

# The $\mathcal{N} = 4$ SYM Integrable Super Spin Chain

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## Abstract

Recently it was established that the one-loop planar dilatation generator of  $\mathcal{N} = 4$  Super Yang-Mills theory may be identified, in some restricted cases, with the Hamiltonians of various integrable quantum spin chains. In particular Minahan and Zarembo established that the restriction to scalar operators leads to an integrable vector  $\mathfrak{so}(6)$  chain, while recent work in QCD suggested that restricting to twist operators, containing mostly covariant derivatives, yields certain integrable Heisenberg XXX chains with non-compact spin symmetry  $\mathfrak{sl}(2)$ . Here we unify and generalize these insights and argue that the *complete* one-loop planar dilatation generator of  $\mathcal{N} = 4$  is described by an integrable  $\mathfrak{su}(2, 2|4)$  super spin chain. We also write down various forms of the associated Bethe ansatz equations, whose solutions are in one-to-one correspondence with the complete set of *all* one-loop planar anomalous dimensions in the  $\mathcal{N} = 4$  gauge theory. We finally speculate on the non-perturbative extension of these integrable structures, which appears to involve non-local deformations of the conserved charges.

# 1 Introduction and Overview

Of all four-dimensional gauge field theories the one with the maximum number  $\mathcal{N} = 4$  [1] of supersymmetries appears to be very special in many ways. At first sight this model appears to be incredibly complicated due to the very large number of fields of different kinds, but upon closer inspection many hidden simplicities appear. It is believed that the model is exactly conformally invariant at the quantum level [2] and thus a non-trivial example (as opposed to free massless field theory) of a *four*-dimensional conformal field theory (CFT). Given the huge successes in understanding CFT's in *two* dimensions, one might hope that at least some of the aspects allowing their treatment in  $D = 2$  might fruitfully reappear in  $D = 4$ . One of the many intriguing features of two-dimensional CFT's is that they are intimately connected to integrable 2+0 dimensional lattice models in statistical mechanics, or, equivalently, to 1+1 dimensional quantum spin chains. Integrable models are even more important for the study of integrable massive deformations of these CFT's. Thus, an optimistic mind could hope that integrability might also play a rôle in the putative simplicity of a theory such as  $\mathcal{N} = 4$  Yang-Mills. Excitingly, a number of recent discoveries lend much support to this idea. One might wonder about standard, naive no-go theorems that seem to suggest that integrability can never exist above  $D = 2$ . These may be potentially bypassed by the fact that there appears to be a hidden “two-dimensionality” in  $SU(N)$  SYM<sub>4</sub> when we look at it at large  $N$ , i.e. when we introduce  $\frac{1}{N}$  as an additional coupling constant in the theory, aside from the 't Hooft coupling  $\lambda = g_{\text{YM}}^2 N$ .

In fact, one might interpret the AdS/CFT correspondence [3], believed to be exact in the  $\mathcal{N} = 4$  case, as one important indication of the validity of this idea: Again following 't Hooft, the worldsheet of the string is a clear candidate for explaining the hidden two dimensions. Interestingly, first indications that the world sheet theory, highly non-trivial due to the curved  $AdS_5 \times S^5$  background, might be integrable have recently appeared [4,5] (for the simpler but related case of pp-wave backgrounds see also [6]).

However, let us get back to the gauge theory and review how integrable structures have recently emerged directly, without resorting to the correspondence. Following the seminal BMN paper [7] it became clear that, despite much work on SYM<sub>4</sub>, not much was known about how to efficiently calculate the anomalous dimensions of local, non-protected conformal operators composed of an arbitrary number of elementary fields. In particular, in [8–10] effective vertex techniques were developed in order to efficiently calculate one-loop two-point functions of operators composed of an arbitrary numbers of scalars. From these two-point correlators the scaling dimensions could then be (indirectly) extracted. The main focus was on the case where the classical dimension of the operators tended to infinity, but the technique was applicable to the finite case as well. In particular [9] contained the full  $\mathfrak{so}(6)$  invariant effective vertex which was used in [11] to find the exact finite dimensional generalizations of the BMN operators. A significant simplification took place when it was realized [12,13] that the effective vertex could be directly interpreted as a *Hamiltonian* acting on a Hilbert space formed by all possible scalar operators of fixed classical dimension. Therefore it was no longer necessary to extract the anomalous dimensions from a two-point function; instead one could directly go about diagonalizing the Hamiltonian. This was done in [13] for a Hilbert

space containing multi-trace gauge-invariant scalar operators; a natural split between a planar “free” part and a non-planar trace-splitting and trace-joining “interaction” part appeared, where the interaction is purely cubic with coupling constant  $\frac{1}{N}$ . This split turned out to be very reminiscent of similar structures appearing in cubic string field theories. In turn, Minahan and Zarembo focused on the planar part of the Hamiltonian and noticed the remarkable fact that it is identical to the one of an  $\mathfrak{so}(6)$  invariant integrable spin chain. The integrability leads, via a hidden Yang-Baxter symmetry, to the appearance of an infinite number of charges commuting with the Hamiltonian. This allows to write down Bethe equations whose solution furnish the spectrum, i.e. the planar anomalous dimensions. For low dimensional operators (i.e. those containing only a few elementary fields) the approach is not necessarily easier than direct, brute force diagonalization of the Hamiltonian (however, it does lead to a very transparent and short derivation of the finite  $J$  BMN operators found in [11]). Its real power lies in the fact that it allows to obtain anomalous dimensions in the limit of a large number of constituent fields.

This power was very recently illustrated in a study of scalar operators belonging to the  $\mathfrak{so}(6)$  representations  $[J_2, J_1 - J_2, J_2]$  with  $\Delta_0 = J_1 + J_2$  in the limit where both charges  $J_1, J_2$  become large [14]. The ground state energy in this representation is argued to be related to the minimal energy solution of a string rotating in two planes in  $S^5$ , which can be calculated exactly [15, 16]. The result for this energy is given by inverting certain elliptic functions, it is a non-trivial *function* of the parameters  $J_1, J_2$ , and agrees on both sides. On the gauge theory side, we believe it can only be obtained by using the Bethe ansatz. This is arguably the most subtle dynamical, quantitative test of AdS/CFT existing to date.

Integrable spin chains had appeared before in gauge theories through the pioneering work of Lipatov on high energy scattering in planar QCD [17]. The model was subsequently identified as a XXX Heisenberg  $\mathfrak{sl}(2)$  spin chain of noncompact spin zero [18]. More recently, and physically closely related to the present study, further integrable structures were discovered by Belitsky, Braun, Derkachov, Korchemsky and Manashov in the computation of planar one-loop anomalous dimensions of various types of quasi-partonic operators in QCD <sup>1</sup> [19]. Subsequently, Kotikov and Lipatov [20] applied the corresponding integrable structure, originally found in the QCD context, to the computation of anomalous dimensions of twist-two operators (i.e. operators containing two scalar fields and an arbitrary number of symmetrized, traceless covariant derivatives acting on them) in  $\text{SYM}_4$ . The result for the anomalous dimension is given in terms of harmonic numbers and was verified by Dolan and Osborn using totally different methods [21]. This result was very useful in (qualitatively) comparing certain string states in  $AdS_5$  to twist-two operators with a large spin in the gauge theory [22]. Very recently Belitsky, Gorsky and Korchemsky [23] considered QCD composite light-cone operators, corresponding to operators of arbitrarily high twist, and discussed the hidden integrability of the associated dilatation operator.

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<sup>1</sup>While QCD is surely not a conformally invariant quantum field theory, it still behaves like one as far as one-loop anomalous dimensions are concerned. We thank A. Belitsky, A. Gorsky and G. Korchemsky for pointing this out to us. In one-loop QCD, integrability is not complete; in particular it requires aligned helicities of the partonic degrees of freedom.

Let us ask the question whether the integrable structures appearing in the planar gauge theory, when one investigates the first radiative corrections to anomalous dimensions, are accidental or rather more generic and thus indicative of a deeper underlying structure. The preliminary evidence from the correspondence with strings [4] certainly points to the second possibility. And indeed a first study of the two-loop corrections to the integrable Hamiltonian yields intriguing evidence that these do *not* break the integrability property [24]. One indirect way to detect the hidden commuting charges is to look for unexpected degeneracies in the energy spectrum. We found that these degeneracies exist in planar SYM<sub>4</sub>, can be explained at one loop by the hidden integrability, and are not broken at the two-loop level. It is important to note that, while the one-loop Hamiltonian for scalar fields is exactly integrable [12], the combined one- and two-loop dilatation operator is not exactly integrable and requires the inclusion of three-loop terms, and so on [24]. Interpreting these corrections in the spin chain language one finds that the higher-loop deformations of the charges are in a certain sense non-local, reminiscent of [4]. The integrable system corresponding to the all-loop planar dilatation operator of  $\mathcal{N} = 4$  Yang-Mills, if it exists, should be some kind of long-range spin chain and has yet to be identified. In particular it is unclear how to extend the Bethe ansatz of [12] to include the higher loop effects.

All this suggests that, maybe, planar  $\mathcal{N} = 4$  Yang-Mills is integrable. If this is the case, the hidden symmetries should extend the known symmetries of SYM<sub>4</sub>. The full symmetry algebra of SYM<sub>4</sub> is neither  $\mathfrak{so}(6)$  nor  $\mathfrak{sl}(2)$ , but the full superconformal algebra  $\mathfrak{psu}(2, 2|4)$ .<sup>2</sup> If the integrable structures discovered to date are not accidental, we should expect that the  $\mathfrak{so}(6)$  one-loop results of Minahan-Zarembo and the  $\mathfrak{sl}(2)$  results suggested from one-loop QCD [19, 23] (see also [20]) can be combined and “lifted” to a full  $\mathfrak{psu}(2, 2|4)$  super spin chain, as first conjectured<sup>3</sup> in [24]. In this paper we will argue that this is indeed the case. In a companion paper, one of us worked out the complete one-loop dilatation operator of  $\mathcal{N} = 4$  Yang-Mills theory [25]. That is, the effective vertex (not necessarily restricted to the planar case) allowing the computation of mixing matrices of composite operators containing arbitrary sequences of elementary fields (scalars, covariant derivatives, field strengths, and fermions) was derived. Upon diagonalization of this matrix (which in practice is however only possible for states of small classical dimension) its eigensystem yields the eigenoperators and their anomalous dimensions. Here we will establish that the restriction of this complete one-loop dilatation operator to the planar case yields the promised  $\mathfrak{su}(2, 2|4)$ <sup>4</sup> spin chain. Our arguments contain some assumptions which should be filled in by experts on integrability; in particular, while we propose an R-matrix, we will assume its existence and uniqueness. At any rate we then proceed to write down the Bethe ansatz equations expected to describe our super spin chain. Again, it is not clear to us whether the validity of the ansatz has been

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<sup>2</sup>The conjugation properties of the algebras are not relevant here, we will always refer to complex algebras, e.g.  $\mathfrak{sl}(4|4) = \mathfrak{su}(2, 2|4)$ ,  $\mathfrak{so}(6) = \mathfrak{sl}(4)$ , etc.

<sup>3</sup>In [23] it was conjectured that the  $\mathfrak{sl}(2)$  and  $\mathfrak{so}(6)$  spin chains combine into a  $\mathfrak{so}(2, 4) \times \mathfrak{so}(6) = \mathfrak{su}(2, 2) \times \mathfrak{su}(4)$  chain. As we shall see, we need the full  $\mathfrak{psu}(2, 2|4)$  algebra in order to achieve this “grand unification”.

<sup>4</sup>At the one-loop level the superconformal algebra  $\mathfrak{psu}(2, 2|4)$  may be extended by a  $\mathfrak{gl}(1)$  charge to the semi-direct product  $\mathfrak{su}(2, 2|4)$ .

rigorously proven in the spin chain literature. However, by analyzing the first few states in our superchain we find complete agreement with the gauge theory results. Therefore we are confident that the details of our Bethe ansatz are indeed correct. A subtlety, to be discussed below, is that for superalgebras one can write down *several* Bethe ansätze, which appear to be different and possess different (pseudo)vacuum states. They nevertheless yield an identical spectrum. We will discuss two specific ansätze which are both natural from different points of view.

The outline of this paper is as follows. We begin with a short review/preview on Bethe ansatz equations, and how they are determined by the Dynkin diagram of the symmetry algebra of the chain in conjunction with the highest weight representation of the degrees of freedom distributed along the chain. We will also introduce some useful, compact notation for writing the Bethe equations. We explain how to employ the latter in order to derive anomalous dimensions in  $\mathcal{N} = 4$  Yang-Mills. Next we present our arguments, via the existence of a unique R-matrix, why one expects the complete one-loop dilatation operator of [25] to become integrable in the large  $N$  limit. The procedure naturally incorporates the  $\mathfrak{sl}(2)$  (twist operators) and the  $\mathfrak{so}(6)$  (Minahan-Zarembo) integrable structures. For super chains we can write seemingly different sets of Bethe ansatz equations. We will pick two examples: The first (which we call “Beauty”) does not correspond to the standard way of choosing the fermionic root(s) of the Dynkin diagram. However, this formulation is particularly useful since the vacuum states of the chain correspond to the half-BPS states of  $\text{SYM}_4$ . To illustrate the freedom of choosing various forms of the Bethe equations we then present a second version (“the Beast”) which employs the standard “distinguished” choice of one fermionic root in the middle of the diagram. As we will see, this shifts the vacuum to a different, high energy state containing only field strengths. It obscures certain well known results (e.g. it becomes hard to see that half-BPS states have vanishing energy) but instead leads to some further interesting spectroscopic results. We end with an outlook on interesting applications and extensions of this work.

## 2 Anomalous dimensions, (super) spin chains, and Bethe ansätze

Let us review how anomalous dimensions in  $\text{YM}_4$  are computed by employing the algebraic Bethe ansatz, once it has been established that the anomalous part of the dilatation operator is identical to the Hamiltonian of an integrable quantum spin chain [12]. Let us explain this in the simplest case of an  $\mathfrak{sl}(2)$  chain, the so-called  $\text{XXX}_{s/2}$  Heisenberg chain. (For a very pedagogical introduction, see [26]). The proposal is that the energy eigenvalues  $E$  of the Hamiltonian are proportional to the anomalous dimensions  $\delta\Delta$  of conformal scaling operators:

$$\delta\Delta = \frac{g_{\text{YM}}^2 N}{8\pi^2} E. \tag{2.1}$$

For spin  $\frac{s}{2} = \frac{1}{2}$  these scaling operators are, in the planar limit, linear combinations of a single trace of sequences of two of the three complex scalars of SYM<sub>4</sub>, say  $Z$  and  $\phi$ :

$$\text{Tr}(ZZZ\phi Z\phi\phi\dots) \quad (2.2)$$

The single trace operator is then interpreted as a chain of spins, where  $Z$  and  $\phi$  represent, respectively, up and down spins. The total number  $L$  of complex scalars inside the trace corresponds to the length of the chain. Each eigenstate (alias scaling operator) of the Hamiltonian is uniquely characterized by a set of Bethe roots  $u_j$ ,  $j = 1, \dots, n$ , and the energy  $E$  of the state is given by

$$E = \pm \sum_{j=1}^n \frac{s}{u_j^2 + \frac{1}{4}s^2}. \quad (2.3)$$

(where for spin  $\frac{s}{2} = \frac{1}{2}$  we should pick the plus sign in front of the sum). The Bethe roots are found by solving the Bethe equations for  $j = 1, \dots, n$

$$\left( \frac{u_j + \frac{i}{2}s}{u_j - \frac{i}{2}s} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{u_j - u_k + i}{u_j - u_k - i}. \quad (2.4)$$

The vacuum state of energy  $E = 0$  is the half-BPS state

$$|0\rangle = \text{Tr} Z^L, \quad (2.5)$$

i.e. it is interpreted as the ferromagnetic ground state (all spins are up, no Bethe roots) of the chain. The excitation number  $n$ , giving the total number of roots, counts the number of  $\phi$ 's or down spins along the chain; it is naturally bounded by  $n \leq L$ . There is an additional constraint on the Bethe roots:

$$1 = \prod_{j=1}^n \frac{u_j + \frac{i}{2}s}{u_j - \frac{i}{2}s}. \quad (2.6)$$

For the spin chain, it means that we have periodic boundary conditions and we are only looking for zero-momentum states. In the gauge theory interpretation (2.2) it expresses the cyclicity of the trace.

The second simplest example concerns twist operators, closely related to quasi-partonic operators in one-loop QCD [19, 20, 23]. Just as the fields in (2.2) form a closed subsector (i.e. they do not mix with any other types of fields) the following operators form another such subsector (see also [25] for further details):

$$\text{Tr}((\mathcal{D}^{m_1} Z)(\mathcal{D}^{m_2} Z) \dots (\mathcal{D}^{m_L} Z)) \quad (2.7)$$

where  $\mathcal{D}$  is the ‘‘light-cone’’ covariant derivative  $\mathcal{D}_{1+i2}$  (as in [23]). This is still a spin chain of length (twist)  $L$ , with non-compact spin  $\frac{s}{2} = -\frac{1}{2}$ , as will be proven in the next chapter. Here the spins at each lattice site  $k$  may take any value  $m_k = 0, 1, 2, \dots$ , as we have an infinite  $s = -1$  representation of  $\mathfrak{sl}(2)$ . Also the total excitation number

$n = \sum m_k$  is not bounded as in the above example. The vacuum is still  $\text{Tr } Z^L$ . Note that in this example the length of the chain is *not* equal to the classical dimension  $L + n$  of the operator. Again, the anomalous dimensions of the scaling operators formed from the states (2.7) are given via eqs. (2.1),(2.3),(2.4),(2.6)! (Here we have to choose the negative sign in the energy formula (2.3).)

In the above example the algebra is  $\mathfrak{sl}(2)$  and thus of rank one. There is a beautiful extension of the Bethe equations to any algebra and any representation of the elementary constituents of the chain at each lattice site [27, 28]. This general form easily extends to the case of super algebras as well, see [29] and references therein. This is what we need when we are looking for a spin chain describing  $\text{SYM}_4$  at one loop, where we expect Bethe equations for the super algebra  $\mathfrak{sl}(4|4)$ . This general equation may be written in the following compact notation, based on knowing the Dynkin diagram of the algebra. The Dynkin diagram of  $\mathfrak{sl}(4|4)$  contains seven dots corresponding to a choice of seven simple roots. Consider a total of  $n$  excitations. For each of the corresponding Bethe roots  $u_i$ ,  $i = 1, \dots, n$ , we specify which of the seven simple roots is excited by  $k_j = 1, \dots, 7$ . The Bethe equations for  $j = 1, \dots, n$  can then be written in the compact form

$$\left( \frac{u_j + \frac{i}{2}V_{k_j}}{u_j - \frac{i}{2}V_{k_j}} \right)^L = \prod_{\substack{l=1 \\ l \neq j}}^n \frac{u_j - u_l + \frac{i}{2}M_{k_j, k_l}}{u_j - u_l - \frac{i}{2}M_{k_j, k_l}}. \quad (2.8)$$

Here,  $M_{kl}$  is the Cartan matrix of the algebra and  $V_k$  are the Dynkin labels of the spin representation. Furthermore, we still consider a cyclic spin chain with zero total momentum. This gives the additional constraint

$$1 = \prod_{j=1}^n \frac{u_j + \frac{i}{2}V_{k_j}}{u_j - \frac{i}{2}V_{k_j}}. \quad (2.9)$$

The energy of a configuration of roots that satisfies the Bethe equations and constraint is now given by <sup>5</sup>

$$E = cL \pm \sum_{j=1}^n \left( \frac{i}{u_j + \frac{i}{2}V_{k_j}} - \frac{i}{u_j - \frac{i}{2}V_{k_j}} \right). \quad (2.10)$$

It is easily seen that restricting these equations to the Dynkin diagram of the algebra  $\mathfrak{so}(6)$  reproduces the Bethe equations of [12]. It will turn out, see below, that these general equations, which are well known in the literature on integrable spin chains, indeed solve the entire problem of computing planar anomalous dimensions in  $\text{SYM}_4$ , once we (i) identify the correct representations of the fundamental fields on the lattice sites, and (ii) after resolving certain subtleties concerning Dynkin diagrams for superalgebras. However, let us first show that we expect planar one-loop  $\text{SYM}_4$  to be completely integrable indeed.

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<sup>5</sup>The Bethe equations determine the energy only up to scale and a shift  $cL$  where  $c$  is a constant to be fixed such that the energy corresponds to planar anomalous dimensions of  $\mathcal{N} = 4$  SYM via (2.1).

### 3 The R-Matrix

In this section we derive the R-matrix for the integrable spin chain considered in this work. For this we make use of a special subsector of the spin chain with residual  $\mathfrak{sl}(2)$  symmetry and show how to lift the universal  $\mathfrak{sl}(2)$  R-matrix to an  $\mathfrak{sl}(4|4)$  invariant R-matrix. The derived Hamiltonian is shown to agree with the complete one-loop planar dilatation generator of  $\mathcal{N} = 4$  SYM, thus proving the integrability of the latter.

**The planar one-loop dilatation operator.** The complete one-loop planar dilatation generator of  $\mathcal{N} = 4$  SYM acting on two fields is given by [25]

$$H_{12} = 2h(J_{12}) := \sum_{j=0}^{\infty} 2h(j) P_{12,j}. \quad (3.1)$$

Here,  $h(j)$  is the  $j$ -th harmonic number defined alternatively by a series or by the digamma function  $\Psi(x) = \partial \log \Gamma(x) / \partial x$

$$h(n) = \sum_{k=1}^n \frac{1}{k} = \Psi(n+1) - \Psi(1). \quad (3.2)$$

The operator  $P_{12,j}$  projects the spins at sites 1, 2 to the module  $V_j$ . Each spin belongs to the ‘fundamental’ module  $V_F$ , consequently the modules  $V_j$  arise in the tensor product of two fundamental modules

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j. \quad (3.3)$$

The primary weights of these modules are given by (see (4.4) for details)

$$\begin{aligned} w_F &= [1; 0, 0; 0, 1, 0; 0, 1], \\ w_0 &= [2; 0, 0; 0, 2, 0; 0, 2], \\ w_1 &= [2; 0, 0; 1, 0, 1; 0, 2], \\ w_j &= [j; j-2, j-2; 0, 0, 0; 0, 2]. \end{aligned} \quad (3.4)$$

**The  $\mathfrak{sl}(2)$  subsector.** A crucial point in the derivation of the full Hamiltonian density [25] was the restriction to a subsector. This subsector is obtained by restricting the allowed weights of states to

$$w = [L + s; s, s; 0, L, 0; 0, L]. \quad (3.5)$$

For the construction of such states we must restrict all spins to the form

$$z^n := \frac{1}{n!} \mathcal{D}^n Z, \quad \text{with } \mathcal{D} := \mathcal{D}_1 + i\mathcal{D}_2, \quad Z := \Phi_5 + i\Phi_6. \quad (3.6)$$

The residual symmetry algebra within the subsector is  $\mathfrak{sl}(2)$  which can be represented by the generators

$$J'_- = \partial, \quad J'_3 = \frac{1}{2} + z\partial, \quad J'_+ = z + z^2\partial \quad (3.7)$$

where  $\partial$  is the derivative with respect to  $z$ . It was then shown that the modules  $V_F, V_j$  of the full theory correspond to the  $\mathfrak{sl}(2)$  modules  $V'_F, V'_j$  with highest weights

$$w'_F = [-1], \quad w'_j = [-2 - 2j]. \quad (3.8)$$

Here, the weights are described by *twice* the spin. The Hamiltonian density (3.1) restricts within this subsector to

$$H'_{12} = 2h(J'_{12}) := \sum_{j=0}^{\infty} 2h(j) P'_{12,j}, \quad (3.9)$$

where  $P'_{12,j}$  projects states of the tensor product  $V'_F \times V'_F$  to the module  $V'_j$ . Interestingly, this is the Hamiltonian of the XXX $_{-1/2}$  Heisenberg spin chain. In other words it is an integrable spin chain with unbroken  $\mathfrak{sl}(2)$  symmetry, where each site transforms in a spin  $w'_F/2 = -\frac{1}{2}$  representation.

**Integrability within the  $\mathfrak{sl}(2)$  subsector.** To prove this statement we make use of the universal R-matrix of  $\mathfrak{sl}(2)$  spin chains. This  $\mathfrak{sl}(2)$  invariant operator can be decomposed into its irreducible components corresponding to the modules  $V'_j$

$$R'_{12}(u) = \sum_{j=0}^{\infty} R'_j(u) P'_{12,j}. \quad (3.10)$$

The eigenvalues  $R'_j(u)$  of the  $\mathfrak{sl}(2)$  universal R-matrix were determined in [30]. In a spin  $w'_j/2 = -1 - j$  representation the eigenvalue is

$$R'_j(u) = (-1)^{j+1} \frac{\Gamma(-j - cu)}{\Gamma(-j + cu)} f(cu). \quad (3.11)$$

The arbitrary function  $f(u)$  and normalization constant  $c$  reflect the trivial symmetries of the Yang-Baxter equation. The induced eigenvalues of the Hamiltonian density are obtained as the logarithmic derivative of  $R'_j(u)$  at  $u = 0$  times the permutation operator  $P_{12}$

$$H_{12} V'_j = P_{12} \left. \frac{\partial \log R'_j(u)}{\partial u} \right|_{u=0} V'_j. \quad (3.12)$$

We note that for even (odd)  $j$  the composite module  $V'_j$  is a (anti)symmetric combination of two  $V'_F$ , consequently the permutation acts as

$$P_{12} V'_j = (-1)^j V'_j. \quad (3.13)$$

We choose the function and constant to be

$$f(cu) = -\frac{\Gamma(-cu)}{\Gamma(cu)}, \quad c = 1. \quad (3.14)$$

Using the fact that  $j$  is exactly integer we find the eigenvalues of the Hamiltonian density

$$(-1)^j \left. \frac{\partial \log R'_j(u)}{\partial u} \right|_{u=0} = 2\Psi(j+1) - 2\Psi(1) = 2h(j). \quad (3.15)$$

This proves that the Hamiltonian density (3.9) is integrable.

**Integrability of the full  $\mathfrak{sl}(4|4)$  R-matrix.** To derive an R-matrix for the full  $\mathfrak{sl}(4|4)$  spin chain we will assume that for given representations of the symmetry algebra there exists a unique R-matrix which satisfies the Yang-Baxter equation (modulo the symmetries of the YBE). We are not aware whether this claim [30] has been proven. Let  $R_{12}$  be this R-matrix for the  $\mathfrak{sl}(4|4)$  integrable spin chain. The R-matrix is an invariant operator, thus it can be reduced to its irreducible components corresponding to the modules  $V_j$

$$R_{12}(u) = \sum_{j=0}^{\infty} R_j(u) P_{12,j}. \quad (3.16)$$

The restriction  $R'$  of the R-matrix to the  $\mathfrak{sl}(2)$  sector must also satisfy the Yang-Baxter equation. The unique solution for the eigenvalues of  $R'$  is (3.11). Due to the one-to-one correspondence of modules  $V_j$  and  $V'_j$ ,

$$V'_j \subset V_j \quad (3.17)$$

the eigenvalues of the unique  $\mathfrak{sl}(4|4)$  R-matrix (3.16) must be

$$R_j(u) = (-1)^j \frac{\Gamma(-j - cu)}{\Gamma(-j + cu)} f(cu). \quad (3.18)$$

As in (3.15) this R-matrix yields  $H_{12} = 2h(J_{12})$  in agreement with (3.1). This in turn shows that the planar one-loop dilatation generator of  $\mathcal{N} = 4$  is integrable. Note however, that this proof is based on the assumption of the existence of a unique R-matrix.

**The Minahan and Zarembo chain.** As an application let us investigate the restriction of the R-matrix to the  $\mathfrak{so}(6)$  subsector investigated by Minahan and Zarembo [12]

$$w = [L; 0, 0; q_1, p, q_2; 0, L], \quad (3.19)$$

i.e. where the spins are given by the  $\mathcal{N} = 4$  SYM scalars  $\Phi_m$ . Two scalars  $\Phi_p, \Phi_q$  can transform in three different irreducible representations of  $\mathfrak{so}(6)$ , symmetric-traceless, antisymmetric and singlet. These correspond to the modules  $V_0, V_1, V_2$ , respectively.

We set the normalization function and constant to

$$f(u) = \frac{\Gamma(3 + cu)}{2\Gamma(1 - cu)}, \quad c = 1. \quad (3.20)$$

and compute the eigenvalues of the R-matrix (3.18) for  $V_0, V_1, V_2$

$$\begin{aligned} R_{12,0}(u) &= \frac{1}{2}(u+1)(u+2), \\ R_{12,1}(u) &= \frac{1}{2}(u-1)(u+2), \\ R_{12,2}(u) &= \frac{1}{2}(u-1)(u-2). \end{aligned} \quad (3.21)$$

We note the projectors to  $V_0, V_1, V_2$  in the scalar sector

$$\begin{aligned} P_{12,0} &= \frac{1}{2} + \frac{1}{2}P_{12} - \frac{1}{6}K_{12}, \\ P_{12,1} &= \frac{1}{2} - \frac{1}{2}P_{12}, \\ P_{12,2} &= \frac{1}{6}K_{12}, \end{aligned} \quad (3.22)$$



**Excitation numbers.** Finally, we need to obtain the number of excitations  $n_k$ ,  $k = 1, \dots, 7$  of the individual simple roots for a state with a given weight

$$w = [\Delta_0; s_1, s_2; q_1, p, q_2; B, L]. \quad (4.4)$$

A weight of (the classical)  $\mathfrak{sl}(4|4)$  is described by various labels. The (bare) dimension is indicated by  $\Delta_0$ . The  $\mathfrak{sl}(4)$  and  $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$  Dynkin labels are given by  $[q_1, p, q_2]$  and  $[s_1, s_2]$ , respectively, where  $s_1, s_2$  equal *twice* the spin. Furthermore,  $B$  describes the  $\mathfrak{gl}(1)$  hypercharge or chirality in the semi-direct product  $\mathfrak{sl}(4|4) = \mathfrak{gl}(1) \ltimes \mathfrak{psl}(4|4)$ . Finally,  $L$ , which is not a label of  $\mathfrak{sl}(4|4)$ , gives the length or number of spins. In  $\mathcal{N} = 4$  SYM the chirality and length of an operator are not good quantum numbers, they are broken at higher loops by the Konishi anomaly. Nevertheless, at one-loop order they are conserved and their leading order values can be used to describe a state.

This is most easily seen in the oscillator picture in [25] using the physical vacuum  $|Z\rangle$ . We write down the action of the generators corresponding to the simple roots in terms of creation and annihilation operators

$$\begin{array}{cccccccc}
 n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 & \\
 \bigcirc & \bigotimes & \bigcirc & \bigcirc & \bigcirc & \bigotimes & \bigcirc & \\
 \mathbf{a}_2^\dagger \mathbf{a}^1 & \mathbf{a}_1^\dagger \mathbf{c}^1 & \mathbf{c}_1^\dagger \mathbf{c}^2 & \mathbf{c}_2^\dagger \mathbf{d}_1^\dagger & \mathbf{d}_2^\dagger \mathbf{d}^1 & \mathbf{b}_1^\dagger \mathbf{d}^2 & \mathbf{b}_2^\dagger \mathbf{b}^1 & 
 \end{array} \quad (4.5)$$

It is now clear that  $n_1 = n_{\mathbf{a}_2^\dagger}$ ,  $n_2 = n_{\mathbf{a}_1^\dagger} + n_{\mathbf{a}_2^\dagger}$  and so on. Using the formulas in [25] we write down the corresponding excitation numbers of the simple roots

$$n_k = \begin{pmatrix} \frac{1}{2}\Delta_0 - \frac{1}{2}(L - B) - \frac{1}{2}s_1 \\ \Delta_0 - (L - B) \\ \Delta_0 - \frac{1}{2}(L - B) - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \\ \Delta_0 - p - \frac{1}{2}q_1 - \frac{1}{2}q_2 \\ \Delta_0 - \frac{1}{2}(L + B) - \frac{1}{2}p - \frac{1}{4}q_1 - \frac{3}{4}q_2 \\ \Delta_0 - (L + B) \\ \frac{1}{2}\Delta_0 - \frac{1}{2}(L + B) - \frac{1}{2}s_2 \end{pmatrix}. \quad (4.6)$$

Not all excitations of the simple roots correspond to physical states. Obviously, the excitation numbers of the oscillators must be non-negative, this gives the bounds <sup>8</sup>

$$0 \leq n_1 \leq n_2 \leq n_3 \leq n_4 \geq n_5 \geq n_6 \geq n_7 \geq 0. \quad (4.7)$$

Furthermore, each fermionic oscillator cannot be excited more than once, this gives the bounds

$$n_2 + 2L \geq n_3 + L \geq n_4 \leq n_5 + L \leq n_6 + 2L. \quad (4.8)$$

---

<sup>8</sup>Superconformal primaries reside in the fundamental Weyl chamber defined by the bounds  $-2n_1 + n_2 > -1$ ,  $n_2 - 2n_3 + n_4 > -1$ ,  $n_3 - 2n_4 + n_5 + L > -1$ ,  $n_4 - 2n_5 + n_6 > -1$ ,  $n_6 - 2n_7 > -1$ . Together with (4.7) this implies, among other relations, (4.8). Solutions of the Bethe equations outside the fundamental domain apparently correspond to mirror images of primary states due to reflections at the chamber boundaries (We thank J. Minahan and K. Zarembo for this insight).

Certainly, we should obtain the  $\mathfrak{so}(6) = \mathfrak{sl}(4)$  subsector studied by Minahan and Zarembo [12] when we remove the outer four simple roots from the Dynkin diagram in (4.1). When we restrict to the states (3.19) of this subsector the number of excitations (4.6) of the outer four roots is trivially zero. They become irrelevant for the Bethe ansatz and can be discarded. Thus all solutions to the  $\mathfrak{so}(6)$  Bethe equations are also solutions to the  $\mathfrak{sl}(4|4)$  Bethe equations. What is more, we can apply this Bethe ansatz to a wider range of operators, in fact, to *all* single-trace operators of  $\mathcal{N} = 4$  SYM.

**Multiplet splitting.** Now we can write down and try to solve the Bethe equations for any state in  $\mathcal{N} = 4$  SYM. Note, however, that the Bethe equations need to be solved only for highest weight states. All descendants of a highest weight state are obtained by adding Bethe roots at infinity,  $u_i = \infty$ . In other words, the solutions to the Bethe equations corresponding to highest weight states are distinguished in that they have no roots  $u_i$  at infinity. Nevertheless, there is one subtlety related to this point which can be used to our advantage. Namely this is multiplet splitting at the unitarity bounds [32]. We assume that the spin chain of  $L$  sites transforms in the tensor product of  $L$  spin representations. The corrections  $\delta\Delta$  to the scaling dimension induced by the Hamiltonian  $H$  are not included in this picture. Thus, in terms of the spin chain only the classical  $\mathfrak{sl}(4|4)$  algebra applies where the scaling dimension is exactly  $\Delta_0$ . Long multiplets of the interacting  $\mathfrak{sl}(4|4)$  algebra close to one or both of the unitarity bounds

$$\begin{aligned} \text{I: } \Delta_0 &= 2 + s_1 + p + \frac{3}{2}q_1 + \frac{1}{2}q_2, \\ \text{II: } \Delta_0 &= 2 + s_2 + p + \frac{1}{2}q_1 + \frac{3}{2}q_2. \end{aligned} \tag{4.9}$$

split up into several semi-short or BPS multiplets. The weights of the additional primary states are offset by

$$\begin{aligned} \delta w_{\text{I}} &= \begin{cases} [+0.5; -1, 0; +1, 0, 0; -0.5, +1], & \text{for } s_1 > 0, \\ [+1.0; 0, 0; +2, 0, 0; 0.0, +1], & \text{for } s_1 = 0, \end{cases} \\ \delta w_{\text{II}} &= \begin{cases} [+0.5; 0, -1; 0, 0, +1; +0.5, +1], & \text{for } s_2 > 0, \\ [+1.0; 0, 0; 0, 0, +2; 0.0, +1], & \text{for } s_2 = 0. \end{cases} \end{aligned} \tag{4.10}$$

The unitarity bounds can also be expressed in terms of excitations of simple roots, we find

$$\begin{aligned} \text{I: } n_1 + n_3 &= n_2 + 1, \\ \text{II: } n_7 + n_5 &= n_6 + 1. \end{aligned} \tag{4.11}$$

The corresponding offsets translate into

$$\begin{aligned} \delta n_{\text{I}} &= \begin{cases} (0, -1, -1, 0, 0, 0, 0), & \text{for } n_2 > n_1, \\ (0, 0, -1, 0, 0, 0, 0), & \text{for } n_2 = n_1, \end{cases} \\ \delta n_{\text{II}} &= \begin{cases} (0, 0, 0, 0, -1, -1, 0), & \text{for } n_6 > n_7, \\ (0, 0, 0, 0, -1, 0, 0), & \text{for } n_6 = n_7, \end{cases} \end{aligned} \tag{4.12}$$

together with an increase of the length  $L$  by one. We thus see that in the case of multiplet shortening the primaries of higher submultiplets have less excitations. In a calculation

this may reduce the complexity of the Bethe equations somewhat as we shall see in an example below.

Multiplet splitting is an extremely interesting issue from the point of view of integrability. Let us consider some operator acting on a spin chain. Assume the operator is invariant under the classical algebra  $\mathfrak{sl}(4|4)$ . In the most general case, this operator can assign a different value to all irreducible multiplets of states. In particular this is so for the submultiplets of a long multiplet at the unitarity bound. Now, if we impose integrability on the operator all these submultiplets become degenerate.<sup>9</sup> A priori, this seems like a miracle. Why should integrability imply *this* degeneracy? It almost seems as if integrability selects that scalar operator which is suitable as a consistent deformation of the dilatation generator! Then, clearly the miracle would turn into the condition for integrability. If this can be made sense of, maybe it also helps to understand higher loop corrections in the light of integrability [24].

**Example.** Let us apply the Bethe ansatz to the twist-two operator with primary weight

$$w = [4; 2, 2; 0, 0, 0; 0, 2]. \quad (4.13)$$

Using (4.6) we find the excitation numbers and length

$$n_{0,k} = (0, 2, 3, 4, 3, 2, 0), \quad L_0 = 2. \quad (4.14)$$

This weight is on both unitarity bounds, (4.11), the excitation numbers of the highest submultiplet, (4.12), are

$$n_k = (0, 1, 2, 4, 2, 1, 0), \quad L = 4. \quad (4.15)$$

We therefore configure the simple roots as follows

$$k_j = (2, 3, 3, 4, 4, 4, 4, 5, 5, 6). \quad (4.16)$$

Now we note that the twist-two operators are unpaired states. Therefore the configurations of Bethe roots must be invariant under the symmetry  $u_j \mapsto -u_j$  of the Bethe equations. This tells us

$$u_1 = u_{10} = 0, \quad u_2 = -u_3, \quad u_4 = -u_5, \quad u_6 = -u_7, \quad u_8 = -u_9, \quad (4.17)$$

which automatically satisfies the momentum constraint (2.9). Furthermore the excitations (4.14) are invariant under flipping the Dynkin diagram,  $n_k \mapsto n_{8-k}$ . This can be used to set

$$u_2 = u_8. \quad (4.18)$$

The Bethe equations (2.8) are then solved exactly by

$$u_2 = \sqrt{\frac{5}{7}}, \quad u_{4,6} = \sqrt{\frac{65 \pm 4\sqrt{205}}{140}}, \quad (4.19)$$

---

<sup>9</sup>This is because the unique integrable operator is equivalent to the one-loop planar correction to the dilatation operator of  $\mathcal{N} = 4$  SYM. In  $\mathcal{N} = 4$  SYM the submultiplets must rejoin into a long multiplet which would be inconsistent if their one-loop anomalous dimensions were different.



usual sense of dualities. To determine the energy of some set of states we apply that Bethe ansatz which seems most suitable for the problem. For example, the Bethe ansatz of the previous section is useful for states with a low energy. In contrast, the Bethe ansatz discussed in this section seems most suitable for states of a large chirality  $B$  or of an energy around  $3L$ .

The number of excitations can be determined as above. We note the action of the simple roots in the oscillator picture of [25]

$$\begin{array}{ccccccc}
n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 \\
\bigcirc & \bigcirc & \bigcirc & \bigotimes & \bigcirc & \bigcirc & \bigcirc \\
\mathbf{b}_2^\dagger \mathbf{b}^1 & \mathbf{a}_1^\dagger \mathbf{b}_1^\dagger & \mathbf{a}_2^\dagger \mathbf{a}^1 & \mathbf{c}_4^\dagger \mathbf{a}_2^\dagger & \mathbf{c}_3^\dagger \mathbf{c}^4 & \mathbf{c}_2^\dagger \mathbf{c}^3 & \mathbf{c}_1^\dagger \mathbf{c}^2
\end{array} \tag{5.5}$$

The vacuum corresponds to the configuration  $\mathbf{a}_1^\dagger \mathbf{a}_1^\dagger |0\rangle$ . The numbers of excitations for a given weight are

$$n_k = \begin{pmatrix} \frac{1}{2}(L-B) + \frac{1}{2}\Delta_0 - \frac{1}{2}s_2 - L \\ \frac{3}{2}(L-B) + \Delta_0 - 2L \\ \frac{3}{2}(L-B) + \frac{1}{2}\Delta_0 - \frac{1}{2}s_1 \\ \frac{1}{2}(L-B) \\ \frac{3}{2}(L-B) - \frac{1}{2}p - \frac{1}{4}q_1 - \frac{3}{4}q_2 \\ \frac{1}{2}(L-B) - p - \frac{1}{2}q_1 - \frac{1}{2}q_2 \\ \frac{1}{2}(L-B) - \frac{1}{2}p - \frac{3}{4}q_1 - \frac{1}{4}q_2 \end{pmatrix} \tag{5.6}$$

We note the bounds

$$0 \leq n_1 \leq n_2, \quad n_3 \geq n_4 \geq n_5 \geq n_6 \geq n_7 \geq 0, \tag{5.7}$$

as well as

$$n_3 - n_2 \leq 2L, \quad n_4 - n_5, n_5 - n_4, n_6 - n_7, n_7 \leq L. \tag{5.8}$$

We defer the discussion of highest weight states and multiplet splitting in the context of the ‘distinguished’ choice of Dynkin diagram to App. A. There we also compare the energies of all dimension four operators to the results of SYM.

**Example.** We see that the vacuum of this choice of Bethe equations has energy  $3L$ . Clearly, this is not the ground state, the ground state is half-BPS and has zero energy. This situation is similar to the antiferromagnetic spin chain, where the ground state is described by a highly excited state (in terms of Bethe roots). In this spin chain the first half-BPS multiplet with primary weight

$$w_0 = [2; 0, 0; 0, 2, 0; 0, 2] \tag{5.9}$$

has the highest weight (in the sense of highest chirality  $B$ )

$$w = [4; 0, 0; 0, 0, 0; 2, 2]. \tag{5.10}$$

This corresponds to the excitations and length

$$n_k = (0, 0, 2, 0, 0, 0, 0), \quad k_j = (3, 3), \quad L = 2. \tag{5.11}$$

The Bethe equations and momentum constraint are solved exactly by

$$u_{1,2} = \pm \frac{i}{\sqrt{3}}. \quad (5.12)$$

Reassuringly, the energy due to the excitations precisely cancels off the vacuum energy

$$E = 0. \quad (5.13)$$

**Spin Waves.** For a low number of excitations but arbitrary length of the spin chain, the Bethe ansatz can quickly provide good approximations to the energies. Especially in the case of two excitations it is often trivial to solve the Bethe equations exactly. We will now discuss these spin wave solutions, which correspond to small fluctuations of the vacuum. The major difference to the previous investigation of spin waves in the context of  $\mathcal{N} = 4$  SYM [12] is the modified vacuum. The solution is quite similar and we will follow along the lines of [12].

Let us consider the following excitations of simple roots

$$n_k = (0, 1, 1, 0, 0, 0, 0), \quad k_j = (2, 3). \quad (5.14)$$

The momentum constraint implies

$$u_1 = \frac{3}{2}u_2. \quad (5.15)$$

and the two Bethe equations then collapse into

$$\left( \frac{i + u_2}{i - u_2} \right)^{L-1} = -1. \quad (5.16)$$

This equation has  $L - 1$  independent solutions

$$u_2 = \cot \frac{\pi n}{L-1}, \quad -\frac{L-1}{2} < n \leq \frac{L-1}{2} \quad (5.17)$$

with energy

$$E = 3L - \frac{2}{3} \sin^2 \frac{\pi n}{L-1}. \quad (5.18)$$

Note that the symmetry  $u_j \mapsto -u_j$  translates into  $n \mapsto -n$ . The states  $n$  and  $-n$  therefore have degenerate energies and form a pair unless  $n = 0$  or  $n = (L-1)/2$ . Then the solutions  $u_{1,2} = \infty$  or  $u_{1,2} = 0$ , respectively are invariant and do not pair up.

We can also consider the thermodynamic limit of a large  $L$ . In leading order we can approximate the positions of the Bethe roots and the corresponding energy. An excitation of simple root 2 yields

$$u_{2,n} \approx \frac{3L}{2\pi n}, \quad \delta E_{2,n} = \frac{4\pi^2 n^2}{3L^2} \quad (5.19)$$

whereas for an excitation of simple root 3 we find

$$u_{3,n} \approx \frac{-2L}{2\pi n}, \quad \delta E_{3,n} = -\frac{4\pi^2 n^2}{2L^2}. \quad (5.20)$$

The sum of all mode numbers  $n$  must vanish due to the momentum constraint. Using this we quickly find an approximation to the above exactly solved spin wave.

$$\begin{aligned}
u_1 &\approx \frac{3L}{2\pi n}, & u_2 &\approx \frac{-2L}{2\pi(-n)}, \\
E &\approx 3L + \frac{4\pi^2 n^2}{3L^2} - \frac{4\pi^2 n^2}{2L^2} = 3L - \frac{2\pi^2 n^2}{3L^2}.
\end{aligned}
\tag{5.21}$$

It would be interesting if some string theory solution describing this Bethe vacuum could be found. The spectrum of fluctuations, somewhat reminiscent of the plane wave spectrum, could then be compared.

## 6 Outlook

We are confident that our proposal furnishes the tool for very comprehensive “spectroscopic” studies of  $\mathcal{N} = 4$  Yang-Mills theory. The recent progress with semi-classical approaches on the string side [22, 34, 16] allows for very interesting dynamical comparisons with gauge theory. In particular our superchain and its associated Bethe equations should be very useful for string motions that also involve the AdS part of the background, or even joint motion in AdS and on the five sphere. In all these applications string theory is able, in principle, to predict anomalous dimensions for large dimension gauge theory operators, to which the Bethe ansatz is ideally suited: On the level of the spin chain these situations correspond to the thermodynamic limit of the chain, in which the Bethe equations often become tractable [14].

It would be helpful to classify all possible ways to write Bethe equations for the above  $\mathfrak{sl}(4|4)$  super spin chain, and identify the corresponding pseudovacua. Correspondingly one should then find further spin wave solutions, describing the simplest fluctuations around the pseudovacua. It would be interesting to see whether the large dimension limits of the pseudovacuum states and their elementary excitations play a special rôle in string theory.

One puzzling aspect of the observed integrable structures is that they really correspond to “hidden” symmetries appearing in large  $N$  SYM<sub>4</sub>. Clearly we would like to understand their presence from the point of view of the gauge theory. Are they a technical consequence of what one could call “planar supersymmetry”, or is there a deeper reason, possibly related to string theory?

However, the most exciting open problem remains to identify the integrable deformation of the super spin chain, with deformation parameter  $\lambda = g_{\text{YM}}^2 N$ , that correctly includes the effects of all quantum loops. This presumably would be a  $\mathfrak{psl}(4|4)$  super spin chain with long-range interactions. A related question concerns the closure of the interacting algebra and the consistency requirements imposed by it. How are these related to integrability and can they be used to fix higher-loop corrections?

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## A Dissection of the Beast

In this appendix we present some more details on the Bethe ansatz of Sec. 5. We will start by an investigation of highest weight states. Next we will apply this to the Konishi state and its descendants, for which we then solve the Bethe equations. Finally, we present the set of all solutions corresponding to states up to classical dimension four.

### A.1 Highest Weights

A primary state is defined as a state annihilated by the boosts  $K, S, \dot{S}$ . This condition is identical to highest weight condition in the sense of Sec. 4. However, when we change the Dynkin diagram of the superalgebra, we also change the notion of positive roots and highest weights. For the ‘distinguished’ Dynkin diagram of Sec. 5 not all of the boosts  $K, S, \dot{S}$  can be positive roots simultaneously. Therefore primary weights (where primary weight shall refer to highest weight in the sense of Sec. 4) are not highest weights. Instead, a highest weight state is annihilated by  $K, S, \dot{Q}$ .

For a given primary weight we need to find the corresponding highest weight. For a generic long multiplet with primary weight  $w$  this is achieved by adding

$$\delta w = [4; 0, 0; 0, 0, 0; 4, 0]. \quad (\text{A.1})$$

It corresponds to acting with  $\dot{Q}^8$  on the primary state. A further application of  $\dot{Q}$ , as well as  $K, S$  which commute with  $\dot{Q}^8$ , will therefore kill the state.

In the classical  $\mathfrak{sl}(4|4)$  there are various shortening conditions that affect the highest weight. For a half-BPS state we add

$$\delta w_{1/2} = [2; 0, 0; 0, -2; 0; 2, 0] \quad (\text{A.2})$$

instead. The fundamental multiplet  $V_F$  with primary weight  $[1; 0, 0; 0, 1, 0; 0, 1]$  is subject to an additional shortening condition. The highest weight is  $[2; 2, 0; 0, 0, 0; 1, 1]$  and corresponds to the field  $\mathcal{F}_{1+i2, 3+i4}$  of  $\mathcal{N} = 4$ . The vacuum is constructed from exactly this field.

There are two shortening conditions for unprotected multiplets, see (4.9). When shortening condition I applies, the multiplet splits in two with highest weights

$$w + \delta w + \delta w_I, \quad w + \delta w_{I+}. \quad (\text{A.3})$$

where

$$\delta w_I = [0; 0, 0; 0, 0, 0; -1, +1],$$

$$\delta w_{I+} = \begin{cases} [+3.5; +1, 0; -1, 0, 0; +3.5, 0], \\ [+3.0; +2, 0; 0, -1, 0; +3.0, 0], \\ [+2.5; +3, 0; 0, 0, -1; +2.5, 0], \\ [+2.0; +4, 0; 0, 0, 0; +2.0, 0]. \end{cases} \quad (\text{A.4})$$

For  $\delta w_{I+}$  it is understood that the topmost line applies for which the resulting labels are positive. When shortening condition II applies, the highest weights are

$$w + \delta w, \quad w + \delta w_{II} + \delta w. \quad (\text{A.5})$$

with

$$\delta w_{II} = \begin{cases} [+0.5; 0, -1; 0, 0, +1; +0.5, +1], \\ [+1.0; 0, 0; 0, 0, +2; 0.0, +1]. \end{cases} \quad (\text{A.6})$$

Finally, if both shortening conditions apply, the four highest weights are

$$\begin{aligned} & w + \delta w + \delta w_I, \\ & w + \delta w_{I+}, \\ & w + \delta w_{II} + \delta w + \delta w_I, \\ & w + \delta w_{II} + \delta w_{I+}. \end{aligned} \quad (\text{A.7})$$

Note that to determine the last highest weight, it is necessary to choose the correct  $\delta w_{I+}$  for  $w + \delta w_{II}$ . The submultiplets join in the interacting theory. In each of the three cases (A.3),(A.5),(A.7) we have listed the highest weight of the long multiplet first.

## A.2 Konishi Submultiplets

The Konishi state has primary weight

$$w_0 = [2; 0, 0; 0, 0, 0; 0, 2]. \quad (\text{A.8})$$

This is at both unitarity bounds. According to the rules of the previous section we find the highest weights of the four submultiplets

$$\begin{aligned} w_1 &= [4.0; 4, 0; 0, 0, 0; 2.0, 2], \\ w_2 &= [6.0; 0, 0; 0, 0, 0; 3.0, 3], \\ w_3 &= [5.5; 3, 0; 0, 0, 1, 2.5, 3], \\ w_4 &= [7.0; 0, 0; 0, 0, 2; 3.0, 4]. \end{aligned} \quad (\text{A.9})$$

The excitation numbers are determined by (5.6)

$$\begin{aligned} n_{1,k} &= (0, 0, 0, 0, 0, 0), & L &= 2, \\ n_{2,k} &= (0, 0, 3, 0, 0, 0), & L &= 3, \\ n_{3,k} &= (0, 0, 2, 1, 0, 0), & L &= 3, \\ n_{4,k} &= (0, 0, 5, 2, 0, 0), & L &= 4. \end{aligned} \quad (\text{A.10})$$

The first configuration is the vacuum, we find the energy  $E = 3L = 6$  straight away.

For the second configuration with  $k_j = (3, 3, 3)$  we note that the roots must be invariant under the map  $u_j \mapsto -u_j$  because the Konishi state is unpaired. Therefore one root is zero while the other two sum to zero. The only way to satisfy the momentum constraint is to pick the singular configuration

$$u_{1,2} = \pm i, \quad u_3 = 0. \quad (\text{A.11})$$

$\Delta_0$	$\mathfrak{sl}(2)^2$	$\mathfrak{sl}(4)$	$B$	$L$	$E^P$
2	[0, 0]	[0, 2, 0]	0	2	$0^+$
	[0, 0]	[0, 0, 0]	0	2	$6^+$
3	[0, 0]	[0, 3, 0]	0	3	$0^-$
	[0, 0]	[0, 1, 0]	0	3	$4^-$
4	[0, 0]	[0, 4, 0]	0	4	$0^+$
	[0, 0]	[0, 2, 0]	0	4	$(5 \pm \sqrt{5})^+$
	[0, 0]	[1, 0, 1]	0	4	$6^-$
	[0, 0]	[0, 0, 0]	0	4	$\frac{1}{2}(13 \pm \sqrt{41})^+$
	[2, 0]	[0, 0, 0]	1	3	$9^-$ + conj.
	[1, 1]	[0, 1, 0]	0	3	$\frac{15}{2}^\pm$
	[2, 2]	[0, 0, 0]	0	2	$\frac{25}{3}^+$

Table 1: All energies  $E$  of primary states with  $\Delta_0 \leq 4$ , see [25]. The parity  $P$  is related to parity under inversion of the spin chain, parity  $\pm$  indicates a degenerate pair. The label ‘+conj.’ indicates conjugate states with  $\mathfrak{sl}(2)^2, \mathfrak{sl}(4)$  labels reversed and opposite chirality  $B$ .

This has to be regularized in the usual way, we again find the energy  $E = 6$ . As an aside, this state has the highest weight for the interacting multiplet. At leading order, it corresponds to the operators  $\varepsilon^{\beta\gamma}\varepsilon^{\delta\varepsilon}\varepsilon^{\eta\alpha} \text{Tr } \mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}\mathcal{F}_{\varepsilon\eta}$  of  $\mathcal{N} = 4$  SYM.

For the third configuration with  $k_j = (3, 3, 4)$  it is straightforward to find and verify

$$u_{1,2} = \pm \frac{1}{\sqrt{3}}, \quad u_3 = 0 \quad (\text{A.12})$$

as a solution to the Bethe equations and momentum constraint with energy  $E = 6$ .

For the last configuration with  $k_j = (3, 3, 3, 3, 3, 4, 4)$  we again require a singular solution. Taking the roots (A.11) as an ansatz, the remaining roots can be found

$$u_{1,2} = \pm i, \quad u_3 = 0, \quad u_{4,5} = \pm \frac{1}{\sqrt{3}}, \quad u_{6,7} = \pm \sqrt{\frac{7}{20}}. \quad (\text{A.13})$$

They satisfy the momentum constraint and yield, not surprisingly,  $E = 6$ .

We have seen that all submultiplet of the Konishi multiplet have the same energy. This is somewhat remarkable, as they have rather different configurations of roots.

### A.3 Bethe Roots for Dimension Four States

Here we will present the Bethe roots for operators up to dimensions four. This demonstrates that also the ‘distinguished’ choice of Dynkin diagram with a pseudovacuum leads to the correct results. We present the complete list of operators and anomalous dimensions in Tab. 1. The Bethe roots which solve the Bethe equations and momentum constraint are given below. The energies agree with Tab. 1, we do not list them again.

[2;0,0;0,2,0;0,2].  $k_j = (3, 3), L = 2$ . See (5.12).

$$u_{1,2} = \pm \frac{i}{\sqrt{3}}. \quad (\text{A.14})$$

[2;0,0;0,0,0;0,2].  $k_j = ()$ ,  $L = 2$ . Trivial.

[3;0,0;0,3,0;0,3].  $k_j = (3, 3, 3, 3, 4, 4, 5)$ ,  $L = 3$ .

$$u_{1,2,3,4} = \pm \sqrt{\frac{-3 \pm i\sqrt{15}}{12}}, \quad u_{5,6} = i\sqrt{\frac{5}{12}}, \quad u_7 = 0. \quad (\text{A.15})$$

[3;0,0;0,1,0;0,3].  $k_j = (3, 3)$ ,  $L = 3$ .

$$u_{1,2} = \pm \frac{i}{\sqrt{5}}. \quad (\text{A.16})$$

[4;0,0;0,4,0;0,4].  $k_j = (3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5)$ ,  $L = 4$ . This is left as an exercise for the reader. Good luck!

[4;0,0;0,2,0;0,4].  $k_j = (3, 3, 3, 3, 4, 4, 5)$ ,  $L = 4$ . Two states distinguished by  $\pm'$ .

$$u_{1,2,3,4} = \pm \sqrt{\frac{\pm' \sqrt{5} \pm i(5 \pm' 3\sqrt{5})}{15}}, \quad u_{5,6} = \pm \sqrt{\frac{-15 \mp' 2\sqrt{5}}{36}}, \quad u_7 = 0. \quad (\text{A.17})$$

[4;0,0;1,0,1;0,4].  $k_j = (3, 3, 3, 3, 4)$ ,  $L = 4$ .

$$u_{1,2,3,4} = \frac{\pm 1 \pm i}{\sqrt{2\sqrt{3}}}, \quad u_5 = 0. \quad (\text{A.18})$$

[4;0,0;0,0,0;0,4]. Two states distinguished by  $\pm'$ .  $k_j = (3, 3, 3, 3)$ ,  $L = 4$ .

$$u_{1,2,3,4} = \pm \sqrt{\frac{-9 \mp' \sqrt{41}}{12} \pm \sqrt{\frac{173 \pm' 33\sqrt{41}}{360}}}. \quad (\text{A.19})$$

[4;2,0;0,0,0;1,3].  $k_j = ()$ ,  $L = 3$ . Trivial.

[4;0,2;0,0,0;-1,3].  $k_j = (2, 2, 3, 3, 3, 3)$ ,  $L = 3$ .

$$u_{1,2} = \pm \sqrt{\frac{17}{60}}, \quad u_{3,4,5,6} = \pm \sqrt{\frac{-39 \pm \sqrt{993}}{30}}. \quad (\text{A.20})$$

[4;1,1;0,1,0;0,3].  $k_j = (2, 3, 3)$ ,  $L = 3$ . A pair of states distinguished by  $\pm'$ .

$$u_1 = \pm' \sqrt{\frac{5}{28}}, \quad u_{2,3} = \mp' \sqrt{\frac{1 \pm i\sqrt{399}}{30}}. \quad (\text{A.21})$$

$$[4;2,2;0,0,0,0,2]. \quad k_j = (2, 2), \quad L = 2.$$

$$u_{1,2} = \pm \sqrt{\frac{9}{28}}. \quad (\text{A.22})$$

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