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A robust and coherent network statistic for detecting gravitational waves from inspiralling compact binaries in non-Gaussian noise

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Abstract
The robust statistic proposed by Creighton (Creighton J D E 1999 *Phys. Rev.* D **60** 021101) and Allen *et al* (Allen *et al* 2001 *Preprint* gr-gc/010500) for the detection of stationary non-Gaussian noise is briefly reviewed. We compute the robust statistic for generic weak gravitational-wave signals in the mixture-Gaussian noise model to an accuracy higher than in those analyses, and reinterpret its role. Specifically, we obtain the coherent statistic for detecting gravitational-wave signals from inspiralling compact binaries with an arbitrary network of earth-based interferometers. Finally, we show that excess computational costs incurred owing to non-Gaussianity is negligible compared to the cost of detection in Gaussian noise.

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1. Introduction

Realistically, detector noise is not Gaussian. Using a network helps by introducing vetoes on instrumental artefacts. Seeking a robust statistic for non-Gaussian noise is aimed at improving these vetoes, leading to higher detection efficiencies.

Use of a network of detectors, as opposed to a single detector, is useful for a variety of reasons. Networks access a larger sky volume and parameter space. They provide better estimates of parameter values. Most significantly, they can provide better detection confidence, especially in the presence of non-stationarity. On the flip side, a network search, be it of the coherent or coincident kind, is much more expensive than a single detector search [1]. For modelled sources, a reduction in this cost may be achieved by way of analytic maximization over source parameters. Indeed, exactly how such a cost reduction can be obtained in detecting Newtonian chirps in Gaussian noise was shown in [2]. One of the main aims of this study is to explore if a similar cost reduction is viable in searches of 2PN chirps in *non-Gaussian noise*.

In this study, we recompute more accurately, the robust statistic of Creighton [3] and Allen *et al* [4] for weak gravitational-wave (GW) signals from a generic source, and reinterpret its role. For this analysis, we model the detector noise as mixture-Gaussian and stationary. Extension to the case of a network of detectors with independent noises is made in a straightforward manner. We then specifically study the case of the 2PN chirp. We obtain the robust statistic for a coherent network search of such signals and show that a cost reduction akin to that discussed in [2] is possible through analytic maximization over a subset of parameters, which includes the polarization-ellipse angle, ψ, and the binary orbit's inclination angle ϵ. Finally, we briefly discuss the computational cost accrued in excess of a search in *Gaussian* noise and show it to be negligible.

2. Noise model

It is well known that the strain $x(t)$ in a detector is typically noisy. Assuming additive noise allows us to express this physical quantity as

$$x(t) = s(t) + n(t) \tag{1}$$

where $s(t)$ is a gravitational-wave signal and the noise, $n(t)$, is generally non-Gaussian. The search challenge here is of filtering signals embedded in non-Gaussian noise. Henceforth, we will often refer to the above quantities by their discrete time-series relatives, $\mathbf{x}$, $\mathbf{s}$ and $\mathbf{n}$, respectively. If the number of points in each times series is *N*, then the quantities in boldface live in an *N*-dimensional vector space. Their counterparts in the frequency domain are denoted by $\tilde{\mathbf{x}}$, $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{n}}$, respectively.

To make progress in defining a detection statistic, we shall assume a certain model for the detector noise in this paper. Following Allen *et al* [4], we take the noise in a given detector to be described by the general probability distribution function (PDF):

$$p(\tilde{\mathbf{n}}) = \prod_{k=1}^{(N-1)/2} \frac{2}{\pi P_k^2} \exp\left[-2g_k\left(\frac{|\tilde{n}_k|^2}{P_k}\right)\right] \tag{2}$$

where the covariance of different frequency components, $\tilde{n}_k$, of the noise is given by

$$\overline{\tilde{n}_k \tilde{n}_{k'}^*} = \tfrac{1}{2}\delta_{kk'} P_k. \tag{3}$$

We take the noise in a detector to have zero mean, i.e., $\bar{\mathbf{n}} = 0$. In the particular case where one has $g_k(y_k) = y_k$, the above PDF describes *Gaussian* noise with zero mean. We assume here that the noise is stationary.

3. Detection statistic

Consider the strain, $\mathbf{x}$, in a given detector. Let the probability that it contains a signal of amplitude A be denoted by $p(\mathbf{x}|A)$. Then the likelihood ratio can be defined as

$$\Lambda = \frac{p(\mathbf{x}|A)}{p(\mathbf{x}|0)}. \tag{4}$$

When $p(\tilde{\mathbf{n}})$ is chosen to be the PDF of Gaussian noise, the Neyman–Pearson criterion yields the decision statistic to be

$$\ln\Lambda = A \times 4\,\Re\left[\sum_{k=1}^{(N-1)/2} \frac{\tilde{s}_k^* \tilde{x}_k}{P_k}\right] + \text{const.} \tag{5}$$

This is the optimal statistic in such noise and for any strength, A, of the signal. For non-Gaussian noise, however, $\ln\Lambda$ is generally a non-trivial function of A, and is not necessarily the optimal statistic for all A.

Since in the first stages of the upcoming detectors, we expect the SNR to be small, we seek a statistic that is *locally optimal* for small A [5]. This is given by

$$L = \left.\frac{\mathrm{d}\ln\Lambda}{\mathrm{d}A}\right|_{A=0} = 4\sum_{k=1}^{(N-1)/2} \Re\left(\frac{\tilde{s}_k^*\tilde{x}_k}{P_k}\right) g_k'\left(\frac{|\tilde{x}_k|^2}{P_k}\right) \tag{6}$$

where the prime ($'$) on a function denotes its derivative with respect to its argument.

As an example of non-Gaussian noise, consider the mixture-Gaussian noise

$$p(\mathbf{n}) = \sum_{i=1}^{K} p_i(\mathbf{n}) = \sum_{i=1}^{K} \frac{w_i \exp\left(-\frac{1}{2}\mathbf{n}^\dagger\cdot\mathbf{\Phi}^i\cdot\mathbf{n}\right)}{(2\pi)^{N/2}\sqrt{\det\mathbf{R}^i}} \tag{7}$$

where $\sum_{i=1}^{K} w_i = 1$, and $w_i > 0$ for all i. $\mathbf{R}^i$ is the autocorrelation matrix of the ith component of the mixture-Gaussian distribution and $\mathbf{\Phi}^i := (\mathbf{R}^i)^{-1}$. It is straightforward to infer $p(\tilde{\mathbf{n}})$ from above. Using the derivative of the associated g_k in the expression for the locally optimal statistic (LOS) above, we find

$$L = \frac{4}{p(\mathbf{x}|0)}\sum_{i=1}^{K} p_i(\mathbf{x}|0)\left[\sum_{k=1}^{(N-1)/2} \Re\left(\frac{\tilde{s}_k^*\tilde{x}_k}{P_k^i}\right) g_k'\left(\frac{|\tilde{x}_k|^2}{P_k^i}\right)\right] \tag{8}$$

which is just the weighted sum of the LOS for each component of the mixture-Gaussian noise.

4. Special case of non-Gaussian noise

When the detector noise can be modelled as an ambient Gaussian noise interspersed with occasional large noise bursts (with a Gaussian distributed amplitude), then

$$\begin{aligned} p(\mathbf{n}) &= p^G(\mathbf{n}) + p^B(\mathbf{n}) \\ &= \frac{w\exp\left(-\frac{1}{2}\mathbf{n}^\dagger\cdot\mathbf{\Phi}^{\mathbf{G}^i}\cdot\mathbf{n}\right)}{(2\pi)^{N/2}\sqrt{\det\mathbf{R}^{\mathbf{G}}}} + \frac{(1-w)\exp\left(-\frac{1}{2}\mathbf{n}^\dagger\cdot\mathbf{\Phi}^{\mathbf{B}^i}\cdot\mathbf{n}\right)}{(2\pi)^{N/2}\sqrt{\det\mathbf{R}^{\mathbf{B}}}}. \end{aligned} \tag{9}$$

Using the $p(\tilde{\mathbf{n}})$ associated with the above distribution in expression (8) for the LOS, we obtain

$$L = \frac{4}{(1+\alpha)}\sum_{k=1}^{(N-1)/2} \Re\left(\frac{\tilde{s}_k^*\tilde{x}_k}{P_k'}\right) \tag{10}$$

where

$$\alpha = \frac{p^B(\mathbf{x}|0)}{p^G(\mathbf{x}|0)} = \frac{(1-w)}{w}\sqrt{\frac{\det\mathbf{R}^{\mathbf{G}}}{\det\mathbf{R}^{\mathbf{B}}}}\exp\left[\tfrac{1}{2}\mathbf{n}^\dagger\cdot(\mathbf{\Phi}^G-\mathbf{\Phi}^B)\cdot\mathbf{n}\right]. \tag{11}$$

The above LOS is very similar to that for Gaussian noise, except for a couple of differences. First, as noted in [4], it is now weighted by the prefactor $(1+\alpha)^{-1}$. Second, we find that the denominator inside the summand in equation (10) is not P_k, but P_k':[1]

$$\frac{1}{P_k'} \equiv \left[\frac{1}{P_k^G} + \frac{\alpha}{P_k^B}\right]. \tag{12}$$

[1] It is important to note that the contribution to the statistic (10) arising from the second term on the right-hand side of equation (12) does not necessarily become negligible under the approximation, $\mathbf{n}^\dagger\cdot\mathbf{\Phi}^{\mathbf{G}^i}\cdot\mathbf{n} \gg \mathbf{n}^\dagger\cdot\mathbf{\Phi}^{\mathbf{B}^i}\cdot\mathbf{n}$, of [4]. In other words, such a term should be present even when this approximation is valid.

Therefore, equation (10) is a small but important generalization of a similar expression derived in [4]: the interpretation of the prefactor $(1+\alpha)^{-1}$ was correctly given in [4] as a factor that 'vetoes' the contribution to the LOS arising from large noise bursts. The factor $P_k'^{-1}$ performs a similar vetoing, but at the level of each frequency band. A new interpretation is the following: if the presence of a signal coincides with that of a noise burst, then the factor $P_k'^{-1}$ can disallow incorrect vetoing based purely on the factor of $(1+\alpha)^{-1}$. This happens when the signal magnitude is such that the contribution of the summand turns out to be relatively large compared to $(1+\alpha)$.

Armed with the general expression for the LOS, (10), we now examine the specific case of the detection statistic for the 2PN binary inspiral signal.

5. The inspiral waveform

The GW strain in the Ith detector (with $\mathbf{o}_{(I)}$ denoting the Euler angles of its orientation relative to a fiducial frame or detector) due to an inspiral chirp is [2]

$$s^I(t) = h_{ij}(t)d^{Iij} = A\,\Re\left[Q_I^* S^I(t)\mathrm{e}^{\mathrm{i}\delta_c}\right] \tag{13}$$

where Q^I are (normalized) functionals of the antenna-pattern functions F^I:

$$Q^I \propto \left[\frac{1+\cos^2\epsilon}{2}\Re(F^I) + \mathrm{i}\cos\epsilon\,\mathrm{Im}(F^I)\right] \tag{14}$$

with $F^I = F\big(\psi, \theta, \phi; \mathbf{o}_{(I)}\big)$ (see [6] for a definition) and (θ, ϕ) being the source-direction angles. The time variation in the chirp is confined to the quantity $S^I(t)$, which we define via its Fourier transform, $\tilde{S}^I(f)$. Denoting f_s as a fiducial frequency, which can be chosen to be the lowest seismic cut-off frequency in a network, the stationary-phase approximation can be shown to yield

$$\tilde{S}^I(f) \propto \left(\frac{f}{f_s}\right)^{-7/6} \exp[-\mathrm{i}\Psi_{(I)}(f)]. \tag{15}$$

For the 2PN waveform, one has

$$\Psi_{(I)}\left(f; t_c, \theta, \phi, \vartheta^1, \vartheta^2\right) = 2\pi f\tau_{(I)}(\theta, \phi) + 2\pi f_s\varphi_\mu(f; \vartheta^1, \vartheta^2)\vartheta^\mu \tag{16}$$

where $\tau_{(I)}$ is the time delay, relative to a fiducial detector, in the arrival of the signal at detector I and

$$\vartheta^\mu = \left(t_c, \vartheta^1, \vartheta^2\right) \qquad \mu = 0, 1, 2 \tag{17}$$

with [7]

$$\vartheta^1 = \tfrac{5}{28}(\pi M f_s)^{-5/3}\eta^{-1} \qquad \vartheta^2 = \frac{\pi}{4}(\pi M f_s)^{-2/3}\eta^{-1}. \tag{18}$$

M in (18) is the total mass and η is the ratio of reduced mass to the total mass. The parameter coefficients are

$$\varphi_\mu = \left(\frac{f}{f_s}, \frac{3}{5}\left(\frac{f}{f_s}\right)^{-5/3}, \varphi_2(f; \vartheta^1, \vartheta^2)\right) \tag{19}$$

where the explicit form of φ_2 will not be required here; the reader interested in this form is referred to [7]. The 2PN chirp, therefore, is completely defined by nine parameters: $(A, \delta_c, \psi, \epsilon, \theta, \phi, t_c, \vartheta^1, \vartheta^2)$, where A depends on the luminosity distance of the source.

6. Network statistic

For uncorrelated noise[2], it is easy to see that the LOS for a network is the sum of the LOS for each detector. In the specific case of the non-Gaussian noise distribution given in equation (9), this implies that in a network of M detectors

$$L_{\mathrm{Net}} = \sum_{I=1}^{M} \frac{4}{(1+\alpha_I)} \sum_{k=1}^{(N-1)/2} \Re\left(\frac{\tilde{s}_k^{I*}\tilde{x}_k^I}{P_k'^I}\right). \tag{20}$$

What is not apparent, however, is the fact that the same analytic maximization over parameters that is possible in Gaussian noise [2] also goes through for non-Gaussian noise. We now illustrate in the specific case of the non-Gaussian noise distribution in equation (9) that this is indeed true. For the 2PN waveform, we have

$$L_{\mathrm{Net}} = \sum_{I=1}^{M} \frac{\left\langle \Re\left[\mathrm{e}^{\mathrm{i}\delta_c} Q_I^* S^I\right], x^I\right\rangle_{(I)}}{(1+\alpha_I)} = \sum_{I=1}^{M} \Re\left[\mathrm{e}^{\mathrm{i}\delta_c} Q_I^* C^I\right] \tag{21}$$

where

$$C^I := \frac{\langle S^I, x^I\rangle_{(I)}}{(1+\alpha_I)} = \frac{4}{(1+\alpha_I)} \sum_{k=1}^{(N-1)/2} \Re\left(\frac{\tilde{S}_k^{I*}\tilde{x}_k^I}{P_k'^I}\right). \tag{22}$$

Maximizing L_{Net} with respect to δ_c gives

$$L_{\mathrm{Net}}\Big|_{\delta_c} = \left|\sum_{I=1}^{M} Q_I^* C^I\right| =: |\mathbf{Q}\cdot\mathbf{C}|. \tag{23}$$

The above statistic has essentially the same form as that in [2] (which addressed the detection of Newtonian chirps in Gaussian noise), with the only exception being that the C^I are now defined differently. The difference arising from the extension to 2PN waveforms is confined to S^I, which appears solely in C^I and not in Q^I. Also, the changes owing to non-Gaussianity appear only in C^I through α_I and $P_k'^I$. The parameters ψ and ϵ, on the other hand, appear in $\mathbf{Q}$ alone. Therefore, the maximization of the above statistic over $\{\psi, \epsilon\}$ can be performed analytically in precisely the same way as was first shown in [2]. Consequently, the statistic resulting from such a maximization is

$$L_{\mathrm{Net}}\Big|_{\delta_c,\psi,\epsilon} = \|\mathbf{C}_{\mathcal{H}}\| \tag{24}$$

where $\mathbf{C}_{\mathcal{H}}$ is the projection of the network cross-correlation vector $\mathbf{C}$ on the helicity plane $\mathcal{H}$, which is defined in terms of the source-direction angles $\{\theta, \phi\}$ and the known orientation angles, $\mathbf{o}^I$, of each detector in a manner identical to that given in [2]. Note that the above statistic can be applied to searches of signals from deterministic sources other than chirps as well, as long as such a source can be modelled (i.e., S^I of such a signal can be obtained) and the signal sought is transient (namely, the beam-pattern functions change negligibly while the signal dwells in the detector bandwidths).

7. Computational costs

Excess cost owing to non-Gaussianity is additive in nature: for detector I, it is determined by the cost of computing the value of $\alpha_{(I)}$ and, therefore, that of

$$\|x^I\|^2 = \langle x^I, x^I\rangle_{(I)}. \tag{25}$$

[2] For treatment of correlated detector noises, see, e.g., [8].

This is just the cost of N complex multiplications. Thus, for M such data trains, it is $4NM$. As an example, the computational cost of a LIGO-I search for 2PN waveforms in Gaussian noise is about 10^{11} flops for $m_{\min} = 0.2M_\odot$ and $N = 10^6$.

This cost is much higher compared to the additional cost in non-Gaussian noise, which for one detector is

$$4N \simeq 4 \times 10^6. \tag{26}$$

Computational costs for coherent and coincident searches in Gaussian noise with a network of detectors is discussed in greater detail in [1].

In conclusion, our robust network statistic for inspiral search in non-Gaussian noise is a simple generalization of the statistic found for Gaussian noise. The excess computational cost necessary to make a search robust is much less than costs for Gaussian searches.

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