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Superalgebra for M-theory on a pp-wave

Nakwoo Kim[a] and Jeong-Hyuck Park[b]

[a] *Max-Planck-Institute für Gravitationsphysik, Albert-Einstein-Institut*
Am Mühlenberg 1, D-14476 Golm, Germany

[b] *School of Physics, Korea Institute for Advanced Study*
207-43, Cheongryangri-Dong, Dongdaemun-Gu, Seoul 130-012, Corea

(kim@aei-potsdam.mpg.de, jhp@kias.re.kr)

We study the superalgebra of the M-theory on a fully supersymmetric pp-wave. We identify the algebra as the special unitary Lie superalgebra, $\mathrm{su}(2|4\,;2,0)$ or $\mathrm{su}(2|4\,;2,4)$, and analyze its root structure. We discuss the typical and atypical representations deriving the typicality condition explicitly in terms of the energy and other four quantum numbers. We classify the BPS multiplets preserving 4, 8, 12, 16 real supercharges and obtain the corresponding spectrum. We show that in the BPS multiplet either the lowest energy floor is a $\mathrm{su}(2)$ singlet or the highest energy floor is a $\mathrm{su}(4)$ singlet.

1 Introduction

Recent advances in string/M-theory [1] show that the matrix model for the M-theory in the infinite momentum frame [2] actually belongs to a one-parameter family of matrix models with the introduction of a mass parameter, μ. The new massive matrix model corresponds to a partonic description of $D0$-branes or alternatively a discretized supermembrane action [3] in the maximally supersymmetric pp-wave background [4, 5]. It is the presence of the massive terms which makes the model more accessible as the mass terms lift up the flat directions completely and the perturbative expansion is possible by powers of μ^{-1} [6, 7].

An interesting feature of the pp-wave matrix model is that the supercharges do not commute with the Hamiltonian because the supersymmetry transformations are explicitly time dependent. Accordingly the bosons and fermions have different masses and numbers as noted in [1]. More thorough understanding requires the complete analysis on the superalgebra itself, including the root structure and the representations. The complete classification of the Lie superalgebras was first done by Kac [8, 9], from which looking at the bosonic symmetry, $\mathrm{su}(2)\oplus\mathrm{su}(4)$, one can easily conclude that the *complexified* superalgebra of the pp-wave matrix model is $\mathrm{A}(1|3)$.

In this paper, we elaborate on the method of Kac and analyze the superalgebra of the M-theory on a fully supersymmetric pp-wave to demonstrate the root structure explicitly and discuss the typical and atypical representations. In particular we show that the actual superalgebra of the pp-wave matrix model is the *special unitary Lie superalgebra*, $\mathrm{su}(2|4\,;2,0)$ or $\mathrm{su}(2|4\,;2,4)$ depending on the sign of μ, and derive the typicality condition explicitly in terms of the energy and other four quantum numbers. We also completely classify the BPS multiplets as 4/16, 8/16, 12/16, 16/16 and obtain the corresponding spectrum.

The organization of the paper is as follows. In section 2, after setting up the gamma matrices and other conventions, we write down the superalgebra of the M-theory on a pp-wave and identify the algebra as the special unitary Lie superalgebra, $\mathrm{su}(2|4\,;2,0)$ or $\mathrm{su}(2|4\,;2,4)$. In section 2, we analyze the root structure of the algebra. We also present the quadratic super-Casimir operator. Section 4 discusses various types of irreducible representations. After describing the general properties, we discuss the typical and atypical irreducible representations, and we derive explicitly the typicality condition in terms of the energy and other four $\mathrm{su}(2)\oplus\mathrm{su}(4)$ quantum numbers. We completely classify the BPS multiplets as a special type of the atypical unitary representation. We find there are BPS multiplets preserving 4, 8, 12, 16 real supercharges, and obtain the corresponding exact spectrum. Further we show that in the BPS multiplet either the "lowest energy" floor is a su(2) singlet or the "highest energy" floor is a su(4) singlet. We list explicit examples in section 5. We conclude in Section 6.

Note Added: Up on finishing this paper, a related work [10] has appeared on the archive but differs in details.

2 Superalgebra

2.1 Gamma Matrices and Spinors

To make the $\mathrm{SO}(3) \times \mathrm{SO}(6)$ structure of the M-theory on a pp-wave manifest, we write the nine dimensional gamma matrices in terms of the three and six dimensional ones, σ^i, γ^a

$$\begin{aligned} \Gamma^i &= \sigma^i \otimes \gamma^{(7)} \qquad \text{for} \quad i = 1,2,3 \\ \Gamma^a &= 1 \otimes \gamma^a \qquad \text{for} \quad a = 4,5,6,7,8,9\,. \end{aligned} \tag{1}$$

With the choice

$$\gamma^{(7)} = i\gamma^4\gamma^5 \cdots \gamma^9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

the six dimensional gamma matrices are in the block diagonal form

$$\gamma^a = \begin{pmatrix} 0 & \rho^a \\ \bar{\rho}^a & 0 \end{pmatrix}, \qquad \rho^a\bar{\rho}^b + \rho^b\bar{\rho}^a = 2\delta^{ab}\,. \tag{3}$$

Note that the fact $\bar{\rho}^a = (\rho^a)^\dagger$ ensures γ^a to be hermitian. Further we set the 4×4 matrices, ρ^a to be anti-symmetric [11]

$$(\rho^a)_{\dot{\alpha}\dot{\beta}} = -(\rho^a)_{\dot{\beta}\dot{\alpha}}\,, \qquad (\bar{\rho}^a)^{\dot{\alpha}\dot{\beta}} = -(\bar{\rho}^a)^{\dot{\beta}\dot{\alpha}}\,. \tag{4}$$

Henceforth $\alpha, \beta = 1,2$ are the su(2) indices, while $\dot{\alpha}, \dot{\beta}$ denote the su(4) indices, $1,2,3,4$.

$\{\rho^{ab} \equiv \frac{1}{2}(\rho^a\bar{\rho}^b - \rho^b\bar{\rho}^a)\}$ forms an orthonormal basis of general 4×4 traceless matrices

$$\mathrm{tr}(\rho^{ab}\rho^{cd}) = 4(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}) \tag{5}$$

having the completeness relation

$$-\tfrac{1}{8}(\rho^{ab})_{\dot{\alpha}}{}^{\dot{\beta}}(\rho_{ab})_{\dot{\gamma}}{}^{\dot{\delta}} + \tfrac{1}{4}\delta_{\dot{\alpha}}{}^{\dot{\beta}}\delta_{\dot{\gamma}}{}^{\dot{\delta}} = \delta_{\dot{\alpha}}{}^{\dot{\delta}}\delta_{\dot{\gamma}}{}^{\dot{\beta}}\,. \tag{6}$$

Similarly for su(2) indices we have

$$-\tfrac{1}{4}(\sigma^{ij})_{\alpha}{}^{\beta}(\sigma_{ij})_{\gamma}{}^{\delta} + \tfrac{1}{2}\delta_{\alpha}{}^{\beta}\delta_{\gamma}{}^{\delta} = \delta_{\alpha}{}^{\delta}\delta_{\gamma}{}^{\beta}\,. \tag{7}$$

The nine dimensional charge conjugation matrix, C is then given by, with $(\sigma^i)^T = -\epsilon^{-1}\sigma^i\epsilon$, $A = 1,2,\cdots,9$,

$$(\Gamma^A)^T = (\Gamma^A)^* = C^{-1}\Gamma^A C\,, \qquad C = \epsilon \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{8}$$

so that Majorana spinor satisfying $\Psi = C\Psi^*$ contains eight independent complex components

$$\Psi = \begin{pmatrix} \psi_{\alpha\dot{\alpha}} \\ \\ \tilde{\psi}_{\alpha}{}^{\dot{\alpha}} \end{pmatrix}, \qquad \tilde{\psi}_{\alpha}{}^{\dot{\alpha}} = \epsilon_{\alpha\beta}(\psi^*)^{\beta\dot{\alpha}}\,. \tag{9}$$

2.2 The Special Unitary Lie Superalgebra, su$(2|4\,;2,0)$ or su$(2|4\,;2,4)$

In terms of the three and six dimensional gamma matrices, the superalgebra of the M-theory in a fully supersymmetric pp-wave background reads*

$$
\begin{array}{c}
[H,Q_{\alpha\dot{\alpha}}]=\frac{\mu}{12}Q_{\alpha\dot{\alpha}}\,,\\[2mm]
[M_{ij},Q_{\alpha\dot{\alpha}}]=i\frac{1}{2}(\sigma_{ij})_{\alpha}{}^{\beta}Q_{\beta\dot{\alpha}}\,,\qquad\qquad [M_{ab},Q_{\alpha\dot{\alpha}}]=i\frac{1}{2}(\rho_{ab})_{\dot{\alpha}}{}^{\dot{\beta}}Q_{\alpha\dot{\beta}}\,,\\[2mm]
[M_{ij},\bar{Q}^{\alpha\dot{\alpha}}]=-i\frac{1}{2}\bar{Q}^{\beta\dot{\alpha}}(\sigma_{ij})_{\beta}{}^{\alpha}\,,\qquad\qquad [M_{ab},\bar{Q}^{\alpha\dot{\alpha}}]=-i\frac{1}{2}\bar{Q}^{\alpha\dot{\beta}}(\rho_{ab})_{\dot{\beta}}{}^{\dot{\alpha}}\,,\\[2mm]
[M_{i},M_{j}]=i\epsilon_{ijk}M_{k}\,,\qquad\qquad M_{i}=\frac{1}{2}\epsilon_{ijk}M_{jk}\,,\\[2mm]
[M_{ab},M_{cd}]=i(\delta_{ac}M_{bd}-\delta_{ad}M_{bc}-\delta_{bc}M_{ad}+\delta_{bd}M_{ac})\,,\\[2mm]
\{Q_{\alpha\dot{\alpha}},\,\bar{Q}^{\beta\dot{\beta}}\}=\delta_{\alpha}{}^{\beta}\delta_{\dot{\alpha}}{}^{\dot{\beta}}H+i\frac{\mu}{6}(\sigma^{ij})_{\alpha}{}^{\beta}\delta_{\dot{\alpha}}{}^{\dot{\beta}}M_{ij}-i\frac{\mu}{12}\delta_{\alpha}{}^{\beta}(\rho^{ab})_{\dot{\alpha}}{}^{\dot{\beta}}M_{ab}\,.
\end{array}
\tag{10}
$$

The sign difference for the so(3) and so(6) generators appearing on the right hand side of the last line is crucial for the consistency. Along with H the so(3) and so(6) generators are hermitian, $(M_{ij})^{\dagger}=M_{ij}$, $(M_{ab})^{\dagger}=M_{ab}$. Note also that su(2) $\equiv$ so(3), su(4) $\equiv$ so(6).

In order to identify the superalgebra in terms of the supermatrices [13], we consider

$$
\mathcal{K}\cdot\mathcal{P}=\phi H+\bar{\theta}^{\alpha\dot{\alpha}}Q_{\alpha\dot{\alpha}}+\bar{Q}^{\alpha\dot{\alpha}}\theta_{\alpha\dot{\alpha}}+\tfrac{1}{2}w^{ij}M_{ij}+\tfrac{1}{2}w^{ab}M_{ab}
\tag{11}
$$

where $\theta_{\alpha\dot{\alpha}},,\bar{\theta}^{\alpha\dot{\alpha}}=(\theta_{\alpha\dot{\alpha}})^{\dagger}$ are Grassmannian "odd" coordinates and

$$
\mathcal{K}=(\phi,\theta,\bar{\theta},w^{ij},w^{ab})\,,\qquad\qquad \mathcal{P}=(H,\bar{Q},Q,M_{ij},M_{ab})\,.
\tag{12}
$$

Setting

$$
[\mathcal{K}_1\cdot\mathcal{P},\mathcal{K}_2\cdot\mathcal{P}]=-i\mathcal{K}_3\cdot\mathcal{P}\,,
\tag{13}
$$

we get

$$
\begin{array}{l}
\phi_3=i(\bar{\theta}_1\theta_2-\bar{\theta}_2\theta_1)\,,\\[2mm]
\theta_3=\frac{1}{4}w_1^{ij}\sigma_{ij}\,\theta_2+\frac{1}{4}w_1^{ab}\rho_{ab}\,\theta_2+i\frac{\mu}{12}\phi_2\theta_1\;-\;(1\leftrightarrow 2)\,,\\[2mm]
w_3^{ij}=w_{1\,k}^{i}w_2^{kj}+\frac{\mu}{3}\bar{\theta}_2\sigma^{ij}\theta_1\;-\;(1\leftrightarrow 2)\,,\\[2mm]
w_3^{ab}=w_{1\,c}^{a}w_2^{cb}-\frac{\mu}{6}\bar{\theta}_2\rho^{ab}\theta_1\;-\;(1\leftrightarrow 2)\,.
\end{array}
\tag{14}
$$

Now if we define a $(2+4)\times(2+4)$ supermatrix, $\mathcal{M}$, as

$$
\mathcal{M}=\left(\begin{array}{cc}
\left(\frac{1}{4}w^{ij}\sigma_{ij}-i\frac{\mu}{6}\phi\right)_{\alpha}{}^{\beta} & \sqrt{\frac{\mu}{3}}\,\theta_{\alpha\dot{\alpha}}\\[4mm]
\sqrt{\frac{\mu}{3}}\,\bar{\theta}^{\beta\dot{\beta}} & \left(\frac{1}{4}w^{ab}(-\rho_{ab})^{T}-i\frac{\mu}{12}\phi\right)^{\dot{\beta}}{}_{\dot{\alpha}}
\end{array}\right)\,,
\tag{15}
$$

*Here we disregard the possible inclusion of the central charges which may appear in the large N limit of the matrix theory [12].

we can show using (6) and (7) that the relation above (14) agrees with the matrix commutator

$$[\mathcal{M}_1, \mathcal{M}_2] = \mathcal{M}_3\,. \tag{16}$$

In general, $\mathcal{M}$ can be defined as a $(2+4)\times(2+4)$ supermatrix subject to

$$\mathrm{str}\,\mathcal{M} = 0\,, \qquad -\mathcal{M}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\mu}{|\mu|} \end{pmatrix} \mathcal{M} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\mu}{|\mu|} \end{pmatrix}. \tag{17}$$

Thus, the superalgebra of the M-theory on a fully supersymmetric pp-wave is the special unitary Lie superalgebra, $\mathrm{su}(2|4\,;2,0)$ for $\mu > 0$ or $\mathrm{su}(2|4\,;2,4)$ for $\mu < 0$ having dimensions $(19|16)$. Our convention here is from Kac [9] such that the bosonic part of $\mathrm{su}(p|q;p',q')$ is given by $\mathrm{su}(p', p-p') \oplus \mathrm{su}(q', q-q')$.

The fact that there are two inequivalent superalgebras for different signs of μ is essentially due to the fact that in nine or odd dimensions there are two inequivalent classes of gamma matrices characterized by $\Gamma^{12\cdots 9} = 1$ or -1 which are not related by the similarity transformations. Rewriting the superalgebra (10) in terms of the charge conjugates as in (9), $S_\alpha{}^{\dot\alpha} \equiv \epsilon_{\alpha\beta}\bar{Q}^{\beta\dot\alpha}$, one can obtain the same superalgebra for the opposite sign of μ with ρ_{ab} replaced by $\bar\rho_{ab} \equiv \frac{1}{2}(\bar\rho_a\rho_b - \bar\rho_b\rho_a)$,

$$\begin{pmatrix} Q_{\alpha\dot\alpha} \\ \\ \mu \\ \\ (\rho_{ab})_{\dot\alpha}{}^{\dot\beta} \end{pmatrix} \qquad \longleftrightarrow \qquad \begin{pmatrix} S_\alpha{}^{\dot\alpha} \\ \\ -\mu \\ \\ (\bar\rho_{ab})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{18}$$

The exchange of $\rho_a \leftrightarrow \bar\rho_a$ corresponds to the inequivalent choices of gamma matrices, $\Gamma^{12\cdots 9} = 1$ or -1. Therefore, sticking to one class of gamma matrices, but allowing both $+$ and $-$ signs for μ we do not lose any generality. Our choice is from (2) $\Gamma^{12\cdots 9} = 1$. Nevertheless after all, we will see that the physics is independent of the sign.

3 The Root Structure of the Superalgebra

In this section we analyze the root structure of the superalgebra. We first start with the following 8×8 representation of the bosonic part, $\mathrm{u}(1)\oplus\mathrm{su}(2)\oplus\mathrm{su}(4)$

$$\Big(R(\tfrac{6}{\mu}H)\,,\ R(M_{ij})\,,\ R(M_{ab})\Big) = \Big(-\tfrac{1}{2}\otimes 1\,,\ -i\tfrac{1}{2}\sigma_{ij}\otimes 1\,,\ 1\otimes -i\tfrac{1}{2}\rho_{ab}\Big)\,, \tag{19}$$

which are orthonormal and hermitian

$$\mathrm{Tr}(R_I^\dagger R_J) = 2\delta_{IJ}\,, \qquad R_I^\dagger = R_I\,, \qquad I,J = 1,2,\cdots,19\,. \tag{20}$$

Our choice of the Cartan subalgebra is

$$\overrightarrow{H} = (\tfrac{6}{\mu}H,\ M_{12},\ M_{45},\ M_{67},\ M_{89})\,. \tag{21}$$

Using the U(4) symmetry, $\rho_a \rightarrow U\rho_a U^T$, $UU^\dagger = 1$, which preserves the anti-symmetric property (4) of ρ_a, we can take the representation of the Cartan subalgebra in a diagonal form. Adopting the bra and ket notation for the su(4) part we set

$$
\begin{aligned}
&R(\tfrac{6}{\mu}H) = -\tfrac{1}{2} \otimes 1\,, \qquad\qquad R(M_{12}) = \tfrac{1}{2}\sigma_3 \otimes 1\,, \\
&R(M_{45}) = 1 \otimes \tfrac{1}{2}\left(\,|1\rangle\langle 1| + |2\rangle\langle 2| - |3\rangle\langle 3| - |4\rangle\langle 4|\,\right)\,, \\
&R(M_{67}) = 1 \otimes \tfrac{1}{2}\left(\,|1\rangle\langle 1| - |2\rangle\langle 2| + |3\rangle\langle 3| - |4\rangle\langle 4|\,\right)\,, \\
&R(M_{89}) = 1 \otimes \tfrac{1}{2}\left(\,|1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3| + |4\rangle\langle 4|\,\right)\,,
\end{aligned}
\tag{22}
$$

which is also compatible with (2).

All the bosonic positive roots are then given by

$$
\begin{aligned}
&R(\mathcal{E}_z) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1\,, && z = (0,1,0,0,0)\,, \\
&R(\mathcal{E}_u) = 1 \otimes |1\rangle\langle 2|\,, && u = (0,0,0,1,1)\,, \\
&R(\mathcal{E}_v) = 1 \otimes |2\rangle\langle 3|\,, && v = (0,0,1,-1,0)\,, \\
&R(\mathcal{E}_w) = 1 \otimes |3\rangle\langle 4|\,, && w = (0,0,0,1,-1)\,, \\
&R(\mathcal{E}_{u+v}) = 1 \otimes |1\rangle\langle 3|\,, && u+v = (0,0,1,0,1)\,, \\
&R(\mathcal{E}_{v+w}) = 1 \otimes |2\rangle\langle 4|\,, && v+w = (0,0,1,0,-1)\,, \\
&R(\mathcal{E}_{u+v+w}) = 1 \otimes |1\rangle\langle 4|\,, && u+v+w = (0,0,1,1,0)\,,
\end{aligned}
\tag{23}
$$

where z and u, v, w are respectively the su(2) and su(4) simple roots. The negative roots follow simply from $R(\mathcal{E}_{-z}) = R(\mathcal{E}_z)^\dagger$, etc.

Just like R_I in (20), $(R(\overrightarrow{H}),\, R(\mathcal{E}_+),\, R(\mathcal{E}_-))$ are also orthonormal. This implies that those two are related by the unitary transformation so that

$$
\begin{aligned}
\tfrac{1}{2}R(M^{ab})M_{ab} = \quad & R(M_{45})M_{45} + R(M_{67})M_{67} + R(M_{89})M_{89} \\
& + \sum_{\chi \in \Delta_4^+} \Big(R(\mathcal{E}_\chi)\mathcal{E}_{-\chi} + R(\mathcal{E}_{-\chi})\mathcal{E}_\chi \Big)\,,
\end{aligned}
\tag{24}
$$

where Δ_4^+ denotes the set of all the su(4) positive roots $u,\ v,\ w,\ u+v,\ v+w,\ u+v+w$.

In terms of the Cartan subalgebra and any su(2) $\oplus$ su(4) root, χ, the superalgebra of the

M-theory on a pp-wave reads up to the complex conjugate

$$
\begin{aligned}
&[\overrightarrow{H},\mathcal{E}_\chi]=\chi\mathcal{E}_\chi\,, && [\mathcal{E}_\chi,\mathcal{E}_{-\chi}]=\chi{\cdot}\overrightarrow{H}\,,\\
&[\mathcal{E}_u,\mathcal{E}_v]=\mathcal{E}_{u+v}\,, && [\mathcal{E}_v,\mathcal{E}_w]=\mathcal{E}_{v+w}\,,\\
&[\mathcal{E}_{u+v},\mathcal{E}_w]=\mathcal{E}_{u+v+w}\,, && [\mathcal{E}_u,\mathcal{E}_w]=0\,,\\
&[\overrightarrow{H},Q]=-R(\overrightarrow{H})Q\,, && [\mathcal{E}_\chi,Q]=-R(\mathcal{E}_\chi)Q\,,
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
\{Q_{\alpha\dot\alpha},\bar{Q}^{\beta\dot\beta}\}=\;&\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}H-\tfrac{\mu}{3}\begin{pmatrix} M_{12} & \sqrt{2}\mathcal{E}_{-z}\\ \sqrt{2}\mathcal{E}_z & -M_{12}\end{pmatrix}_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}\\
&+\tfrac{\mu}{3}\delta_\alpha{}^\beta\begin{pmatrix} f_1 & \mathcal{E}_{-u} & \mathcal{E}_{-u-v} & \mathcal{E}_{-u-v-w}\\ \mathcal{E}_u & f_2 & \mathcal{E}_{-v} & \mathcal{E}_{-v-w}\\ \mathcal{E}_{u+v} & \mathcal{E}_v & f_3 & \mathcal{E}_{-w}\\ \mathcal{E}_{u+v+w} & \mathcal{E}_{v+w} & \mathcal{E}_w & f_4\end{pmatrix}_{\dot\alpha}{}^{\dot\beta}\,,
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
&f_1=\tfrac{1}{2}(M_{45}+M_{67}+M_{89})\,, && f_2=\tfrac{1}{2}(M_{45}-M_{67}-M_{89})\,,\\
&f_3=\tfrac{1}{2}(-M_{45}+M_{67}-M_{89})\,, && f_4=\tfrac{1}{2}(-M_{45}-M_{67}+M_{89})\,.
\end{aligned}
\tag{27}
$$

In particular from (25), the unique fermionic simple root, q and other fermionic positive roots are given as

$$
\begin{aligned}
&[\overrightarrow{H},Q_{11}]=qQ_{11}\,, && q=(\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})\,,\\
&[\overrightarrow{H},Q_{12}]=(q+u)Q_{12}\,, && Q_{12}=-[\mathcal{E}_u,Q_{11}]\,,\\
&[\overrightarrow{H},Q_{13}]=(q+u+v)Q_{13}\,, && Q_{13}=-[\mathcal{E}_{u+v},Q_{11}]\,,\\
&[\overrightarrow{H},Q_{14}]=(q+u+v+w)Q_{14}\,, && Q_{14}=-[\mathcal{E}_{u+v+w},Q_{11}]\,,\\
&[\overrightarrow{H},Q_{21}]=(q+z)Q_{21}\,, && Q_{21}=-[\sqrt{2}\mathcal{E}_z,Q_{11}]\,,\\
&[\overrightarrow{H},Q_{22}]=(q+z+u)Q_{22}\,, && Q_{12}=-[\sqrt{2}\mathcal{E}_z,Q_{12}]\,,\\
&[\overrightarrow{H},Q_{23}]=(q+z+u+v)Q_{23}\,, && Q_{23}=-[\sqrt{2}\mathcal{E}_z,Q_{13}]\,,\\
&[\overrightarrow{H},Q_{24}]=(q+z+u+v+w)Q_{24}\,, && Q_{24}=-[\sqrt{2}\mathcal{E}_z,Q_{14}]\,.
\end{aligned}
\tag{28}
$$

The complexification of su(2|4 ; 2, 0) or su(2|4 ; 2, 4) corresponds to A(1|3), and its Dynkin diagram is with the simple roots

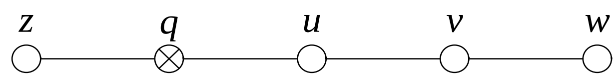

Figure 1: The Dynkin diagram of A(1|3)

Finally the second order Casimir operator, $\mathcal{C}_M$ is

$$\mathcal{C}_M = \tfrac{12}{\mu}H^2 - \tfrac{\mu}{3}\mathcal{C}_4 + \tfrac{2\mu}{3}\mathcal{C}_2 + \bar{Q}^{\alpha\dot{\alpha}}Q_{\alpha\dot{\alpha}} - Q_{\alpha\dot{\alpha}}\bar{Q}^{\alpha\dot{\alpha}}\,, \tag{29}$$

where $\mathcal{C}_4$ and $\mathcal{C}_2$ are su(4) and su(2) Casimirs

$$\begin{aligned} \mathcal{C}_4 &= \tfrac{1}{2}M^{ab}M_{ab} = M_{45}^2 + M_{67}^2 + M_{89}^2 + \sum_{\chi\in\Delta_4^+}\Big(\mathcal{E}_\chi\mathcal{E}_{-\chi} + \mathcal{E}_{-\chi}\mathcal{E}_\chi\Big)\,, \\ \mathcal{C}_2 &= M_{12}^2 + \mathcal{E}_z\mathcal{E}_{-z} + \mathcal{E}_{-z}\mathcal{E}_z\,. \end{aligned} \tag{30}$$

4 Irreducible Representations

Starting with an eigenstate of H, by acting $\bar{Q}^{\alpha\dot{\alpha}}$'s as many as possible maximally eight times surely, one can obtain a state which is annihilated by all the $\bar{Q}^{\alpha\dot{\alpha}}$'s. Now under the action of the bosonic operators, the state opens up an irreducible representation of $\mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(4)$ or the zeroth floor multiplet. Further from (25), any state in the multiplet is annihilated by all the fermionic negative roots. Consequently there exists a unique *superlowest weight*, $|\Lambda_L\rangle$, in the supermultiplet annihilated by all the negative roots

$$\bar{Q}^{11}|\Lambda_L\rangle = 0\,, \qquad \mathcal{E}_{-\chi}|\Lambda_L\rangle = 0\,, \quad \chi = z, u, v, w. \tag{31}$$

The superlowest weight vector is specified by an arbitrary energy value, E_0 and four non-negative integers, J_z, J_u, J_v, J_w

$$\Lambda_L = \Big(\tfrac{6}{\mu}E_0,\, -\tfrac{1}{2}J_z\,,\, -\tfrac{1}{2}(J_u + 2J_v + J_w)\,,\, -\tfrac{1}{2}(J_u + J_w)\,,\, -\tfrac{1}{2}(J_u - J_w)\Big) \tag{32}$$

satisfying for the $\mathrm{su}(2) \oplus \mathrm{su}(4)$ simple roots, $\chi = z, u, v, w$,

$$-2\frac{\chi\cdot\Lambda_L}{\chi^2} = J_\chi\,, \qquad (\mathcal{E}_\chi)^{J_\chi+1}|\Lambda_L\rangle = 0\,. \tag{33}$$

All the other states are generated by repeated applications of the positive roots on $|\Lambda_L\rangle$, and without loss of generality one can safely restrict on the simple roots only, Q_{11}, $\mathcal{E}_\chi$, $\chi = z, u, v, w$. In general the different ordering of multiplication may result in the degeneracy for the states of a definite weight vector. Using the commutator relations $[\mathcal{E}_\chi, Q] \sim Q$ in (25), one can always move all the Q_{11}'s appearing to either far right or far left allowing other fermionic positive roots. Therefore the whole supermultiplet is exhibited by

$$\mathcal{E}_{\chi_n} \cdots \mathcal{E}_{\chi_1} Q_{2\dot{\beta}_l} \cdots Q_{2\dot{\beta}_1} Q_{1\dot{\alpha}_m} \cdots Q_{1\dot{\alpha}_1} |\Lambda_L\rangle \tag{34}$$

or equivalently

$$Q_{2\dot{\beta}_l} \cdots Q_{2\dot{\beta}_1} Q_{1\dot{\alpha}_m} \cdots Q_{1\dot{\alpha}_1} \mathcal{E}_{\chi_n} \cdots \mathcal{E}_{\chi_1} |\Lambda_L\rangle \,. \tag{35}$$

The latter is essentially the repeated applications of the fermionic positive roots on the zeroth floor. As the zeroth floor multiplet has dimension [14]

$$d_0 = (J_z + 1) \times \left[\tfrac{1}{12}(J_u+1)(J_v+1)(J_w+1)(J_u+J_v+2)(J_v+J_w+2)(J_u+J_v+J_w+3)\right], \tag{36}$$

Eq.(35) implies that the supermultiplet has a finite dimension, d_s

$$d_s \leq 256 \times d_0 \,. \tag{37}$$

The application of a $Q_{\alpha\dot{\alpha}}$ changes the u(1) $\oplus$ su(2) $\oplus$ su(4) multiplets jumping from an irreducible representation to another. In particular, the number of the applied fermionic roots determines the floor number, zero to eight at most. Each floor has the energy

$$E_N = E_0 + \tfrac{\mu}{12} N \,, \qquad N = 0, 1, 2, \cdots, 8. \tag{38}$$

Apparently the zeroth floor corresponds to the lowest energy states for $\mu > 0$ or to the highest energy states for $\mu < 0$.

Each of the zeroth and the highest floors corresponds to an irreducible representation of u(1) $\oplus$ su(2) $\oplus$ su(4), while others are in general reducible representations. Each irreducible representation is specified by its lowest weight, λ_L, annihilated by all the bosonic negative roots of the form

$$\lambda_L = \left(\tfrac{6}{\mu}E_0 + \tfrac{1}{2}N,\, -\tfrac{1}{2}j_z\,,\, -\tfrac{1}{2}(j_u + 2j_v + j_w)\,,\, -\tfrac{1}{2}(j_u + j_w)\,,\, -\tfrac{1}{2}(j_u - j_w)\right)\,. \tag{39}$$

The corresponding highest weight is then [15]

$$\lambda_H = \left(\tfrac{6}{\mu}E_0 + \tfrac{1}{2}N,\, \tfrac{1}{2}j_z\,,\, \tfrac{1}{2}(j_w + 2j_v + j_u)\,,\, \tfrac{1}{2}(j_w + j_u)\,,\, \tfrac{1}{2}(j_w - j_u)\right)\,, \tag{40}$$

while the dimension is given by (36) with $J \leftrightarrow j$.

Now if we define

$$T^l_\alpha \equiv Q_{\alpha l} \cdots Q_{\alpha 2} Q_{\alpha 1} \qquad \text{for} \quad l = 1, 2, 3, 4 \tag{41}$$

then all the "naturally ordered" products of the fermionic roots are given by [16]

$$T^m_1\,, \qquad T^l_2 T^m_1\,, \qquad 1 \leq l \leq m \leq 4\,. \tag{42}$$

There are fourteen of them and they are all the possible products of $Q_{\alpha\dot{\alpha}}$'s which commute with any bosonic negative root

$$[\mathcal{E}_{-\chi}, T_1^m] = 0\,, \qquad [\mathcal{E}_{-\chi}, T_2^l T_1^m] = 0\,, \qquad 1 \leq l \leq m \leq 4\,. \tag{43}$$

Apparently we have the following possible lowest weights for $\mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(4)$ multiplets on each floor

$$\begin{array}{lc}
N = 8\ : & T_2^4 T_1^4 |\Lambda_L\rangle \\
N = 7\ : & T_2^3 T_1^4 |\Lambda_L\rangle \\
N = 6\ : & T_2^2 T_1^4 |\Lambda_L\rangle\,, \qquad T_2^3 T_1^3 |\Lambda_L\rangle \\
N = 5\ : & T_2^1 T_1^4 |\Lambda_L\rangle\,, \qquad T_2^2 T_1^3 |\Lambda_L\rangle \\
N = 4\ : & T_1^4 |\Lambda_L\rangle\,, \qquad T_2^1 T_1^3 |\Lambda_L\rangle\,, \qquad T_2^2 T_1^2 |\Lambda_L\rangle \\
N = 3\ : & T_1^3 |\Lambda_L\rangle\,, \qquad T_2^1 T_1^2 |\Lambda_L\rangle \\
N = 2\ : & T_1^2 |\Lambda_L\rangle\,, \qquad T_2^1 T_1^1 |\Lambda_L\rangle \\
N = 1\ : & T_1^1 |\Lambda_L\rangle \\
N = 0\ : & |\Lambda_L\rangle\,.
\end{array} \tag{44}$$

A few remarks are in order. They are truly lowest weights only if they do not decouple. The dimension of the corresponding $\mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(4)$ irreducible representation can be easily obtained from (28), (32) and (36). One finds that the sum of the dimensions is equal to $256 \times d_0$ only if the zeroth floor is a $\mathrm{su}(2) \oplus \mathrm{su}(4)$ singlet, i.e. $d_0 = 1$, and for the non-singlet zeroth floor it is strictly less. Thus in general, apart from (44), there are hidden lowest weights, of which the annihilation by the bosonic negative roots requires the explicit information of the superlowest weight vector (32). For example, a possible hidden lowest weight is

$$(J_u Q_{12} + Q_{11} \mathcal{E}_u) |\Lambda_L\rangle\,, \tag{45}$$

which vanishes identically if $J_u = 0$.

4.1 Typical and Atypical Representations

Typical representation is defined to have the possible maximal dimension, $d_s = 256 \times d_0$, while atypical representation has less dimension. Namely, in a typical representation all the possible states which could appear will appear. Consequently for typical representation, $T_2^4 T_1^4 |\Lambda_L\rangle$ does not decouple

$$T_2^4 T_1^4 |\Lambda_L\rangle \neq 0. \tag{46}$$

Proposition : An irreducible representation is typical if and only if Eq.(46) holds.
Proof : We only need to prove the sufficiency. First consider an arbitrary state on the zeroth

floor, $|\lambda\rangle$, and the corresponding bosonic operator, $\mathcal{B}_-$ which is essentially a product of bosonic negative roots taking $|\lambda\rangle$ to the superlowest weight

$$\mathcal{B}_-|\lambda\rangle = |\Lambda_L\rangle\,. \tag{47}$$

Now consider any state, $|\upsilon\rangle$ in the supermultiplet and write it as (35)

$$|\upsilon\rangle = Q_{2\dot{\beta}_l}\cdots Q_{2\dot{\beta}_1}Q_{1\dot{\alpha}_m}\cdots Q_{1\dot{\alpha}_1}|\lambda\rangle\,. \tag{48}$$

Acting the complementary fermionic positive roots and $\mathcal{B}_-$ successively we obtain using (43)

$$\mathcal{B}_-Q_{2\dot{\beta}_4}\cdots Q_{2\dot{\beta}_{l+1}}Q_{1\dot{\alpha}_4}\cdots Q_{1\dot{\alpha}_{m+1}}|\upsilon\rangle = \mathcal{B}_-T_2^4T_1^4|\lambda\rangle = T_2^4T_1^4|\Lambda_L\rangle\,. \tag{49}$$

Thus (46) implies the nonvanishing of $|\upsilon\rangle$. This completes our proof.

To see whether $T_2^4T_1^4|\Lambda_L\rangle$ decouples or not we need the following recurrent relations

$$\begin{aligned}
&(T_1^l)^\dagger T_1^l|\Lambda_L\rangle = c_l(T_1^{l-1})^\dagger T_1^{l-1}|\Lambda_L\rangle\,,\\
&(T_2^lT_1^m)^\dagger T_2^lT_1^m|\Lambda_L\rangle = c_{l+4}(T_2^{l-1}T_1^m)^\dagger T_2^{l-1}T_1^m|\Lambda_L\rangle\,,
\end{aligned} \tag{50}$$

where $1\leq l\leq m\leq 4$, $T_\alpha^0=1$ and c_{l+4} is independent of m

$$\begin{aligned}
c_1 &= E_0+\tfrac{\mu}{12}(2J_z-3J_u-2J_v-J_w)\,,\\
c_2 &= E_0+\tfrac{\mu}{12}(2J_z+J_u-2J_v-J_w+4)\,,\\
c_3 &= E_0+\tfrac{\mu}{12}(2J_z+J_u+2J_v-J_w+8)\,,\\
c_4 &= E_0+\tfrac{\mu}{12}(2J_z+J_u+2J_v+3J_w+12)\,,\\
c_5 &= E_0-\tfrac{\mu}{12}(2J_z+3J_u+2J_v+J_w+4)\,,\\
c_6 &= E_0-\tfrac{\mu}{12}(2J_z-J_u+2J_v+J_w)\,,\\
c_7 &= E_0-\tfrac{\mu}{12}(2J_z-J_u-2J_v+J_w-4)\,,\\
c_8 &= E_0-\tfrac{\mu}{12}(2J_z-J_u-2J_v-3J_w-8)\,.
\end{aligned} \tag{51}$$

Consequently

$$(T_2^4T_1^4)^\dagger T_2^4T_1^4|\Lambda_L\rangle = \prod_{k=1}^{8}c_k|\Lambda_L\rangle\,. \tag{52}$$

Thus for all $k=1,2,\cdots,8$ when $c_k\neq 0$, $T_2^4T_1^4|\Lambda_L\rangle$ can not decouple and the supermultiplet is typical. Otherwise the representation is shortened or atypical and the energy value is quantized.

Similarly it is also interesting to note that if $T_1^4|\Lambda_L\rangle\neq 0$, then all the states of the form

$$Q_{1\dot{\alpha}_m}\cdots Q_{1\dot{\alpha}_1}|\lambda\rangle\,,\qquad 0\leq m\leq 4 \tag{53}$$

do not decouple, where $|\lambda\rangle$ is an arbitrary state of the zeroth floor.

4.2 BPS Multiplets

BPS multiplets are naturally associated to a superspace with lower number of "odd" coordinates [17]. Namely they are a kind of atypical unitary representation of which *either* the superlowest weight is annihilated by a certain number of supercharges, $Q_{\alpha\dot{\alpha}}|\Lambda_L\rangle = 0$ *or* the superhighest weight is so, $\bar{Q}^{\alpha\dot{\alpha}}|\Lambda_H\rangle = 0$. First we focus on the case for the superlowest weight. Acting the bosonic negative roots we further get

$$Q_{\beta\dot{\beta}}|\Lambda_L\rangle = 0 \qquad \text{for} \;\; 1 \leq \beta \leq \alpha\,, \quad 1 \leq \dot{\beta} \leq \dot{\alpha}\,. \tag{54}$$

Hence the constraint also removes $Q_{\beta\dot{\beta}}$, $\beta \leq \alpha$, $\dot{\beta} \leq \dot{\alpha}$ completely from the supermultiplet written in the form (34) implying the elimination of the corresponding "odd" coordinates. By unitarity we mean the positive definite norm, the physically relevant condition. For generic $Q_{\beta\dot{\beta}}$

$$\begin{aligned} ||Q_{\beta\dot{\beta}}|\Lambda_L\rangle||^2 = \;\; & E_0 + \tfrac{\mu}{6} J_z (\sigma_3)_\beta{}^\beta \\ & + \tfrac{\mu}{12}\mathrm{diag}(-3J_u - 2J_v - J_w,\, J_u - 2J_v - J_w,\, J_u + 2J_v - J_w,\, J_u + 2J_v + 3J_w)_{\dot{\beta}}{}^{\dot{\beta}}\,, \end{aligned} \tag{55}$$

so that

$$\begin{aligned} E_0 &\geq \tfrac{\mu}{12}(2J_z + 3J_u + 2J_v + J_w) \qquad \text{for} \;\; \mu > 0\,, \\ E_0 &\geq \tfrac{|\mu|}{12}(2J_z + J_u + 2J_v + 3J_w) \qquad \text{for} \;\; \mu < 0\,. \end{aligned} \tag{56}$$

The saturation, if ever happens, must occur at least with Q_{21} for $\mu > 0$ and Q_{14} for $\mu < 0$.

Combining the results above we get the following classification of the BPS multiplets.

SU(2) Singlet BPS $(\mu > 0)$

4/16 BPS :	$E_0 = \frac{\mu}{12}(3J_u + 2J_v + J_w)$,	$J_z = 0,\ J_u \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 1)$
8/16 BPS :	$E_0 = \frac{\mu}{12}(2J_v + J_w)$,	$J_z = J_u = 0,\ J_v \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 2)$
12/16 BPS :	$E_0 = \frac{\mu}{12}J_w$,	$J_z = J_u = J_v = 0,\ J_w \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 3)$
Vacua :	$E_0 = 0$,	$J_z = J_u = J_v = J_w = 0$,	$(\alpha, \dot{\alpha}) = (2, 4)$

SU(4) Singlet BPS $(\mu < 0)$

8/16 BPS :	$E_0 = \frac{\lvert\mu\rvert}{6}J_z$,	$J_u = J_v = J_w = 0,\ J_z \neq 0$,	$(\alpha, \dot{\alpha}) = (1, 4)$
Vacua :	$E_0 = 0$,	$J_z = J_u = J_v = J_w = 0$,	$(\alpha, \dot{\alpha}) = (2, 4)$

where $(\alpha, \dot{\alpha})$ refers the highest fermionic root in (54).

Although it appears here that the sign of μ would matter, it is not completely asymmetric, since instead of the superlowest weight one can construct the BPS multiplet from the *superhighest weight*, $|\Lambda_H\rangle$, to obtain the exactly same BPS spectrum as above but for the opposite sign. More explicitly

$$\Lambda_H = \left(\tfrac{6}{\mu}E_H,\, \tfrac{1}{2}J'_z\,,\, \tfrac{1}{2}(J'_u + 2J'_v + J'_w)\,,\, \tfrac{1}{2}(J'_u + J'_w)\,,\, \tfrac{1}{2}(J'_u - J'_w)\right)\,, \tag{57}$$

$$\bar{Q}^{\alpha\dot{\alpha}}|\Lambda_H\rangle = 0 \qquad \Longrightarrow \qquad \bar{Q}^{\beta\dot{\beta}}|\Lambda_H\rangle = 0 \quad \text{ for } \; 1 \leq \beta \leq \alpha\,, \quad 1 \leq \dot{\beta} \leq \dot{\alpha}\,, \tag{58}$$

and

$$\begin{aligned} ||\bar{Q}^{\beta\dot{\beta}}|\Lambda_H\rangle||^2 = \;& E_H - \tfrac{\mu}{6}J'_z(\sigma_3)_\beta{}^\beta \\ & -\tfrac{\mu}{12}\text{diag}(-3J'_u - 2J'_v - J'_w,\, J'_u - 2J'_v - J'_w,\, J'_u + 2J'_v - J'_w,\, J'_u + 2J'_v + 3J'_w)_{\dot{\beta}}{}^{\dot{\beta}}\,, \end{aligned} \tag{59}$$

where, compared to (55), the sign of μ is flipped while J replaced by J'. The BPS multiplets and the energy spectrum are identical to the previous result

SU(2) Singlet BPS $\quad (\mu < 0)$

4/16 BPS :	$E_H = \frac{\lvert\mu\rvert}{12}(3J'_u + 2J'_v + J'_w)$,	$J'_z = 0,\ J'_u \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 1)$
8/16 BPS :	$E_H = \frac{\lvert\mu\rvert}{12}(2J'_v + J'_w)$,	$J'_z = J'_u = 0,\ J'_v \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 2)$
12/16 BPS :	$E_H = \frac{\lvert\mu\rvert}{12}J'_w$,	$J'_z = J'_u = J'_v = 0,\ J'_w \neq 0$,	$(\alpha, \dot{\alpha}) = (2, 3)$
Vacua :	$E_H = 0$,	$J'_z = J'_u = J'_v = J'_w = 0$,	$(\alpha, \dot{\alpha}) = (2, 4)$

SU(4) Singlet BPS $\quad (\mu > 0)$

8/16 BPS :	$E_H = \frac{\mu}{6}J'_z$,	$J'_u = J'_v = J'_w = 0,\ J'_z \neq 0$,	$(\alpha, \dot{\alpha}) = (1, 4)$
Vacua :	$E_H = 0$,	$J'_z = J'_u = J'_v = J'_w = 0$,	$(\alpha, \dot{\alpha}) = (2, 4)$

In summary, there are two ways of defining the BPS multiplets, one from the superlowest weight and the other from the superhighest weight. Whichever we choose, if it corresponds to the lowest energy, the "lowest energy" floor is a su(2) singlet, while if it corresponds to the highest energy, the "highest energy" floor is a su(4) singlet. After all the physics is independent of the sign of μ.

It is worth noting that the su(2) singlet on the "lowest energy" floor does not necessarily imply the su(4) singlet on the "highest energy" floor, and *vice versa*.

5 Examples

In this section we provide examples of typical, atypical and the BPS representations discussed in the previous sections. We use two different notations for su(2) $\oplus$ su(4) in this section. For a few simple representations we just give the dimensions for su(2) $\oplus$ su(4), like (d, d'). But as we come to study larger multiplets it is better to use the Dynkin labels. Instead of using the full Dynkin labels we mostly give the Dynkin labels of the su(2) $\oplus$ su(4) lowest weight only, and the states of different energies are given in separate lines. Our convention for su(4) is that $4 = (1, 0, 0)$ and the positive fermionic roots Q's are in $(2, 4)$. For larger multiplets of su(4) we give here a short dictionary

$$\begin{gathered} 4 = (1,0,0)\,, \qquad \overline{4} = (0,0,1)\,, \\ 6 = (0,1,0)\,, \\ 10 = (2,0,0)\,, \qquad \overline{10} = (0,0,2) \\ 15 = (1,0,1)\,, \\ 20 = (1,1,0)\,, \qquad 20' = (0,2,0)\,, \qquad \overline{20} = (0,1,1)\,, \\ 36 = (2,0,1)\,, \qquad \overline{36} = (1,0,2)\,. \end{gathered} \tag{60}$$

5.1 Typical and atypical representations from su$(2) \oplus$ su(4) singlet

First we consider the supermultiplets where the superlowest weight is a su(2) $\oplus$ su(4) singlet. The construction is especially simple in this case since there are no possible hidden su(2) $\oplus$ su(4) lowest weights and all the possible lowest weights are those listed in (44). For generic values of $E_0 \neq 0, \pm\frac{1}{3}\mu, -\frac{2}{3}\mu, -\mu$, the representation becomes typical having the minimal dimension, 256. In this case the supermultiplets are of the form presented in Table 1.

The general typical representation can be obtained from (35) as the direct product of the minimal typical supermultiplet above with an arbitrary su(2) $\oplus$ su(4) multiplet. Surely the su(2) $\oplus$ su(4) multiplet becomes the zeroth as well as the eighth floors in the generic typical supermultiplet.

Now we consider the atypical representations where the superlowest weight is a su(2)$\oplus$su(4) singlet. Using the atypicality condition (51), it is easy to see that the representation becomes short when $E_0 = 0, \pm\frac{1}{3}\mu, -\frac{2}{3}\mu, -\mu$. For the unitary representations we choose the sign of μ such that E_0 is nonnegative. All together there are five short unitary representations which have $(1, 1)$ as the lowest or the highest "energy" floor. They are presented in Table 2.

Floors	
8	$(1,1)$
7	$(2,\overline{4})$
6	$(3,6)\oplus(1,\overline{10})$
5	$(4,4)\oplus(2,\overline{20})$
4	$(5,1)\oplus(3,15)\oplus(1,20')$
3	$(4,\overline{4})\oplus(2,20)$
2	$(3,6)\oplus(1,10)$
1	$(2,4)$
0	$(1,1)$

Table 1: An example of typical representation constructed from $(1,1)$

Energy$\times(12/\lvert\mu\rvert)$					
12					$(1,1)$
11					$(2,4)$
10					$(3,6)\oplus(1,10)$
9					$(4,\overline{4})\oplus(2,20)$
8			$(5,1)$	$(1,1)$	$(3,15)\oplus(1,20')$
7			$(4,\overline{4})$	$(2,4)$	$(2,\overline{20})$
6			$(3,6)$	$(3,6)\oplus(1,10)$	$(1,\overline{10})$
5			$(2,4)$	$(2,20)$	
4		$(1,1)$	$(1,1)$	$(1,20')$	
3		$(2,4)$			
2		$(1,10)$			
1					
0	$(1,1)$				

Table 2: Atypical unitary representations from $(1,1)$

5.2 Adjoint representation

Here we consider the adjoint representation of the superalgebra which is also a kind of atypical representations. We already know the contents of the adjoint representation: It has 19 bosonic and 16 fermionic generators, and all the bosonic generators commute with the Hamiltonian while the 16 fermionic generators have eigenvalues, $\pm\frac{\mu}{12}$.

The zeroth floor is $(2,\overline{4})$ with the superlowest weight, $|\Lambda_L\rangle=\bar{Q}^{24}$ of the Dynkin label, $(1;0,0,1)$. Now acting Q, which is in $(2,4)$, the first floor decomposes into

$$(2,4)\otimes(2,\overline{4})=(1\oplus 3,\,1\oplus 15)\,. \tag{61}$$

Among them $(3,15)$ decouples since the lowest weight vanishes, $T^1_1|\Lambda_L\rangle=0$ because $c_1=0$ with $E_0=-\frac{\mu}{12}$, $J_z=1$, $J_u=J_v=0$, $J_w=1$ implying the atypicality. Other three hidden lowest weights are constructed from the superlowest states as follows,

$$
\begin{aligned}
(1,1) \quad &: \quad (2Q_{11}\mathcal{E}_z - \mathcal{E}_z Q_{11})\mathcal{E}_{u+v+w}|\Lambda_L\rangle\,, \\
(3,1) \quad &: \quad Q_{11}\mathcal{E}_{u+v+w}|\Lambda_L\rangle\,, \\
(1,15) \quad &: \quad Q_{11}\mathcal{E}_z|\Lambda_L\rangle\,.
\end{aligned}
\tag{62}
$$

Surely they correspond to u(1), so(3) and so(6) generators respectively.

Similarly from Table 1 the second floor decomposes into

$$
\begin{aligned}
(3,6)\otimes(2,\overline{4}) \quad &= \quad (2\oplus 4,\ 4\oplus\overline{20})\,, \\
(1,10)\otimes(2,\overline{4}) \quad &= \quad (2,\ 4\oplus 36)\,,
\end{aligned}
\tag{63}
$$

and the possible lowest weights are

$$
\begin{aligned}
(2,4) \quad &: \quad Q_{12}Q_{11}\mathcal{E}_z\mathcal{E}_{v+w}|\Lambda_L\rangle\,, \\
(2,\overline{20}) \quad &: \quad Q_{12}Q_{11}\mathcal{E}_z|\Lambda_L\rangle\,, \\
(4,4) \quad &: \quad Q_{12}Q_{11}\mathcal{E}_{v+w}|\Lambda_L\rangle\,, \\
(4,\overline{20}) \quad &: \quad Q_{12}Q_{11}|\Lambda_L\rangle\,, \\
(2,4) \quad &: \quad Q_{21}Q_{11}\mathcal{E}_{u+v+w}|\Lambda_L\rangle\,, \\
(2,36) \quad &: \quad Q_{21}Q_{11}|\Lambda_L\rangle\,.
\end{aligned}
\tag{64}
$$

As before one should check whether they decouple or not. It turns out that all of them decouple except only one lowest weight, namely $Q_{21}Q_{14}|\Lambda_L\rangle = Q_{21}Q_{11}\mathcal{E}_{u+v+w}|\Lambda_L\rangle = -\frac{\mu}{3}Q_{11}$, as it should be. The complete adjoint representation is presented in Table 3.

Energy	
$+\mu/12$	$(2,4)$
0	$(1,1)\oplus(3,1)\oplus(1,15)$
$-\mu/12$	$(2,\overline{4})$

Table 3: Adjoint representation of A(1|3)

5.3 BPS multiplets

In this subsection we construct several examples of the BPS multiplets. In addition to Vacua which have the trivial representation given by a single state, there are $3/4, 1/2, 1/4$ BPS multiplets.

We first consider the SU(2) singlet BPS multiplets. For simplicity we consider the cases where the superlowest weights have only one nonzero entry for the su(4) Dynkin labels, (J_u, J_v, J_w). The method is essentially the same as the one employed previously for the construction of the adjoint representation. We just report the results here.

For 3/4 BPS multiplets we already showed that only Q_{14}, Q_{24} act nontrivially on the superlowest weight. So the nonvanishing lowest weights on higher floors are only $Q_{14}|\Lambda_L\rangle, Q_{24}Q_{14}|\Lambda_L\rangle$. The complete 3/4 BPS multiplets are presented in Table 4.

Energy$\times(12/|\mu|)$	
$J+2$	$(0;0,0,J-2)$
$J+1$	$(1;0,0,J-1)$
J	$(0;0,0,J)$

Table 4: 3/4 BPS multiplets

Note that when $J_w = 1$ the second floor vanishes and the BPS multiplet becomes the fundamental representation. With the notation denoting the $\mathrm{su}(2), \mathrm{su}(4)$ dimensions, we present the fundamental representation in Table 5.

Energy	
$|\mu|/6$	$(2,1)$
$|\mu|/12$	$(1,\overline{4})$

Table 5: Fundamental representation of A(1|3)

For 1/2 BPS multiplets, the Dynkin label of the superlowest weights are in general $(0;0,J_v \neq 0, J_w)$. Here for simplicity we treat the cases with $J_w = 0$. The complete 1/2 BPS multiplets are in Table 6.

Energy $\times(12/|\mu|)$	
$2J+4$	$(0;0,J-2,0)$
$2J+3$	$(1;0,J-2,1)$
$2J+2$	$(0;0,J-2,2)\oplus(2;0,J-1,0)$
$2J+1$	$(1;0,J-1,1)$
$2J$	$(0;0,J,0)$

Table 6: 1/2 BPS multiplets

Note that for $J_v = 1$ the construction stops at the second floor so we have a shorter multiplet as in Table 7.

Energy	
$\|\mu\|/3$	$(3,1)$
$\|\mu\|/4$	$(2,\overline{4})$
$\|\mu\|/6$	$(1,6)$

Table 7: 1/2 BPS multiplet with $J_v = 1$

Now we consider the 1/4 BPS multiplets. Again for simplicity we start with $(0; J, 0, 0)$ as the superlowest weight. The complete 1/4 BPS multiplets are presented in Table 8.

Energy $\times(12/\|\mu\|)$	Representations
$3J+6$	$(0; J-2, 0, 0)$
$3J+5$	$(1; J-2, 0, 1)$
$3J+4$	$(0; J-2, 0, 2) \oplus (2; J-2, 1, 0)$
$3J+3$	$(1; J-2, 1, 1) \oplus (3; J-1, 0, 0)$
$3J+2$	$(0; J-2, 2, 0) \oplus (2; J-1, 0, 1)$
$3J+1$	$(1; J-1, 1, 0)$
$3J$	$(0; J, 0, 0)$

Table 8: 1/4 BPS multiplets

The case of $J_u = 2$ reads Table 9.

Energy	
$\|\mu\|/2$	$(4,1)$
$5\|\mu\|/12$	$(3,\overline{4})$
$\|\mu\|/3$	$(2,6)$
$\|\mu\|/4$	$(1,4)$

Table 9: 1/4 BPS multiplet with $J_u = 1$.

Finally we consider the SU(4) singlet BPS multiplets. In addition to the superlowest weight with the Dynkin label, $(J_z \neq 0; 0, 0, 0)$, all the possible $\mathrm{su}(2) \oplus \mathrm{su}(4)$ lowest weights are given by $T_2^l|\Lambda_L\rangle$, $l = 1, 2, 3, 4$. The complete SU(4) singlet BPS multiplets are presented in Table 10.

Energy$\times(12/|\mu|)$	Representations
$2J_z$	$(J_z; 0, 0, 0)$
$2J_z - 1$	$(J_z - 1; 1, 0, 0)$
$2J_z - 2$	$(J_z - 2; 0, 1, 0)$
$2J_z - 3$	$(J_z - 3; 0, 0, 1)$
$2J_z - 4$	$(J_z - 4; 0, 0, 0)$

Table 10: SU(4) singlet BPS multiplets

From

$$(T_1^l)^\dagger T_1^l|\Lambda_L\rangle = \tfrac{|\mu|}{3}(J_z + 1 - l)(T_1^{l-1})^\dagger T_1^{l-1}|\Lambda_L\rangle\,, \qquad l = 1, 2, 3, 4\,, \tag{65}$$

it is obvious that when $J_z = 1, 2, 3$ the supermultiplets truncate consistently.

6 Conclusion

We have identified the superalgebra of the M-theory in the fully supersymmetric pp-wave background as the special unitary Lie superalgebra, $\mathrm{su}(2|4\,; 2, 0)$ or $\mathrm{su}(2|4\,; 2, 4)$ depending on the sign of the mass parameter, $\mu > 0$ or $\mu < 0$ separately. The reason why we have two inequivalent superalgebras for different signs is essentially due to the fact that in odd dimensions there are two classes of gamma matrices which are not related by the similarity transformations. Nevertheless, despite the mathematical distinction, all the physical results we obtained are independent of the sign.

We have analyzed the root structure of the algebra and presented the second order super-Casimir operator, in the expression of which the su(2) and su(4) Casimirs appear with opposite signs.

We wrote down the general building blocks for the states in any irreducible representation of the superalgebra. The generic supermultiplet consists of several floors, from zero to eight at most, characterized by the discrete energy difference, $\mu/12$. Each of the zeroth and the highest floors corresponds to an irreducible representation of $\mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(4)$, while others are in general reducible representations. We identified all the "naturally ordered" products of the fermionic roots which give the possible lowest weights for the $\mathrm{u}(1) \oplus \mathrm{su}(2) \oplus \mathrm{su}(4)$ irreducible representations. Apart from them there are hidden lowest weights depending on the explicit information of the superlowest weight vector. We have derived explicitly the typicality condition which consists of eight equations in terms of the energy and other four $\mathrm{su}(2) \oplus \mathrm{su}(4)$ quantum numbers.

We have completely classified the BPS multiplets as a special type of the atypical unitary representation. There are two ways of defining the BPS multiplet compensating the mathematical distinction of the different signs for μ. One is from the superlowest weight and the other from the superhighest weight. We find there are 4/16, 8/16, 12/16, 16/16 BPS multiplets preserving 4, 8, 12, 16 real supercharges and obtain the corresponding exact spectrum. We show that, irrespective of the sign of μ, either the "lowest energy" floor of the BPS multiplet is a su(2) singlet or the "highest energy" floor of the BPS multiplet is a su(4) singlet.

One can apply the results of the present paper directly to the spectrum of the matrix model on a pp-wave. The physical states are grouped into the unitary irreducible representations. In [7, 10] it was pointed out that the half BPS multplets with the superlowest weight Dynkin label, $(0; 0, J_v, 0)$ indeed exist in the massive model implying the protected energy from the pertubative corrections to all orders. It is plausible that there exist more protected BPS or atypical supermultiplets. Finding such multiplets will be of much interest.

Given the supersymmetry transformation rule for the fermions in the matrix model on a pp-wave, one can in principle obtain the classical BPS equations for the bosons to satisfy. One of the authors recently developed a systematic method to derive all the BPS equations studying the "projection matrix" into the space of the Killing spinors [18]. In the matrix model for M-theory on a pp-wave the spinors are in the Majorana representation. Consequently the dimension of the space of the Killing spinors can be arbitrary from 1 to 16 meaning 1/16, 2/16, $\cdots$, 16/16 BPS equations. Our analysis on the BPS multiplets implies that among them only 4/16, 8/16, 12/16, 16/16 BPS equations admit finite energy solutions. Apart from the known 1/2, 1 BPS solutions [1, 19, 20], it will be interesting to look for the 4/16, 12/16 finite energy solutions.

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