \Omega_\omega, \mathcal{H}_\omega$ be the GNS data for ω [21], that is,

$$\Omega_\omega := [1], \pi_\omega(a)\Omega_\omega := [a], \langle \pi_\omega(a)\Omega_\omega, \pi_\omega(b)\Omega_\omega \rangle := \omega(b^*a) \quad (4.9)$$

where $[a]$ is the equivalence class $\{a + b; \omega(b^*b) = 0\}$. Then

$$U_\omega(\varphi)\pi_\omega(a)\Omega_\omega := \pi_\omega(\alpha_\varphi(a))\Omega_\omega \quad (4.10)$$

represents the diffeomorphism group unitarily as inner automorphisms of $\mathcal{B}(\mathcal{H}_\omega)$ with Ω_ω as cyclic invariant vector.

Using the language of the present paper, in [11] the following result was established.

Theorem 4.1.

Suppose that

- 1) $G = U(1)$.
 - 2) π is cyclic already for Cyl_b so that necessarily $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu)$ with cyclic vector $\Omega = 1$ by lemma 3.1.
 - 3) π is diffeomorphism invariant with Ω as invariant cyclic vector where the diffeomorphisms act by pull back.
 - 4) The one parameter subgroups $t \mapsto \pi(W_t^n(S))$ are weakly continuous.
 - 5) Ω is in the domain of any self-adjoint generator $-i[\frac{d}{dt}]_{t=0}\pi(W_t^n(s))$.
- Then necessarily $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu_0) = \mathcal{H}_0$ is the Ashtekar – Lewandowski representation.

Several of the assumptions of theorem 4.1 are unsatisfactory: First of all, the restriction to $U(1)$ makes it of limited physical relevance since in particular loop quantum gravity would need such a result for general compact groups. Next, it is not natural to require that already Cyl_b is cyclic for the representation, the most general interesting representations will be those for which only the full algebra \mathfrak{A} is cyclic. Furthermore, while it is natural to assume that the constants are in the domain of the self-adjoint generators of the Weyl elements (because the unit function is a cyclic vector for Cyl_b), one has no intuition whether there are not more general representations which violate this assumption.

On the other hand, if one does not assume weak continuity of the fluxes then the requirements will be too weak to limit the number of possible representations. This is already the case for the Schrödinger representation of ordinary quantum mechanics: If one gives up weak continuity of the Weyl elements then many more representations exist which are not captured by the Stone – von Neumann theorem. In fact, the Stone – von Neumann theorem not only requires the representation to be cyclic but even to be irreducible (that is, every vector is cyclic), otherwise also more representations result. We thus expect to find a strong result also only in the irreducible case. Irreducibility is actually more physical than cyclicity since then no non-trivial invariant subspaces exist and moreover, there are no distinguished cyclic elements. We do not know at present whether cyclicity is actually enough for the result to be proved below.

There is one more unnatural assumption in theorem 4.1: Why should it be the case that the vector 1 is left invariant by $U_\pi(\varphi)$? If we have only cyclicity of \mathfrak{A} then it is also not clear why it should be the vector 1 which is cyclic. Actually, this discussion leads to the representation theory of $\text{Diff}^\omega(\Sigma)$ as the following discussion reveals:

Suppose that we do have a diffeomorphism invariant representation π . Then the action of $U_\pi(\varphi)$ is known on the whole representation space \mathcal{H} provided we know it on the 1^ν because

$$U_\pi(\varphi)\psi = U_\pi(\varphi) \sum_\nu \pi(\psi_\nu)1^\nu = \sum_\nu \pi(\alpha_\varphi(\psi_\nu))U_\pi(\varphi)1^\nu \quad (4.11)$$

The natural pull – back representation of definition 4.1 would assign $U_\pi(\varphi)1^\nu = 1^\nu$. But can we say more about the possible representations $U_\omega(\varphi)$?

Suppose that we are interested in asymptotically flat situations. Then, if we include among $\text{Diff}^\omega(\Sigma)$ also symmetries of the asymptotically Minkowskian metric, then $\text{Diff}^\omega(\Sigma)$ will contain the asymptotic Poincaré group as a subgroup and one concludes that U_π is in particular a unitary representation of the asymptotic Poincaré group. The *continuous*⁴, irreducible, unitary representations of that group have been classified by Wigner and it seems that one is in a good position. However, it is unclear if and how these representations can be extended to all of $\text{Diff}^\omega(\Sigma)$ and, moreover, it turns out that generic representations will even violate the continuity assumption: For instance, the pull-back representation of one – parameter subgroups of $\text{Diff}^\omega(\Sigma)$ on the Ashtekar – Lewandowski space \mathcal{H}_0 is not weakly continuous [1].

To see that there really is an abundance of unitarily inequivalent representations of $\text{Diff}^\omega(\Sigma)$, suppose that we start from a representation π of \mathfrak{A} on a Hilbert space \mathcal{H} with unitary pull-back representation U_π of $\text{Diff}^\omega(\Sigma)$. Let $W \in \mathcal{B}(\mathcal{H})$ be any bounded operator with bounded inverse which we consider as being of

⁴Here we mean continuity of one parameter subgroups.

the form $W = \pi(a)$ for some $a \in \mathfrak{A}$. Let us also denote $\alpha_\varphi(W) := \pi(\alpha_\varphi(a))$. We claim that

$$U'_\pi(\varphi) := W^{-1}\alpha_\varphi(W)U_\pi(\varphi) \quad (4.12)$$

defines a representation of $\text{Diff}^\omega(\Sigma)$ on \mathcal{H} . We have

$$\begin{aligned} U'_\pi(\varphi)U'_\pi(\varphi') &= W^{-1}\alpha_\varphi(W)U_\pi(\varphi)W^{-1}\alpha_{\varphi'}(W)U_\pi(\varphi') \\ &= W^{-1}\alpha_\varphi(W)[U_\pi(\varphi)W^{-1}U_\pi(\varphi)^{-1}][U_\pi(\varphi)\alpha_{\varphi'}(W)U_\pi(\varphi)^{-1}]U_\pi(\varphi)U_\pi(\varphi') \\ &= W^{-1}\alpha_\varphi(W)\alpha_\varphi(W^{-1})\alpha_\varphi(\alpha_{\varphi'}(W))U_\pi(\varphi \circ \varphi') \\ &= W^{-1}\alpha_{\varphi \circ \varphi'}(W)U_\pi(\varphi \circ \varphi') = U'_\pi(\varphi \circ \varphi') \end{aligned} \quad (4.13)$$

Since the two representations are equivalent,

$$WU'_\pi(\varphi)W^{-1} = U_\pi(\varphi) \quad (4.14)$$

the requirement that also $U'_\pi(\varphi)$ is a unitary representation leads to the condition

$$\alpha_\varphi(WW^\dagger) = WW^\dagger \quad (4.15)$$

and can be satisfied, for instance, if WW^\dagger is a constant matrix. In that case the polar decomposition gives $W = CV$ where C is an arbitrary constant self-adjoint and positive operator while V is unitary. Of course, the question is whether the representation U'_π on \mathcal{H} is unitarily equivalent to U_π . As (4.15) reveals, this will be the case if and only if W is a unitary operator, that is, $C = \text{id}_{\mathcal{H}}$.

One might think that one can bring more structure into the analysis by requiring that the representation U_π to be irreducible as well (not only π) because then it follows from Schur's lemma that $W = \lambda \text{id}_{\mathcal{H}}$ and unitary equivalence requires $|\lambda| = 1$. However, it is well known that interesting representations of the diffeomorphism group are generically quite reducible. For instance, the pull back representation on \mathcal{H}_0 is extremely reducible [1], we have a countably (under suitable superselection criteria [22]) infinite direct sum decomposition

$$\mathcal{H}_0 = \bigoplus_{[\gamma]} \mathcal{H}_0^{[\gamma]} \quad (4.16)$$

where the non-separable, mutually orthogonal, invariant subspaces $\mathcal{H}_0^{[\gamma]}$ are spanned by spin network functions [23] over graphs γ' which belong to one and same (generalized) knot class $[\gamma]$. None of these subspaces alone captures interesting physics and thus it seems to be an unphysical requirement to restrict to irreducible representations of $\text{Diff}^\omega(\Sigma)$.

These cautionary remarks are just to indicate that there are a priori many inequivalent, unitary representations of $\text{Diff}^\omega(\Sigma)$ available and their classification goes beyond the scope of the present paper. The selection of one of them might be comparable to the selection of a definite spin representation of the Poincaré group, however, *it is much more complicated* (the diffeomorphism group is an infinite dimensional group!). Accordingly, we must be modest and specify the representation of $\text{Diff}^\omega(\Sigma)$ in the statement of our theorem below. Obviously, we will choose the pull-back representation which is natural because it is available in any representation of \mathfrak{A} as shown in lemma 3.1.

Before we state our theorem, let us define the notion of a spin network function on $\overline{\mathcal{A}}$.

Definition 4.2.

Choose precisely one representative ρ from each equivalence class of irreducible representations of G , denote by d_ρ the dimension of the representation space of ρ and denote for any $h \in G$ and $M, N = 1, \dots, d_\rho$ by $\rho_{MN}(h)$ the matrix elements of the unitary matrix $\rho(h)$. Consider a graph γ together with a labeling of each of its edges $e \in E(\gamma)$ with label ρ_e, M_e, N_e ; $M_e, N_e = 1, \dots, d_{\rho_e}$ and collect them into a **spin network**

$$s = (\gamma, \vec{\rho} = \{\rho_e\}_{e \in E(\gamma)}, \vec{M} = \{M_e\}_{e \in E(\gamma)}, \vec{N} = \{N_e\}_{e \in E(\gamma)}) \quad (4.17)$$

Then the **spin network function** with label s is given by

$$T_s(A) = \prod_{e \in E(\gamma)} \{\sqrt{d_{\rho_e}} [\rho_e(A(e))]_{M_e N_e}\} \quad (4.18)$$

As one can show, they provide an orthonormal basis for \mathcal{H}_0 .

We can then state our main result.

Theorem 4.2.

Let G be a compact, connected gauge group, Σ an oriented D -manifold with fixed analytic structure and associated diffeomorphism group $\text{Diff}^\omega(\Sigma)$. Let \mathfrak{A} be the Weyl algebra generated from bounded cylindrical functions on the space \mathcal{A} of smooth G -connections over Σ and exponentiated electric fluxes.

Let π be a representation of \mathfrak{A} on a Hilbert space \mathcal{H} with corresponding representation U_π of $\text{Diff}^\omega(\Sigma)$.

Suppose that

i) **Irreducibility**

π is irreducible.

ii) **Continuity**

The one - parameter groups $t \mapsto \pi(W_t^n(S))$ are weakly continuous.

iii) **Diffeomorphism Invariance**

The representation U_π of $\text{Diff}^\omega(\Sigma)$ is unitary and coincides with the natural representation.

If G is not Abelian we also must require⁵:

iv) **Domains**

The vectors 1^ν are in the common dense domain of the operators $\pi(Y_n(S))$, $\pi(Y_n(S))\pi(Y_{n'}(S))$ for any S, n, n' .

Then π is unitarily equivalent to the Ashtekar - Lewandowski representation π_0 .

As we have discussed above, the only weak (that is, possibly overly restrictive) assumption in this theorem left is, as compared to theorem 4.1, that we restrict ourselves to the natural representation of the diffeomorphism group (and the natural representation of the gauge group in the non-Abelian case). In particular, there is no longer a restriction to a particular gauge group, to particular domains or to particular subalgebras which should already be cyclic.

Proof of theorem 4.2:

Before we go into the technical details, let us explain the strategy of the proof:

Step 1: Continuity \Rightarrow Dense Domain \mathcal{D} of Individual Fluxes

The self-adjoint generators $\pi(E_n(S))$ of the unitary groups $t \mapsto \pi(W_t^n(S))$ defined by

$$-i\pi(E_n(S)) = \left[\frac{d}{dt}\right]_{t=0}\pi(W_t^n(S)) \tag{4.19}$$

which exist due to our continuity assumption, are not everywhere defined, however, it turns out that one can always find, without additional assumptions, a suitable dense domain \mathcal{D}_S^n of $\pi(E_n(S))$ on which we will be able to work out the consequences of diffeomorphism invariance and unitarity of the Weyl elements.

Step 2: Analytic Diffeomorphisms

This step is technical and prepares for step 3 in which existence and properties of certain analytic diffeomorphisms are needed. If the assumptions i), ii), iii) and iv) displayed in theorem 4.2 are supplemented by an assumption v) about the gauge group then this step can be avoided as we show in appendix A.

Step 3: Unitarity + Diffeomorphism Invariance $\Rightarrow \mu_\nu = \mu_0$

The domain \mathcal{D}_S^n contains (multiples of) the so-called spin-network functions [23]. These are labeled, among other things, by a graph γ and an irreducible representation π_e for each edge $e \in E(\gamma)$. One can show that linear combinations of those are eigenfunctions of (polynomials of) the $\pi(E_n(S))$ with an eigenvalue which depends on the π_e and on the number of intersections N_e of S with e . Using unitarity of the Weyl elements or the corresponding symmetry of the fluxes one can establish a relation between this eigenvalue,

⁵At least presently. Our results for the Abelian case indicate that this condition can be abolished, however, we were not able to circumvent it for now.

which diverges as $N_e \rightarrow \infty$, and the expectation value of the spin network functions (considered as bounded operators on \mathcal{H}_ν), which is bounded, because due to diffeomorphism invariance that expectation value is actually independent of N_e . This leads to a contradiction unless $\mu_\nu = \mu_0$ is the Ashtekar – Lewandowski measure.

Step 4: Diffeomorphism Invariance + Irreducibility $\Rightarrow \mathcal{H} = \mathcal{H}_0$

The results of step 3 do not yet exclude the possibility that $\pi(W_t^n(S))$ has off-diagonal action on the direct sum of the $\mathcal{H}_\nu \cong \mathcal{H}_\nu$. Making use of our already available knowledge that all the Hilbert spaces are Ashtekar Lewandowski Hilbert spaces with explicitly known spin network basis, it is possible to show that $\pi(W_t^n(S)) = W_t^n(S)\text{id}_{\mathcal{H}}$ is diagonal and all entries are equal. This contradicts the irreducibility condition unless there is only one copy of \mathcal{H}_0 .

From the structure of the proof it is clear that all assumptions are used in an essential way, in particular, there would be no result if weak continuity is given up. Let us now go to the details.

Step 1:

Let us write $S_n := (n, S)$ in what follows and similarly $W_t(S_n) := W_t^n(S)$. The following trick for how to construct a dense domain for all the fluxes follows the proof of Stone’s theorem that establishes a one – to one correspondence between self-adjoint operators on a Hilbert space and weakly continuous one parameter unitary groups.

Lemma 4.1.

Let $\phi \in C_c^\infty(\mathbb{R})$ be a smooth test function For any $\psi \in \mathcal{H}$ and any S_n define

$$\psi_{\phi, S_n} := \int_{\mathbb{R}} dt \phi(t) \pi(W_t(S_n)) \psi \tag{4.20}$$

Then for each S_n the finite linear combinations of the vectors $\pi(T_s)1_{\phi, S_n}^\nu$ form a dense set $\mathcal{D}(S_n)$ in \mathcal{H} as s, ϕ, ν vary.

Proof of lemma 4.1:

The functions $C(\overline{\mathcal{A}})$ are dense in $\mathcal{H}_\nu = L_2(\overline{\mathcal{A}}, d\mu_\nu)$ for any ν . Finite linear combinations of spin network functions (which form a unital $*$ -subalgebra of $\text{Cyl}^\infty \subset C(\overline{\mathcal{A}})$ which separates the points of $\overline{\mathcal{A}}$) are dense in $C(\overline{\mathcal{A}})$ with respect to the sup norm $\|\cdot\|_\infty$ on $C(\overline{\mathcal{A}})$ by the Weierstrass theorem. Since $\|\cdot\|_2 \leq \|\cdot\|_\infty$ it follows that finite linear combinations of spin network functions are dense in any \mathcal{H}_ν , hence finite linear combinations of the functions $\pi(T_s)1^\nu$ are dense in \mathcal{H} as s, ν vary.

Now for any $\psi \in \mathcal{H}$

$$\|\psi_{\phi, S_n} - \psi\| \leq \int dt \phi(t) \|[\pi(W_t(S_n)) - \text{id}_{\mathcal{H}}]\psi\| \tag{4.21}$$

can be made arbitrarily small due to the assumed weak continuity of the Weyl elements by suitably restricting

the support of ϕ . Hence the $[\pi(T_s)1^\nu]_{\phi, S_n}$ lie dense as s, ν, ϕ vary for any S_n . Thus

$$\begin{aligned}
& \| \pi(T_s)1^\nu_{\phi, S_n} - [\pi(T_s)1^\nu]_{\phi, S_n} \| \\
&= \| \int dt \phi(t) [\pi(T_s)\pi(W_t(S_n)) - \pi(W_t(S_n))\pi(T_s)]1^\nu \| \\
&\leq \int dt \phi(t) \| [\pi(T_s)\pi(W_t(S_n)) - \pi(W_t(S_n))\pi(T_s)]1^\nu \| \\
&= \int dt \phi(t) \| [\pi(W_t(S_n))^{-1}\pi(T_s)\pi(W_t(S_n)) - \pi(T_s)]1^\nu \| \\
&= \int dt \phi(t) \| \pi(W_t(S_n))^{-1}T_s W_t(S_n) - T_s \|1^\nu \| \\
&= \int dt \phi(t) \| W_t(S_n)^{-1}T_s W_t(S_n) - T_s \|_\nu \\
&\leq \int dt \phi(t) \| W_t(S_n)^{-1}T_s W_t(S_n) - T_s \|_\infty
\end{aligned} \tag{4.22}$$

where in the third step we have used unitarity of the Weyl elements, in the fourth we have used the representation property, in the fifth we have used that the argument of π is just a function which acts as a multiplication operator and in the sixth step we have again used continuity with respect to the sup norm $\|\cdot\|_\nu \leq \|\cdot\|_\infty$.

Since T_s is a continuous (on \bar{A}) cylindrical function over $\gamma(s)$ and right translation $A(e) \mapsto e^{tn^j \tau_j} A(e)$ is continuous in t , it follows that (4.22) can be made arbitrarily small by suitably restricting the support of ϕ . \square

Corollary 4.1.

The set of vectors $\pi(f)1^\nu_{\phi, S_n}$, as $f \in \text{Cyl}^\infty, \nu$ vary, form a dense set of C^∞ vectors for the self-adjoint generator $\pi(E(S_n))$ of $\pi(W_t(S_n))$, more precisely

$$-i\pi(E(S_n))\pi(f)1^\nu_{\phi, S_n} := \left[\frac{d}{dt}\right]_{t=0}\pi(W_t(S_n))\pi(f)1^\nu_{\phi, S_n} = \pi(Y(S_n)f)1^\nu_{\phi, S_n} + \pi(f)1^\nu_{-\dot{\phi}, S_n} \tag{4.23}$$

In particular, $\mathcal{D}(S_n)$ is a dense invariant domain for $\pi(E(S_n))$.

Proof of corollary 4.1:

We have

$$\begin{aligned}
\pi(W_t(S_n))\pi(f)1^\nu_{\phi, S_n} &= \pi(W_t(S_n)fW_t(S_n)^{-1}) \int ds \phi(s)\pi(W_{t+s}(S_n))1^\nu \\
&= \pi(W_t(S_n)fW_t(S_n)^{-1}) \int ds \phi(s-t)\pi(W_s(S_n))1^\nu
\end{aligned} \tag{4.24}$$

Observing the Weyl relations, differentiation of (4.24) in the strong sense yields (4.23) (for details see [24]). \square

Step 2:

Lemma 4.2.

Let S be an analytic, oriented, open surface and e an oriented, analytic path. Let p_1, \dots, p_m , $m \geq 1$ be fixed interior points of e and choose $\sigma_k \in \{-1, +1\}$, $k = 1, \dots, m$. Then there exists an analytic diffeomorphism $\varphi_{m, \vec{\sigma}}$ such that $\varphi_{m, \vec{\sigma}}^{-1}(S)$ intersects e precisely in the points p_1, \dots, p_m and such that

$$\sigma([\varphi_{m, \vec{\sigma}}^{-1}(S)]_{p_k}, [e]_{p_k}) = \sigma_k \tag{4.25}$$

Proof of lemma 4.2:

We certainly find a smooth diffeomorphism $\varphi_{m,\bar{\sigma}}^\infty$ with the required properties. Consider a compact set C containing e and the algebra of real valued, continuous functions S_C generated by the functions

$$\{\varphi|_C^a; \varphi \in \text{Diff}^\omega(\sigma), a = 1, \dots, D\} \quad (4.26)$$

Then S_C separates the points of C (choose $\varphi = \text{id}_\sigma$) and does not leave any point $x_0 \in C$ invariant. Hence by the Weierstrass theorem S_C is dense in the set $C(C)$ of continuous functions on C and since $\varphi_{m,\bar{\sigma}}^\infty$ is continuous we find an analytic diffeomorphism $\varphi_{m,\bar{\sigma}}^0$ that approximates it uniformly on C in the sup norm. While the intersection points p'_k of $[\varphi_{m,\bar{\sigma}}^0]^{-1}(S)$ with e may not yet coincide with the p_k (although they are arbitrarily close), it is nevertheless true that $\sigma([\varphi_{m,\bar{\sigma}}^0]^{-1}(S))|_{p'_k}, [e]_{p'_k} = \sigma_k$ since the σ functions take only discrete values.

We will now construct successively analytic diffeomorphisms φ_k , $k = 1, \dots, m$ which preserve e such that $\varphi_k^{-1}(p_l) = p_l$, $l = 0, \dots, k-1$, $\varphi_k^{-1}(p'_k) = p_k$, $\varphi_k^{-1}(p_{m+1}) = p_{m+1}$ where $p_0 = b(e)$, $p_{m+1} = f(e)$ and such that the σ_k are not changed. Then

$$\varphi_{m,\bar{\sigma}} := \varphi_{m,\bar{\sigma}}^0 \circ \varphi_1 \circ \dots \circ \varphi_m \quad (4.27)$$

provides the searched for diffeomorphism.

To construct φ_k explicitly, choose w.l.g. an analytic coordinate system such that e coincides with the interval $[0, 1]$ of the x^1 -axis (if e does not lie entirely within the domain of a chart, replace e by a closed segment of it that does in what follows). Then $p_k = (x_k, 0, \dots, 0)$ and $p'_k = (y_k, 0, \dots, 0)$ are the coordinates of the points in question and we label them in such a way that $x_0 = 0 < x_1 < \dots < x_m < x_{m+1} = 1$, $0 < y_1 < \dots < y_m < 1$. The situation for φ_k is such that $y_l = x_l$, $l = 1, \dots, k-1$ already while the y_l , $l = k, \dots, m$ are unspecified. Thus, the idea is to construct an analytic vector field $x \mapsto v_k(x)$ on \mathbb{R} which has zeroes at the points $x_0 = 0, x_1, \dots, x_{k-1}, x_{m+1} = 1$ and whose flow maps y_k to x_k . Consider the analytic vector field on Σ defined in our coordinate system by $\vec{v}_k(\vec{x}) = (v_k(x^1), 0, \dots, 0)$. The integral curves $c_{\vec{x}}^{\vec{v}_k}(t)$ it generates defines a one parameter family of analytic diffeomorphisms $\varphi_t^{\vec{v}_k}(\vec{x}) := c_{\vec{x}}^{\vec{v}_k}(t)$ of the form

$$\varphi_t^{\vec{v}_k}(\vec{x}) = (\varphi_t^{v_k}(x^1), x^2, \dots, x^D) \quad (4.28)$$

where $\varphi_t^{v_k}(x^1)$ is the corresponding one parameter group in \mathbb{R} . It follows that $\varphi_t^{\vec{v}_k}(\vec{x})$ for any t is just an x^1 dependent translation along the x^1 -axis so that the σ_k are left invariant. It remains to construct $v_k(x)$ and to choose t with the required properties. Notice that if v_k has a zero at some x then x is a fixed point of $\varphi_t^{v_k}$.

Our ansatz is given for $\delta > 0$ by

$$v_k(x) = \text{sgn}(x_k - y_k) [1 - e^{-\frac{(x-1)^2}{2\delta^2}}] \prod_{l=0}^{k-1} [1 - e^{-\frac{(x-x_l)^2}{2\delta^2}}] \quad (4.29)$$

where $\text{sgn}(x) = 1, -1, 0$ if $x >, <, = 0$ is the sign function. The flow thus by construction preserves $x = 0, x_1, \dots, x_{k-1}, 1$ and moves, apart from the fix points, into positive or negative x^1 -direction respectively if $x_k - y_k > / < 0$ respectively (if $x_k = y_k$ already we can choose $v_k = 0$ obviously). We set $\alpha = \text{sgn}(x_k - y_k)$ in the remainder of this proof.

Consider the function $f(t, x) := v_k(x)$ which does not depend explicitly on t . We have

$$|f(t, x) - f(t, x')| = \left| \int_x^{x'} dy v'_k(y) \right| \leq \int_x^{x'} dy |v'_k(y)| \leq \frac{m+2}{\delta} |x - x'| \quad (4.30)$$

where we have used that for the function $g(x) = 1 - e^{-x^2/(2\delta^2)}$ holds $0 \leq g(x) \leq 1$ and $|g'(x)| \leq e^{-1/2}/\delta$. Thus the function $f(t, x)$ satisfies a global Lipschitz condition in the open domain $G = \mathbb{R} \times [a, b]$ where $[0, 1] \subset [a, b]$ defines the boundaries of our chart in x^1 direction. By the existence and uniqueness theorem

of Picard – Lindelöf for the differential equation $\dot{x}(t) = f(t, x)$, for any $(t_0, x_0) \in G$ there exists $\epsilon > 0$ and a unique solution in $t \in [t_0 - \epsilon, t_0 + \epsilon]$ with initial condition $x(t_0) = x_0$. The number ϵ is bounded by $\min(r, \frac{c}{L})$ where r is the maximal number such that $V_r := (t_0 - r, t_0 + r) \times (x_0 - r, x_0 + r)$ is still in G while $c = \sup_{(t,x) \in V_r} |f(t, x)|$. In our case $t_0 = 0$, $x_0 = y_k$, $r = \min(y_k - a, b - y_k)$, $c = 1$ so that $\epsilon = r$ is arbitrarily large by choosing coordinates in which $|a|, |b|$ become arbitrarily large and the segment of e considered remains the image of $[0, 1]$. It follows that we find a solution of the differential equation $\dot{c}_{y_k}(t) = v_k(c_{y_k}(t))$ with $c_{y_k}(0) = y_k$ which exists for $t \in [-r, r]$ for arbitrarily large but finite r .

Notice that after applying the $k - 1$ th diffeomorphism φ_{k-1} we have already achieved that $y_l = x_l$, $l = 1, \dots, k-1$ (the order of the points y_k cannot be changed by a diffeomorphism) and our job is now to construct φ_k which has to move $y_k > y_{k-1} = x_{k-1}$ to x_k while leaving $x = 0, x_1, \dots, x_{k-1}, 1$ fixed. Define now

$$\mu := \frac{1}{2} \min(y_k - x_{k-1}, x_k - x_{k-1}, 1 - \frac{1+\alpha}{2}x_k - \frac{1-\alpha}{2}y_k) \quad (4.31)$$

and the interval $I_k = [x_{k-1} + \mu, 1 - \mu]$. Consider the differential equation $\dot{x}(t) = v_k(x(t))$, $x(0) = y_k$. As long as $x(t) \in I_k$ we have $|x(t) - 1| \geq \mu$, $|x(t) - x_l| \geq \mu$; $l = 0, \dots, k-1$ and thus $\alpha v_k(x(t)) \geq [1 - \exp(-\frac{\mu^2}{2\delta^2})]^{k+1}$. On the other hand, $\alpha v_k(x(t)) \leq 1$ for all $t \in \mathbb{R}$. From the integral equation

$$x(t) = y_k + \int_0^t ds v(x(s)) \quad (4.32)$$

we thus conclude for $t \geq 0$

$$y_k + t \geq x(t) \geq y_k + [1 - \exp(-\frac{\mu^2}{2\delta^2})]^{k+1} t \quad (4.33)$$

for $\alpha = 1$ and inequality signs reversed in case $\alpha = -1$, as long as $x(t) \in I_k$. Now $y_k - (x_{k-1} + \mu) \geq \frac{y_k - x_{k-1}}{2} \geq 0$ and $1 - \mu - y_k \geq \frac{1+x_k}{2} - y_k \geq 0$ if $\alpha = 1$ because $x_k, 1 \geq y_k$ or $1 - \mu - y_k \geq \frac{1+y_k}{2} - y_k \geq 0$ if $\alpha = -1$. Hence $y_k \in I_k$ and since all tree terms in (4.33) are monotonously increasing (decreasing) with t for $\alpha = 1$ ($\alpha = -1$) we can guarantee $x(t) \in I_k$ by requiring $t \leq T$ where $y_k + T = 1 - \mu$ for $\alpha > 0$ and $y_k - T = x_{k-1} + \mu$ for $\alpha < 0$. We conclude that

$$x(T) \geq y_k + [1 - \exp(-\frac{\mu^2}{2\delta^2})]^{k+1} (1 - \mu - y_k) \quad (4.34)$$

for $\alpha > 0$ and

$$x(T) \leq y_k + [1 - \exp(-\frac{\mu^2}{2\delta^2})]^{k+1} (y_k - \mu - x_{k-1}) \quad (4.35)$$

for $\alpha < 0$. We claim that we can choose δ small enough such that the right hand side of (4.34) ((4.35)) is bigger (lower) than x_k . Thus we must satisfy

$$\ln(1 - e^{-\frac{\mu^2}{2\delta^2}}) \geq \frac{1}{k+1} \ln\left(\frac{x_k - y_k}{1 - \mu - y_k}\right) \quad (4.36)$$

for $\alpha > 0$ and

$$\ln(1 - e^{-\frac{\mu^2}{2\delta^2}}) \geq \frac{1}{k+1} \ln\left(\frac{y_k - x_k}{y_k - \mu - x_{k-1}}\right) \quad (4.37)$$

for $\alpha < 0$. Notice that the argument of the logarithm on the right hand side of (4.36) and (4.37) respectively is smaller than one since $x_k < 1 - \mu$ and $\mu + x_{k-1} < x_k$ respectively. Since both hand sides of (4.36), (4.37) are negative we must show

$$|\ln(1 - e^{-\frac{\mu^2}{2\delta^2}})| \leq \frac{1}{k+1} |\ln\left(\frac{x_k - y_k}{1 - \mu - y_k}\right)| \quad (4.38)$$

for $\alpha > 0$ and similar for $\alpha < 0$. Let $h(x) = -\ln(1 - x)$, $0 \leq x < 1$. We claim that $h(x) \leq \sqrt{x}$. For $k(x) := \sqrt{x} + \ln(1 - x)$ we find $k(0) = 0$, $k'(x) > 0$ for $x < (\sqrt{2} - 1)^2$. Hence, (4.38) holds provided that

$$e^{-\frac{\mu^2}{4\delta^2}} \leq \frac{1}{k+1} |\ln\left(\frac{x_k - y_k}{1 - \mu - y_k}\right)| \quad (4.39)$$

holds and δ is small enough so that $e^{-\frac{\mu^2}{2\delta^2}} < 3 - \sqrt{8} < 1$. We conclude

$$-\frac{\mu^2}{4\delta^2} \leq \ln\left(\frac{1}{k+1} \left| \ln\left(\frac{x_k - y_k}{1 - \mu - y_k}\right) \right| \right) \quad (4.40)$$

which is trivially satisfied if the right hand side is positive (which it will not be when k is large). If the right hand side is negative we find

$$\delta^2 < \min\left(\frac{\mu^2}{2 \ln(3 - \sqrt{8})}, \frac{\mu^2}{4 \left| \ln\left(\frac{1}{k+1} \left| \ln\left(\frac{x_k - y_k}{1 - \mu - y_k}\right) \right| \right) \right|}\right) \quad (4.41)$$

which can always be satisfied. The case $\alpha < 0$ is similar.

From the continuity of the solution $x(t)$ we conclude that there exists $t_k \in [0, T]$ such that $x(t) = x_k$. Thus, if $t \mapsto \varphi_{k,t}$ is the one parameter family of diffeomorphisms generated by $-v_k$ we define $\varphi_k := \varphi_{k,t_k}$ and have $\varphi_k(x_l) = x_l$, $l = 0, \dots, k-1$, $l = m$ and $\varphi_k(y_k) = x_k$ as desired, which concludes our induction step. \square

Step 3:

This step contains the main argument in our proof. The self-adjoint generator $\pi(Y_n(S))$ is symmetric and has dense domain $\mathcal{D}(S_n)$ for any n . From (4.23) we find the symmetry condition

$$\begin{aligned} & \langle \pi(f)1_{\phi, S_n}^\nu, \pi(E(S_n))\pi(f')1_{\phi', S_n}^{\nu'} \rangle \\ &= i \langle \pi(f)1_{\phi, S_n}^\nu, \pi(Y(S_n)f')1_{\phi', S_n}^{\nu'} + \pi(f')1_{-\phi', S_n}^{\nu'} \rangle \\ &= \langle \pi(E(S_n))\pi(f)1_{\phi, S_n}^\nu, \pi(f')1_{\phi', S_n}^{\nu'} \rangle \\ &= -i \langle \pi(Y(S_n)f)1_{\phi, S_n}^\nu + \pi(f)1_{-\phi, S_n}^\nu, \pi(f')1_{\phi', S_n}^{\nu'} \rangle \end{aligned} \quad (4.42)$$

Choose $f = 1$ then we obtain the master condition

$$\begin{aligned} & - \langle 1_{\phi, S_n}^\nu, \pi(Y(S_n)f')1_{\phi', S_n}^{\nu'} \rangle \\ &= \langle 1_{\phi, S_n}^\nu, \pi(f')1_{-\phi', S_n}^{\nu'} \rangle + \langle 1_{-\phi, S_n}^\nu, \pi(f')1_{\phi', S_n}^{\nu'} \rangle \end{aligned} \quad (4.43)$$

We now split the proof into the Abelian case and the non-Abelian case because we are able to use less assumptions in the Abelian case. We will also indicate what goes wrong when trying to repeat the Abelian proof for the non-Abelian case which might lead the ambitious reader to a method for how to circumvent the obstacle. Also, we offer two different proof methods in the non-Abelian case.

Abelian Case

We will carry out the proof just for one copy of $U(1)$, the general case is similar. In that case $n_j = 1$ and we can drop the label n from $Y_n(S)$ and $W_t^n(S)$. We employ some of the ideas already used in [11].

Choose $f' = \alpha_{\varphi_m^{e,S}}(T_s)$ where T_s is a spin-network function and $\varphi_m^{e,S}$ is an analytic diffeomorphism such that $(\varphi_m^{e,S})^{-1}(\gamma(s))$ intersects S in precisely m distinct points p_1, \dots, p_m which are interior points of the edge $e \in E(\gamma(s))$ and such that $\sigma([S]_{p_k}, [(\varphi_m^{e,S})^{-1}(e)]_{p_k}) = 1$. The existence of such a diffeomorphism can be deduced from the first half of lemma 4.2. Notice that we only need one of the 2^m diffeomorphisms constructed in lemma 4.2 for the Abelian case. The irreducible representations of $U(1)$ are one dimensional so that π_e is just determined by an integer λ_e . Moreover $Y(S)f' = m\lambda_e f'$ is an eigenfunction. Thus (4.43) becomes

$$-m\lambda_e \langle 1_{\phi, S}^\nu, \pi(f')1_{\phi', S}^{\nu'} \rangle = \langle 1_{\phi, S}^\nu, \pi(f')1_{-\phi', S}^{\nu'} \rangle + \langle 1_{-\phi, S}^\nu, \pi(f')1_{\phi', S}^{\nu'} \rangle \quad (4.44)$$

We now estimate the left hand side and the right hand side of (4.44). For the right hand side we have

$$\begin{aligned}
& | \langle 1_{\phi,S}^\nu, \pi(f') 1_{-\dot{\phi},S}^{\nu'} \rangle + \langle 1_{-\dot{\phi},S}^\nu, \pi(f') 1_{\phi,S}^{\nu'} \rangle | \\
& \leq \|1_{\phi,S}^\nu\| \|\pi(f') 1_{-\dot{\phi},S}^{\nu'}\| + \|1_{-\dot{\phi},S}^\nu\| \|\pi(f') 1_{\phi,S}^{\nu'}\| \\
& \leq \|f'\|_\infty [\|1_{\phi,S}^\nu\| \|1_{-\dot{\phi},S}^{\nu'}\| + \|1_{-\dot{\phi},S}^\nu\| \|1_{\phi,S}^{\nu'}\|] \\
& \leq \|T_s\|_\infty [\|\phi\|_1 \|\dot{\phi}'\|_1 + \|\dot{\phi}\|_1 \|\phi'\|_1]
\end{aligned} \tag{4.45}$$

where in the second step we have used the Cauchy-Schwarz inequality, in the third the elementary continuity of the operator norm with respect to the C^* -norm which is diffeomorphism invariant (so the dependence on $\varphi_m^{e,S}$ drops out) and finally $\|\psi_\phi\| \leq \int dt |\phi(t)| \quad \|\pi(W_t(S))\psi\| \leq \|\phi\|_1 \|\psi\|$ due to unitarity. For the left hand side of (4.44) we have

$$\begin{aligned}
& | -m\lambda_e \langle 1_{\phi,S}^\nu, \pi(f') 1_{\phi',S}^{\nu'} \rangle | \\
& \geq m|\lambda_e| [| \langle 1^\nu, \pi(f') 1^{\nu'} \rangle | - | \langle 1_{\phi,S}^\nu - 1^\nu, \pi(f') 1^{\nu'} \rangle | - | \langle 1_{\phi,S}^\nu, \pi(f') (1_{\phi',S}^{\nu'} - 1^{\nu'}) \rangle |] \\
& \geq m|\lambda_e| [| \langle 1^\nu, \pi(f') 1^{\nu'} \rangle | - \|T_s\|_\infty \{ \|1_{\phi,S}^\nu - 1^\nu\| + \|1_{\phi,S}^{\nu'} - 1^{\nu'}\| \}]
\end{aligned} \tag{4.46}$$

Suppose now that for $\lambda_e \neq 0$ we have

$$\delta = | \langle 1^\nu, \pi(f') 1^{\nu'} \rangle | = | \langle 1^\nu, \pi(T_s) 1^{\nu'} \rangle | > 0 \tag{4.47}$$

where we have made use of diffeomorphism invariance of $1^\nu, 1^{\nu'}$ so that the only dependence on the intersection number m in (4.46) is just the prefactor. Due to weak continuity of the fluxes, we can restrict the support of ϕ, ϕ' to $|t| \leq \epsilon(\nu, \delta/4, S, s)$ and $|t| \leq \epsilon(\nu', \delta/4, S, s)$ respectively such that $\|1_{\phi,S}^\nu - 1^\nu\| \leq \delta/(4\|T_s\|_\infty)$ and $\|1_{\phi',S}^{\nu'} - 1^{\nu'}\| \leq \delta/(4\|T_s\|_\infty)$ respectively. Now choose for the so chosen ϕ, ϕ' the intersection number

$$m > 2 \frac{\|f'\|_\infty [\|\phi\|_1 \|\dot{\phi}'\|_1 + \|\dot{\phi}\|_1 \|\phi'\|_1]}{\delta}$$

Then we have produced a contradiction, thus $\langle 1^\nu, \pi(T_s) 1^{\nu'} \rangle = 0$ unless $\lambda_e = 0$. Since $e \in E(\gamma(s))$ was arbitrary and since $1^\nu, \pi(f) 1^{\nu'} = \mu_\nu(f) \delta_{\mu\nu}$ we conclude

$$\mu_\nu(T_s) = \begin{cases} 1 & : s \text{ trivial} \\ 0 & : \text{otherwise} \end{cases} \tag{4.48}$$

which is one of the equivalent definitions of the Ashtekar-Lewandowski measure.

Non-Abelian Case

Before we proceed to the proof under the additional assumption iv) let us first outline why the strategy for the Abelian case does not work here. The basic problem is that no $f' = T_s$ spin-network function is an eigenfunction $Y_n(S) T_s \not\propto T_s$ of the fluxes. Without this being the case, we have no chance to get something like $Y_n(S) \alpha_{\varphi_m^{e,S}}(T_s) \propto m$ which was crucial in the Abelian case. Thus, in order to get this proportionality the idea is to consider something squares of the form $Y_n(S)^2 \alpha_{\varphi_m^{e,S}}(T_s)$ which have better chances because if we take suitable linear combinations then we obtain Laplacians which do have the T_s as eigenfunctions.

In order to get such squares we just have to iterate (4.43) or (B.9) by choosing $f' = Y_n(S)f$ or $f' = Y_j^+(S)f$ for some f to be suitably chosen. This results in

$$\begin{aligned}
& \langle 1_{\phi,S_n}^\nu, \pi(Y(S_n)^2 f) 1_{\phi',S_n}^{\nu'} \rangle \\
& = \langle 1_{\phi,S_n}^\nu, \pi(f) 1_{\phi',S_n}^{\nu'} \rangle + 2 \langle 1_{-\dot{\phi},S_n}^\nu, \pi(f) 1_{-\dot{\phi}',S_n}^{\nu'} \rangle + \langle 1_{\dot{\phi},S_n}^\nu, \pi(f) 1_{\phi',S_n}^{\nu'} \rangle
\end{aligned} \tag{4.49}$$

Let us now choose $f = \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)$ for any spin network function T_s where $\varphi_{m,\sigma}^{e,S}$ is the diffeomorphism constructed in lemma 4.2 such that $\varphi_{m,\sigma}^{e,S}(S)$ intersects $\gamma(s)$ precisely in m interior points p_k of $e \in E(\gamma(s))$ with relative orientation σ_k . Let us write $e = f_1^{-1} \circ e_1 \circ f_2^{-1} \circ e_2 \circ \dots \circ f_m^{-1} \circ e_m$ where $p_k = f_k \cap e_k$. Then

$$\begin{aligned} & \alpha_{\varphi_{m,\sigma}^{e,S}}(Y_n(S)^2)T_s \\ &= \sum_{I,J=1}^m \sigma_I \sigma_J (R_{e_I}^n - R_{f_I}^n)(R_{e_J}^n - R_{f_J}^n)T_s \\ &= \sum_{I=1}^m [2(R_{e_I}^n)^2 + 2(R_{f_I}^n)^2]T_s + 8 \sum_{I < J}^m \sigma_I \sigma_J R_{e_I}^n R_{e_J}^n T_s \end{aligned} \quad (4.50)$$

where in the second step we have used gauge invariance of T_s at p_k , that is, $(R_{e_I}^n + R_{f_I}^n)T_s = 0$. Now the idea would be to choose $n^k = \delta_{jk}$, to sum over j and to average over the 2^m possible choices for $\sigma = (\sigma_1, \dots, \sigma_m)$. Thus

$$2^{-m} \sum_j \sum_{\sigma} \alpha_{\varphi_{m,\sigma}^{e,S}}(Y_j(S)^2)T_s = -8m \lambda_{\pi_e} T_s \quad (4.51)$$

where $\pi_e = \pi_e(s)$ is the irreducible representation of G labeling e and $-\lambda_{\pi_e} \leq 0$ is the corresponding eigenvalue of the Laplacian. In order to exploit this, we write (4.49) as

$$\begin{aligned} & 2^{-m} \sum_j \sum_{\sigma} \langle 1_{\phi,S_j}^{\nu}, \pi(Y(S_j)^2 \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)) 1_{\phi',S_j}^{\nu'} \rangle \\ &= 2^{-m} \sum_j \sum_{\sigma} \langle 1_{\phi,(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu}, \pi(\alpha_{\varphi_{m,\sigma}^{e,S}}(Y_j(S)^2)T_s) 1_{\phi',(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu'} \rangle \\ &= 2^{-m} \sum_j \sum_{\sigma} [\langle 1_{\phi,(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu}, \pi(T_s) 1_{\phi',(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu'} \rangle + 2 \langle 1_{-\phi,(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu}, \pi(T_s) 1_{-\phi',(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu'} \rangle \\ & \quad + \langle 1_{\phi,(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu}, \pi(T_s) 1_{\phi',(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu'} \rangle] \end{aligned} \quad (4.52)$$

Now the trouble is that a (j, σ) -dependence has entered the states $1_{\phi,(\varphi_{m,\sigma}^{e,S})^{-1}(S_j)}^{\nu}$ which prevents us from using (4.51). In order to get rid of the (j, σ) -dependence of the states, we would need to construct states which are common C^∞ -vectors for all the $\alpha_{\varphi_{m,\sigma}^{e,S}}(Y_j(S))$. That is certainly possible: First of all we pass to the $Y_j^+(S)$ of appendix B and choose as S a surface which is the disjoint union of m pieces S_I . Then define

$$1_{\phi,S}^{\nu} := \int_G d\mu_H(g_1) \dots \int_G d\mu_H(g_m) \phi(g_1, \dots, g_m) \prod_{I=1}^m \pi(W_{g_I}(S_I)) 1^{\nu} \quad (4.53)$$

and if we can arrange that $S_{\sigma} = \varphi_{m,\sigma}^{e,S}{}^{-1}(S) = \cup_I \sigma_I S_I$ then

$$Y_j^+(S_{\sigma}) 1_{\phi,S}^{\nu} = 1_{-\sum_I \sigma_I R_I^j \phi, S}^{\nu} \quad (4.54)$$

for all j, σ . However, the sum over I involved in (4.54) destroys our estimates performed in the Abelian case for which it was crucial that the right hand side of (4.44) was already independent of m . This is the obstacle that prevents us from using the idea employed for the Abelian case. Thus, in order to proceed, let us make the additional assumption iv). While we feel that this is not necessary, we could so far not find a way to circumvent the obstacle just mentioned.

According to assumption iv) we can take the limit $\phi(t) \rightarrow \delta(t)$ in (4.52). Then the (j, σ) -dependence

disappears from the states, $1^\nu_{\phi, (\varphi_{m,\sigma}^{e,S})^{-1}(S_j)} \rightarrow 1^\nu$, and we find

$$\begin{aligned}
& 2^{-m} \sum_j \sum_\sigma \langle 1^\nu, \pi(Y(S_j)^2 \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)) 1^{\nu'} \rangle \\
&= -8\lambda_{\pi_e} m \langle 1^\nu, \pi(T_s) 1^{\nu'} \rangle \\
&= \sum_j [\langle 1^\nu, \pi(2^{-m} \sum_\sigma \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)) \pi(Y_j(S)^2) 1^{\nu'} \rangle + 2 \langle \pi(Y_j(S)) 1^\nu, \pi(2^{-m} \sum_\sigma \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)) \pi(Y_j(S)) 1^{\nu'} \rangle \\
&\quad + \langle \pi(Y_j(S)^2) 1^\nu, \pi(2^{-m} \sum_\sigma \alpha_{\varphi_{m,\sigma}^{e,S}}^{-1}(T_s)) 1^{\nu'} \rangle] \tag{4.55}
\end{aligned}$$

Estimating the right hand side of (4.56) from above we have

$$\begin{aligned}
& 8\lambda_{\pi_e} m |\langle 1^\nu, \pi(T_s) 1^{\nu'} \rangle| \\
&\leq \|T_s\|_\infty \sum_j [\|\pi(Y_j(S)^2) 1^{\nu'}\| + \|\pi(Y_j(S)) 1^\nu\| \|\pi(Y_j(S)) 1^{\nu'}\| + \|\pi(Y_j(S)^2) 1^\nu\|] \tag{4.56}
\end{aligned}$$

Due to the diffeomorphism invariance of 1^ν the right hand side of (4.56) no longer depends on m in contrast to the left hand side which implies as in the Abelian case that $\mu_\nu = \mu_0$ is the Ashtekar Lewandowski measure.

Step 4:

We stress that the additional requirement iv) is only necessary in step 3 of the proof. The remainder is again independent of that. It rests crucially on our already available knowledge that all μ_ν equal the Ashtekar-Lewandowski measure.

We make the general ansatz

$$\pi(W_t(S_n)) = [M_t(S_n)] [W_t(S_n) \otimes \pi(1)] \tag{4.57}$$

where the second factor acts diagonally in each entry ψ_ν of $\psi = \oplus \psi_\nu$ by left translation on $\overline{\mathcal{A}}$ and $M_t(S_n)$ is an operator valued matrix of the form

$$M_t(S_n) \psi = \sum_{\nu, \nu'} ([M_t(S_n)]_{\nu\nu'} \cdot \psi_{\nu'}) 1^\nu \tag{4.58}$$

Using the representation property

$$\pi(W_t(S_n)) \pi(f) \pi(W_t(S_n))^{-1} = \pi(W_t(S_n) f W_t(S_n)^{-1}) \tag{4.59}$$

and that f is arbitrary we conclude that

$$[(M_t(S_n))_{\nu\nu'}, f] = 0 \tag{4.60}$$

for any $f \in \text{Cyl}_b$ and any ν, ν' . It follows that $M_t(S_n)$ is a multiplication operator valued matrix.

Due to the unitarity of $\pi(W_t(S_n))$ we compute

$$\begin{aligned}
1 &= \|\pi(W_t(S_n)) 1^\nu\|^2 = \|M_t(S_n) 1^\nu\|^2 = \|\sum_{\nu'} ([M_t(S_n)]_{\nu'\nu} \cdot 1) 1^{\nu'}\|^2 \\
&= \sum_{\nu'} \|[M_t(S_n)]_{\nu'\nu}\|_{\mu_{\nu'}}^2 \tag{4.61}
\end{aligned}$$

for any ν . Thus all the matrix entries $[M_t(S_n)]_{\nu\nu'}$ are $L_2(\overline{\mathcal{A}}, d\mu_0)$ functions and we can expand them in terms of spin network functions

$$[M_t(S_n)]_{\nu\nu'} = \sum_s [Z_t(S_n, s)]_{\nu\nu'} T_s \tag{4.62}$$

for some complex valued coefficients $[M_t(S_n, s)]_{\nu\nu'}$ of which all but countably many must vanish.

From diffeomorphism covariance (remember that U_π is the natural representation of the diffeomorphism group) we have for $n_j(x) = n_j = \text{const}$.

$$\begin{aligned} U_\pi(\varphi)\pi(W_t^n(S))U_\pi(\varphi)^{-1} &= \pi(\alpha_\varphi(W_t^n(S))) = \pi(W_t^n(\varphi^{-1}(S))) = M_t^n(\varphi^{-1}(S))[W_t^n(\varphi^{-1}(S)) \otimes \pi(1)] \\ &= [U_\pi(\varphi)M_t(S_n)U_\pi(\varphi)^{-1}] [U_\pi(\varphi)[W_t^n(S) \otimes \pi(1)]U_\pi(\varphi)^{-1}] \\ &= [U_\pi(\varphi)M_t(S_n)U_\pi(\varphi)^{-1}] [W_t^n(\varphi^{-1}(S)) \otimes \pi(1)] \end{aligned} \quad (4.63)$$

and

$$U_\pi(\varphi)\pi(T_s)U_\pi(\varphi)^{-1} = \pi(\alpha_\varphi(T_s)) = \pi(T_{\varphi(s)}) \quad (4.64)$$

where $\varphi(s) = (\varphi^{-1}(\gamma(s)), \pi(\vec{s}), \vec{M}(s), \vec{N}(s))$, we conclude by comparing coefficients that

$$[Z_t^n(\varphi^{-1}(S), s)]_{\nu,\nu'} = [Z_t^n(S, \varphi(s))]_{\nu\nu'} \quad (4.65)$$

for any s .

Suppose now that $\gamma(s) \neq \emptyset$ for $D > 3$ or $\gamma(s) \neq \emptyset, \partial S$ for $D = 3$ (neither the empty graph nor the graph formed by the boundary of the closure of S , that is $\gamma(s) \neq \bar{S} - \text{Int}(S)$) or $\gamma(s) \neq \emptyset, \bar{S}$ for $D = 2$. Then we find a countably infinite number of analytic diffeomorphisms φ_k which leave S invariant but such that the $\varphi_k(\gamma(s))$ are mutually different. To construct such a diffeomorphism for $D > 1$, simply take any analytical vector field which is everywhere tangent to S and tangent to ∂S (e.g. vanishes on (non differentiable points of) ∂S). Then $S, \partial S$ are left invariant as sets, but not pointwise, by the one parameter group of analytical diffeomorphisms generated by that vector field. Thus for $D > 3$ even a graph which lies completely within the closure \bar{S} can be mapped non-trivially, for $D = 2$ the graph cannot be mapped non-trivially only if $\gamma(s) = S$ and for $D = 3$ we must have $\gamma(s) = \partial S$. Thus, unless one of the cases indicated holds, we always find a one parameter group of analytical diffeomorphisms $t \mapsto \varphi^t$ which preserve S but move $\gamma(s)$ non-trivially for each t and we just need to take $\varphi_k = \varphi^{1/k}$. But this implies that

$$[Z_t^n(S, s)]_{\nu\nu'} = [Z_t^n(S, \varphi_k(s))]_{\nu\nu'} \quad (4.66)$$

for all $k = 0, 1, 2, \dots$. Due to the mutual orthogonality of spin network functions over mutually different graphs, (4.66) contradicts normalizability (4.61) unless $[Z_t^n(S, s)]_{\nu\nu'} = 0$ for such s since

$$\|M_t^n(S)_{\nu\nu'}\|^2 \geq \sum_{s', \gamma(s')=\gamma(s)} |[Z_t^n(S, s')]_{\nu\nu'}|^2 \geq \sum_{k=0}^{\infty} |[Z_t^n(S, \varphi_k(s))]_{\nu\nu'}|^2 = |[Z_t^n(S, s)]_{\nu\nu'}|^2 \sum_{k=0}^{\infty} 1 \quad (4.67)$$

We conclude that $M_t^n(S)$ is a matrix of cylindrical L_2 -functions over the graph ∂S in $D = 3$ or over S in $D = 2$ and it is a constant function in $D > 3$. Hence we may write it in the form

$$M_t^n(S)_{\nu\nu'} = K_t^n(S)_{\nu\nu'} + \sum_I K_t^n(S, I)_{\nu\nu'} T_{\gamma_S, I} \quad (4.68)$$

where $K_t^n(S)_{\nu,\nu'}$, $K_t^n(S, I)_{\nu,\nu'}$ are constants, $\gamma_S = \bar{S}$ in $D = 2$, $\gamma_S = \partial S$ in $D = 3$, I denotes a sum over spin network labels other than the graph and of course $K_t^n(S, I)_{\nu,\nu'} = 0$ for $D > 3$. Since $\partial\varphi(S) = \varphi(\partial S)$ we find that these constant matrices only depend on the diffeomorphism type \hat{S} of the surface S , that is,

$$K_t^n(S)_{\nu,\nu'} = K_t^n(\hat{S})_{\nu,\nu'} \text{ and } K_t^n(S, I)_{\nu,\nu'} = K_t^n(\hat{S}, I)_{\nu,\nu'} \quad (4.69)$$

Let us now consider the cases $D = 2, 3$ more closely. Since by construction our Weyl algebra of fluxes is built from the fluxes through a disjoint union of cubes \square , the associated $\pi(W_t^n(\square))$ are mutually commuting and it will be sufficient to consider each \square separately. We may write

$$[M_t^n(\square)](A) = \begin{cases} \rho_t^n(A(\square)) & D = 2 \\ \rho_t^n(A(\partial\square)) & D = 3 \end{cases} \quad (4.70)$$

where we have dropped the label \square since the diffeomorphism class types of all our \square coincide. Now we can subdivide $\square = \square_1 \cup \square_2$ into two disjoint pieces. Since the corresponding Weyl operators commute and since $[W_t^n(\square_i), M_t^n(\square_j)_{\nu\nu'}] = 0$ because the edges of the graph \square_j or $\partial\square_j$ are either of the “in” or “out” type with respect to \square_i , we easily find

$$\begin{aligned} D = 2 & : \rho_t^n(A(\square_1))\rho_t^n(A(\square_2)) = \rho_t^n(A(\square)) = \rho_t^n(A(\square_1 \circ \square_2)) = \rho_t^n(A(\square_1)A(\square_2)) \\ D = 3 & : \rho_t^n(A(\partial\square_1))\rho_t^n(A(\partial\square_2)) = \rho_t^n(A(\partial\square)) = \rho_t^n(A(\partial\square_1 \circ \partial\square_2)) = \rho_t^n(A(\partial\square_1)A(\partial\square_2)) \end{aligned} \quad (4.71)$$

where we have chosen appropriate starting points of the edges \square and loops $\partial\square$ respectively.

Since $A \in \overline{\mathcal{A}}$ is arbitrary we find that for arbitrary $h_1, h_2 \in G$

$$\rho_t^n(h_1)\rho_t^n(h_2) = \rho_t^n(h_2)\rho_t^n(h_1) = \rho_t^n(h_1h_2) \quad (4.72)$$

where commutativity follows from the commutativity of the corresponding Weyl operators. Setting e.g. $h_1 = 1_G$ we see that $\rho_t^n(1_G) = \pi(1)$ is the identity operator. It follows that $h \mapsto \rho_t^n(h)$ is a commutative representation of G on the not necessarily separable Hilbert space $\ell_2(\mathcal{N})$ where \mathcal{N} denotes the countable index set of the labels ν . Let us denote the inner product on $\ell_2(\mathcal{N})$ by $(\cdot, \cdot)'$, that is $(v, v')' = \sum_{\nu} \bar{v}_{\nu} v'_{\nu}$. Consider the new inner product

$$(v, v') := \int_G d\mu_H(g) (\rho_t^n(g)v, \rho_t^n(g)v')' \quad (4.73)$$

We must check whether (4.73) is well defined. The vectors 1^{ν} form a complete orthonormal basis in ℓ_2 with respect to the inner product $(\cdot, \cdot)'$ and the new Hilbert space is the completion with respect to the new inner product of the finite linear combinations of the 1^{ν} , so it suffices to check that $(1^{\nu}, 1^{\nu}) < \infty$ for all ν . We have

$$(1^{\nu}, 1^{\nu}) = \int_G d\mu_H(g) \sum_{\nu'} |(\rho_t^n)_{\nu'\nu}(g)|^2 \leq \sum_{\nu'} \|M_t^n(\square)_{\nu'\nu}\|_{\mu_0}^2 = 1 \quad (4.74)$$

by (4.61) because $M_t^n(S)$ depends on the connection only through the edge \square or the loop $\partial\square$ respectively. Hence (4.73) is well-defined and $g \mapsto \rho_t^n(g)$ is a unitary representation of G on this Hilbert space with inner product (\cdot, \cdot) .

Since G is compact, ρ_t^n , represented on that Hilbert space is unitarily equivalent to a (possibly uncountably) direct sum of irreducible, finite dimensional representations [25] (proposition 2.5 and theorem 3.1) all of which must be commutative. If G is not Abelian, then the only commutative irreducible representations are trivial and it follows immediately $\rho_t^n(h) = \pi(1)$ for all $h \in G$. If G is Abelian then $G = U(1)^N$ for some N and every irreducible representation is of the form $(u_1, \dots, u_N) \mapsto (u_1^{z_1}, \dots, u_N^{z_N})$ for some integers z_k and any $u_k \in U(1)$. In our case the representation of every $U(1)$ factor that occurs in the decomposition of $\rho_t^n(h)$ into irreducibles is therefore of the form $u \mapsto u^{z_t^n}$ where $z_t^n \in \mathbb{Z}$ and $u \in U(1)$. Due to the representation property $\pi(W_s^n(\square))\pi(W_t^n(\square)) = \pi(W_{s+t}^n(\square))$ for all $s, t \in \mathbb{R}$ and due to the fact that all edges in question are of the “in” or “out” type with respect to S we infer that $\rho_s^n(h)\rho_t^n(h) = \rho_{s+t}^n(h)$ is a one-parameter group of representations. This implies that $z_{s+t}^n = z_s^n + z_t^n$ for any $s, t \in \mathbb{R}$. Due to weak continuity we have $\rho_t^n(h) \rightarrow \pi(1)$ as $t \rightarrow 0$. Since z_t^n is an integer, there exists $\epsilon^n > 0$ such that $z_t^n = 0$ for all $|t| < \epsilon^n$. But then for any $t \in \mathbb{R}$ we find $m \in \mathbb{N}$ such that $|t/m| < \epsilon^n$ and thus $z_t^n = m z_{t/m}^n = 0$. Thus, also in the Abelian case the only occurring representation is trivial and we also get here that $\rho_t^n(h) = \pi(1)$.

It remains to discuss the case $D > 3$. Since in this case $M_t^n(\square) = M_t^n(\square)$ is just a constant we have by splitting $\square = \square_1 \cup \square_2$ into disjoint pieces that

$$M_t^n(\square) = M_t^n(\square) = M_t^n(\square_1 \cup \square_2) = M_t^n(\square_1) M_t^n(\square_2) = M_t^n(\square)^2 \quad (4.75)$$

and since $M_t^n(\square)$ is invertible we find $M_t^n(\square) = \pi(1)$.

We conclude that $M_t^n(S) = \pi(1)$ for $n^j(x) = n^j = \text{const.}$ and any (allowed) surface S . For an arbitrary unit vector $n_j(x)$ we find a constant unit vector n_j^0 and an element $g_{n,n^0} \in \text{Fun}(\Sigma, G)$ such that $\alpha_{g_{n,n^0}}(W_t^n(S)) = W_t^{n^0}(S)$. Then

$$\begin{aligned} \pi(W_t^n(S)) &= \pi(\alpha_{g_{n,n^0}^{-1}}(W_t^{n^0}(S))) = U_\pi(g_{n,n^0})^{-1} \pi(W_t^{n^0}(S)) U_\pi(g_{n,n^0}) \\ &= U_\pi(g_{n,n^0})^{-1} [W_t^{n^0}(S) \otimes \pi(1)] U_\pi(g_{n,n^0}) = [\alpha_{g_{n,n^0}^{-1}}(W_t^{n^0}(S)) \otimes \pi(1)] = [W_t^n(S) \otimes \pi(1)] \end{aligned} \quad (4.76)$$

so that $M_t^n(S) = \pi(1)$ also in the general case.

We thus have shown that $\pi(W_t^n(S)) = W_t^n(S) \otimes \pi(1)$. We can now finally invoke irreducibility: If the representation is to be irreducible, then every vector is cyclic, in particular any of the 1^ν is cyclic. But the algebra of operators generated by $\pi(f), \pi(W_t^n(S))$ never leaves the sector $\mathcal{H}_\nu = \mathcal{H}_0 \otimes 1^\nu$. It follows that we can allow only one copy of the Ashtekar-Lewandowski Hilbert space. That \mathcal{H}_0 itself is the representation space of an irreducible representation of \mathfrak{A} will be shown in [26].

This finishes the proof.

□

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A A simpler proof under stronger assumptions

In this appendix we present a somewhat simpler proof of theorem 4.2 which works under the following assumption in addition to i), ii), iii) and iv) specified there:

v) Gauge Invariance

The representation U_π of $\text{Fun}(\Sigma, G)$ is unitary and coincides with the natural representation.

Under this assumption we can avoid step 2 in the proof of the theorem altogether by means of the following observation:

While the proof in the main text never uses gauge invariance, the following replacement of the non-Abelian part of step 3 heavily relies on gauge invariance of the 1^ν which follows from our assumption v) made above.

According to assumption iv) we may let $\phi(t) \rightarrow \delta(t, 0)$ in (4.42) and (4.43) and in this limit the (S, n) –dependence of all the states $1'_{\phi, S_n} \rightarrow 1^\nu$ vanishes.

Let $\varphi_m^{S,e}$ be an analytic diffeomorphism that makes $\alpha_{\varphi_m^{S,e}}(T_s)$ intersect with S in precisely m interior

points p_k of $\varphi_m^{S,e}(e)$, say with positive relative orientation. Then (4.42) implies

$$\begin{aligned}
& \sum_j \langle 1^\nu, \pi(Y(S_j)^2 \alpha_{\varphi_m^{S,e}}^{-1}(T_s)) 1^{\nu'} \rangle \\
&= 8 \langle 1^\nu, \pi(-m\lambda_{\pi_e} T_s + \sum_j \sum_{I < J}^m R_{e_I}^n R_{e_J}^n T_s) 1^{\nu'} \rangle \\
&= \sum_j [\langle 1^\nu, \pi(\alpha_{\varphi_m^{S,e}}^{-1}(T_s)) \pi(Y_j(S)^2) 1^{\nu'} \rangle + 2 \langle \pi(Y_j(S)) 1^\nu, \pi(\alpha_{\varphi_m^{S,e}}^{-1}(T_s)) \pi(Y_j(S)) 1^{\nu'} \rangle \\
&\quad + \langle \pi(Y_j(S)^2) 1^\nu, \pi(\alpha_{\varphi_m^{S,e}}^{-1}(T_s)) 1^{\nu'} \rangle] \tag{A.1}
\end{aligned}$$

Let $G_p := \text{Fun}(\{p_k\}_{k=1}^m, \sigma)$ be the subgroup of the gauge group which is trivial except at the points p_k . Due to the gauge invariance of the 1^ν and the normalization of the Haar measure we have for the left hand side of (A.1)

$$\begin{aligned}
& 8 \langle 1^\nu, \pi(-m\lambda_{\pi_e} T_s + \sum_j \sum_{I < J}^m R_{e_I}^j R_{e_J}^j T_s) 1^{\nu'} \rangle \\
&= 8 \int_{G_p} d\mu_0(g) \langle U_\pi(g) 1^\nu, \pi(-m\lambda_{\pi_e} T_s + \sum_j \sum_{I < J}^m R_{e_I}^j R_{e_J}^j T_s) U_\pi(g) 1^{\nu'} \rangle \\
&= 8 \int_{G_p} d\mu_0(g) \langle 1^\nu, \pi(\alpha_{g^{-1}}(-m\lambda_{\pi_e} T_s + \sum_j \sum_{I < J}^m R_{e_I}^j R_{e_J}^j T_s)) 1^{\nu'} \rangle \\
&= 8 \int_{G^m} d\mu_H(g_1) \dots d\mu_H(g_m) \langle 1^\nu, \pi(-m\lambda_{\pi_e} T_s + \sum_j \sum_{I < J}^m \text{ad}_{j_k}(g_I) \text{ad}_{j_l}(g_J) R_{e_I}^k R_{e_J}^l T_s) 1^{\nu'} \rangle \\
&= -8m\lambda_{\pi_e} \langle 1^\nu, \pi(T_s) 1^{\nu'} \rangle \tag{A.2}
\end{aligned}$$

where in the third step we have used G_p -invariance of T_s and in the fourth the Peter&Weyl theorem. Here μ_0 is the induced Haar measure on G_p . Hence, since the right hand side of (A.1) is again independent of m , we conclude $\mu_\nu = \mu_0$.

Thus, averaging over the gauge group has the same effect as averaging over the discrete subset of the diffeomorphism group that was constructed in step 2 which therefore becomes redundant.

In conclusion, this way of proving theorem 4.2 uses step 1, avoids step 2 altogether, modifies the non-Abelian part of step 3 as just displayed and leaves step 4 unmodified.

B Construction of more general C^∞ -vectors for the Fluxes

In order to supplement the discussion in the main text around the non-Abelian part of step 3 in the proof of theorem 4.2 we list here some additional information about the construction of more general C^∞ -vectors for the flux operators⁶.

A) If $n_j(x) = \text{const.}$ we may also get rid of the n -dependence of the ψ_{ϕ, S_n} in (4.20) as follows: We notice that each flux vector field can be uniquely split as $Y_n(S) = Y_n^+(S) - Y_n^-(S)$ where

$$Y_n^\pm(S) = \sum_{x \in S} n_j(x) \sum_{[e] \in \mathcal{E}_x^\pm(S)} R_{x,[e]}^j =: \sum_{x \in S} n_j(x) R_{x,S,\pm}^j \tag{B.1}$$

⁶In order to use them one would need to replace in assumption iv) of theorem 4.2 the $Y_n(S)$ by the $Y_n^+(S)$ constructed below.

and $\mathcal{E}_x^\pm(S) = \{[e] \in \mathcal{E}; \sigma([S]_x, [e]) = \pm 1\}$. It is easy to check that for the semisimple generators

$$Y_j^\pm(S) = \frac{1}{2}([Y_k(S), Y_l(S)]f_{klj} \pm Y_j(S)) \quad (\text{B.2})$$

where $Y_j(S) = Y_n(S)$ for $n_k = \delta_{jk}$ so that the algebra generated by the $Y_n(S)$ allows us to isolate the pieces $Y_j^\pm(S)$. For the Abelian generators we define $Y_j^+(S) = Y_j(S)$. The $Y_j^+(S)$ satisfy the simple algebra

$$[Y_j^+(S), Y_k^+(S)] = f_{jkl}Y_l^+(S) \quad (\text{B.3})$$

(Actually, since $\text{Lie}(S)$ is semisimple, any $A \in \text{Lie}(S)$ can be written as $A = [B, C]$ so that we can also isolate the $Y_n^\pm(S)$ for any n such that $n^j\tau_j \in \text{Lie}(S)$.) Since Vec contains them, we are allowed to construct the corresponding Weyl elements

$$W_t(S) := \exp(t^j Y_j^+(S)) \quad (\text{B.4})$$

and since the $Y_j^+(S)$ have the same commutation relations as the Lie algebra basis elements τ_j we conclude

$$W_t(S)W_{t'}(S) := W_{c(t,t')}(S) \quad (\text{B.5})$$

where the composition function c is defined as follows: For any compact, connected gauge group the exponential map $t^j \mapsto g(t) := \exp(t^j\tau_j)$ is surjective, hence we find a subset $R \subset \mathbb{R}^{|\dim(G)|}$ so that it becomes a bijection. We then may define $c: R \times R \rightarrow R$ uniquely by

$$e^{c^j(t,t')\tau_j} := e^{t^j\tau_j} e^{t'^j\tau_j} \quad (\text{B.6})$$

Then (B.5) follows. Consider now for any $\psi \in \mathcal{H}$ and $\phi \in C^\infty(G)$ the vector

$$\psi_{\phi,S} := \int_G d\mu_H(g) \phi(g) \pi(W_g(S)) \psi \quad (\text{B.7})$$

where $W_{g(t)}(S) := W_t(S)$. Then (B.1) is a C^∞ vector for all $\dim(G)$ operators $\pi(Y_j^+(S))$, namely

$$-i\pi(Y_j^+(S))\psi_{\phi,S} = \psi_{-R^j\phi,S} \quad (\text{B.8})$$

due to the invariance of the Haar measure. It follows by similar arguments as those displayed in corollary 4.1 that the vectors $\pi(f)1_{\phi,S}^\nu$ provide a common dense domain $\mathcal{D}(S)$ of C^∞ -vectors for all $Y_j^+(S)$.

B)
By methods similar to those displayed in equations (4.42) and (4.43) one can show that symmetry of the fluxes implies for $n^j(x) = \text{const}$.

$$- \langle 1_{\phi,S}^\nu, \pi(Y_n^+(S)f')1_{\phi',S}^{\nu'} \rangle = \langle 1_{\phi,S}^\nu, \pi(f')1_{-R_n\phi',S}^{\nu'} \rangle + \langle 1_{-R_n\phi,S}^\nu, \pi(f')1_{\phi',S}^{\nu'} \rangle \quad (\text{B.9})$$

and

$$\begin{aligned} & \langle 1_{\phi,S}^\nu, \pi((Y_n^+(S))^2 f)1_{\phi',S}^{\nu'} \rangle \\ &= \langle 1_{\phi,S}^\nu, \pi(f)1_{(R_n)^2\phi',S}^{\nu'} \rangle + 2 \langle 1_{-R_n\phi,S}^\nu, \pi(f)1_{-R_n\phi',S}^{\nu'} \rangle + \langle 1_{R_n^2\phi,S}^\nu, \pi(f)1_{\phi',S}^{\nu'} \rangle \end{aligned} \quad (\text{B.10})$$

where now the n -independent C^∞ -vectors of A) are used.

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