

Non-protected operators in $\mathcal{N} = 4$ SYM and multiparticle states of AdS₅ SUGRA

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ABSTRACT

We study a class of non-protected local composite operators which occur in the R symmetry singlet channel of the OPE of two stress-tensor multiplets in $\mathcal{N} = 4$ SYM. At tree level these are quadrilinear scalar dimension four operators, two single-traces and two double-traces. In the presence of interaction, due to a non-trivial mixing under renormalization, they split into linear combinations of conformally covariant operators. We resolve the mixing by computing the one-loop two-point functions of all the operators in an $\mathcal{N} = 1$ setup, then diagonalizing the anomalous dimension matrix and identifying the quasiprimary operators. We find one operator whose anomalous dimension is negative and suppressed by a factor of $1/N^2$ with respect to the anomalous dimensions of the Konishi-like operators. We reveal the mechanism responsible for this suppression and argue that it works at every order in perturbation theory. In the context of the AdS/CFT correspondence such an operator should be dual to a multiparticle supergravity state whose energy is less than the sum of the corresponding individual single-particle states.

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1 Introduction

One of the fundamental ideas driving much of the progress in theoretical high energy physics over the past decades is the perceived deep relation between gauge fields and strings. Perhaps the most concrete realization of this idea is the AdS/CFT correspondence [1–3] that conjectures a duality between, for example, type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills in four dimensions. A lot of evidence for this conjecture has been accumulated during the past few years (for an exhaustive list of references, see e.g. [4, 5]).

A large part of the tests for the validity of the AdS/CFT correspondence are based on the comparison of perturbative (weak coupling) calculations in $\mathcal{N} = 4$ SYM to results obtained from $\text{AdS}_5 \times S^5$ supergravity, the latter supposedly describing the theory at strong t’Hooft coupling $\lambda = g_{YM}^2 N / (4\pi^2)$.¹

It is natural to start by studying quantities having the simplest possible behavior at weak and strong coupling. These include, for instance, two- and three-point correlation functions of chiral primary operators (CPOs), n -point correlators of the “extremal” and “near-extremal” type, as well as conformal and R-current anomalies. These quantities are essentially related to BPS states (short, or protected operators) in either the gauge or the supergravity/string sectors, and therefore they render only a part of the correspondence between gauge theories and strings.

If we would like to go further and use the AdS/CFT duality to better understand the relation between gauge and string theories, we should consider non-protected quantities, non-BPS operators in particular. In string theory there is a clear distinction between supergravity (light) and string (heavy) modes. From this point of view non-BPS operators are naturally interpreted as falling into two different classes: operators from the first class are dual to “gravity multiparticle states”², while operators from the second class are dual to “string single/multiparticle states”. To which class a concrete operator belongs can be decided by looking at the behavior of its scaling dimension at strong coupling since the latter is related by duality to the mass of the corresponding particle. This is a nice picture but hardly operational because of our present inability to compute scaling dimensions of non-protected operators *at strong coupling* by QFT methods. It is therefore of certain interest to see if one can classify non-BPS operators by their different *weak-coupling* behaviour and make contact with their string theory interpretation.

In the present work we focus on a small but controllable part of the non-protected sector of operators of $\mathcal{N} = 4$ SYM. These are superconformal primaries that appear in the OPE of two CPOs with dimension two in the $[0, 2, 0]$ representation of the $SU(4) \sim SO(6)$

¹Strictly speaking, such calculations test the weak version of the AdS/CFT correspondence that conjectures a duality between the large N , large-t’Hooft coupling limit of $\mathcal{N} = 4$ SYM and the SUGRA limit of IIB string theory on $\text{AdS}_5 \times S^5$. Recently there have been very interesting attempts to include massive strings modes of IIB theory in such tests [6, 7].

²Multiparticle supergravity states constitute a wide class including, e.g., states dual to 1/4 BPS operators of the gauge theory.

R symmetry group (the lowest components of the stress-tensor multiplet). It has been shown [8, 9] that non-protected superconformal primaries appear *only* in the singlet channel of that OPE. In free field theory, this infinite set of non-protected operators naturally breaks into two classes. One of them, which we call the K -class, consists of operators which are bilinear single-traces in the elementary scalar fields while the other, the O -class, consists of operators that are quadrilinear double-traces. An important property of the operators in the K -class is that their canonical dimensions saturate the unitarity bound [10–12], i.e., they are twist-two operators, and in free field theory they are conserved tensors (except the scalar K -class operators) [13, 14]. The operators in the O -class, instead, do not obey any conservation condition in free field theory and their canonical dimensions are above the unitarity bound, i.e., they are higher-twist operators.

It is one of the important features of AdS/CFT that the operators in the above two classes are clearly distinguished by the large N strong coupling dynamics of the $\mathcal{N} = 4$ SYM. Namely, the operators in the K -class should acquire large anomalous dimensions in the $\lambda \rightarrow \infty$ limit and, as a consequence, they decouple in the supergravity regime. This is thought to be a manifestation of the fact that the K -class operators are dual to massive string modes [2]. On the other hand, in what appears to be one of the more intriguing non-trivial AdS/CFT results, the contributions of *some of* the operators in the O -class survive in the supergravity induced four-point function of stress-tensor multiplets, while at the same time these operators receive non-trivial corrections to their canonical dimensions both perturbatively and in the supergravity regime [15–22]. The exact reason which stays behind this behavior remains unclear. It is commonly stated that the operators in the O -class which do not drop out in the strong coupling limit are associated with multiparticle supergravity states. No matter how appealing such a point of view might be, it is at the same time rather puzzling since it is not immediately clear how non-trivial tree level corrections to the energy of multiparticle states could arise in a theory like the $\text{AdS}_5 \times S^5$ supergravity where the energy of all single-particle states is protected.

In the present work, we intend to shed more light on the above question by studying the simplest scalar operators in the O -class, i.e. (free field) dimension 4 scalar operators occurring in the singlet R -symmetry channel in the OPE of two stress-tensor multiplets. Our interest in looking at this particular example is motivated by the knowledge of the OPE structure in the supergravity approximation. As was found in Ref. [20], the $SU(4)$ -singlet channel of the corresponding OPE contains a scalar operator of dimension $\Delta = 4 + \gamma$, where the strong-coupling anomalous dimension is given by $\gamma = -16/N^2$. Thus, this operator, which survives in the supergravity approximation, is a natural candidate to be identified with the gravity two-particle bound state. It is therefore interesting to see if one can find the weak coupling counterpart of this operator by studying the corresponding OPE in perturbation theory.

In the free field limit, the dimension four scalar operators we are interested in are suitable linear combinations of the four possible independent quadrilinear structures, two single traces and two double traces, which can be chosen to be orthonormal.

Switching on the interaction, we study their quantum properties by evaluating their one-loop two-point functions. UV infinities which arise at this order require a non-trivial renormalization of the four operators and the consequent appearance of anomalous dimensions. The evaluation of the anomalous dimensions is complicated by a non-trivial mixing of the operators under renormalization. The complete anomalous dimensions matrix has to be determined and from its diagonalization we can read off the anomalous dimensions of the four “pure CFT states”. The latter correspond to the correct *quasiprimary* operators in the O -class. Our main result is that *one* of the above four quasiprimary operators acquires an anomalous dimension which is suppressed by a factor of $1/N^2$ with respect to the anomalous dimensions of *all* the operators in the K -class, as well as with respect to the remaining three quasiprimary operators in the O -class. The mechanism behind this suppression is tied to the fact that, at large N , the dominant sector in the two-point function calculation for the operator in question is given by planar disconnected Feynman graphs. The latter factorize into two-point functions of CPOs, i.e., of operators *protected from renormalization*. Consequently, the leading contributions vanish and one is left with a mixture of planar and non-planar graphs, which in our particular case are of the same order in $1/N$. This “partial non-renormalization” argument applies to all orders in perturbation theory. Remarkably, the anomalous dimension of the above quasiprimary operator is negative (at one loop, at least) and can be interpreted as a binding energy. This is in accordance with the interpretation of this operator as corresponding to a two-particle bound state with energy smaller than the sum of the energies of the respective single-particle states. We point out that the strong coupling anomalous dimension of the surviving O -class operator(s) is non-zero and negative which can also be interpreted as a binding energy of the corresponding two-particle supergravity state. Our results indicate that both planar and non-planar graphs are responsible for the non-trivial anomalous dimensions of supergravity states. This raises an interesting question regarding the nature of the binding energies of multiparticle supergravity states. Similar questions have been recently discussed in [23].

The organization of the paper is as follows. In Section 2 we discuss the free field theory OPE of two lowest weight CPOs. In Section 3 we present the general formalism for calculating perturbative anomalous dimensions of composite operators in CFT and for dealing with the problem of operator mixing and splitting. In Section 4 we recall the $\mathcal{N} = 1$ superspace approach [24, 25] that we use to calculate the one-loop two-point functions of the four free-field operators and fix our notations and conventions. This formalism is then used in Section 5 to compute the anomalous dimension matrix. The diagonalization procedure is then performed to obtain the corresponding one-loop quasiprimary operators. We test our results for consistency with certain constraints obtained from the independent OPE analysis of one- and two-loop four-point calculations. In Section 6 we discuss our results in connection with the AdS/CFT duality. In Appendix A we provide some relevant information about the perturbative four-point functions of the lowest weight CPOs and the corresponding OPE. Appendix B contains some technical details of the calculations.

2 Non-protected operators in $\mathcal{N} = 4$ SYM

There is by now an exhaustive list of gauge invariant protected operators in $\mathcal{N} = 4$ SYM, both of the BPS type [26] and of the so-called “semishort” type (see, e.g., [27]). Non-protected gauge invariant operators are much less well understood, even if they form perhaps the most important part of the spectrum. In this work we focus on a set of non-protected operators of $\mathcal{N} = 4$ SYM that is controllable and relatively well known. These are the superconformal primaries that appear in the OPE of two CPOs with lowest dimension. The latter are the lowest components of the supercurrent multiplet and are formed by single trace bilinears of the six real scalars $\phi^L(x)$, $L = 1, 2, \dots, 6$ of $\mathcal{N} = 4$ SYM transforming under the $\mathbf{20} \equiv [0, 2, 0]$ irrep of $SU(4)$. Explicitly, up to a normalization constant, they are defined as

$$O_{\mathbf{20}}^{LM}(x) \sim \text{Tr}[\phi^L(x)\phi^M(x)] - \frac{1}{6}\delta^{LM}\text{Tr}[\phi^N(x)\phi^N(x)] \quad (2.1)$$

where the trace is over the $SU(N)$ indices. The OPE of two such operators involves all operators in the decomposition

$$\begin{aligned} [0, 2, 0] \times [0, 2, 0] &= [0, 0, 0] + [0, 2, 0] + [0, 4, 0] + [2, 0, 2] \\ &\quad + [1, 0, 1] + [1, 2, 1] \end{aligned} \quad (2.2)$$

The irreps in the first line are realized as even spin symmetric traceless tensors while those in the second line as odd spin ones. A general non-renormalization theorem then shows that among the superconformal primaries that may appear in the OPE of the two CPOs (2.1), *only* the ones in the singlet $[0, 0, 0]$ channel are non-protected by superconformal invariance [8, 9]. For the latter operators the only constraint imposed by superconformal invariance is the unitarity condition [10, 12]

$$\Delta \geq 2 + s \quad (2.3)$$

where Δ is the scaling dimension of the corresponding operator while $s = 2n$, $n = 0, 1, 2, \dots$ is the spin.

An explicit expression for the singlet channel of the free OPE can then be written as follows:

$$\begin{aligned} O_{\mathbf{20}}^{LM}(x_1)O_{\mathbf{20}}^{LM}(x_2) &= \frac{1}{x_{12}^4} + a\frac{1}{x_{12}^2}\delta_{LM} : \text{Tr} [\phi^L(x_1)\phi^M(x_2)] : + b : O_{\mathbf{20}}^{LM}(x_1)O_{\mathbf{20}}^{LM}(x_2) : \\ &= \frac{1}{x_{12}^4} + \frac{1}{x_{12}^2}C_K(x_{12}, \partial_2) * [K](x_2) + C_O(x_{12}, \partial_2) * [O](x_2) \end{aligned} \quad (2.4)$$

where a and b are constants dependent on the normalization of the operators. $[K](x)$ and $[O](x)$ denote the K - and O -class operators, respectively and $C(x_{12}, \partial_2)$ denote the corresponding OPE coefficients [20]. The important assumption behind a conformal OPE such as (2.4) is that the operators appearing in the r.h.s. are “pure CFT states” or else

quasiprimary operators. This means that under conformal transformations they behave in a well-defined way determined uniquely by their spin and scaling dimension. In practice, quasiprimary operators form an orthogonal basis of the operator algebra such that all two-point functions between different operators vanish identically. This latter property, together with the explicit expressions for all the OPE coefficients $C(x_{12}, \partial_2)$ [28] allows one to study conformally invariant four-point functions in terms of the OPE.

It should be pointed out, however, that sometimes the conformal labels (spin and dimension) are not sufficient to distinguish all the operators of a given type. For instance, later on in this paper we will have to deal with the degeneracy of the free quadrilinear scalar operators of dimension four: We will find four such structures, all with the same conformal labels. It is generally believed that the conformal interactions lift such degeneracies by creating *anomalous dimensions*. Indeed, this is what will happen in our case.

From (2.4) we see that the free-field realization of the K -class operators is given in terms of single-trace bilinears while that of the O -class operators in terms of double-trace quadrilinears in the ϕ 's. Consequently, the canonical dimensions of the operators in the K -class are of the form

$$\Delta_K = 2 + s, \quad s = 0, 2, 4, \dots \quad (2.5)$$

while those in the O -class are of the form

$$\Delta_O = 4 + s, \quad s = 0, 2, 4, \dots \quad (2.6)$$

In other words, the operators in the K -class are of twist two while those in the O -class are of twist four. Furthermore, since the canonical dimensions of the free K -class operators saturate the unitarity bound (2.3), these operators form an infinite set of free short supermultiplets containing conserved currents. However, the corresponding supermultiplets become long in the interacting theory since there is no mechanism to protect their dimensions from radiative corrections [10]. The operators in the O -class do not saturate the unitarity bound (2.3) and they also acquire in general anomalous dimensions.

Now, if the conformal OPE (2.4) is valid as an operator statement one should be able to define the operators in the K - and O -classes, both in perturbation theory as well as non-perturbatively, and study their properties. This is clearly a formidable task. For the operators in the K -class this would be very interesting in view of the recent conjecture regarding the behavior of the anomalous dimension of operators with twist two with large canonical dimension and large spin [7].³ The study of higher-twist composite operators is

³The one-loop anomalous dimensions $\eta_K = \Delta_K - 2 - s$ of the operators in the K -class have been calculated in [29] by applying conformal OPE techniques to the one-loop four-point function of CPOs:

$$\eta_K = \frac{\lambda}{2\pi^2} \sum_{k=0}^{s+2} \frac{1}{k} \sim \frac{\lambda}{2\pi^2} \ln s \quad \text{for } s \rightarrow \infty$$

Using the results of [30] one could extend the above analysis to two loops and test the conjecture of [7] about the absence of $\ln^k s$ terms in the k -loop anomalous dimension of the K -class operators.

equally interesting and its the purpose of the present work to initiate the investigations in this direction.

3 Anomalous dimensions and operator mixing in CFT

The study of composite operators in a renormalizable quantum field theory is among the most important and complicated subjects [31]. Even if such studies simplify considerably in CFTs due to the absence of beta functions, they remain quite elaborate due to operator mixing under renormalization. In this Section we present the general framework for the study of composite operators in CFT in the presence of operator mixing, setting up the explicit calculations of the following Sections.

3.1 Anomalous dimensions of composite operators in CFT

Given a renormalizable QFT, we consider a renormalized composite operator $\mathcal{O}(x)$ (scalars only, for simplicity) constructed from the elementary fields of the theory. The fact that the operator is renormalized means that the insertions of $\mathcal{O}(x)$ into renormalized correlation functions are finite. In particular, the n -point functions of $\mathcal{O}(x)$ are finite. If the theory is at its fixed point (CFT, the beta functions vanish) the renormalized n -point functions satisfy Ward identities which become the configuration space analog of the Callan-Symanzik RG equations at the fixed point [32]

$$\left[\sum_{k=1}^n x_k^\mu \frac{\partial}{\partial x_k^\mu} + n\Delta \right] \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \dots \rangle_R = 0. \quad (3.1)$$

The parameter Δ is the scaling dimension of the operator $\mathcal{O}(x)$ and determines its transformation properties under scale transformations. In particular, for the two-point function we have

$$\left[x^\mu \frac{\partial}{\partial x^\mu} + 2\Delta \right] \langle \mathcal{O}(x) \mathcal{O}(0) \rangle_R = 0 \quad (3.2)$$

In a free field theory, the dimension Δ appearing in (3.2) coincides with the *canonical* dimension Δ_0 of the operator which can be inferred from the canonical dimensions of the elementary fields entering the definition of \mathcal{O} . Translation invariance together with the Ward identity (3.2) then constrain the two-point functions to be power-like. After turning on the interaction, the perturbative evaluation of the two-point function leads in general to divergences which emerge from the explicit calculation of Feynman integrals. Since the subtraction of these divergences introduces a non-trivial dependence on the UV scale, the two-point function loses its power-like behavior and does not satisfy (3.2) anymore. However, order by order in the perturbative expansion we can recover the conformal Ward identity by modifying the dimension as $\Delta = \Delta_0 + \gamma$. The deviation γ from the canonical dimension is the *anomalous dimension* which is given as a power series in the perturbation parameter.

To be more explicit and to show how the general calculation works, we consider the bare (non-renormalized) field $\mathcal{O}^{(0)}(x)$ and evaluate $\langle \mathcal{O}^{(0)}(x)\mathcal{O}^{(0)}(0) \rangle$ in perturbation theory. At tree level we recover the power-like behavior⁴. The assumption that the theory is a *non-trivial* renormalized CFT requires the existence of a dimensionless renormalized coupling λ . Using dimensional regularization ($d = 4 - 2\epsilon$) to regularize the integrals, we can present the UV divergences as simple poles in ϵ , e.g., at one loop

$$\langle \mathcal{O}^{(0)}(x)\mathcal{O}^{(0)}(0) \rangle|_{0+1} = \frac{1}{(x^2)^{\Delta_0}} \left[1 + \lambda \frac{a}{\epsilon} + O(\epsilon^0) \right] \quad (3.3)$$

where a is a constant independent of λ and ϵ . We also work henceforth with normalized conformal operators. To cancel the divergence in (3.3) we perform the usual multiplicative renormalization of the composite operator by introducing a divergent renormalization constant $Z(\lambda, \epsilon)$

$$\mathcal{O}(x) \equiv Z(\lambda, \epsilon)\mathcal{O}^{(0)}(x) \quad (3.4)$$

which is determined by requiring $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle$ to be finite at this order. Using the minimal subtraction scheme, this immediately gives

$$Z(\lambda, \epsilon) = 1 - \frac{\lambda a}{2} \frac{1}{\epsilon} \quad (3.5)$$

Since $Z(\lambda, \epsilon)$ depends on the UV scale, the one-loop renormalized two-point function does not satisfy the conformal Ward identity (3.2). To remedy this we postulate that the conformal Ward identity (3.2) is satisfied order by order in perturbation theory and at the same time the canonical dimension of the operator $\mathcal{O}(x)$ is modified as

$$\Delta^{(1)} = \Delta_0 + \gamma^{(1)} \quad (3.6)$$

Then, $\gamma^{(1)}$ is determined by requiring

$$\left[\mu \frac{\partial}{\partial \mu} + 2\Delta^{(1)} \right] \langle \mathcal{O}(x)\mathcal{O}(0) \rangle|_{0+1} = O(\lambda^2) \quad (3.7)$$

where μ is the mass scale of dimensional regularization. To leading first order in λ , we obtain from (3.7)

$$\gamma^{(1)} = -\mu \frac{\partial}{\partial \mu} \ln Z \quad (3.8)$$

The μ -dependence of Z comes from dimensional transmutation since in $(4-2\epsilon)$ dimensions the physical coupling λ_* becomes dimensionful and we need to define a dimensionless coupling as $\lambda = \mu^{2\epsilon} \lambda_*$. As a consequence, we find

$$\gamma^{(1)} = \lambda a \quad (3.9)$$

⁴Notice that it is possible that renormalized conformal two-point functions may contain divergences even at tree level. These divergences are due to *ultralocal* short distance singularities in the two-point function (see, e.g., [33]). They are insensitive to the perturbative expansion [34] and give rise to *conformal anomalies* which should not be confused with the *anomalous* dimensions of the operators.

This is the first-order version of the more general, well-known result that the anomalous dimension is read off from the coefficient of the simple pole in the renormalization constant Z . The solution of (3.7) at this order in perturbation theory is

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_{0+1} = \frac{1}{x^{2\Delta_0}} [1 + C\lambda - \gamma^{(1)} \ln(x^2\mu^2)] \quad (3.10)$$

where the constant C encodes possible finite renormalization (finite counterterms) of the two-point function. This relation is an explicit proof of the statement that the modification of the canonical dimensions of quantum operators is due to renormalization and is signaled by logarithmic terms appearing in the two-point functions.

3.2 Operator mixing in CFT

The general discussion presented above cannot be immediately applied when the free theory contains more than one linearly independent composite operators having the same canonical dimensions. In fact, in such a case one can start in the free field theory with a set of mutually orthogonal operators which correspond to a basis of free quasiprimary operators. However, in general nothing prevents them from mixing at the quantum level. Therefore, in perturbation theory one has to orthonormalize the basis order by order, as we are going to describe in detail ⁵.

Suppose that we have a set of scalar operators \mathcal{O}_i , $i = 1, \dots, p$ of equal free dimension Δ_0 and with identical quantum numbers. By suitable rescalings of the operators we can choose this set to be orthonormal

$$\langle \mathcal{O}_i^{(0)}(x)\mathcal{O}_j^{(0)}(0) \rangle_0 = \frac{\delta_{ij}}{x^{2\Delta_0}} \quad (3.11)$$

Note that this choice of basis is not unique as we can always make an orthogonal transformation $\mathcal{O}'_i = o_{ij}\mathcal{O}_j$, $o^T o = \mathbb{I}$ which preserves (3.11). Now, let us switch on the interaction and compute the two-point functions at order λ . In general, at one-loop we expect to find something like

$$\langle \mathcal{O}_i^{(0)}(x)\mathcal{O}_j^{(0)}(0) \rangle_{0+1} = \frac{1}{x^{2\Delta_0}} [\delta_{ij} + \lambda\rho_{ij} + \lambda\frac{\omega_{ij}}{\epsilon} + O(\epsilon)] \quad (3.12)$$

where $\rho = \rho^T$ and $\omega = \omega^T$ are constant symmetric matrices⁶. In general they are not diagonal, as a result of the fact that at one loop the operators are not orthogonal anymore and develop mixed, possibly divergent, two-point functions.

⁵In the context of $\mathcal{N} = 4$ SYM, the mixing of composite operators has been discussed in a number of recent papers [16, 15, 20, 36, 37].

⁶Here we make the discussion more general by considering also finite contributions ρ which correspond to finite renormalizations of the two-point function.

To cancel the $1/\epsilon$ poles we introduce a renormalization matrix Z_{ij} as

$$\mathcal{O}_i \equiv Z_{ij} \mathcal{O}_j^{(0)}, \quad Z_{ij} = \delta_{ij} - \frac{1}{\epsilon} \frac{\lambda \omega_{ij}}{2} \quad (3.13)$$

which is determined by requiring $\langle \mathcal{O}_i \mathcal{O}_j \rangle_{0+1}$ to be finite. If we formally follow the same reasoning which in the case of a single operator brought us to (3.10), we eventually arrive at the set of renormalized one-loop two-point functions

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle_{0+1} = \frac{1}{x^{2\Delta_0}} [\delta_{ij} + \lambda(\rho_{ij} - \omega_{ij} \ln(x^2 \mu^2))] + O(\lambda^2) \quad (3.14)$$

The above result shows that in general the one-loop renormalized operators in (3.14) fail to be quasiprimary, since the two-point functions of the latter should be diagonal,

$$\langle \tilde{\mathcal{O}}_i(x) \tilde{\mathcal{O}}_j(0) \rangle_{0+1} = \frac{\delta_{ij}}{x^{2\Delta_0}} [1 + \lambda \rho_i - \lambda \gamma_i \ln(x^2 \mu^2)] + O(\lambda^2) \quad (3.15)$$

(no summation on repeated indices is assumed). Therefore, in order to find the explicit realization of the one-loop quasiprimary operators we need to bring eq. (3.14) in the form (3.15). This can be done by performing the linear transformation

$$\tilde{\mathcal{O}} = (L_0 + \lambda L_1) \mathcal{O} \quad (3.16)$$

which implies

$$\begin{aligned} \langle \tilde{\mathcal{O}}(x) \tilde{\mathcal{O}}(0) \rangle &= (L_0 + \lambda L_1) \langle \mathcal{O} \mathcal{O} \rangle (L_0 + \lambda L_1)^T \\ &= \frac{1}{(x^2)^{\Delta_0}} [L_0 L_0^T + \lambda(L_1 L_0^T + L_0 L_1^T + L_0 \rho L_0^T - L_0 \omega L_0^T \ln(x^2 \mu^2))] + O(\lambda^2) \end{aligned} \quad (3.17)$$

Comparing (3.15) with (3.17) term by term, we obtain the following three equations:

$$L_0 L_0^T = \mathbb{I} \quad (3.18)$$

$$L_0 \omega L_0^T = \Gamma \quad (3.19)$$

$$L_1 L_0^T + L_0 L_1^T + L_0 \rho L_0^T = P \quad (3.20)$$

where Γ denotes a diagonal matrix with eigenvalues γ_i and P a diagonal matrix with eigenvalues ρ_i . The corresponding eigenvectors $\tilde{\mathcal{O}}_i$ are the one-loop quasiprimary operators having one-loop anomalous dimensions γ_i .

From (3.18) we see that the matrix L_0 must be orthogonal and from (3.19) we see that it must diagonalize the matrix ω . If all the eigenvalues γ_i are different, $\gamma_i \neq \gamma_j$ (i.e., all the quasiprimary operators acquire different anomalous dimensions), then the solution of eq. (3.19) is *unique*. In this case, the one-loop renormalization and diagonalization of the anomalous dimension matrix *uniquely fixes* the form of the tree level quasiprimary operators. Finally, eq. (3.20) involves the finite counterterms ρ_i . These can be fixed by choosing a suitable normalization for the one-loop renormalized correlators (3.15).

Choosing for example $\rho_i = 0$ ($P = 0$) and setting $L_1 = \ell L_0$, eq. (3.20) can be rewritten as

$$\ell + \ell^T = -L_0 \rho L_0^T \quad (3.21)$$

We notice that this equation leaves some freedom that may affect the explicit expression of the one-loop quasiprimary operators to $O(\lambda)$. To fix this freedom one would need to discuss the two-point functions (3.14) at order $O(\lambda^2)$, but this is outside the scope of the present work. A similar pattern is expected to be reproduced order by order.

We conclude that the diagonalization of the set of operators at one loop requires a particular orthogonal transformation L_0 already at tree level. This means that the choice of the free basis is not arbitrary, but is completely determined by the one-loop $\ln(x^2 \mu^2)$ terms in the full matrix of two-point functions. Therefore, when discussing an operator at the free level with an eye to its quantum properties, we need to resolve the mixing as explained above.

3.3 Operator splitting in CFT

Let us now discuss what are the consequences of the above mixing phenomenon for the OPE and the four-point functions. Considering for clarity only scalar fields, the generic form of a free OPE is [20]

$$A(x_1)A(x_2) = \frac{1}{x_{12}^{2\Delta_A}} + g_O \frac{1}{x_{12}^{2(\Delta_A - \frac{1}{2}\Delta_O)}} [O(x_2) + \dots] \quad (3.22)$$

where the dots represent derivatives of O and Δ_A , Δ_O are the canonical dimensions of A and O respectively. The parameter g_O is connected to the three-point function of O as

$$\langle A(x_1)A(x_2)O(x_3) \rangle = \frac{g_O}{x_{12}^{2(\Delta_A - \frac{1}{2}\Delta_O)} (x_{13}^2 x_{23}^2)^{\frac{1}{2}\Delta_O}} \quad (3.23)$$

The important point in free field theory is that given the explicit expression for A in terms of the elementary fields, one uniquely obtains the expressions of all the fields on the r.h.s. of the OPE (3.22) using for example a Taylor expansion of the l.h.s. [35, 20]. In fact, to any given dimension and spin corresponds *one* explicit expression of elementary fields, such as O in (3.22). However, complications arise when there exists at the free theory level more than one linearly independent expressions in terms of the elementary fields corresponding to conformal operators of a given dimension and spin. In this case the fields appearing on the r.h.s. of the OPE (3.22), such as O , represent a particular linear combination of the different, linearly independent expressions. This phenomenon is termed the *free-field theory degeneracy* of the conformal OPE.

In the presence of such a degeneracy one is forced to study the mixing under renormalization of all the linearly independent expressions, following the discussion presented before. In this way, at one-loop, one ends up with a set of quasiprimary operators which

should be the correct CFT states appearing in the OPE (3.22). Finally, inverting relations such as (3.16) one can express O in terms of an orthonormal basis of one-loop quasiprimary operators Σ_i

$$O(x) = \sum_{i=1}^p a_i \Sigma_i(x) \quad (3.24)$$

where p is the number of the linearly independent free field expressions. At one loop the coefficients a_i in (3.24) are just numbers (they do not depend on the coupling) that are uniquely determined by the diagonalization procedure explained before. The relation (3.24) is the *operator splitting* discussed in [20, 21] in the case of the stress tensor of $\mathcal{N} = 4$ SYM.

Once the mixing of the various conformal operators described in the previous subsection has been resolved, we can make contact with the independent four-point function approach to the OPE. In it one tries to extract information about the spectrum and the anomalous dimensions of the operators involved in the OPE by making a conformal partial wave expansion of the four-point amplitude of, e.g., four operators $O_{\mathbf{20}}$. Matching the terms $\ln v$, $\ln^2 v$, etc. in this expansion ($v = (x_{12}^2 x_{34}^2)/(x_{14}^2 x_{23}^2)$ is one of the conformal cross-ratios), one can find consistency conditions on the mixing coefficients a_i in (3.24) and on the anomalous dimensions. The above four-point amplitude has been computed up to two loops [18], which allows one to find three such conditions. This procedure has already been discussed in [30, 36], but the full details needed for its application are worked out in Appendix A. Assuming that the quasiprimary operators Σ_i have one-loop anomalous dimensions γ_i and are normalized, we can write down the following consistency relations

$$\sum_{i=1}^p a_i^2 = 1 \quad (3.25)$$

$$\sum_{i=1}^p a_i^2 \gamma_i = -4 \frac{\lambda}{N^2} \quad (3.26)$$

$$\sum_{i=1}^p a_i^2 \gamma_i^2 = 18 \frac{\lambda^2}{N^2} \quad (3.27)$$

The first of these conditions can be viewed merely as a conventional normalization of the operator O in (3.24). The second is derived from the one-loop and the third from the two-loop four-point amplitude (at order $1/N^2$). Although these equations are not sufficient to determine all the eight parameters a_i, γ_i (in our case $p = 4$, see the next section), they nevertheless put some non-trivial constraints on the parameters. For instance, since the right-hand side of eq. (3.26) is negative and is of order $1/N^2$, we expect to find in our calculation that: i) at least one anomalous dimension is negative; ii) all the quasiprimary operators corresponding to finite (in the large N limit) anomalous dimensions enter in the linear combination for O_1 with coefficients of order at least $1/N$. In addition, conditions (3.25)–(3.27) provide a very non-trivial cross-check both on our two-point one-loop

calculations and on the existing perturbative four-point ones. In Section 5 we show that our results do indeed verify these conditions.

4 $\mathcal{N} = 1$ superspace approach

We are interested in the calculation of the anomalous dimensions of scalar composite operators of $\mathcal{N} = 4$ SYM theory which have canonical dimension 4 and belong to the $[0, 0, 0]$ representation of the R-symmetry group. In particular, we can construct four possible dimension 4 quadrilinear operators in terms of the six elementary scalar fields of the theory

$$\begin{aligned}\mathcal{A}_1 &= \text{Tr}(\phi^L \phi^M) \text{Tr}(\phi^L \phi^M) \\ \mathcal{A}_2 &= [\text{Tr}(\phi^L \phi^L)]^2 \\ \mathcal{A}_3 &= \text{Tr}(\phi^L \phi^M \phi^L \phi^M) \\ \mathcal{A}_4 &= \text{Tr}(\phi^L \phi^L \phi^M \phi^M),\end{aligned}\tag{4.1}$$

As explained in Subsection 3.3, the free OPE (2.4) contains one particular combination - named \mathcal{O}_1 - of the above operators. This is obtained by explicit calculations as [21]

$$O_{\mathbf{20}}^{LM}(x_1) O_{\mathbf{20}}^{LM}(x_2) \sim \frac{1}{x_{12}^4} + [\mathcal{O}_1(x_2) + (\partial \mathcal{O}_1(x_2))] + \dots\tag{4.2}$$

with

$$\mathcal{O}_1 \sim \mathcal{A}_1 - \frac{1}{6} \mathcal{A}_2\tag{4.3}$$

We may then choose this operator as the first vector in the orthonormal free basis (3.11) and the remaining three basis vectors can be constructed as linear combinations of the fields in (4.1) that are orthogonal to (4.3). However, as already explained, when moving on to the perturbative evaluation of two-point functions, we expect these operators to mix non-trivially. Therefore, we are forced to evaluate all the ten different two-point functions among them and solve the mixing problem.

To perform quantum calculations, we find it convenient to work in a $\mathcal{N} = 1$ setup where the field content of the theory is given in terms of one real vector superfield V and three chiral superfields Φ^i containing the six scalars organized into the $\mathbf{3} \times \bar{\mathbf{3}}$ of $SU(3) \subset SU(4)$:

$$\Phi^i|_{\theta=0} = \phi^i + i\phi^{i+3} \quad \bar{\Phi}_i|_{\theta=0} = \phi^i - i\phi^{i+3} \quad i = 1, \dots, 3\tag{4.4}$$

The classical action is (we use the notations of [24, 25, 38])

$$\begin{aligned}S &= \int d^8z \text{Tr} (e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i) + \frac{1}{2g^2} \int d^6z \text{Tr} W^\alpha W_\alpha \\ &\quad + \frac{ig}{3!} \text{Tr} \int d^6z \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{ig}{3!} \text{Tr} \int d^6\bar{z} \epsilon^{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k]\end{aligned}\tag{4.5}$$

where $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$, and $V = V^a T^a$, $\Phi^i = \Phi^{ia} T^a$, T^a being $SU(N)$ matrices in the fundamental representation. Since the theory is at its conformal fixed point, possible divergences can arise only in correlators of composite operators.

In terms of the (anti)chiral $\mathcal{N} = 1$ superfields the four free operators given in (4.1) are⁷

$$\begin{aligned}\mathcal{A}_1 &= \frac{1}{2} [\text{Tr}(\Phi^i \Phi^j) \text{Tr}(\bar{\Phi}_i \bar{\Phi}_j) + \text{Tr}(\Phi^i \bar{\Phi}_j) \text{Tr}(\Phi^j \bar{\Phi}_i)] \\ \mathcal{A}_2 &= \text{Tr}(\Phi^i \bar{\Phi}_i) \text{Tr}(\Phi^j \bar{\Phi}_j) \\ \mathcal{A}_3 &= \text{Tr}(\Phi^i \Phi^j \bar{\Phi}_i \bar{\Phi}_j) \\ \mathcal{A}_4 &= \frac{1}{4} [\text{Tr}(\Phi^i \bar{\Phi}_i \Phi^j \bar{\Phi}_j) + 2\text{Tr}(\Phi^i \Phi^j \bar{\Phi}_j \bar{\Phi}_i) + \text{Tr}(\bar{\Phi}_i \Phi^i \bar{\Phi}_j \Phi^j)]\end{aligned}\quad (4.6)$$

When we turn on the interaction, in order to guarantee gauge invariance, we have to insert extra exponential factors inside the traces. The gauge invariant superfields are then given by

$$\begin{aligned}\mathcal{A}_1 &= \frac{1}{2} [\text{Tr}(\Phi^i \Phi^j) \text{Tr}(\bar{\Phi}_i \bar{\Phi}_j) + \text{Tr}(e^{gV} \Phi^i e^{-gV} \bar{\Phi}_j) \text{Tr}(e^{gV} \Phi^j e^{-gV} \bar{\Phi}_i)] \\ \mathcal{A}_2 &= \text{Tr}(e^{gV} \Phi^i e^{-gV} \bar{\Phi}_i) \text{Tr}(e^{gV} \Phi^j e^{-gV} \bar{\Phi}_j) \\ \mathcal{A}_3 &= \text{Tr}(e^{gV} \Phi^i \Phi^j e^{-gV} \bar{\Phi}_i \bar{\Phi}_j) \\ \mathcal{A}_4 &= \frac{1}{4} [\text{Tr}(e^{gV} \Phi^i e^{-gV} \bar{\Phi}_i e^{gV} \Phi^j e^{-gV} \bar{\Phi}_j) + 2\text{Tr}(e^{gV} \Phi^i \Phi^j e^{-gV} \bar{\Phi}_j \bar{\Phi}_i) \\ &\quad + \text{Tr}(e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i e^{-gV} \bar{\Phi}_j e^{gV} \Phi^j)]\end{aligned}\quad (4.7)$$

To compute correlation functions for these superfield operators, we introduce the Euclidean generating functional

$$W[\vec{J}] = \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \mathcal{D}V e^{S+J^i \mathcal{A}_i} \quad (4.8)$$

for n -point functions

$$\langle \mathcal{A}_{i_1}(z_1) \cdots \mathcal{A}_{i_n}(z_n) \rangle = \left. \frac{\delta^n W}{\delta J^{i_1}(z_1) \cdots \delta J^{i_n}(z_n)} \right|_{\vec{J}=0} \quad (4.9)$$

where $z \equiv (x, \theta, \bar{\theta})$. To evaluate perturbative contributions to n -point functions it is sufficient to determine the contributions to $W[\vec{J}]$ at order n in the sources. In particular, for two-point functions we need evaluate the quadratic terms

$$W[\vec{J}] \rightarrow \int d^4x_1 d^4x_2 d^4\theta J^i(x_1, \theta, \bar{\theta}) \frac{\rho_{ij}(g^2, N)}{[(x_1 - x_2)^2]^{\Delta_0 + \gamma}} \mathcal{P}(D_\alpha, \bar{D}_{\dot{\alpha}}) J^j(x_2, \theta, \bar{\theta}) \quad (4.10)$$

where $\mathcal{P}(D_\alpha, \bar{D}_{\dot{\alpha}})$ is an operatorial expression built up from spinorial derivatives. As discussed in the previous Section, the particular dependence on the bosonic coordinates

⁷To be more precise, the operators (4.1) are the lowest components of these superfields. In a small abuse of language we give the same names both to the superfields and to their lowest components.

is fixed order by order by conformal invariance and brings to the determination of the anomalous dimensions. The only freedom left is in the appearance of possible perturbative multiplicative corrections encoded in the matrix $\rho_{ij}(g^2, N)$.

In order to compute the first-order contributions to (4.10), we first draw all possible diagrams with two external sources at first order in $\lambda \equiv g^2 N/4\pi^2$. In the Feynman gauge, the relevant propagators read (the ghosts do not contribute at this order)

$$\langle V^a(z)V^b(0) \rangle = -\frac{\delta^{ab}}{4\pi^2} \frac{1}{x^2} \delta^{(4)}(\theta) \quad \langle \Phi^{ia}(z)\bar{\Phi}_j^b(0) \rangle = \frac{\delta^{ab}}{4\pi^2} \frac{\delta_j^i}{x^2} \delta^{(4)}(\theta) \quad (4.11)$$

while the vertices we need from the action are

$$\begin{aligned} V_1 &= igf_{abc}\delta_j^i\bar{\Phi}_i^aV^b\Phi^{jc} \\ V_2 &= -\frac{g}{3!}\epsilon_{ijk}f_{abc}\Phi^{ia}\Phi^{jb}\Phi^{kc} & \bar{V}_2 &= -\frac{g}{3!}\epsilon^{ijk}f_{abc}\bar{\Phi}_i^a\bar{\Phi}_j^b\bar{\Phi}_k^c \end{aligned} \quad (4.12)$$

with additional \bar{D}^2 , D^2 factors for chiral, antichiral lines respectively.

Our basic conventions to deal with color structures are as follows. For a general simple Lie algebra we have

$$[T^a, T^b] = if^{abc}T^c \quad (4.13)$$

where T^a are the generators and f^{abc} the structure constants. The matrices T^a are normalized as

$$\text{Tr}(T^a T^b) = \delta^{ab} \quad (4.14)$$

We specialize to the case of $SU(N)$ Lie algebra whose generators T^a , $a = 1, \dots, N^2 - 1$ are taken in the fundamental representation, i.e. they are $N \times N$ traceless matrices. The basic relation which allows to deal with products of T^a 's is the following

$$T_{pq}^a T_{rs}^a = \left(\delta_{ps}\delta_{qr} - \frac{1}{N}\delta_{pq}\delta_{rs} \right). \quad (4.15)$$

From this identity, together with (4.13), we can easily obtain all the identities used to compute the color structures associated to the Feynman diagrams relevant for the two-point correlation functions.

As a second step one needs to perform the superspace D -algebra to reduce each diagram to an ordinary multiloop integral. To deal with possible divergences in these integrals we perform massless dimensional regularization by analytically continuing the theory to d dimensions, $d = 4 - 2\epsilon$. We evaluate the integrals using the general formula

$$\int d^d x \frac{1}{(x^2)^a [(x-y)^2]^b} = \pi^2 \frac{\Gamma(a+b-\frac{d}{2}) \Gamma(\frac{d}{2}-a) \Gamma(\frac{d}{2}-b)}{\Gamma(a)\Gamma(b)} \frac{1}{(x^2)^{a+b-\frac{d}{2}}} \quad (4.16)$$

and extracting the $\frac{1}{\epsilon}$ pole. The leading behavior of the integrals we use in the calculation is

$$\int d^d x \frac{1}{(x^2)^{1-\epsilon} [(x-y)^2]^{2-2\epsilon}} \sim \frac{\pi^2}{\epsilon} \frac{1}{x^2} \quad (4.17)$$

$$\int d^d x \frac{1}{(x^2)^{2-2\epsilon} [(x-y)^2]^{2-2\epsilon}} \sim 2 \frac{\pi^2}{\epsilon} \frac{1}{(x^2)^2} \quad (4.18)$$

5 One-loop anomalous dimension matrix and quasiprimary composite operators in $\mathcal{N} = 4$ SYM

In this section we compute the full one-loop anomalous dimension matrix for the operators (4.1) using the $\mathcal{N} = 1$ superspace approach described in the preceding section. Let us first write the four gauge invariant superfields (4.7) by expanding the exponentials inside the traces up to order gV

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{2} \Phi^{ia} \Phi^{jb} \bar{\Phi}_i^c \bar{\Phi}_j^d [(\delta_{ab} \delta_{cd} + \delta_{ad} \delta_{bc}) + igV^e (f^{dea} \delta^{bc} + f^{ceb} \delta^{ad})] \\ \mathcal{A}_2 &= \Phi^{ia} \Phi^{jb} \bar{\Phi}_i^c \bar{\Phi}_j^d [(\delta_{ac} \delta_{bd} + igV^e (f^{cea} \delta^{bd} + f^{deb} \delta^{ac}))] \\ \mathcal{A}_3 &= \Phi^{ia} \Phi^{jb} \bar{\Phi}_i^c \bar{\Phi}_j^d [\text{Tr}(T_a T_b T_c T_d) + igV^e (f^{gea} \text{Tr}(T_g T_b T_c T_d) + f^{heb} \text{Tr}(T_a T_h T_c T_d))] \\ \mathcal{A}_4 &= \frac{1}{4} \Phi^{ia} \Phi^{jb} \bar{\Phi}_i^c \bar{\Phi}_j^d \left\{ \text{Tr}(T_a T_c T_b T_d) + 2\text{Tr}(T_a T_b T_d T_c) + \text{Tr}(T_a T_d T_b T_c) \right. \\ &\quad \left. + igV^e [f^{gea} (\text{Tr}(T_g T_c T_b T_d) + 2\text{Tr}(T_g T_b T_d T_c) + \text{Tr}(T_g T_d T_b T_c)) \right. \\ &\quad \left. + f^{heb} (\text{Tr}(T_a T_c T_h T_d) + 2\text{Tr}(T_a T_h T_d T_c) + \text{Tr}(T_a T_d T_h T_c))] \right\} \end{aligned} \quad (5.1)$$

First of all, we are interested in the calculation of the tree-level two-point functions of these operators. Neglecting for the moment their color and flavor structures, we concentrate on the two-point function of the generic operator $\Phi\Phi\bar{\Phi}\bar{\Phi}$. Its diagram is shown in Figure 1 where the superspace derivatives acting on the propagators are explicitly indicated. In this case the D-algebra is performed rather easily and the result is given by

$$W[J]_{tree} \rightarrow \int d^4 x_1 d^4 x_2 d^4 \theta J(x_1, \theta, \bar{\theta}) \frac{1}{(x_1 - x_2)^8} \left(D^2 \bar{D}^2 + \frac{i}{2} \partial_{\alpha\dot{\alpha}} D^\alpha \bar{D}^{\dot{\alpha}} + \frac{1}{6} \square \right) J(x_2, \theta, \bar{\theta}) \quad (5.2)$$

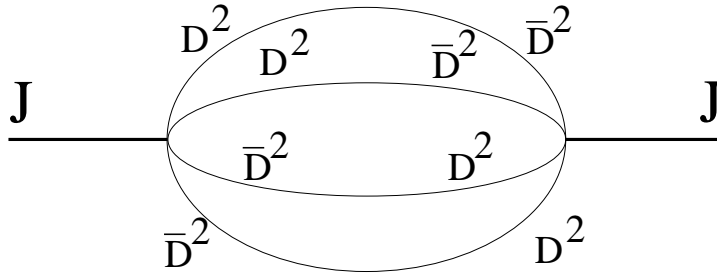


Figure 1: Tree level contribution to $\langle A(z_1)A(z_2) \rangle$

To evaluate the full result for $\langle \mathcal{A}_i \mathcal{A}_j \rangle_0$ we have to take into account the combinatorics coming from the sum over flavor indices and contractions of the color tensor structures

appearing in (5.1). This is a straightforward algebraic calculation which finally gives

$$\langle \mathcal{A}_i(z) \mathcal{A}_j(0) \rangle_0 = \frac{3(N^2 - 1)}{(4\pi^2)^4} \begin{pmatrix} \frac{7N^2+2}{2} & N^2 + 6 & \frac{7N^2-8}{N} & \frac{9N^2-16}{2N} \\ N^2 + 6 & 2(3N^2 - 2) & \frac{2(N^2-4)}{N} & \frac{7N^2-8}{N} \\ \frac{7N^2-8}{N} & \frac{2(N^2-4)}{N} & \frac{3N^4-8N^2+24}{N^2} & \frac{N^4-16N^2+48}{2N^2} \\ \frac{9N^2-16}{2N} & \frac{7N^2-8}{N} & \frac{N^4-16N^2+48}{2N^2} & \frac{7N^4-32N^2+96}{4N^2} \end{pmatrix} \times \left(D^2 \bar{D}^2 + \frac{i}{2} \partial_{\alpha\dot{\alpha}} D^\alpha \bar{D}^{\dot{\alpha}} + \frac{1}{6} \square \right) \frac{1}{x^8} \delta^{(4)}(\theta) \quad (5.3)$$

We notice that the operators \mathcal{A}_i do not form an orthogonal basis at tree level. Diagonalizing the previous matrix one can find a suitable orthonormal basis. This is discussed in details in Appendix B where the diagonalization is performed for large N .

Let us now move to the computation of the one-loop two-point function. The various contributions are shown in Figure 2. To draw the diagrams 2a, 2b, 2c and 2f, 2g, 2h we have used only the interaction vertices given in (4.12), while diagrams 2d and 2e contain also external vertices with one vector field, as given in (5.1).

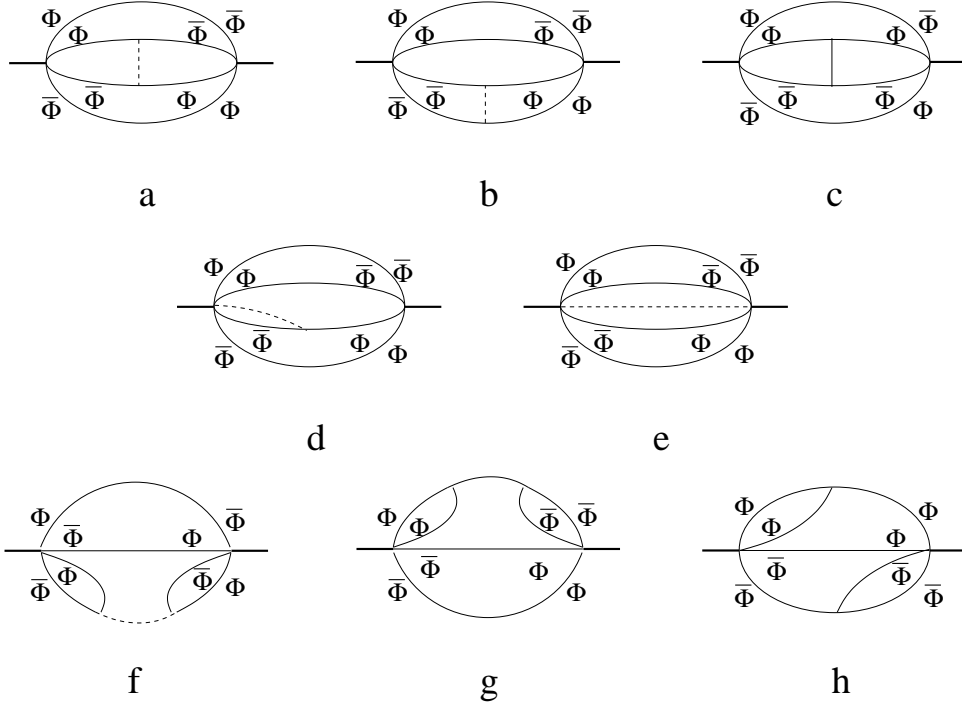


Figure 2: One-loop contributions to $\langle A(z_1)A(z_2) \rangle$

Since we are interested merely in the anomalous dimensions of the operators, in accordance with (3.14) we consider only the logarithmic divergences which, in dimensional regularization, appear as $1/\epsilon$ poles⁸. Therefore, we only concentrate on $1/\epsilon$ divergent

⁸Note that, in a momentum space approach, as the one used in [38], these terms are instead the ones appearing as $1/\epsilon^2$ poles.

diagrams. After performing the D -algebra and looking at the structure of the integrals which arise, it is not difficult to see that graphs 2a, 2c, 2d, 2f, and 2g are the only divergent diagrams, whereas graphs 2b, 2e and 2h give only finite contributions which contribute to the ρ matrix as defined in (3.14).

After D -algebra, the structure of superspace derivatives we are left with is the same as the one at tree-level (see eq. (5.2)). Along the calculation, we can then concentrate only on the contributions proportional to $D^2\bar{D}^2$, the other ones following by supersymmetry. The graphs 2a, 2c and 2d then reduce to the ordinary Feynman diagram in Figure 3a, while the reduction of graphs 2f and 2g produce the diagram in Figure 3b. The corresponding integrals are (4.17) and (4.18), respectively. They both contribute with a simple pole divergence, the integral of Figure 3b giving twice the result of the one in Figure 3a.

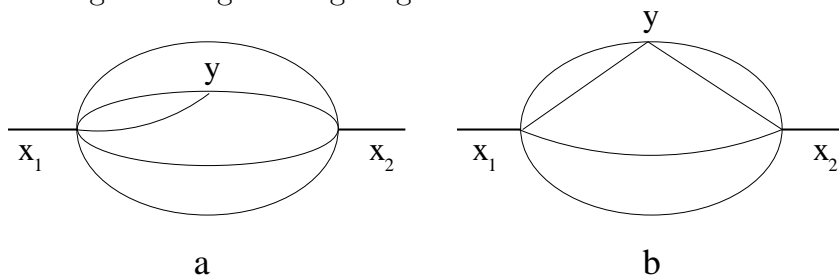


Figure 3: Bosonic integrals resulting after D -algebra

At this point we have to take into account the factors coming from the flavor and color combinatorics. This is the most tedious and tiring part of the calculation, even if rather straightforward. To perform the calculation we have first done the flavor contractions by hand, while to deal with the color contractions we have taken advantage of a computer program we have implemented with *Mathematica* [39]. As a result, the anomalous dimension matrix of the four operators $\mathcal{A}_1, \dots, \mathcal{A}_4$ is given by

$$-\frac{3}{2}\lambda(N^2 - 1) \begin{pmatrix} -2N^2 + 13 & -6(2N^2 + 7) & \frac{21N^2+16}{N} & -\frac{53N^2-32}{2N} \\ -6(2N^2 + 7) & -12(6N^2 + 1) & \frac{6(N^2+16)}{N} & -\frac{3(33N^2-32)}{N} \\ \frac{21N^2+16}{N} & \frac{6(N^2+16)}{N} & -\frac{11N^4-96N^2+128}{N^2} & \frac{3N^4+112N^2-256}{2N^2} \\ -\frac{53N^2-32}{2N} & -\frac{3(33N^2-32)}{N} & \frac{3N^4+112N^2-256}{2N^2} & -\frac{59N^4-64N^2+512}{4N^2} \end{pmatrix} \quad (5.4)$$

From the results (5.3, 5.4), we see that the four operators do mix non-trivially at tree and one-loop level. As discussed in section 3.2, we need to solve the mixing in order to find the anomalous dimensions of the quasiprimary operators belonging to the singlet sector of the theory. This amounts to building first of all an orthonormal basis at tree level out of the four operators $\mathcal{A}_1, \dots, \mathcal{A}_4$, and then to writing the anomalous dimension matrix in this new basis. It is from this matrix (once we have diagonalized it) that we can read off the pure conformal operators of the theory and the values of their anomalous dimensions up to order g^2 .

The procedure is described in some details in Appendix B, where we have computed the values of the anomalous dimensions up to order $1/N^2$, and have identified the corresponding quasiprimary operators $\Sigma_1, \dots, \Sigma_4$. The explicit expressions for the latter are given in (B.9) and their corresponding anomalous dimensions are

$$\begin{aligned}
\gamma_1 &= -10 \frac{\lambda}{N^2} \\
\gamma_2 &= \left(6 + \frac{20}{N^2}\right) \lambda \\
\gamma_3 &= \left[\frac{13 + \sqrt{41}}{4} - \frac{5}{41N^2}(41 + 19\sqrt{41}) \right] \lambda \\
\gamma_4 &= \left[\frac{13 - \sqrt{41}}{4} - \frac{5}{41N^2}(41 - 19\sqrt{41}) \right] \lambda
\end{aligned} \tag{5.5}$$

In terms of these operators, the (normalized) operator \mathcal{O}_1 appearing in the OPE (4.2) is

$$\mathcal{O}_1 = \Sigma_1 - \frac{1}{N} [\alpha_3 \Sigma_3 + \alpha_4 \Sigma_4] + \frac{1}{N^2} \left[-\frac{21}{16} \Sigma_1 + \frac{3\sqrt{5}}{4} \Sigma_2 \right] + O\left(\frac{1}{N^3}\right) \tag{5.6}$$

where α_3 and α_4 are given in (B.12). Therefore, at leading order in N , the operator which appears in the singlet channel of the OPE of the two CPOs \mathcal{O}_{20} is a quasiprimary operator with vanishing anomalous dimension. It becomes a mixture of pure states when subleading corrections in N are taken into account. The meaning of this result will be discussed on general grounds in the following Section.

To conclude this Section, we check our result against the four-point OPE analysis mentioned in Section 3.3 and discussed in detail in Appendix A. From (5.6) we see that the condition (3.25) is satisfied up to order $1/N^2$:

$$1 - \frac{21}{8N^2} + \frac{1}{N^2} [\alpha_3^2 + \alpha_4^2] = 1 + O\left(\frac{1}{N^3}\right) \tag{5.7}$$

while by virtue of the results (5.5), the condition (3.26) reads

$$-\frac{10\lambda}{N^2} + \frac{1}{N^2} [\alpha_3^2 \gamma_3 + \alpha_4^2 \gamma_4] = -\frac{4\lambda}{N^2} + O\left(\frac{1}{N^3}\right) \tag{5.8}$$

which can also hold at order $\frac{1}{N^2}$.

Finally, condition (3.27) is also satisfied at this same order as can be easily checked

$$\frac{1}{N^2} [\alpha_3^2 \gamma_3^2 + \alpha_4^2 \gamma_4^2] = 18 \frac{\lambda^2}{N^2} + O\left(\frac{1}{N^3}\right). \tag{5.9}$$

6 Non-protected operators and multi-particle states in supergravity

In the previous two sections we have studied in detail the four possible scalar quasiprimary operators which are $SU(4)$ singlets of canonical dimension four in $\mathcal{N} = 4$ SYM. At one loop, we identified them explicitly in terms of the elementary superfields (B.9) whose anomalous dimensions are given in (5.5).

The first important observation that follows from our results is connected with the anomalous dimensions (5.5). We see that one of the four anomalous dimensions, γ_1 , is suppressed by a factor of $1/N^2$ with respect to the remaining three, $\gamma_{2,3,4}$. Although this is merely a one-loop result, it points at a fundamental difference between the quasiprimary operator Σ_1 on the one hand, and $\Sigma_{2,3,4}$ on the other. Namely, the anomalous dimensions of the latter three operators behave like the anomalous dimensions of the K -class operators, that is, they are positive and of order $O(\lambda)$. Therefore, it is justified to believe that these three operators are dual to string modes. Their anomalous dimensions become infinite when $\lambda, N \rightarrow \infty$, so that they decouple in the supergravity limit. On the other hand, the anomalous dimension of the operator Σ_1 , being of order $1/N^2$ at one loop, has a chance to remain finite in the supergravity limit.

The above observations are in agreement with the known strong coupling calculations of the four-point function of the CPOs (2.1). In particular, in [20] it was shown that the contribution of *all* the operators in the K -class is absent in the supergravity limit of that four-point function. However, the contribution of scalar fields with canonical dimension four was shown to be non-trivial. Our result above indicates that *only one* of the four possible quasiprimary operators with canonical dimension four might survive in the supergravity limit and give a non-trivial contribution to the four-point function of the CPOs (2.1). This is the quasiprimary operator Σ_1 which therefore should correspond to a two-particle supergravity bound state.

Further support for our interpretation of Σ_1 as a two-particle supergravity bound state comes from the usual identification of the conformal anomalous dimension with the energy of a state in a radial quantization (see e.g. [3]). A negative anomalous dimension would then correspond to a *binding energy*. In this sense, it is quite suggestive for the nature of the state corresponding to the quasiprimary operator Σ_1 that both its one-loop as well as its strong coupling anomalous dimensions are negative. It would be interesting to find out if this property persists at higher orders in perturbation theory.

Our next observation concerns the operator Σ_2 . From (5.5) we see that in the large N limit its anomalous dimension $\gamma_2 = 6\lambda$ of this operator equals twice the one-loop anomalous dimension $\gamma_{\mathcal{K}} = 3\lambda$ of the Konishi scalar $\sim \text{Tr}(e^{gV}\Phi^i e^{-gV}\bar{\Phi}_i)$ [40, 15, 21]. Alternatively, one can also deduce from (B.13) that for large N $\Sigma_2 = \mathcal{K}^2 \sim \mathcal{A}_2$, i.e., the operator Σ_2 coincides with the canonically normalized square of the Konishi scalar. In other words, we observe that in the large N limit the anomalous dimensions of the individual constituents of the composite operator Σ_1 just add up. Owing to this property

it is then natural to identify Σ_2 with an operator dual to a two-particle string state.

Another intriguing consequence of our results comes when we compare our formula (5.6) and the free OPE (4.2). In principle, there is no reason why the free operator \mathcal{O}_1 that appears in the OPE (4.2) and the quasiprimary operator Σ_1 might bare any resemblance. Nevertheless, it is easily seen from (5.6) or from (B.9) and (4.3) that in the large N limit \mathcal{O}_1 coincides with the quasiprimary operator Σ_1 . Indeed, to calculate the two-point function of \mathcal{O}_1 one should use the decomposition (5.6) into quasiprimary operators. However, as we have argued above, we expect that in the supergravity limit the operators $\Sigma_{2,3,4}$ decouple since they are dual to massive string modes. Then, since we are at the same time considering the large N limit, we see from (5.6) and (B.9) that \mathcal{O}_1 coincides with the quasiprimary operator Σ_1 dual to a two-particle supergravity state in the supergravity limit. It is remarkable that the OPE itself somehow *knows* the special rôle of the operator \mathcal{O}_1 and gives us its explicit expression in terms of elementary superfields without the need to go through the process of diagonalization of all the possible mixed fields.

Now we want to argue that that the suppression of the anomalous dimension of Σ_1 should work to all orders in perturbation theory. This is due to the special nature of the operator \mathcal{O}_1 which can be written as a direct product of two CPOs:

$$\mathcal{O}_1(x) = \mathcal{O}_{\mathbf{20}}^{LM}(x)\mathcal{O}_{\mathbf{20}}^{LM}(x). \quad (6.1)$$

One can show that all of its two-point functions, with itself and with the remaining three operators $\mathcal{A}_{2,3,4}$ from (4.1) give logarithmic terms that are at least $1/N$ suppressed.

Let us first discuss the two-point function $\langle \mathcal{O}_1 \mathcal{O}_1 \rangle$. From Figure 4 we see that at tree level it includes a connected and a disconnected parts. The connected part is $1/N^2$ suppressed as it is *non-planar*.

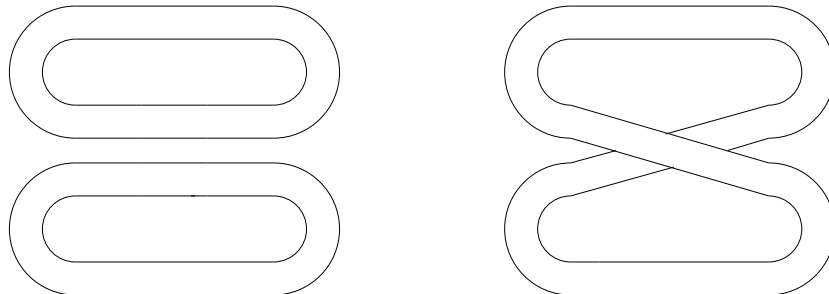


Figure 4: Contributions to the tree-level two-point function of \mathcal{O}_1 in the double line notation.

The perturbative corrections to the above picture can be of three types (Figure 5). Firstly, there can be corrections to *each* of the disconnected graphs separately. Secondly,

there can be lines that *connect the disconnected* graphs. Finally, there can be perturbative corrections to the *non-planar* connected graphs. Among them, the leading contribution to the $1/N$ expansion is supplied by the first type of graphs, where all the corrections are confined to the disconnected parts. Now, the crucial point is that these graphs can be factorized as follows:

$$\langle \mathcal{O}_1(x)\mathcal{O}_1(y) \rangle_1^{\text{disc.}} = 2 \langle \mathcal{O}_{\mathbf{20}}^{LM}(x)\mathcal{O}_{\mathbf{20}}^{NP}(y) \rangle_0 \langle \mathcal{O}_{\mathbf{20}}^{LM}(x)\mathcal{O}_{\mathbf{20}}^{NP}(y) \rangle_1, \quad (6.2)$$

where the subscript $\langle \rangle_{0,1}$ indicates the perturbation level. For this subset the trace over the $SO(6)$ indices in (6.1) is of no relevance, so we can treat each factor in the right-hand side of (6.2) as an independent two-point function of the *protected operator* $\mathcal{O}_{\mathbf{20}}$. The protected operators have a fixed scaling dimension determined by their nature of short superconformal representations. Consequently, all such divergent⁹ graphs, which could possibly contribute to the anomalous dimension of $\mathcal{O}_{\mathbf{20}}$, should add up to zero. All the remaining *planar* and *non-planar connected* graphs are subleading, at most of order $1/N^2$.

Clearly, this mechanism works to *all orders* in perturbation theory. Indeed, at order n eq. (6.2) can be replaced by

$$\langle \mathcal{O}_1(x)\mathcal{O}_1(y) \rangle_n^{\text{disc.}} = \sum_{m=0}^n C_{nm} \langle \mathcal{O}_{\mathbf{20}}^{LM}(x)\mathcal{O}_{\mathbf{20}}^{NP}(y) \rangle_m \langle \mathcal{O}_{\mathbf{20}}^{LM}(x)\mathcal{O}_{\mathbf{20}}^{NP}(y) \rangle_{n-m}. \quad (6.3)$$

In this sum, for a given perturbative correction of order m to the first disconnected part we have collected all the corrections of order $n - m$ to the second disconnected part. The latter correspond to the protected two-point function at order $n - m$, so they give no logarithmic terms. The same argument can then be applied to the first part of order m .

It remains to show that the two-point functions of \mathcal{O}_1 with the operators $\mathcal{A}_{2,3,4}$ from (4.1) give logarithmic terms that are at least $1/N$ suppressed. In the case of $\mathcal{A}_{3,4}$ this is due to the fact that they are single-trace operators, so we have one color trace fewer in the graphs. The disconnected (leading) sector of the two-point function of the double-trace operators $\mathcal{O}_1 \sim \mathcal{O}_{\mathbf{20}}\mathcal{O}_{\mathbf{20}}$ and $\mathcal{A}_2 \sim \mathcal{K}\mathcal{K}$ is empty, since it is not possible to factorize it into two-point functions matching an $\mathcal{O}_{\mathbf{20}}$ factor with a singlet \mathcal{K} factor (see Figures 4 and 5). Again, this is true to all orders in perturbation theory. Thus, all the entries in the first row of the anomalous dimension matrix (B.7) are at least $1/N$ suppressed. When diagonalizing such a matrix in the large N limit, the operator \mathcal{O}_1 effectively becomes the eigenvector Σ_1 with a subleading eigenvalue. This is the mechanism which is responsible for the $1/N^2$ suppression of the anomalous dimension of Σ_1 and thus for the existence of two-particle supergravity states with non-zero binding energy.

It is important to mention that the same mechanism can be applied to any multitrace operator which is defined as a product of any number of protected operators, not necessarily identical. Then we may conjecture that such operators constitute a substantial part,

⁹In fact, the $1/2$ BPS short operator $\mathcal{O}_{\mathbf{20}}$ is completely non-renormalized, including the finite quantum corrections. However, here we are only interested in logarithmic terms which come from the divergent graphs.

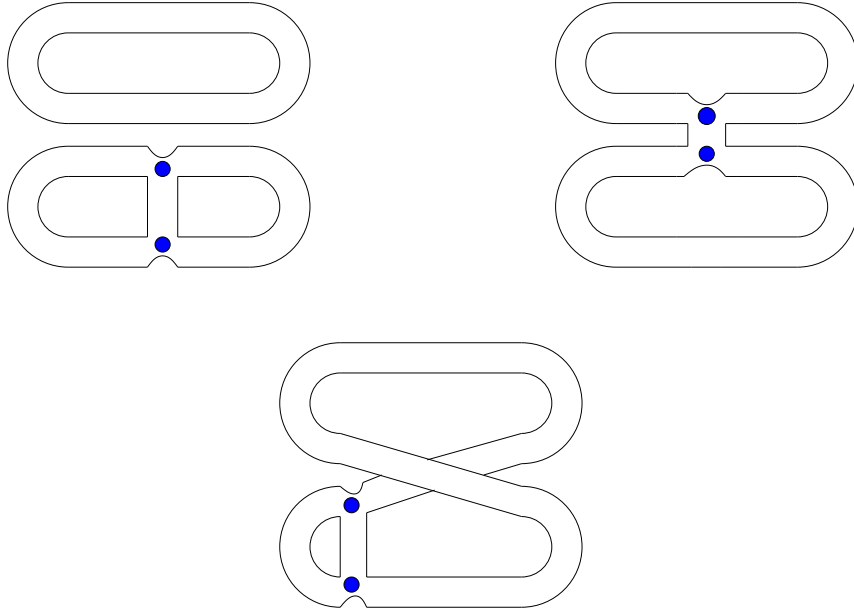


Figure 5: Contributions to the one-loop two-point function of \mathcal{O}_1 in the double line notation.

if not all, of the supergravity multiparticle (scalar) states. Whether one can construct multiparticle states with non-vanishing spin in this way is not immediately clear. The point is that BPS short operators like \mathcal{O}_{20} are always scalars. However, we know another type of protected operators which can have arbitrary spin [8, 9]. It is possible that they, together with the BPS operators, play an important rôle in constructing multiparticle states.

In conclusion, we have studied the dimension 4 scalar quadrilinear operators that appear in the $SU(4)$ singlet channel of the OPE between two lowest weight CPOs in $\mathcal{N} = 4$ SYM. At one loop, due to the four-fold degeneracy at the free level, we had to diagonalize a 4×4 matrix of two-point functions. We were able to identify the one-loop quasiprimary operators, to find their anomalous dimensions and to give explicit expressions for them in terms of the elementary superfields. We found that one of the four quasiprimary operators is a possible candidate to correspond to a two-particle supergravity state, since its anomalous dimension is $1/N^2$ suppressed with regards to the anomalous dimensions of operators that are dual to string modes. We pointed out that the special structure of the operator \mathcal{O}_1 is connected to the mechanism responsible for such a $1/N^2$ suppression. Curiously, the free conformal OPE seems to provide the explicit expression for the quasiprimary operator dual to the two-particle supergravity state, to leading order in N .

Although our results are one-loop and restricted to the simplest scalar quadrilinear operators, we believe that the general pattern we sketched remains valid both at higher loops as well as for non-scalar quadrilinear operators. It would be desirable to extend our results to two loops in order to get a clearer picture. The mechanism for the $1/N^2$ suppression that we have revealed affects *both planar and non-planar* graphs. It seems therefore possible that non-planar graphs survive in the supergravity limit and play an

important rôle in the formation of multiparticle supergravity states. It would be interesting to understand better the strong coupling supergravity limit that seems to be more complicated than initially thought, in order to resolve the puzzle of the multiparticle supergravity states.

We have discussed the general approach to solve the mixing and splitting of operators. Our strategy works in any CFT and can be applied to the study of composite operators in a variety of cases, such as $\mathcal{N} = 2$ and $\mathcal{N} = 1$ superconformal theories, as well as in studies of multitrace operators in the pp-wave limit.

Note added When our paper was ready for submission to the e-archive, we noticed a new paper [41] where the operator mixing problem is discussed for scalar operators of approximate dimension 4 but belonging to the **20** of $SU(4)$. Our operators are $SU(4)$ singlets.

Note added in proof In addition to the four linearly independent operators that we considered in the text, Eq. (4.1), there exist two more operators with the same quantum numbers (apart from fermionic operators which do not contribute to the two-point functions at level $O(\lambda)$). Their lowest components are $\text{Tr}[D_\mu \phi^I D^\mu \phi^I]$ and $\text{Tr}[F_{\mu\nu} F^{\mu\nu}]$. The first of them is the leading term in the conformal descendant of the Konishi scalar $\square K$, and a combination of the two is the leading term in the $\mathcal{N} = 4$ SYM Lagrangian which belongs to a protected multiplet. Hence, these operators can be identified as pure states. In addition, at tree level they are orthogonal to all four operators in Eq. (4.1). Then it is easy to see that their presence can only modify the one-loop pure state operators in Eq. (B.9) by terms of order $O(\lambda)$ which are beyond the scope of this work.

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A Four-point functions and the OPE

Here we recall the relevant facts about conformal partial wave expansions required to carry out an independent verification of our results at the OPE level (see [20, 21, 30] for details) and show how to extract some information about the quasiprimary operators Σ_i encoded in the two-loop four-point function of $O_{\mathbf{20}}^{LM}$.

Every irreducible representation of the R symmetry algebra in the tensor product decomposition (2.2) appears in the OPE of $O_{\mathbf{20}}^{LM}$ as an infinite tower of operators $O_{\Delta,l}$, where Δ is the conformal dimension and l is the Lorentz spin. We are interested only in the singlet R symmetry channel for which the corresponding contribution to the four-point function can be viewed as an expansion of the type

$$F = \langle O_{\mathbf{20}}^{LM}(x_1) O_{\mathbf{20}}^{LM}(x_2) O_{\mathbf{20}}^{NP}(x_3) O_{\mathbf{20}}^{NP}(x_4) \rangle = \sum_{\Delta,l} A_{\Delta,l} \mathcal{H}_{\Delta,l}(x_1, x_2, x_3, x_4). \quad (\text{A.1})$$

Here $\mathcal{H}_{\Delta,l}(x_1, x_2, x_3, x_4)$ is the canonically normalized conformal partial wave amplitude (CPWA) representing the individual contribution (exchange) of an operator $O_{\Delta,l}$ and $A_{\Delta,l}$ is a normalization coefficient. We treat the CPWA as a double series of the type

$$\mathcal{H}_{\Delta,l}(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^4 x_{34}^4} v^{(\Delta-l)/2} \sum_{n,m} c_{mn}^{\Delta,l} v^n Y^m, \quad (\text{A.2})$$

where $Y = 1 - (x_{13}^2 x_{24}^2)/(x_{14}^2 x_{23}^2)$ is the other conformally invariant cross-ratio. In particular, the CPWA of a scalar and of a second-rank tensor have the following leading terms

$$\mathcal{H}_{\Delta,0}(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^4 x_{34}^4} v^{\frac{\Delta}{2}} \left(1 + \frac{\Delta}{4} Y + \frac{\Delta^3}{16(\Delta-1)(\Delta+1)} v + \dots \right) \quad (\text{A.3})$$

$$\mathcal{H}_{\Delta,2}(x_1, x_2, x_3, x_4) = \frac{1}{x_{12}^4 x_{34}^4} v^{(\Delta-2)/2} \left(\frac{1}{4} Y^2 - \frac{1}{4} v \dots \right) \quad (\text{A.4})$$

Assuming that the dimension Δ takes the form $\Delta = \Delta_0 + \gamma$, where γ is the one-loop anomalous dimension, we see that the non-analytic term $v^{\Delta/2}$ gives rise to the logarithmic terms of perturbation theory:

$$v^{\gamma/2} = 1 + 1/2\gamma \ln v + 1/8\gamma^2 \ln^2 v + \dots \quad (\text{A.5})$$

Thus, comparing the analytic terms in (A.3) and (A.4) against the ones in a concrete four-point function allows one to identify the canonical dimensions and spins of the contributing operators, while the logarithmic terms store the information about their anomalous dimensions. Let us see how this works for our operators Σ_i of canonical dimension $\Delta_0 = 4$.

The operator with the lowest canonical dimension contributing to the OPE of two operators $O_{\mathbf{20}}^{LM}$ is the Konishi scalar K_s . Among the operators with $\Delta_0 = 4$ one finds the stress-energy tensor T , the Konishi tensor K_t , a new superconformal primary operator Ξ and finally the operators Σ_i [21]. The operators K_s , K_t and Ξ are from the K -class discussed in the Introduction and the stress-energy tensor is protected. Thus, schematically we may write

$$O_{\mathbf{20}}^{LM} O_{\mathbf{20}}^{LM} = K_s + K_t + \Xi + T + \sum_{i=1}^4 \Sigma_i + \text{higher dim.} \quad (\text{A.6})$$

Our quasiprimary operators Σ_i are Lorentz scalars with $\Delta_0 = 4$ and, as is clear from (A.3), the lowest-power monomial occurring in their CPWAs is v^2 . Hence, looking for the coefficients of v^2 , $v^2 \ln v$ and $v^2 \ln^2 v$ allows us to read off some information about their normalization coefficients A_i and anomalous dimensions γ_i . One should bear in mind, however, that in principle the coefficients of the above mentioned structures receive contributions from *all* operators participating in (A.6). Thus, expanding Eq. (A.1) in powers of γ up to the second order, we find (only the relevant terms are indicated)

$$\begin{aligned} F &= \frac{v^2}{x_{12}^4 x_{34}^4} \left[\left(\sum_{i=1}^4 A_i + \frac{1}{6} A_{K_s} - \frac{1}{4} A_{K_t} - \frac{1}{4} A_{\Xi} - \frac{1}{4} A_T \right) \right. \\ &+ \frac{1}{2} \ln v \left(\sum_{i=1}^4 A_i \gamma_i + \frac{1}{6} A_{K_s} \gamma_{K_s} - \frac{1}{4} A_{K_t} \gamma_{K_t} - \frac{1}{4} A_{\Xi} \gamma_{\Xi} \right) \\ &\left. + \frac{1}{8} \ln^2 v \left(\sum_{i=1}^4 A_i \gamma_i^2 + \frac{1}{6} A_{K_s} \gamma_{K_s}^2 - \frac{1}{4} A_{K_t} \gamma_{K_t}^2 - \frac{1}{4} A_{\Xi} \gamma_{\Xi}^2 \right) \right], \end{aligned} \quad (\text{A.7})$$

where we have also taken into account that the stress-energy tensor is protected.

Since in the large N limit the normalization coefficients and anomalous dimensions of all operators in (A.6) except Σ_i are already known (see e.g. [30]):

$$A_{K_s} = \frac{4}{3N^2}, \quad A_{K_t} = \frac{16}{63N^2}, \quad A_{\Xi} = \frac{16}{35N^2}, \quad A_T = \frac{8}{45N^2}, \quad (\text{A.8})$$

and

$$\gamma_{K_s} = \gamma_{K_t} = 3\lambda, \quad \gamma_{\Xi} = \frac{25}{6}\lambda \quad (\text{A.9})$$

one can substitute these quantities¹⁰ in (A.7).

The CPWA expansion (leading in $1/N^2$) of the four point function of the operators $O_{\mathbf{20}}^{LM}$ up to two loops was constructed in Ref. [30]. Thus, the part of this four-point function relevant to our analysis can be extracted from Ref. [30] and it reads

$$F = \frac{v^2}{x_{12}^4 x_{14}^4} \left[\frac{1}{10} \left(1 + \frac{2}{3N^2} \right) - \frac{1}{5N^2} \lambda \ln v + \frac{7}{45N^2} \lambda^2 \ln^2 v \right]. \quad (\text{A.10})$$

Finally, comparing eq. (A.7) (with (A.8) and (A.9) inserted) and eq. (A.10), we obtain the following set of equations for the normalization constants and the one-loop anomalous dimensions of the operators Σ_i in leading order in $1/N^2$:

$$\sum_{i=1}^4 A_i \equiv g_0^2 = \frac{1}{10} \left(1 + \frac{2}{3N^2} \right), \quad \sum_{i=1}^4 A_i \gamma_i = -\frac{2}{5} \frac{\lambda}{N^2}, \quad \sum_{i=1}^4 A_i \gamma_i^2 = \frac{9}{5} \frac{\lambda^2}{N^2}. \quad (\text{A.11})$$

It remains to note that for canonically normalized operators Σ_i the coefficients A_i coincide with the square of the normalization coefficient of their three-point function with two $O_{\mathbf{20}}$. As a consequence, we obtain the consistency conditions (3.25)–(3.27) on the diagonalization coefficients $a_i^2 = g_0^{-2} A_i$ in (3.24). They were used in Section 5 to verify the agreement of our findings with the OPE structure implied by one- and two-loop four-point functions of CPOs.

B Diagonalization procedure

The four operators defined in (4.1) are not orthonormal at tree level. In fact, by computing their two-point functions (for simplicity we are now restricting our attention to the lowest components) we find

$$\langle \mathcal{A}_i(x) \mathcal{A}_j(0) \rangle_0 = \frac{1}{(4\pi^2)^4} \frac{C_{ij}}{(x^2)^4} \quad (\text{B.1})$$

where

$$C_{ij} = 3(N^2 - 1) \begin{pmatrix} \frac{7N^2+2}{2} & N^2 + 6 & \frac{7N^2-8}{N} & \frac{9N^2-16}{2N} \\ N^2 + 6 & 2(3N^2 - 2) & \frac{2(N^2-4)}{N} & \frac{7N^2-8}{N} \\ \frac{7N^2-8}{N} & \frac{2(N^2-4)}{N} & \frac{3N^4-8N^2+24}{N^2} & \frac{N^4-16N^2+48}{2N^2} \\ \frac{9N^2-16}{2N} & \frac{7N^2-8}{N} & \frac{N^4-16N^2+48}{2N^2} & \frac{7N^4-32N^2+96}{4N^2} \end{pmatrix} \quad (\text{B.2})$$

The diagonal 2×2 matrices are leading order in N plus corrections order $\frac{1}{N^2}$ and $\frac{1}{N^4}$, whereas the off-diagonal matrices contain only subleading contributions order $\frac{1}{N}$ and $\frac{1}{N^3}$.

¹⁰One can see that both combinations $\frac{1}{6}A_{K_s} - \frac{1}{4}A_{K_t} - \frac{1}{4}A_{\Xi} - \frac{1}{4}A_T$ and $\frac{1}{6}A_{K_s}\gamma_{K_s} - \frac{1}{4}A_{K_t}\gamma_{K_t} - \frac{1}{4}A_{\Xi}\gamma_{\Xi}$ in fact vanish.

Therefore, at leading order in N the double trace operators $\mathcal{A}_1, \mathcal{A}_2$ do not mix with the single traces \mathcal{A}_3 and \mathcal{A}_4 and a block diagonalization allows to easily obtain an orthogonal basis. A possible choice might be $(\mathcal{A}_1 - \frac{1}{6}\mathcal{A}_2), \mathcal{A}_2, (\mathcal{A}_3 - \frac{2}{7}\mathcal{A}_4)$ and \mathcal{A}_4 .

At order $\frac{1}{N^2}$, keeping contributions from the matrix (B.2) up to this order, we can define the orthonormal basis

$$\begin{aligned}
\mathcal{O}_1 &= \frac{1}{\sqrt{10}N^2} \left(1 + \frac{2}{3N^2}\right) \left[\mathcal{A}_1 - \frac{1}{6}\mathcal{A}_2\right] \\
\mathcal{O}_2 &= \frac{1}{\sqrt{18}N^2} \left(1 + \frac{217}{84N^2}\right) \left[\mathcal{A}_2 - \frac{1}{N} \left(\frac{3}{2}\mathcal{A}_3 + 3\mathcal{A}_4\right) + \frac{4}{N^2} \left(\mathcal{A}_1 - \frac{1}{6}\mathcal{A}_2\right)\right] \\
\mathcal{O}_3 &= \sqrt{\frac{7}{60}} \frac{1}{N^2} \left(1 + \frac{67}{28N^2}\right) \left[\mathcal{A}_3 - \frac{2}{7}\mathcal{A}_4 + \frac{1}{N} \left(\mathcal{A}_2 - \frac{12}{7}\mathcal{A}_1\right)\right] \\
\mathcal{O}_4 &= \frac{2}{\sqrt{21}N^2} \left(1 + \frac{61}{14N^2}\right) \left[\mathcal{A}_4 - \frac{1}{N}\mathcal{A}_1 + \frac{5}{2N^2} \left(\mathcal{A}_3 - \frac{2}{7}\mathcal{A}_4\right)\right]
\end{aligned} \tag{B.3}$$

which satisfies

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle_0 = \frac{1}{(4\pi^2)^4} \frac{1}{(x^2)^4} \left[\delta_{ij} + O\left(\frac{1}{N^3}\right) \right] \tag{B.4}$$

This is the basis we are going to use in order to solve the mixing problem at one-loop.

We now move to the one-loop calculation. As explain in the main text, divergent contributions arise in all the two-point functions between any pair of operators $\mathcal{A}_1, \dots, \mathcal{A}_4$ given in (4.7). Using *Mathematica* we can then compute

$$\langle \mathcal{A}_i(x) \mathcal{A}_j(0) \rangle_1 = \frac{1}{\epsilon} \frac{\omega_{ij}}{(4\pi^2)^4} \frac{1}{(x^2)^4} \tag{B.5}$$

where

$$\omega = -\frac{3}{2}\lambda(N^2 - 1) \begin{pmatrix} -2N^2 + 13 & -6(2N^2 + 7) & \frac{21N^2+16}{N} & -\frac{53N^2-32}{2N} \\ -6(2N^2 + 7) & -12(6N^2 + 1) & \frac{6(N^2+16)}{N} & -\frac{3(33N^2-32)}{N} \\ \frac{21N^2+16}{N} & \frac{6(N^2+16)}{N} & -\frac{11N^4-96N^2+128}{N^2} & \frac{3N^4+112N^2-256}{2N^2} \\ -\frac{53N^2-32}{2N} & -\frac{3(33N^2-32)}{N} & \frac{3N^4+112N^2-256}{2N^2} & -\frac{59N^4-64N^2+512}{4N^2} \end{pmatrix} \tag{B.6}$$

is the anomalous dimensions matrix. It is not diagonal since there is mixing among the operators. We then face the problem to solve the mixing perturbatively in $\frac{1}{N}$, for large N . We will determine the anomalous dimensions up to order $\frac{1}{N^2}$ and the corresponding quasiprimary operators up to $\frac{1}{N}$.

First of all, we have to diagonalize the matrix of the two-point functions at tree level. The new basis, up to the order we are interested in, is given in (B.3). On this basis the matrix ω is given by

$$\omega = \lambda \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & \frac{16}{7} & \frac{4\sqrt{5}}{7} \\ 0 & 0 & \frac{4\sqrt{5}}{7} & \frac{59}{14} \end{pmatrix} + \frac{1}{N} \sqrt{\frac{6}{7}} \begin{pmatrix} 0 & 0 & -4 & -\sqrt{5} \\ 0 & 0 & \sqrt{5} & -\frac{15}{2} \\ -4 & \sqrt{5} & 0 & 0 \\ -\sqrt{5} & -\frac{15}{2} & 0 & 0 \end{pmatrix} \right]$$

$$+ \frac{1}{N^2} \left(\begin{array}{cccc} -4 & 2\sqrt{5} & 0 & 0 \\ 2\sqrt{5} & -10 & 0 & 0 \\ 0 & 0 & \frac{431}{49} & -\frac{215\sqrt{5}}{98} \\ 0 & 0 & -\frac{215\sqrt{5}}{98} & \frac{255}{49} \end{array} \right) \Big] + O\left(\frac{1}{N^3}\right) \quad (\text{B.7})$$

from which it is clear that we have a zero eigenvalue at leading order in N .

Again by using *Mathematica* we have computed its eigenvalues up to order $\frac{1}{N^2}$

$$\begin{aligned} \gamma_1 &= -10\frac{\lambda}{N^2} \quad , \quad \gamma_2 = \left(6 + \frac{20}{N^2}\right) \lambda \\ \gamma_3 &= \left[\frac{13 + \sqrt{41}}{4} - \frac{5}{41N^2}(41 + 19\sqrt{41}) \right] \lambda \\ \gamma_4 &= \left[\frac{13 - \sqrt{41}}{4} - \frac{5}{41N^2}(41 - 19\sqrt{41}) \right] \lambda \end{aligned} \quad (\text{B.8})$$

and the corresponding eigenvectors

$$\begin{aligned} \Sigma_1 &= \left(1 - \frac{21}{16N^2}\right) \mathcal{O}_1 + \frac{\sqrt{42}}{4N} \mathcal{O}_3 - \frac{7\sqrt{5}}{12N^2} \mathcal{O}_2 + O\left(\frac{1}{N^3}\right) \\ \Sigma_2 &= \left(1 - \frac{45}{4N^2}\right) \mathcal{O}_2 - \frac{1}{N} \sqrt{\frac{3}{14}} \left(\sqrt{5}\mathcal{O}_3 + 10\mathcal{O}_4\right) + \frac{4\sqrt{5}}{3N^2} \mathcal{O}_1 + O\left(\frac{1}{N^3}\right) \\ \Sigma_3 &= \sqrt{\frac{287 - 27\sqrt{41}}{574}} \left[\mathcal{O}_3 + \frac{27 + 7\sqrt{41}}{16\sqrt{5}} \mathcal{O}_4 - \frac{\sqrt{42}}{4N} \left(\mathcal{O}_1 - \sqrt{5} \frac{\sqrt{41} + 5}{4} \mathcal{O}_2 \right) \right] + O\left(\frac{1}{N^2}\right) \\ \Sigma_4 &= \sqrt{\frac{287 + 27\sqrt{41}}{574}} \left[\mathcal{O}_3 + \frac{27 - 7\sqrt{41}}{16\sqrt{5}} \mathcal{O}_4 - \frac{\sqrt{42}}{4N} \left(\mathcal{O}_1 + \sqrt{5} \frac{\sqrt{41} - 5}{4} \mathcal{O}_2 \right) \right] + O\left(\frac{1}{N^2}\right) \end{aligned} \quad (\text{B.9})$$

These are one-loop quasiprimary operators with anomalous dimensions γ_i . They are uniquely defined, because the anomalous dimensions are all different (cf. the general discussion in Section 3.2). At this order these operators satisfy

$$\langle \Sigma_i(x) \Sigma_j(0) \rangle_{0+1} \sim \frac{1}{(4\pi^2)^4} \frac{1}{(x^2)^4} \delta_{ij} \left[1 - \gamma_i \ln(x^2 \mu^2) + O\left(\frac{1}{N^2}\right) \right] \quad (\text{B.10})$$

up to a finite renormalization constant which we have not considered.

Inverting eq. (B.9) we can express the operator \mathcal{O}_1 in terms of pure states

$$\mathcal{O}_1 = \Sigma_1 - \frac{1}{N} [\alpha_3 \Sigma_3 + \alpha_4 \Sigma_4] + \frac{1}{N^2} \left[-\frac{21}{16} \Sigma_1 + \frac{3\sqrt{5}}{4} \Sigma_2 \right] + O\left(\frac{1}{N^3}\right) \quad (\text{B.11})$$

where

$$\begin{aligned}\alpha_3 &= -\frac{27 - 7\sqrt{41}}{64} \sqrt{42 \left(\frac{1}{10} + \frac{27}{70\sqrt{41}} \right)} \\ \alpha_4 &= \frac{27 + 7\sqrt{41}}{64} \sqrt{42 \left(\frac{1}{10} - \frac{27}{70\sqrt{41}} \right)}\end{aligned}\tag{B.12}$$

Similarly, from (B.3) and (B.9) we can determine the (normalized) operator \mathcal{A}_2 , which we call \mathcal{K}^2 , in terms of pure states

$$\mathcal{K}^2 \equiv \frac{1}{\sqrt{18N^2}} \left(1 + \frac{5}{6N^2} \right) \mathcal{A}_2 = \Sigma_2 + \frac{1}{N} [\beta_3 \Sigma_3 + \beta_4 \Sigma_4] + \frac{1}{N^2} \left[\frac{5\sqrt{5}}{6} \Sigma_1 - \frac{162}{7} \Sigma_2 \right] + O\left(\frac{1}{N^3}\right),\tag{B.13}$$

where

$$\begin{aligned}\beta_3 &= \frac{287 + 85\sqrt{41}}{4592} \sqrt{861 + 81\sqrt{41}} \\ \beta_4 &= \frac{287 - 85\sqrt{41}}{4592} \sqrt{861 - 81\sqrt{41}}\end{aligned}\tag{B.14}$$

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