

Conformal geodesics on vacuum space-times

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Abstract

We discuss properties of conformal geodesics on general, vacuum, and warped product space-times and derive a system of conformal deviation equations. The results are used to show how to construct on the Schwarzschild-Kruskal space-time global conformal Gauss coordinates which extends smoothly and without degeneracy to future and past null infinity.

1 Introduction

Geometrically defined coordinate systems are convenient in the local analysis of fields but they often degenerate on larger domains. It is well known, for instance, that Gauss coordinates are of limited use because the underlying geodesics tend to develop caustics. In [7] conformal Gauss coordinates have been introduced and successfully employed in local studies of conformally rescaled space-times. Here the time-like coordinate lines are generated by conformal geodesics, curves which are associated with conformal structures in a similar way as geodesics are associated with metrics. In this case the danger of a coordinate break-down is even more severe, because the conformal geodesics generating the system can not only develop envelopes or intersections, as occur in caustics of metric geodesics, but they may even become tangent to each other. However, it will be shown in this article that the structure responsible for this difficulty also has useful aspects.

Subsequent analyses of conformal Gauss coordinates revealed that conformal geodesics possess various remarkable properties. In [4] they have been studied systematically in the context of the conformal field equations. Though this at first introduces additional complications into the equations because it requires the use of Weyl connections, it leads to unexpected simplifications. While the Bianchi equation satisfied by the rescaled conformal Weyl tensor, which is given by $d^i{}_{jkl} = \Theta^{-1} C^i{}_{jkl}$ where Θ denotes the conformal factor, always implies hyperbolic reduced systems of partial differential equations, the generalized conformal field equations (in vacuum, possibly with a cosmological constant) admit in the new gauge hyperbolic reductions in which the frame coefficients, the connection coefficients and the components of the conformal Ricci tensor are governed by equations of the form

$$\partial_\tau e^\mu{}_k = \dots, \quad \partial_\tau \hat{\Gamma}_i{}^j{}_k = \dots, \quad \partial_\tau \hat{R}_{jk} = \dots,$$

where τ denotes the time variable in the given gauge and the right hand sides are given by algebraic functions of the unknowns $e^\mu{}_k$, $\hat{\Gamma}_i{}^j{}_k$, \hat{R}_{jk} , $d^i{}_{jkl}$ and known functions on the solution manifold.

Moreover, associated with a conformal geodesic is a function along that curve, the *conformal factor*, which is determined up to a constant factor that can be fixed on a given initial hypersurface. This function necessarily has zeros at points where the conformal geodesic crosses null infinity. It turns out that for given data on the initial hypersurface this function can be determined explicitly and acquires a form which allows us to prescribe to null infinity by a suitable choice of initial data a finite coordinate location.

In [4] these facts are used to provide a rather complete discussion of Anti-de Sitter-type solutions, including a characterization of these solutions in terms of initial and boundary data. In [5] these properties form a basic ingredient of a detailed investigation of the behaviour of asymptotically flat solutions near space-like and null infinity under certain assumptions on the Cauchy data for the solutions. In particular, the *cylinder at space-like infinity* introduced in [5] to remove the conformal singularity at space-like infinity is obtained as a

limit set of conformal geodesics. The results obtained there strongly suggest that even under the most stringent smoothness assumptions on the conformal data near space-like infinity the solutions will in general develop logarithmic singularities at null infinity. However, they also suggest that these singularities can be avoided if the data satisfy certain *regularity conditions* at space-like infinity.

The conversion of these conjectures into facts requires the proof of a certain existence result for the *regular finite initial value problem at space-like infinity* formulated in [5]. While the basic difficulty of this proof has nothing to do with the specific features of the underlying conformal Gauss system, only the proof will show that the latter is doing what we expect it to do: to extend near space-like infinity smoothly and without degeneracy to null infinity if null infinity admits a smooth differentiable structure at all.

An independent proof that conformal Gauss systems behave this way on general space-times is difficult, because it requires the information on the asymptotic structure of the solution which we first hope to obtain by completing the analysis of [5]. The question whether conformal Gauss systems can be used globally is even more difficult.

In the context of the hyperboloidal initial value problem with smooth initial data it can be shown that these coordinates remain good for a while and that they can in fact be globally defined for data sufficiently close to Minkowskian hyperboloidal data. However, nothing is known for data which deviate strongly from Minkowskian once. There remains the possibility to study the behaviour of conformal Gauss systems on specific solutions. It has been shown in [5] that there exist conformal Gauss systems on the Schwarzschild space-time which smoothly cover neighbourhoods of space-like infinity including parts of null infinity.

One may be tempted to think that this is only possible due to the weakness of the field in that domain (although it doesn't look weak in the conformal picture) while in regimes of strong curvature a break-down of the coordinates will occur. It turns out that this is not true. It is the main result of this article that there exist conformal Gauss coordinates on the Schwarzschild-Kruskal space-time which smoothly cover the whole space-time and which extend smoothly (in fact analytically) and without degeneracy through null infinity.

This raises hopes that conformal Gauss systems stay regular under much wider circumstances than expected so far and that we may be able to exploit the simplicity of the reduced equations and the fact that conformal Gauss systems provide by themselves the conformal compactification in time directions under quite general assumptions. How robust these systems really are, whether they will stay regular under non-linear perturbations of the Schwarzschild-Kruskal space, remains to be seen. Though a general analytical investigation of these equations appears difficult at present, important information about them may be obtained by numerical calculations. Conversely, there are good reasons to believe that it should be possible to give strong analytical support to numerical work based on the use of conformal Gauss systems because the latter are amenable to a geometric analysis.

We begin the article by discussing properties of conformal geodesics on general space-times and then specialize to vacuum space-times, using also some of the results which have been obtained already in the articles referred to above. For later application we work out the conformal geodesic equations on warped product space-times. Various features of conformal Gauss systems are then illustrated by explicit examples on Minkowski space-time.

After specializing further to warped product vacuum space-times a conformal Gauss system on the Schwarzschild-Kruskal space-time will be analysed. Expressions for the conformal geodesics determined by suitably chosen initial data are derived in terms of elliptic and theta functions. It is shown that the curves extend smoothly through null infinity and that the associated conformal factor defines a smooth conformal extension if the congruence is regular. We do not attempt to derive the explicit expression of the rescaled metric in terms of the new coordinates, because this cannot be done anyway in more general situations. Instead, the regularity of the coordinate system and the conformal extension is established by analysing a system of equations which is the analogue for conformal geodesics of the Jacobi equation for metric geodesics. While specific features of the Schwarzschild solution are used here, it is clear that similar techniques apply in more general cases. We end the article by indicating as a possible application the numerical calculation of entire asymptotically flat vacuum solutions.

2 Conformal geodesics on pseudo-Riemannian manifolds

Before we introduce conformal geodesics we recall a few concepts and formulae of conformal geometry. On a pseudo-Riemannian manifold (M, \tilde{g}) of dimension $n \geq 3$ we consider two operations preserving the conformal structure defined by \tilde{g} : (i) conformal rescalings of the metric

$$\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad (1)$$

with smooth conformal factor $\Omega > 0$, (ii) transitions $\nabla \rightarrow \hat{\nabla}$ of the Levi-Civita connection ∇ of a metric g in the conformal class of \tilde{g} into (torsion free) *Weyl connections* $\hat{\nabla}$ with respect to \tilde{g} . In terms of the Christoffel symbols Γ and the connection coefficients $\hat{\Gamma}$, defined by $\nabla_{\partial_\mu} \partial_\nu = \Gamma_{\mu}{}^{\rho}{}_{\nu} \partial_\rho$ and $\hat{\nabla}_{\partial_\mu} \partial_\nu = \hat{\Gamma}_{\mu}{}^{\rho}{}_{\nu} \partial_\rho$ respectively, the transition is described by

$$\Gamma_{\mu}{}^{\rho}{}_{\nu} \rightarrow \hat{\Gamma}_{\mu}{}^{\rho}{}_{\nu} = \Gamma_{\mu}{}^{\rho}{}_{\nu} + S(f)_{\mu}{}^{\rho}{}_{\nu}, \quad (2)$$

$$S(f)_{\mu}{}^{\rho}{}_{\nu} \equiv \delta^{\rho}{}_{\mu} f_{\nu} + \delta^{\rho}{}_{\nu} f_{\mu} + g_{\mu\nu} g^{\rho\lambda} f_{\lambda},$$

with a smooth 1-form f . For given f the difference tensor $S(f)$ depends only on the conformal structure of g . The transformation of the Levi-Civita connection

$\tilde{\Gamma}$ of \tilde{g} under 1 is a special case of (2) in which the 1-form is exact

$$\tilde{\Gamma}_{\mu}{}^{\rho}{}_{\nu} \rightarrow \Gamma_{\mu}{}^{\rho}{}_{\nu} = \tilde{\Gamma}_{\mu}{}^{\rho}{}_{\nu} + S(f)_{\mu}{}^{\rho}{}_{\nu} \quad \text{with} \quad f = \Omega^{-1} d\Omega. \quad (3)$$

Conversely, if the 1-form f is closed (2) arises locally from a rescaling of g with a suitable conformal factor. By (2) we have

$$\hat{\nabla}_{\rho} g_{\mu\nu} = -2 f_{\rho} g_{\mu\nu}, \quad (4)$$

It follows from this that $\hat{\nabla}$ preserves the conformal structure of \tilde{g} in the sense that with a function $\theta > 0$ satisfying on a given curve $x(\tau)$ in M

$$\hat{\nabla}_{\dot{x}} \theta = \theta \langle f, \dot{x} \rangle, \quad (5)$$

the metric $\theta^2 g_{\mu\nu}$ is parallelly transported along $x(\tau)$ with respect to $\hat{\nabla}$.

The curvature tensors of the connections $\hat{\nabla}$ and ∇ , defined by $(\hat{\nabla}_{\lambda} \hat{\nabla}_{\rho} - \hat{\nabla}_{\rho} \hat{\nabla}_{\lambda}) X^{\mu} = \hat{R}^{\mu}{}_{\nu\lambda\rho} X^{\nu}$ etc., are related by

$$\hat{R}^{\mu}{}_{\nu\lambda\rho} - R^{\mu}{}_{\nu\lambda\rho} = 2 \{ \nabla_{[\lambda} S_{\rho]}{}^{\mu}{}_{\nu} + S_{\delta}{}^{\mu}{}_{[\lambda} S_{\rho]}{}^{\delta}{}_{\nu} \} \quad (6)$$

where indices are raised or lowered with respect to g . The tensor

$$\hat{L}_{\mu\nu} = \frac{1}{n-2} \{ \hat{R}_{(\mu\nu)} - \frac{n-2}{n} \hat{R}_{[\mu\nu]} - \frac{1}{2(n-1)} g_{\mu\nu} \hat{R} \},$$

which occurs in the decomposition $\hat{R}^{\mu}{}_{\nu\lambda\rho} = 2 \{ g^{\mu}{}_{[\lambda} \hat{L}_{\rho]\nu} - g^{\mu}{}_{\nu} \hat{L}_{[\lambda\rho]} - g_{\nu[\lambda} \hat{L}_{\rho]}{}^{\mu} \} + C^{\mu}{}_{\nu\lambda\rho}$ of the curvature tensor of $\hat{\nabla}$ into its trace parts and the trace-free, conformally invariant conformal Weyl tensor $C^{\mu}{}_{\nu\lambda\rho}$, is related to the tensor

$$L_{\mu\nu} = \frac{1}{n-2} \{ R_{\mu\nu} - \frac{1}{2(n-1)} R g_{\mu\nu} \},$$

by the equation

$$\nabla_{\nu} f_{\mu} - f_{\mu} f_{\nu} + g_{\mu\nu} \frac{1}{2} f_{\lambda} f^{\lambda} = L_{\mu\nu} - \hat{L}_{\mu\nu}. \quad (7)$$

2.1 Conformal geodesics

A conformal geodesic associated with the conformal structure defined by \tilde{g} is given by a curve $x(\tau)$ in M and a 1-form $b(\tau)$ along $x(\tau)$ such that the equations

$$(\tilde{\nabla}_{\dot{x}} \dot{x})^{\mu} + S(b)_{\lambda}{}^{\mu}{}_{\rho} \dot{x}^{\lambda} \dot{x}^{\rho} = 0, \quad (8)$$

$$(\tilde{\nabla}_{\dot{x}} b)_{\nu} - \frac{1}{2} b_{\mu} S(b)_{\lambda}{}^{\mu}{}_{\nu} \dot{x}^{\lambda} = \tilde{L}_{\lambda\nu} \dot{x}^{\lambda}, \quad (9)$$

hold on $x(\tau)$. For given initial data $x_* \in M$, $\dot{x}_* \in T_{x_*} M$, $b_* \in T_{x_*}^* M$ there exists a unique conformal geodesic $x(\tau)$, $b(\tau)$ near x_* satisfying for given $\tau_* \in \mathbb{R}$

$$x(\tau_*) = x_*, \quad \dot{x}(\tau_*) = \dot{x}_*, \quad b(\tau_*) = b_*. \quad (10)$$

From (8) it follows that the sign of $\tilde{g}(\dot{x}, \dot{x})$ is preserved along $x(\tau)$, since we have

$$\tilde{\nabla}_{\dot{x}}(\tilde{g}(\dot{x}, \dot{x})) = -2 \langle b, \dot{x} \rangle \tilde{g}(\dot{x}, \dot{x}). \quad (11)$$

In particular, if $\tilde{g}(\dot{x}, \dot{x}) = 0$ holds at one point it holds everywhere on $x(\tau)$ and (8) implies that $x(\tau)$ can be reparametrized to coincide with a null geodesic of \tilde{g} .

Let $x(\tau)$, $b(\tau)$ and $\bar{x}(\bar{\tau})$, $\bar{b}(\bar{\tau})$ be two solutions to (8), (9) with $\tilde{g}(\dot{x}, \dot{x}) \neq 0$. We derive now the conditions under which the curves $x(\tau)$, $\bar{x}(\bar{\tau})$ coincide locally as point sets such that there exists a local reparameterization $\tau = \tau(\bar{\tau})$ with $x(\tau(\bar{\tau})) = \bar{x}(\bar{\tau})$. The latter relation and (8) imply $\dot{x} \partial_{\bar{\tau}}^2 \tau + 2 \langle \bar{b} - b, \dot{x} \rangle \dot{x} (\partial_{\bar{\tau}} \tau)^2 - \tilde{g}(\dot{x}, \dot{x}) (\bar{b} - b) (\partial_{\bar{\tau}} \tau)^2 = 0$ which is equivalent to the equations

$$\bar{b} - b = \alpha \dot{x}^{\flat}, \quad \partial_{\bar{\tau}}^2 \tau + \alpha \tilde{g}(\dot{x}, \dot{x}) (\partial_{\bar{\tau}} \tau)^2 = 0, \quad (12)$$

with some function α . In the first of these equations the index of \dot{x} is lowered with the metric \tilde{g} . From the first equation and (9) follows

$$\dot{\alpha} = 2 \alpha \langle b, \dot{x} \rangle + \frac{1}{2} \alpha^2 \tilde{g}(\dot{x}, \dot{x}), \quad (13)$$

which together with (12) is equivalent to our requirement. Equations (11), (13) imply $\partial_{\tau}(\alpha \tilde{g}(\dot{x}, \dot{x})) = 1/2 (\alpha \tilde{g}(\dot{x}, \dot{x}))^2$ which has the solutions

$$\alpha \tilde{g}(\dot{x}, \dot{x}) = \frac{2 \alpha_* \tilde{g}(\dot{x}_*, \dot{x}_*)}{2 - \alpha_* \tilde{g}(\dot{x}_*, \dot{x}_*) \Delta \tau},$$

where $\Delta \tau = \tau - \tau_*$ and $\Delta \bar{\tau} = \bar{\tau} - \bar{\tau}_*$. From these solutions follows finally with (12)

$$\Delta \tau = \frac{4 e \Delta \bar{\tau}}{1 + 2 e \alpha_* \tilde{g}(\dot{x}_*, \dot{x}_*) \Delta \bar{\tau}}, \quad \bar{b} = b + \frac{1}{\tilde{g}(\dot{x}, \dot{x})} \frac{2 \alpha_* \tilde{g}(\dot{x}_*, \dot{x}_*)}{2 - \alpha_* \tilde{g}(\dot{x}_*, \dot{x}_*) \Delta \bar{\tau}} \dot{x}, \quad (14)$$

$$e, \alpha_*, \tau_*, \bar{\tau}_* \in \mathbb{R}, \quad e \neq 0.$$

Thus, the changes of the initial data (10) which locally preserve the point set spread out by the curve $x(\tau)$ are given by

$$\dot{x}_* \rightarrow 4 e \dot{x}_*, \quad b_* \rightarrow b_* + \alpha_* \dot{x}, \quad e, \alpha_* \in \mathbb{R}, \quad e \neq 0. \quad (15)$$

The 1-form remains unchanged and an affine parameter transformation results if $\alpha_* = 0$. With suitable choices of the free constants, however, any fractional linear transformations of the parameter can be obtained and it can be arranged that $\tau \rightarrow \infty$ at any prescribed value of $\bar{\tau}$.

This fact is related to an important difference between conformal geodesics and metric geodesics. It may happen that the parameter τ on a conformal geodesic takes any value in \mathbb{R} while the curve still acquires endpoints in M as $\tau \rightarrow \pm\infty$. If the transformation in (14) has a pole it may further happen that

the point sets spread out by $x(\tau)$, $\bar{x}(\bar{\tau})$ coincide only partly, namely at the points where the transformation is defined, but each of the curves extends into a region not entered by the other one. In fact, there may occur an arbitrary number of such overlaps (we will encounter and make use of this phenomenon below). In certain contexts it may be preferable to call the union of the corresponding point sets a conformal geodesic. However, it will be more convenient for us to preserve this name for a solution of the conformal geodesic equation with its distinguished parameter.

If the 1-form b is used to define a connection $\hat{\nabla}$ along $x(\tau)$ by requiring that its difference tensor with respect to $\tilde{\nabla}$ is given by

$$\hat{\nabla} - \tilde{\nabla} = S(b), \quad (16)$$

equation (8) can be written in the form $\hat{\nabla}_{\dot{x}}\dot{x} = 0$. Thus $x(\tau)$ is an autoparallel with respect to $\hat{\nabla}$. Equation (9), which determines the connection $\hat{\nabla}$ along that autoparallel, acquires some meaning by comparing it with (7). A congruence of conformal geodesics, covering smoothly and without caustics an open set U , defines a smooth 1-form field b and thus a connection $\tilde{\nabla}$ on U . If ∇ , f , L are replaced in (7) by $\tilde{\nabla}$, b , \tilde{L} respectively and transvect with \dot{x} , we find that equation (9) takes the form of a restriction on the connection $\hat{\nabla}$ in the direction of the congruence, given by $\hat{L}_{\mu\nu}\dot{x}^\mu = 0$. Here it turns out important that (7) does not contain the contraction of $\nabla_\nu f_\mu$.

Let f be an arbitrary 1-form and $\hat{\nabla} = \tilde{\nabla} + S(f)$ the associated Weyl connection. Observing (7) with ∇ , L replaced by $\tilde{\nabla}$, \tilde{L} , we find for any curve $x(\tau)$ in M and 1-form $b(\tau)$ along $x(\tau)$ the identities

$$(\tilde{\nabla}_{\dot{x}}\dot{x})^\mu + S(b)_{\lambda}{}^\mu{}_\rho \dot{x}^\lambda \dot{x}^\rho = (\hat{\nabla}_{\dot{x}}\dot{x})^\mu + S(b-f)_{\lambda}{}^\mu{}_\rho \dot{x}^\lambda \dot{x}^\rho, \quad (17)$$

$$(\tilde{\nabla}_{\dot{x}}b)_\nu - \frac{1}{2} b_\mu S(b)_{\lambda}{}^\mu{}_\nu \dot{x}^\lambda - \tilde{L}_{\lambda\nu} \dot{x}^\lambda \quad (18)$$

$$= (\hat{\nabla}_{\dot{x}}(b-f))_\nu - \frac{1}{2} (b-f)_\mu S(b-f)_{\lambda}{}^\mu{}_\nu \dot{x}^\lambda - \hat{L}_{\lambda\nu} \dot{x}^\lambda.$$

It follows that conformal geodesics are invariant under transitions to general Weyl connections and, in particular, under conformal rescalings of \tilde{g} in the following sense: if $x(\tau)$, $b(\tau)$ is a solution of these equations with respect to $\tilde{\nabla}$, then $x(\tau)$, $b(\tau) - f|_{x(\tau)}$ is a solution of the conformal geodesic equation with respect to $\hat{\nabla} = \tilde{\nabla} + S(f)$. Note that the parameter τ is an invariant of the conformal class of \tilde{g} . It is determined by the initial conditions (10) but it does not depend on the Weyl connection and the metric in the conformal class chosen to write the conformal geodesics equations.

Let e_k , $k = 0, 1, \dots, n-1$, be a frame field which is parallel along $x(\tau)$ for the connection $\hat{\nabla}$ of (16) and satisfies $\tilde{g}(e_i, e_k) = \Theta^{-2} \text{diag}(1, \dots, 1, -1, \dots, -1)$ at $x(\tau_*)$ with some $\Theta = \Theta_* > 0$. It follows then from (4), (5), that this relation is preserved along $x(\tau)$ with the function $\Theta = \Theta(\tau)$ satisfying

$$\tilde{\nabla}_{\dot{x}}\Theta = \Theta \langle b, \dot{x} \rangle, \quad \Theta(\tau_*) = \Theta_*. \quad (19)$$

Consider again a congruence of conformal geodesics as above and initial data Θ_* chosen such that the function Θ satisfying (19) is smooth and positive on U . Then the associated 1-form is given in terms of the Levi-Civita connection ∇ of the metric $g = \Theta^2 \tilde{g}$, for which the frame is orthonormal, by

$$h(\tau) = b(\tau) - \Theta^{-1} d\Theta|_{x(\tau)}, \quad (20)$$

along $x(\tau)$. Equations (19) and (11) imply $x(\tau)$

$$\langle h, \dot{x} \rangle = 0, \quad g(\dot{x}, \dot{x}) = \Theta^2 \tilde{g}(\dot{x}, \dot{x}) = \Theta_*^2 \tilde{g}(\dot{x}_*, \dot{x}_*). \quad (21)$$

We assume the congruence, the function Θ , and the frame e_k to be constructed by a suitable choice of the initial data such that

$$g(\dot{x}, \dot{x}) = 1, \quad e_0 = \dot{x}. \quad (22)$$

From (21) and $\nabla = \tilde{\nabla} + S(\Theta^{-1} d\Theta) = \hat{\nabla} - S(h)$ follows $\nabla_{\dot{x}} e_k = -\langle h, e_k \rangle \dot{x} + g(\dot{x}, e_k) h^\sharp$, where the index of h is raised with g . This implies that

$$\begin{aligned} F_{\dot{x}} e_k &\equiv \nabla_{\dot{x}} e_k - g(\dot{x}, e_k) \nabla_{\dot{x}} \dot{x} + g(\nabla_{\dot{x}} \dot{x}, e_k) \dot{x} \\ &= -\langle h, e_k \rangle \dot{x} + g(\dot{x}, e_k) h^\sharp - g(\dot{x}, e_k) h^\sharp + g(h^\sharp, e_k) \dot{x} = 0. \end{aligned}$$

Thus the frame e_k , which is parallelly propagated with respect to $\hat{\nabla}$, is in general not parallelly but always Fermi-propagated with respect to ∇ .

The following general observation, which follows by a direct calculation, describes the sense in which Fermi-transport is conformally invariant. Let $\theta > 0$ be some conformal factor and \tilde{g}, g metrics on M such that $g = \theta^2 \tilde{g}$. Denote by \tilde{F}, F the respective Fermi-transport. Let $x(\tau)$ be any curve in M such that $g(\dot{x}, \dot{x}) = 1$. For any vector field X along the curve we then have $\tilde{F}_{\theta \dot{x}}(\theta X) = \theta^2 F_{\dot{x}} X$, where $\theta \dot{x}$ is the tangent vector of the curve parametrized in terms of \tilde{g} -arc length.

Applying this to the situation considered above, we see that the \tilde{g} -orthonormal frame Θe_k is Fermi-transported along the conformal geodesics, if the latter are parametrized in terms of \tilde{g} -arc length.

Let $f : M \rightarrow M$ be a conformal diffeomorphism of \tilde{g} such that $f^* \tilde{g} = \Omega^2 \tilde{g}$ with some function Ω on M . Since conformal geodesics are invariants of the conformal structure, it is not surprising that f maps conformal geodesics into conformal geodesics. If K is a conformal Killing vector field its flow defines a 1-parameter family of local conformal diffeomorphism and thus maps the set of conformal geodesics into itself. Though we will occasionally make use of this fact, it will not be demonstrated it here and we refer to [10] for the formal argument.

2.2 The conformal deviation equations

Following the discussion of the Jacobi equation for metric geodesics, we derive now an analogous system of equations for conformal geodesics. Let $x(\tau, \lambda)$,

$b(\tau, \lambda)$ be a family of solutions to (8), (9) depending smoothly on a parameter λ . We denote the tangent vector field of the conformal geodesics, the *deviation vector field* of the congruence, and the *deviation 1-form* by

$$X = \partial_\tau x = \dot{x}, \quad Z = \partial_\lambda x, \quad B = \tilde{\nabla}_Z b,$$

respectively. Considering $x = x(\tau, \lambda)$ as a map from some open subset of \mathbb{R}^2 into \tilde{M} and denoting by Tx its tangent map, gives $Tx([\partial_\tau, \partial_\lambda]) = 0$, whence $[X, Y] = \tilde{\nabla}_X Z - \tilde{\nabla}_Z X = 0$. This, (8), and $(\tilde{\nabla}_X \tilde{\nabla}_Z - \tilde{\nabla}_Z \tilde{\nabla}_X) X - \tilde{\nabla}_{[X, Z]} Z = \tilde{R}(X, Z) X$ imply the *conformal Jacobi equation*

$$\tilde{\nabla}_X \tilde{\nabla}_X Z = \tilde{R}(X, Z) X - S(B; X, X) - 2S(b; X, \tilde{\nabla}_X Z), \quad (23)$$

where $(S(b; X, Y))^\mu = S(b)_{\rho \nu} X^\rho Y^\nu$ for given 1-form b and vector fields X, Y . The last two terms on the right hand side of (23) indicate why conformal geodesics are potentially more useful than metric geodesics for the construction of coordinate systems. Under suitable circumstances the acceleration induced by the 1-form b may counteract curvature induced tendencies of the curves to develop caustics.

Caustics of conformal geodesics can be more complicated than caustics of metric geodesics because for a given tangent vector there exists (essentially) a 3-parameter family of conformal geodesics with the same tangent vector. Moreover, to analyse them we need to complement the conformal Jacobi equation by a equation which governs the behaviour of B . Writing $(\tilde{\nabla}_X \tilde{\nabla}_Z - \tilde{\nabla}_Z \tilde{\nabla}_X) b - \tilde{\nabla}_{[X, Z]} b = -b \tilde{R}(X, Z)$, we get from (9) the *1-form deviation equation*

$$\tilde{\nabla}_X B = -b \tilde{R}(X, Z) + (\tilde{\nabla}_Z \tilde{L})(X, \cdot) + \tilde{L}(\tilde{\nabla}_X Z, \cdot) \quad (24)$$

$$+ \frac{1}{2} (B \cdot S(b; X, \cdot) + b \cdot S(B; X, \cdot) + b \cdot S(b; \tilde{\nabla}_X Z, \cdot)),$$

where $(B \cdot S(b; X, \cdot))_\nu = B_\mu S(b)_{\rho \nu} X^\rho$. We refer to the equations (23), (24) as to the system of *conformal deviation equations*. They form a linear system of ODE's for Z and B along the curves $x(\tau)$.

2.3 Conformal geodesic equations on warped products

For later applications we work out the simplifications of the conformal geodesic equations which are obtained in the case where \tilde{g} can be written as a warped product. Thus we assume that there exist coordinates x^μ for which the set $I = \{0, 1, \dots, n-1\}$ in which the index μ takes its values can be decomposed into two non-empty subsets I_1, I_2 with $I = I_1 \cup I_2, \emptyset = I_1 \cap I_2$, such that in terms of the coordinates $x^A, A \in I_1$, and $x^a, a \in I_2$, the metric takes the form

$$\tilde{g} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = h_{AB} dx^A dx^B + f^2 k_{cd} dx^c dx^d, \quad (25)$$

$$h_{AB} = h_{AB}(x^C), \quad k_{ab} = k_{ab}(x^c), \quad 1 \leq p \equiv k_{ab} k^{ab} \leq n-1, \quad f = f(x^A) > 0.$$

The Christoffel coefficients are then given by

$$\tilde{\Gamma}_B^A{}_C = \frac{1}{2} h^{AD} (h_{BD,C} + h_{CD,B} - h_{BC,D}), \quad \tilde{\Gamma}_b^A{}_c = -h^{AD} f_{f,D} k_{bc},$$

$$\tilde{\Gamma}_B^A{}_c = 0, \quad \tilde{\Gamma}_B^a{}_C = 0,$$

$$\tilde{\Gamma}_B^a{}_c = \tilde{\Gamma}_c^a{}_B = \frac{1}{f} f_{,B} k^a{}_c, \quad \tilde{\Gamma}_b^a{}_c = \frac{1}{2} k^{ad} (k_{bd,c} + k_{cd,b} - k_{bc,d}).$$

Those which cannot be derived by symmetry considerations from the coefficients above, vanish. The curvature tensor has components

$$R^A{}_{BCD}[\tilde{g}] = R^A{}_{BCD}[h], \quad R^a{}_{BCD}[\tilde{g}] = -k^a{}_c \frac{1}{f} D_B D_D f, \quad (26)$$

$$R^a{}_{bcd}[\tilde{g}] = R^a{}_{bcd}[k] - 2 h^{CD} f_{,C} f_{,D} k^a{}_{[c} k_{d]b}, \quad (27)$$

where D denotes the h -Levi-Civita connection with connection coefficients $\tilde{\Gamma}_A^B{}_C$. Components which cannot be deduced by symmetry considerations from those above, vanish. In particular

$$R^A{}_{BCd}[\tilde{g}] = 0, \quad R^A{}_{bcd}[\tilde{g}] = 0, \quad R^a{}_{BCD}[\tilde{g}] = 0. \quad (28)$$

This implies $L_{Ac}[\tilde{g}] = 0$,

$$L_{cd}[\tilde{g}] = \frac{1}{2} R_{cd}[k] -$$

$$\frac{1}{12} k_{cd} R[k] - \left\{ \frac{1}{12} R[h] + \frac{3-p}{6} f D_A D^A f + \frac{(p-1)(6-p)}{12} D_A f D^A f \right\} k_{cd},$$

$$\begin{aligned} L_{AB}[\tilde{g}] &= \frac{1}{2} R_{AB}[h] - \frac{1}{12} h_{AB} R[h] - \frac{p}{2f} \{ D_A D_B f - \frac{1}{3} h_{AB} D_C D^C f \} \\ &\quad - \frac{1}{12 f^2} \{ R[k] - p(p-1) D_C f D^C f \} h_{AB}. \end{aligned}$$

The conformal geodesic equations take the form

$$\begin{aligned} \ddot{x}^A + \tilde{\Gamma}_B^A{}_C \dot{x}^B \dot{x}^C + \tilde{\Gamma}_b^A{}_c \dot{x}^b \dot{x}^c = \\ -2(b_C \dot{x}^C + b_c \dot{x}^c) \dot{x}^A + (h_{BC} \dot{x}^B \dot{x}^C + f^2 k_{bc} \dot{x}^b \dot{x}^c) h^{AD} b_D, \end{aligned} \quad (29)$$

$$\ddot{x}^a + \tilde{\Gamma}_b{}^a{}_c \dot{x}^b \dot{x}^c + \tilde{\Gamma}_B{}^a{}_c \dot{x}^B \dot{x}^c + \tilde{\Gamma}_b{}^a{}_C \dot{x}^b \dot{x}^C = \quad (30)$$

$$-2(b_C \dot{x}^C + b_c \dot{x}^c) \dot{x}^a + (h_{BC} \dot{x}^B \dot{x}^C + f^2 k_{bc} \dot{x}^b \dot{x}^c) \frac{1}{f^2} k^{ac} b_c,$$

$$\dot{b}_A - \tilde{\Gamma}_C{}^D{}_A \dot{x}^C b_D - \tilde{\Gamma}_c{}^d{}_A \dot{x}^c b_d = \quad (31)$$

$$(b_C \dot{x}^C + b_c \dot{x}^c) b_A - \frac{1}{2} (h^{BC} b_B b_C + \frac{1}{f^2} k^{bc} b_b b_c) h_{AD} \dot{x}^D + \tilde{L}_{AB}[\tilde{g}] \dot{x}^B,$$

$$\dot{b}_a - \tilde{\Gamma}_c{}^D{}_a \dot{x}^c b_D - \tilde{\Gamma}_C{}^d{}_a \dot{x}^C b_d - \tilde{\Gamma}_c{}^d{}_a \dot{x}^c b_d = \quad (32)$$

$$(b_C \dot{x}^C + b_c \dot{x}^c) b_a - \frac{1}{2} (h^{BC} b_B b_C + \frac{1}{f^2} k^{bc} b_b b_c) f^2 k_{ac} \dot{x}^c + \tilde{L}_{ab}[\tilde{g}] \dot{x}^b.$$

3 Conformal geodesics on vacuum fields

Though many of the subsequent discussions apply to more general situations, we shall assume from now on that $\dim M = 4$, $\text{sign}(\tilde{g}) = (1, -1, -1, -1)$ and that \tilde{g} satisfies Einstein's vacuum field equations $\tilde{R}_{\mu\nu} = 0$. With the resulting simplification and the notation introduced above the conformal geodesic equations and the equation for the frame e_k now take the form

$$\tilde{\nabla}_{\dot{x}} \dot{x} + 2 \langle b, \dot{x} \rangle \dot{x} - \tilde{g}(\dot{x}, \dot{x}) b^\sharp = 0, \quad (33)$$

$$\tilde{\nabla}_{\dot{x}} b - \langle b, \dot{x} \rangle b + \frac{1}{2} \tilde{g}^\sharp(b, b) \dot{x}^\flat = 0, \quad (34)$$

$$\tilde{\nabla}_{\dot{x}} e_k + \langle b, \dot{x} \rangle e_k + \langle b, e_k \rangle \dot{x} - \tilde{g}(\dot{x}, e_k) b^\sharp = 0. \quad (35)$$

Since these equations admit solutions with vanishing 1-form b , metric geodesics form on vacuum space-times a subclass of conformal geodesics.

3.1 Conformal geodesics on Minkowski space

Denote by x^μ coordinates on Minkowski space (M, η) in which the metric takes the form $\eta = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ and let $\epsilon_k = \delta^\mu{}_k \partial_\mu$ be the orthonormal standard frame. There are various possibilities to determine the solutions to (33), (34), (35). The framework of the normal conformal Cartan connection (cf. [4]), in which conformal geodesics are defined as ‘‘horizontal’’ curves on a certain bundle allows a purely algebraic calculation of these curves, because the bundle admits in the case of the conformally compactified

Minkowski space a purely group theoretical description. Since the curves on the bundle space are known explicitly there only remains the task to calculate their projection onto the base space. We just give the result of the calculation.

Besides the initial data (10) we prescribe the value $\Theta_* > 0$ and a Lorentz transformation A_* which determines the initial value $e_{k*} = \Theta_*^{-1} A_*^j{}_k \epsilon_j$ of the frame e_k . The corresponding solution is given by

$$\Theta(\tau) = \Theta_* \left(1 + \Delta\tau \langle b_*, \dot{x}_* \rangle + \frac{1}{4} (\Delta\tau)^2 \eta(\dot{x}_*, \dot{x}_*) \eta^\sharp(b_*, b_*) \right), \quad (36)$$

$$x^\mu(\tau) = x_*^\mu + \frac{\Theta_*}{\Theta(\tau)} \left(\Delta\tau \dot{x}_*^\mu + \frac{1}{2} (\Delta\tau)^2 \eta(\dot{x}_*, \dot{x}_*) b_*^\mu \right), \quad (37)$$

$$b_\nu(\tau) = (1 + \Delta\tau \langle b_*, \dot{x}_* \rangle) b_{*\nu} - \frac{1}{2} \Delta\tau \eta^\sharp(b_*, b_*) \dot{x}_{*\nu}, \quad (38)$$

$$e_k = \frac{1}{\Theta(\tau)} A^j{}_k(\tau) \epsilon_j, \quad (39)$$

with a Lorentz transformation $A(\tau) = (A^j{}_k(\tau))$ given by

$$A(\tau) = A_* + \Delta\tau b_*^\sharp \dot{x}_*^b - \frac{\Theta_*}{\Theta(\tau)} \left(\Delta\tau \dot{x}_* + \frac{1}{2} (\Delta\tau)^2 \eta(\dot{x}_*, \dot{x}_*) b_*^\sharp \right) (b_* \cdot A_* + \frac{1}{2} \Delta\tau \eta^\sharp(b_*, b_*) \dot{x}_*^b).$$

Indices are moved in this subsection with the metric η .

Conformal geodesics which satisfy $\eta(\dot{x}, \dot{x}) = 0$ at one point coincide (as point sets) with null geodesics, those for which $b_* = 0$ are Minkowski geodesics. If \dot{x}_* is space- or time-like we can assume, possibly after a reparametrization, that $\langle b_*, \dot{x}_* \rangle = 0$. It follows that the remaining conformal geodesics fall into one of the following classes. If \dot{x}_* and b_*^\sharp generate a space-like 2-surface in the tangent space of x_* , the corresponding conformal geodesics is a metric circle in the plane tangent to that 2-surface. If \dot{x}_* and b_*^\sharp generate a time-like 2-surface and \dot{x}_* is space-like, the corresponding conformal geodesics is a space-like metric hyperbola in the plane tangent to that 2-surface. If \dot{x}_* is space-like and b_* is null, the conformal geodesics is a space-like curve in a null planed.

We shall mainly be interested in the last case in which \dot{x}_* and b_*^\sharp generate a time-like 2-surface and \dot{x}_* is time-like. An example of such a conformal geodesic is given by the curve

$$x(\tau) = \left(\frac{\tau}{1 - \frac{a^2 \tau^2}{4}}, \frac{1}{a} + \frac{\frac{a \tau^2}{2}}{1 - \frac{a^2 \tau^2}{4}}, 0, 0 \right), \quad |\tau| < \frac{2}{|a|},$$

which is a time-like hyperbola satisfying $\eta_{\mu\nu} x^\mu(\tau) x^\nu(\tau) = -a^{-2}$.

We shall use now time-like conformal geodesics to construct coordinate systems on Minkowski space where the parameter τ on the curves defines a time coordinate while the other coordinates are obtained by dragging along spatial coordinates on \tilde{S} . Most important for us is the fact that the construction provides a conformal rescaling and extension of Minkowski space to which the coordinates are adapted in a natural way.

We denote by r_* the restriction of the standard radial coordinate r on Minkowski space to the hypersurface $\tilde{S} = \{x^0 = 0\}$. In spherical coordinates θ, ϕ the inner metric induced on \tilde{S} then takes the form $\tilde{h} = -(dr_*^2 + r_*^2 d\sigma^2)$ with $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$ the standard line element on the 2-sphere. If we write $r_* = \tan(\frac{\chi}{2})$ with $0 \leq \chi < \pi$, we find that the conformal factor $\Theta_* = 2(1 + r_*^2)^{-1} = 1 + \cos\chi$ allows us to realize the conformal embedding of \tilde{S} into the 3-sphere $S = \tilde{S} \cup \{i\}$ with line element $\Theta_*^2 \tilde{h} = -(d\chi^2 + \sin^2\chi d\sigma^2)$, where i denotes the point $\{\chi = \pi\}$ at space-like infinity.

To define a congruence of conformal geodesics and a conformal factor, we choose initial data on \tilde{S} as follows. At $x_* = (0, r_* u) \in \tilde{S}$, $u \in \mathbb{R}^3$ with $|u| = 1$, we set $b_*^c = a u^c \epsilon_c$ (sum over $c = 1, 2, 3$) and $\dot{x}_* = \Theta_*^{-1} \epsilon_0$ such that $\Theta_*^2 \eta(\dot{x}, \dot{x}) = 1$. The function a on \tilde{S} is determined by the following consideration. With the data above and $\tau_* = 0$ we get $\Theta(\tau) = \Theta_*(1 - \frac{1}{4}\tau^2 a^2 \Theta_*^{-2})$ from (36). To obtain, if possible, a 1-form b which is exact, we require $\Theta^{-1} d\Theta = b$ at x_* which gives $a = 2r_*(1 + r_*^2)^{-1}$. Our data are then spherically symmetric and we can express the conformal factor (36) and the curves (37) in terms of the time coordinate $t = x^0$ and the radial coordinate r to obtain

$$\Theta = \Theta_* \left(1 - \frac{1}{4}\tau^2 r_*^2\right), \quad t = \frac{2(1 + r_*^2)\tau}{4 - \tau^2 r_*^2}, \quad r = \frac{r_*^2(4 + \tau^2)}{4 - \tau^2 r_*^2}. \quad (40)$$

To relate these expressions to known facts, we use $r_* = \tan \frac{\chi}{2}$ and set $\frac{\tau}{2} = \tan \frac{s}{2}$ to obtain

$$\Theta = \Omega \omega, \quad \text{with} \quad \Omega = \cos s + \cos \chi, \quad \omega = 1 + \left(\frac{\tau}{2}\right)^2 = \frac{1}{\cos^2 \frac{s}{2}}, \quad (41)$$

$$t = \frac{\sin s}{\cos \chi + \cos s}, \quad r = \frac{\sin \chi}{\cos \chi + \cos s}. \quad (42)$$

Reading the last equation as a coordinate transformation, we get

$$\Omega^2 \eta = \Omega^2 (dt^2 - dr^2 - r^2 d\sigma^2) = g_E \equiv ds^2 - d\chi^2 - \sin^2\chi d\sigma^2.$$

The set $M_E = \mathbb{R} \times S$ endowed with the metric g_E defines the Einstein cosmos. We see that (42) realizes the well known conformal embedding of Minkowski space into the Einstein cosmos, which maps the former onto the subset $|s \pm \chi| < \pi$, $0 \leq \chi < \pi$ of M_E . The boundary ∂M of this set in M_E supplies representations of future and past null infinity, future and past time-like infinity, and space-like infinity of Minkowski space, which are given by the subsets $\mathcal{J}^\pm = \{s \pm \chi = \pm\pi\}$, $i^\pm = \{s = \pm\pi, \chi = 0\}$, $i^0 = \{s = 0, \chi = \pi\}$ of ∂M respectively.

In terms of τ and Θ we get from (40) the metric

$$\Theta^2 \eta = \omega^2 \left(\frac{1}{\omega^2} d\tau^2 - d\chi^2 - \sin^2 \chi d\sigma^2 \right), \quad (43)$$

which extends smoothly to \mathcal{J}^\pm but does not extend to i^\pm where $\omega \rightarrow \infty$. This is not due to an unfortunate choice of initial data on \tilde{S} . The curve $\tau \rightarrow z(\tau) = (s = 2 \arctan \frac{\tau}{2}, \chi = 0)$ is a conformal geodesic on the Einstein cosmos which approaches i^\pm as $\tau \rightarrow \pm\infty$. The freedom to perform parameter transformations, characterized by (14), (15), does not allow us to find a parametrization for which curve extends simultaneously to i^- and i^+ . Thus the separation of i^- and i^+ realizes a conformal invariant.

It is possible, however, to find reparametrizations under which the rescaled metric extends smoothly either to i^+ or to i^- . The curve $\bar{\tau} \rightarrow \bar{z}(\bar{\tau}) = (s = s_* + 2 \arctan(\frac{\bar{\tau}}{2} - \tan \frac{s_*}{2}), \chi = 0)$ is again a conformal geodesic, because ∂_s is a Killing vector field for g_E . We choose s_* to be a constant with $0 < s_* < \pi$. Then $\bar{z}(\bar{\tau})$ is related to $z(\tau)$ by the parameter transformation $\tau = \bar{\tau} (1 + \tan^2 \frac{s_*}{2} - \frac{1}{2} \bar{\tau} \tan \frac{s_*}{2})^{-1}$. Comparing with (14) we are led to set $\tau_* = 0$, $\bar{\tau}_* = 0$, $4e = (1 + \tan^2 \frac{s_*}{2})^{-1}$, and $\alpha_* \eta(\dot{x}_*, \dot{x}_*) = -\tan \frac{s_*}{2}$ and to consider the initial data

$$\dot{\hat{x}}_* = \frac{1}{1 + \tan^2 \frac{s_*}{2}} \dot{x}_*, \quad \bar{\Theta}_* = (\eta(\dot{\hat{x}}_*, \dot{\hat{x}}_*))^{1/2} = \frac{1}{\cos^2 \frac{s_*}{2}} \Theta_*, \quad \bar{b}_* = b_* - \frac{\tan \frac{s_*}{2}}{\eta(\dot{x}_*, \dot{x}_*)} \dot{x}_*. \quad (44)$$

Observing these data and setting $\frac{\bar{\tau}}{2} = \tan \frac{s-s_*}{2} + \tan \frac{s_*}{2}$, we get from (37) again equations (42), while (36) yields now

$$\bar{\Theta} = \bar{\Theta}_* \left(1 - \bar{\tau} \frac{\tan \frac{s_*}{2}}{1 + \tan^2 \frac{s_*}{2}} + \frac{1}{4} \bar{\tau}^2 \frac{\tan^2 \frac{s_*}{2} - \tan^2 \frac{\chi}{2}}{(1 + \tan^2 \frac{s_*}{2})^2} \right) = \Omega \bar{\omega},$$

with Ω as in (41) and $\bar{\omega} = \cos^{-2} \frac{(s-s_*)}{2}$. It follows that the metric

$$\bar{\Theta}^2 \eta = \bar{\omega}^2 (ds^2 - d\chi^2 - \chi^2 d\sigma^2) = \bar{\omega}^2 \left(\frac{1}{\bar{\omega}^2} d\bar{\tau}^2 - d\chi^2 - \chi^2 d\sigma^2 \right),$$

extends regularly onto the domain $|s - s_*| < \pi$ containing i^+ .

If we insist on a conformal factor Θ which defines a conformal compactification of (\tilde{S}, \tilde{h}) and initial data ensuring $b = \Theta^{-1} d\Theta$ on \tilde{S} , invariance under $t \rightarrow -t$ implies the existence of a point on \tilde{S} where b vanishes. The conformal geodesic through such a point will then be a metric geodesic. In our first example such a point is given by $r_* = 0$ and it follows from (40) that the parameter τ takes values in \mathbb{R} and the conformal factor is constant on the curve $r = 0$. Thus, in order to achieve a compactification which extends smoothly to i^+ for a finite value of the parameter we have to choose non-time-symmetric initial data such as (44).

Notice that s_* could have been chosen to be a function on \tilde{S} . The possibility to change the parametrization while leaving the curves as point sets unchanged offers a large freedom to select slices of constant parameter value.

3.2 The conformal factor and the 1-form on general vacuum fields

In the following we shall derive some general, explicit information on the solutions to (33), (34), (35) (cf. the more general discussion in [4] on solutions to the vacuum field equations $\tilde{R}_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}$ with a cosmological constant λ).

From (33), (34) follows

$$\tilde{\nabla}_{\dot{x}} \langle b, \dot{x} \rangle = - \langle b, \dot{x} \rangle^2 + \frac{1}{2} \tilde{g}(\dot{x}, \dot{x}) \tilde{g}^\sharp(b, b), \quad (45)$$

$$\tilde{\nabla}_{\dot{x}} \tilde{g}^\sharp(b, b) = \langle b, \dot{x} \rangle \tilde{g}^\sharp(b, b), \quad (46)$$

where the index *sharp* on the symbol of a metric indicates here and in the following the contravariant version of that metric.

Assume again that the congruence of conformal geodesics, Θ , and the frame have been chosen such that (22), whence $\tilde{g}(\dot{x}, \dot{x}) = \Theta^{-2}$ holds along the congruence. Equation (11) then implies $\tilde{\nabla}_{\dot{x}} \Theta = \Theta \langle b, \dot{x} \rangle$. Taking a derivative and observing (45) yields $\tilde{\nabla}_{\dot{x}}^2 \Theta = 1/2 \tilde{g}^\sharp(b, b) \Theta^{-1}$. Taking another derivative and observing (46) gives finally

$$\tilde{\nabla}_{\dot{x}}^3 \Theta = 0. \quad (47)$$

In terms of the initial data (10) this yields with $\Theta_* = \sqrt{\tilde{g}(\dot{x}_*, \dot{x}_*)}$ the explicit expression

$$\begin{aligned} \Theta(\tau) &= \Theta_* + \Delta\tau \dot{\Theta}_* + \frac{1}{2} (\Delta\tau)^2 \ddot{\Theta}_* \\ &= \Theta_* \left(1 + \Delta\tau \langle b_*, \dot{x}_* \rangle + \frac{1}{4} (\Delta\tau)^2 \tilde{g}(\dot{x}_*, \dot{x}_*) \tilde{g}^\sharp(b_*, b_*) \right). \end{aligned} \quad (48)$$

Similar steps lead to $\tilde{\nabla}_{\dot{x}} (\tilde{g}(\dot{x}, \dot{x}) (\tilde{g}^\sharp(b, b))^2) = 0$, whence

$$\Theta^{-1} \tilde{g}^\sharp(b, b) = \Theta_*^{-1} \tilde{g}^\sharp(b_*, b_*) = 2 \ddot{\Theta}_*. \quad (49)$$

In particular, the sign of $\tilde{g}^\sharp(b, b)$ is preserved as long as $\Theta > 0$. From (34), (35) follows $\tilde{\nabla}_{\dot{x}} (\Theta \langle b, e_k \rangle) = 1/2 \Theta \tilde{g}^\sharp(b, b) \tilde{g}(\dot{x}, e_k)$. From this equation and (19), (22) the following explicit expression can be derived for the components $b_k \equiv \langle b, e_k \rangle$ of the 1-form b in the frame e_k

$$b_0 = \Theta^{-1} \dot{\Theta}, \quad b_a = \Theta^{-1} d_a \quad \text{with} \quad d_a = \langle b_*, \Theta_* e_{k*} \rangle. \quad (50)$$

This implies with (49) the relation

$$\dot{\Theta}^2 - \delta^{ab} d_a d_b = \Theta^2 \eta^{jk} b_j b_k = \Theta^2 g^\sharp(b, b) = \tilde{g}^\sharp(b, b) = \Theta \Theta_*^{-1} \tilde{g}^\sharp(b_*, b_*) = 2 \Theta \ddot{\Theta}_*. \quad (51)$$

While the expressions (48), (50) depend on invariants built from the initial data, they are universal in the sense that their form does not depend on the solution \tilde{g} .

Using the representation of the 1-form with respect to the connection of g , i.e. (20), we can write by (50) on the congruence $e_k(\Theta) = \Theta \langle b, e_k \rangle - \Theta \langle h, e_k \rangle$. If there is a point on a given curve of the congruence where Θ vanishes and Θb and h remain bounded on the curve, (50), (51) imply

$$\eta^{jk} e_j(\Theta) e_k(\Theta) \rightarrow 0 \quad \text{as} \quad \Theta \rightarrow 0. \quad (52)$$

3.3 The \tilde{g} -adapted form of the conformal geodesic equation

We write $b = \hat{b} + \zeta \dot{x}^b$ (where here and below indices of 1-forms and vectors are moved with \tilde{g}) with \hat{b} such that

$$\langle \hat{b}, \dot{x} \rangle = 0, \quad \text{whence} \quad \zeta = \frac{\langle b, \dot{x} \rangle}{\tilde{g}(\dot{x}, \dot{x})}, \quad g^\sharp(b, b) = \langle b, \dot{x} \rangle^2 + g^\sharp(\hat{b}, \hat{b}). \quad (53)$$

Equations (33), (34) are then equivalent to $\tilde{\nabla}_{\Theta \dot{x}} \Theta \dot{x} = \hat{b}^\sharp$, $\tilde{\nabla}_{\Theta \dot{x}} \hat{b} = -\tilde{g}^\sharp(\hat{b}, \hat{b}) \Theta \dot{x}^b$. We introduce the parameter transformation

$$\bar{\tau}(\tau) = \bar{\tau}_* + \int_{\tau_*}^{\tau} \frac{d\tau'}{\Theta(\tau')}, \quad (54)$$

with inverse $\tau = \tau(\bar{\tau})$ and set $\bar{x}(\bar{\tau}) = x(\tau(\bar{\tau}))$. Then $\bar{x}' \equiv \partial_{\bar{\tau}} \bar{x} = \Theta \dot{x}$ satisfies $\tilde{g}(\bar{x}', \bar{x}') = 1$ and we obtain the \tilde{g} -adapted conformal geodesic equations

$$\tilde{\nabla}_{\bar{x}'} \bar{x}' = \hat{b}^\sharp, \quad \tilde{\nabla}_{\bar{x}'} \hat{b} = \beta^2 \bar{x}'^b, \quad (55)$$

where, by (50),

$$\beta^2 \equiv -\tilde{g}^\sharp(\hat{b}, \hat{b}) = \delta^{ab} d_a d_b = \text{const}. \quad (56)$$

along the conformal geodesic. These equations bring out the important role of \hat{b} . If \hat{b} vanishes at a point it vanishes along $\bar{x}(\bar{\tau})$ and the curve is a \tilde{g} -geodesic.

We determine the transformation (54) under the assumption that

$$g^\sharp(b_*, b_*) = \tilde{g}(\dot{x}_*, \dot{x}_*) \tilde{g}^\sharp(b_*, b_*) < 0. \quad (57)$$

It follows then from (48) that $\Theta(\tau)$ vanishes at

$$\tau_{\pm} = \tau_* - \frac{2\Theta_*}{\Theta_* \langle b_*, \dot{x}_* \rangle \mp |\beta|}, \quad (58)$$

with $\tau_+ \neq \tau_-$, such that

$$\Theta = \frac{1}{4} \Theta_* g^\sharp(b_*, b_*) (\tau - \tau_+) (\tau - \tau_-), \quad (59)$$

and the integration of (54) gives

$$\bar{\tau}(\tau) = \bar{\tau}_* + \frac{1}{|\beta|} \log \frac{(\tau_* - \tau_+)(\tau - \tau_-)}{(\tau - \tau_+)(\tau_* - \tau_-)}, \quad (60)$$

whence

$$\Delta\tau = \frac{2\Theta_* \sinh(\frac{|\beta|}{2} \Delta\bar{\tau})}{|\beta| \cosh(\frac{|\beta|}{2} \Delta\bar{\tau}) - \Theta_* \langle b_*, \dot{x}_* \rangle \sinh(\frac{|\beta|}{2} \Delta\bar{\tau})}. \quad (61)$$

In terms of the parameter $\bar{\tau}$ we also get

$$\Theta = \frac{\Theta_* \beta^2}{\left(|\beta| \cosh(\frac{|\beta|}{2} \Delta\bar{\tau}) - \Theta_* \langle b_*, \dot{x}_* \rangle \sinh(\frac{|\beta|}{2} \Delta\bar{\tau})\right)^2}. \quad (62)$$

Suppose that the solution admits a smooth conformal extension for which $\Theta = 0$ on \mathcal{J}^+ , that the extension admits a point i^+ such that \mathcal{J}^+ coincides with the past light cone of i^+ , and that one of our conformal geodesics, $x(\tau)$ say, passes through i^+ for a finite value τ_i of τ . Then condition (57) precludes a discussion of the zero of Θ on $x(\tau)$. If (58) describes the zeros of Θ on the conformal geodesics covering a neighbourhood of $x(\tau)$, then, approaching $x(\tau)$, we should find $\tau_{\pm} \rightarrow \tau_i$, and consequently $\beta = 0$, $\langle b_*, \dot{x}_* \rangle \neq 0$ on $x(\tau)$ and $\tau_i = \tau_* - \frac{2}{\langle b_*, \dot{x}_* \rangle}$.

3.4 The \tilde{g} -adapted conformal deviation equations

Denote by $\bar{x}(\bar{\tau}, \lambda)$, $\hat{b}(\bar{\tau}, \lambda)$ a smooth family of solutions to (55) with family parameter λ . As before, we write

$$X = \partial_{\bar{\tau}} \bar{x} = \bar{x}', \quad Z = \partial_{\lambda} \bar{x}, \quad \hat{B} = \tilde{\nabla}_Z \hat{b}. \quad (63)$$

Following the derivation of the conformal deviation equations and observing that vacuum field equations $\tilde{L}_{\mu\nu} = 0$, we obtain the \tilde{g} -adapted conformal deviation equations

$$\tilde{\nabla}_X \tilde{\nabla}_X Z = C(X, Z)X + \hat{B}^{\sharp}. \quad (64)$$

$$\tilde{\nabla}_X \hat{B} = -\hat{b}C(X, Z) + (\tilde{\nabla}_Z \tilde{g}^{\sharp}(\hat{b}, \hat{b}))X^{\flat} + \tilde{g}^{\sharp}(\hat{b}, \hat{b})\tilde{\nabla}_X Z^{\flat}, \quad (65)$$

where C denotes the conformal Weyl tensor of \tilde{g} .

While in general $\tilde{\nabla}_Z(\tilde{g}^{\sharp}(\hat{b}, \hat{b})) \neq 0$, we have by (56) always $\tilde{g}^{\sharp}(\hat{b}, \hat{b}) = \text{const.}$ along $\bar{x}(\bar{\tau})$, which implies $\tilde{\nabla}_X(\tilde{\nabla}_Z \tilde{g}^{\sharp}(\hat{b}, \hat{b})) = \tilde{\nabla}_Z(\tilde{\nabla}_X \tilde{g}^{\sharp}(\hat{b}, \hat{b})) = 0$. Thus the coefficients of X^{\flat} and $\tilde{\nabla}_X Z^{\flat}$ in the second line of 65 are constant and known along $\bar{x}(\bar{\tau})$ by their initial data at some $\bar{\tau}_*$.

3.5 Conformal geodesics and the \tilde{g} -adapted conformal deviation equations on warped product vacuum fields

It will be shown now that on warped product vacuum fields the discussion of the conformal geodesic equations reduces under suitable assumptions to the analysis of one equation and that a similar statement is true for the \tilde{g} -adapted conformal deviation equations.

If \tilde{g} can be written in the form (25), the vacuum field equations take the form

$$R_{AC}[h] = \frac{p}{f} D_A D_C f, \quad R_{ac}[k] = f^{-(p-2)} D_A (f^{(p-1)} D^A f) k_{ac}, \quad (66)$$

and the dependence of the various fields in the second equation on x^a implies

$$R_{ac}[k] = \frac{R[k]}{p} k_{ac}, \quad R[k] = p f^{-(p-2)} D_A (f^{(p-1)} D^A f) = \text{const.} \quad (67)$$

We shall assume from now on that $p = 2$ and that the indices A, a take values $0, 1$ and $2, 3$ respectively. Then $R_{ABCD}[h] = R[h] h_{A[C} h_{D]B}$ and $R_{AC}[h] = \frac{1}{2} R[h] h_{AC}$ and thus $2 D_A D_B f = h_{AB} D_C D^C f$ by the first of equations (66). Contracting with D^A , commuting derivatives yields $D_B (f^2 D_A D^A f) = 0$ if (66) is used again. It follows

$$2c \equiv f^2 D_A D^A f = \text{const.}, \quad D_A D_B f = \frac{c}{f^2} h_{AB}, \quad R_{ABCD}[h] = \frac{4c}{f^3} h_{A[C} h_{D]B}. \quad (68)$$

These equations imply with (26), (27)

$$R^A{}_{BCD}[\tilde{g}] = \frac{4c}{f^3} h^A{}_{[C} h_{D]B}, \quad R^a{}_{bcd}[\tilde{g}] = \frac{4c}{f} k^a{}_{[c} k_{d]b}, \quad R^a{}_{BcD}[\tilde{g}] = -\frac{c}{f^3} k^a{}_{c} h_{BD}. \quad (69)$$

It follows that $R^\mu{}_{\nu\lambda\rho} = 0$ unless $c \neq 0$.

If Einstein's vacuum field equation $\tilde{L}_{\mu\nu} = 0$ holds, equations (29), (30), (31), (32) admit solutions satisfying $\dot{x}^a = 0$, $b_c = 0$, and these solutions obey equations which can be written in the form

$$D_{\dot{x}} \dot{x} = -2 \langle b, \dot{x} \rangle \dot{x} + h(\dot{x}, \dot{x}) b^\sharp, \quad D_{\dot{x}} b = \langle b, \dot{x} \rangle b - \frac{1}{2} h^\sharp(b, b) \dot{x}^\flat, \quad (70)$$

where all quantities are derived from h . We shall discuss now the \tilde{g} -adapted version of (70), assuming that $\Delta \equiv \det(h_{AB}) < 0$. With

$$\epsilon_h = \sqrt{|\Delta|} dx^0 \wedge dx^1, \quad (71)$$

it follows that $\epsilon_h(\dot{x}, \cdot) = \sqrt{|\Delta|}(\dot{x}^0 dx^1 - \dot{x}^1 dx^0)$ and $h^\sharp(\epsilon_h(\dot{x}, \cdot), \epsilon_h(\dot{x}, \cdot)) = -h(\dot{x}, \dot{x})$. Since the vectors \dot{x} , \hat{b}^\sharp are contained in the 2-dimensional space

spanned by ∂_A , $A = 0, 1$, and $\langle \hat{b}, \dot{x} \rangle = 0$, the equations above and (56) imply the representation

$$\hat{b} = \pm \beta \epsilon_h(\Theta \dot{x}, \cdot) = \pm \beta \epsilon_h(\bar{x}', \cdot), \quad (72)$$

where the sign is determined by the initial conditions and the convention for β . Using this expression in the second of equations 55, gives $\beta^2 \bar{x}' = D_{\bar{x}'} \hat{b} = \pm \beta \epsilon_h(D_{\bar{x}'} \bar{x}', \cdot)$, which can be rewritten in the form

$$D_{\bar{x}'} \bar{x}' = \hat{b}^\sharp. \quad (73)$$

Thus, the \tilde{g} -adapted forms of the two equations (70) are equivalent to each other and we only have to solve (73) with \hat{b} as in (72) to obtain a solution to (70).

Observing the formulae for the connection coefficients, the results (28) for the curvature tensor of warped products, and

$$C^A{}_{BCD}[\tilde{g}] = \frac{4c}{f^3} h^A{}_{[C} h_{D]B}, \quad (74)$$

equations (64), (65) are seen to be equivalent to each other and to the equation

$$D_X D_X Z = \frac{2c}{f^3} \epsilon_h(X, Z) \epsilon_h(X, \cdot)^\sharp \pm (D_Z \beta \epsilon_h(X, \cdot)^\sharp + \beta \epsilon_h(D_X Z, \cdot)^\sharp). \quad (75)$$

The sign here is chosen as in (72) and use has been made of the identities

$$\epsilon_h(\epsilon_h(X, \cdot)^\sharp, \cdot) = X^\flat, \quad X h(Z, X) - Z = \epsilon_h(X, Z) \epsilon_h(X, \cdot)^\sharp.$$

4 Conformal geodesics on the Schwarzschild space-time

For our discussions various forms of the Schwarzschild line element will be needed. Its standard form with mass m is given for $\bar{r} > 2m$ by

$$\tilde{g} = \left(1 - \frac{2m}{\bar{r}}\right) dt^2 - \left(1 - \frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2 d\sigma^2. \quad (76)$$

In terms of the retarded and advanced null coordinates

$$w = t - (\bar{r} + 2m \log(\bar{r} - 2m)), \quad v = t + (\bar{r} + 2m \log(\bar{r} - 2m)), \quad (77)$$

the line element (76) is obtained in the forms

$$\tilde{g} = \left(1 - \frac{2m}{\bar{r}}\right) dw^2 + 2 dw d\bar{r} - \bar{r}^2 d\sigma^2, \quad \tilde{g} = \left(1 - \frac{2m}{\bar{r}}\right) dv^2 - 2 dv d\bar{r} - \bar{r}^2 d\sigma^2, \quad (78)$$

respectively. These extend analytically into regions where $\bar{r} \leq 2m$. The retarded null coordinate w extends smoothly through \mathcal{J}^+ and through the past

horizon to the past singularity, while the advanced null coordinate v extends smoothly through \mathcal{J}^- and the future horizon to the future singularity.

The transformation

$$s = \left(\frac{\bar{r}}{2m} - 1\right)^{1/2} \exp\left(\frac{\bar{r}}{4m}\right) \sinh\left(\frac{t}{4m}\right), \quad \rho = \left(\frac{\bar{r}}{2m} - 1\right)^{1/2} \exp\left(\frac{\bar{r}}{4m}\right) \cosh\left(\frac{t}{4m}\right),$$

applied to (76) with $m > 0$ produces the Schwarzschild-Kruskal line element

$$\tilde{g} = K (ds^2 - d\rho^2) - \bar{r}^2 d\sigma^2 \quad \text{with} \quad K = \frac{32m^3}{\bar{r}} \exp\left(-\frac{\bar{r}}{2m}\right), \quad \bar{r} = k(\rho^2 - s^2), \quad (79)$$

where k denotes the inverse of the map $]0, \infty[\ni \bar{r} \longrightarrow \left(\frac{\bar{r}}{2m} - 1\right) \exp\left(\frac{\bar{r}}{2m}\right) \in]-1, \infty[$. This line element extends analytically to a non-degenerate line element on the domain $\rho^2 - s^2 > -1$.

The coordinate r , given for $\bar{r} > 2m$ by

$$\bar{r} = \frac{1}{r} \left(r + \frac{m}{2}\right)^2 \quad \text{resp.} \quad r = \frac{1}{2} \left(\bar{r} - m + \sqrt{\bar{r}(\bar{r} - 2m)}\right), \quad (80)$$

yields the isotropic Schwarzschild line element

$$\tilde{g} = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\sigma^2), \quad r > \frac{m}{2}. \quad (81)$$

On the hypersurface $\{t = 0\}$ it induces initial data which can be extended analytically to an initial data set

$$(\tilde{S} = \mathbb{R}^3 \setminus \{0\}, \tilde{h} = -(1 + \frac{m}{2r})^4 (dr^2 + r^2 d\sigma^2), \tilde{\chi} = 0), \quad (82)$$

where $\tilde{\chi}$ denotes the second fundamental form. These data may be identified isometrically with the initial data induced by the Schwarzschild-Kruskal metric 79 on the hypersurface $\{s = 0\}$ by the transformation

$$\rho = \frac{r - \frac{m}{2}}{\sqrt{2mr}} \exp\left(\frac{(r + \frac{m}{2})^2}{4mr}\right).$$

In the following discussions one may think of the conformal geodesics as being constructed on the Schwarzschild-Kerr solution (79) with initial data being prescribed on the hypersurface $\tilde{S} = \{s = 0\}$. It will be convenient, however, to specify them in terms of the coordinate r in (82). To discuss the different aspects of these curves we will use those coordinates which will appear to give the simplest formulae. Because of the symmetries $s \rightarrow -s$ and $\rho \rightarrow -\rho$, it will be sufficient to consider the region $\{s \geq 0, \rho \geq 0\}$ if we prescribe data for the conformal geodesics which respect these symmetries. The data will also be spherically symmetric.

The function \bar{r} extends by the first of equations (80) to an analytic function on \tilde{S} which takes its minimum $\bar{r}_{\min} = 2m$ at the ‘‘throat’’ $\{r = r_h \equiv \frac{m}{2}\}$. The

space (\tilde{S}, \tilde{h}) has two asymptotically flat ends. In terms of (82) the end i_1 at $r = \infty$ has the “usual” representation, while the end i_2 at $r = 0$ has a coordinate representation adapted to the metric $(1 + \frac{m}{2r})^{-4} \tilde{h}$ which may be thought of as a conformal compactification of \tilde{h} at the end i_2 . The map

$$r \rightarrow \left(\frac{m}{2}\right)^2 \frac{1}{r}, \quad (83)$$

which corresponds to the isometry $\rho \rightarrow -\rho$ allows us to show the equivalence of the spatial infinities i_1 and i_2 . It leaves \bar{r} invariant and has $\{r = r_h\}$ as its fixed point set.

4.1 Conformally compactified Schwarzschild-Kruskal data and initial data for a congruence of conformal geodesics

We shall in the following prescribe initial data on \tilde{S} for a congruence of conformal geodesics whose \tilde{g} -adapted initial vectors \tilde{x}' coincide with the future directed unit normals of \tilde{S} . By \bar{r}_* will be denoted the restriction to \tilde{S} of the function \bar{r} on the Schwarzschild-Kruskal space-time and r will be considered as a coordinate on \tilde{S} . For the conformal factor we choose

$$\Theta_* = \frac{1}{\bar{r}_*^2} = \frac{r^2}{(r + \frac{m}{2})^4} \quad (84)$$

which implies

$$-\Theta_*^2 \tilde{h} = (r + \frac{m}{2})^{-4} (dr^2 + r^2 d\sigma^2). \quad (85)$$

It follows that the transformation $r = \tan \frac{\chi}{2}$, $0 < \chi < \pi$ (and, to get simple equations, $\frac{m}{2} = \tan \frac{\mu}{2}$) could be used to make the conformal compactification achieved by Θ_* manifest in terms of coordinates, since it realizes an embedding of \tilde{S} into the unit 3-sphere S^3 with the poles at $\chi = 0$, $\chi = \pi$ corresponding to the ends i_2 , i_1 respectively.

The choice (84) is particularly well adapted to the Schwarzschild-Kruskal geometry and implies simple formulae. However, it is important to note that our basic results do not depend on it. For the analysis of the field near space-like infinity other choices might be preferable (cf. [5]), because the metric (85) does not extend smoothly to space-like infinity.

We set furthermore

$$b_* = \hat{b}_* = \Theta_*^{-1} d\Theta_* = -\frac{2(r - \frac{m}{2})}{r(r + \frac{m}{2})} dr, \quad (86)$$

and, observing $F \equiv 1 - 2m/\bar{r}_* = (r - \frac{m}{2})^2 (r + \frac{m}{2})^{-2}$, choose

$$\beta = \frac{2r(r - \frac{m}{2})}{(r + \frac{m}{2})^3}, \quad (87)$$

such that β is analytic on \tilde{S} and $\beta = \sqrt{-\tilde{g}^\sharp(\hat{b}_*, \hat{b}_*)} = \frac{2}{\bar{r}_*} \sqrt{F(\bar{r}_*)} > 0$ near the end i_1 in analogy to our procedure on Minkowski space. If \tilde{h} is used to raise indices, \hat{b}_*^\sharp is outward pointing at the ends i_1, i_2 , as it should. With the choices above the general formula (59) reduces to

$$\Theta = F(\bar{r}_*) \left(\left(\frac{2\Theta_*}{\beta} \right)^2 - \tau^2 \right), \quad (88)$$

where functions with subscript $*$ are assumed to be constant along the conformal geodesics. Note that our initial data and gauge conditions are preserved by the isometry (83), and are adapted to the spherical symmetry and the time reflection symmetry. We have $\beta \rightarrow -\beta$ under (83) and $\beta(r) = 0$ precisely for $r = \frac{m}{2}$. This means that conformal geodesic satisfying the initial data above at points with $r = \frac{m}{2}$ will be metric geodesics in the hypersurface $\{\rho = 0\}$ of the Schwarzschild-Kruskal space-time.

4.2 The conformal geodesic equations on the Schwarzschild space-time

To discuss conformal geodesics on the Schwarzschild space-time we write the \tilde{g} -adapted form of the conformal geodesic equation in terms of the line element (76), which can be written in the form (25) with the metric h given by

$$h = F dt^2 - \frac{1}{F} d\bar{r}^2, \quad F = \left(1 - \frac{2m}{\bar{r}}\right).$$

Initial data for the conformal geodesics will be given on the hypersurface $\tilde{S} = \{t = 0\}$ with $\tau_* = 0, \bar{\tau}_* = 0$ on \tilde{S} .

Because of (72) (where due to our conventions we have to use the minus sign), equations (73) take the form

$$t'' + \frac{F_{,\bar{r}}}{F} \bar{r}' t' = \frac{1}{F} \beta \bar{r}', \quad (89)$$

$$\bar{r}'' - \frac{F_{,\bar{r}}}{2F} (\bar{r}')^2 + \frac{F F_{,\bar{r}}}{2} (t')^2 = F \beta t', \quad (90)$$

with β satisfying (56). Assuming on \tilde{S} that the initial vector is the future directed unit normal to \tilde{S} , the initial data on \tilde{S} are given by

$$t_* = 0, \quad \bar{r}_* > 2m, \quad t'_* = \frac{1}{\sqrt{F(\bar{r}_*)}}, \quad \bar{r}'_* = 0, \quad \hat{b}_{t_*} = 0, \quad \hat{b}_{\bar{r}_*} = -\beta(\bar{r}_*) \frac{1}{\sqrt{F(\bar{r}_*)}}.$$

The \tilde{g} -normalization gives

$$F (t')^2 - \frac{1}{F} (\bar{r}')^2 = 1. \quad (91)$$

Solving for $t' > 0$ and inserting the result into equation (90) gives

$$0 = \bar{r}'' + \frac{1}{2} F_{,\bar{r}} - \beta \sqrt{F + (\bar{r}')^2}. \quad (92)$$

Dividing by the square root and multiplying with \bar{r}' leads to $d(\sqrt{F + (\bar{r}')^2} - \beta \bar{r})/d\bar{r} = 0$, whence

$$\sqrt{F + (\bar{r}')^2} - \beta \bar{r} = \gamma, \quad (93)$$

with the constant of integration given by $\gamma = \sqrt{F(\bar{r}_*)} - \beta(\bar{r}_*) \bar{r}_*$. It follows that

$$\bar{r}' = \pm \sqrt{(\gamma + \beta \bar{r})^2 - F(\bar{r})}, \quad (94)$$

with a sign which depends on the value of \bar{r}_* .

Given $\bar{r}(\bar{\tau})$, we obtain $t(\bar{\tau})$ by integrating $(t')^2 = (\beta \bar{r} + \gamma)^2 F^{-2}$ for the given initial conditions. However, the integration of t is in general not particularly interesting, because t does neither extend smoothly through \mathcal{J}^+ nor through the future horizon.

To study the extension of the conformal geodesics through \mathcal{J}^+ it is convenient to use the first of the line elements (78). The conformal geodesic equations then read

$$w'' - \frac{1}{2} F_{,\bar{r}} (w')^2 = -\beta w', \quad \bar{r}'' + \frac{1}{2} F F_{,\bar{r}} (w')^2 + F_{,\bar{r}} \bar{r}' w' = \beta (\bar{r}' + F w').$$

These equations imply for \bar{r} the equations obtained above. The normalization of the tangent vector reads $F (w')^2 + 2 w' \bar{r}' = 1$. Solving for w' and requiring $w' > 0$ on \tilde{S} gives

$$w' = \frac{1}{F} (\sqrt{F + (\bar{r}')^2} - \bar{r}') = \frac{1}{\sqrt{F + (\bar{r}')^2} + \bar{r}'}, \quad (95)$$

which has to be integrated with the initial conditions $w_* = -\bar{r}_* - 2m \log(\bar{r}_* - 2m)$. Thus $w(\bar{\tau})$ can be obtained by a simple integration once $\bar{r}(\bar{\tau})$ has been determined.

To study the extension through the horizon, it is convenient to use the second of the line elements (78). The conformal geodesic equations then read

$$v'' + \frac{1}{2} F_{,\bar{r}} (v')^2 = \beta v', \quad \bar{r}'' + \frac{1}{2} F F_{,\bar{r}} (v')^2 - F_{,\bar{r}} \bar{r}' v' = -\beta (\bar{r}' - F v').$$

The normalization of the tangent vector gives now $F (v')^2 - 2 v' \bar{r}' = 1$, which leads to the equation

$$v' = \frac{1}{F} (\sqrt{F + (\bar{r}')^2} + \bar{r}'), \quad (96)$$

which has to be integrated with the initial conditions $v_* = \bar{r}_* + 2m \log(\bar{r}_* - 2m)$.

4.2.1 Conformal geodesics on which \bar{r} is constant

The explicit solution of the key equation (94) requires the discussion of three different cases. We first consider the borderline solution.

A change of the sign in (94) should occur near conformal geodesics along which \bar{r} is constant. Requiring $\bar{r}' = 0$ in (92) gives $1/2 F' = \beta \sqrt{F}$. With β given by (87) this condition has the solution

$$\bar{r}_* = \hat{r} \equiv \frac{5}{2} m \quad \text{resp.} \quad r = r_{\pm} \equiv \frac{3 \pm \sqrt{5}}{4} m.$$

Inserting $\bar{r}' = 0$ into (91), using (60) with $\tau_* = 0$, $\bar{\tau}_* = 0$, and observing (84), (87) with $r = r_+$ gives on the conformal geodesics on which $\bar{r} = \hat{r}$

$$t = \frac{\bar{\tau}}{\sqrt{F(\hat{r})}} = \frac{25}{4} m \log \left(\frac{2\Theta_* + \beta\tau}{2\Theta_* - \beta\tau} \right),$$

and thus $t \rightarrow \infty$ as $\tau \rightarrow \tau_i \equiv \frac{2\Theta_*(r_+)}{\beta(r_+)}$.

It follows now from equations (92), (94) that each of the conformal geodesics specified by our data falls into one of the following four classes: (i) the conformal geodesics passing through points with $r = \frac{m}{2}$, which coincide with metric geodesics in the Schwarzschild-Kruskal space-time and approach the singularity, (ii) the conformal geodesics passing through points with $r = r_{\pm}$, which are tangent to the static Killing field ∂_t and approach time-like infinity for the finite value τ_i of their parameters, (iii) the conformal geodesics passing through points with $0 < r < r_-$ or with $r_+ < r < \infty$, for which \bar{r} is monotonically increasing, (iv) the conformal geodesics passing through points with $r_- < r < \frac{m}{2}$ or with $\frac{m}{2} < r < r_+$, for which \bar{r} is monotonically decreasing. Though it will be seen below that the integration procedure for (94) depends on the classes specified above, it is clear that our data, which are smooth on \tilde{S} , determine a smooth congruence of conformal geodesics which is free of caustics near \tilde{S} .

4.2.2 Conformal geodesics on which \bar{r} is increasing

Because of the symmetry of the data it is sufficient to discuss the case $r_+ < r < \infty$. Under this assumption the “+” sign holds in (94). The radicand in this equation factorizes as

$$(\gamma + \beta\bar{r})^2 - F(\bar{r}) = \frac{\beta^2}{\bar{r}} (\bar{r} - \bar{r}_*) (\bar{r} - \alpha_+) (\bar{r} - \alpha_-). \quad (97)$$

Because $\gamma = -\sqrt{F(\bar{r}_*)}$, it follows that

$$\alpha_{\pm} = \pm \alpha \quad \text{with} \quad \alpha = \sqrt{\frac{m\bar{r}_*}{2F(\bar{r}_*)}}.$$

Since $\alpha < \bar{r}_*$ and $\alpha \rightarrow \hat{r}$ as $\bar{r}_* \rightarrow \hat{r}$, the polynomial on the right hand side of (97) has under our assumption three different zeros while in the limit above two zeros will coincide and the integration procedure needs to be changed.

The use of the notation

$$t \equiv \left(\frac{(\alpha + \bar{r}) \bar{r}_*}{(\alpha + \bar{r}_*) \bar{r}} \right)^{\frac{1}{2}}, \quad k \equiv \left(\frac{1}{2} \left(1 + \frac{\alpha}{\bar{r}_*} \right) \right)^{\frac{1}{2}} = \left(\frac{1}{2} \left(1 + \sqrt{\frac{m}{2\bar{r}_* - 4m}} \right) \right)^{\frac{1}{2}}, \quad e \equiv \sqrt{2} k,$$

in (94) gives, after an integration,

$$\beta \bar{\tau} = \frac{2}{e} \sqrt{(e^2 - 1)(e^2 - k^2)} \int_1^{\frac{1}{e} \sqrt{1 + \frac{\alpha}{\bar{r}}}} \frac{dt}{(1 - e^2 t^2) \sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

where, by our assumptions, $1 < e < \sqrt{2}$, $\frac{1}{\sqrt{2}} < k < 1$. It is well known how to express the elliptic integral of the third kind which occurs above in terms of theta functions (for all manipulations involving elliptic integrals, elliptic functions, theta functions, etc. we refer to [9]). The substitution $t = sn(u, k)$ with Jacobi's elliptic function sn yields

$$\begin{aligned} \beta \bar{\tau} &= \frac{2}{e} \sqrt{(e^2 - 1)(e^2 - k^2)} \int_K^u \frac{dv}{(1 - e^2 sn^2 v)} \\ &= -2 Z(c)(u - K) + \log \left(\frac{\theta_1(\frac{\pi}{2K}(u + c))}{\theta_1(\frac{\pi}{2K}(u - c))} \right). \end{aligned} \quad (98)$$

Here Z denotes Jacobi's zeta function

$$Z(u) = \frac{d}{du} \left(\log \theta_4\left(\frac{\pi}{2} \frac{u}{K}\right) \right) = \frac{2\pi}{K} \sin\left(\pi \frac{u}{K}\right) \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - 2q^{2n-1} \cos\left(\pi \frac{u}{K}\right) + q^{4n-2}}.$$

Thus Z is an analytic function on the real line with $Z(u) > 0$ for $0 < u < K$ and $Z(0) = Z(K) = 0$. The theta functions are given in their product representations by

$$\theta_1(z) = 2q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2z + q^{4n}),$$

$$\theta_2(z) = 2q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2z + q^{4n}),$$

$$\theta_4(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos 2z + q^{4n-2}),$$

with $q = \exp(-\pi \frac{K'}{K})$. The constant $K > 0$ is the complete integral of the first kind

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

$K' = K(k')$ with $k'^2 = 1 - k^2$, and c is the (unique) number in the open interval $]0, K[$ satisfying $sn(c, k) = 1/e$.

With $\tau_* = 0$, $\bar{\tau}_* = 0$, (84), and (87) we get from equation (60)

$$\beta \bar{\tau} = -\log g(\tau) \quad \text{with} \quad g(\tau) \equiv \frac{2\Theta_* - \beta\tau}{2\Theta_* + \beta\tau}. \quad (99)$$

Setting now $x = K - u$ to exhibit the symmetries of the solution, we obtain

$$\tau = G(x, k) \equiv \frac{2\Theta_*}{\beta} \frac{\theta_2\left(\frac{\pi(c-x)}{2K}\right) e^{xZ(c)} - \theta_2\left(\frac{\pi(c+x)}{2K}\right) e^{-xZ(c)}}{\theta_2\left(\frac{\pi(c-x)}{2K}\right) e^{xZ(c)} + \theta_2\left(\frac{\pi(c+x)}{2K}\right) e^{-xZ(c)}}.$$

Since the denominator of the function G is positive in an open neighbourhood of the closed interval $[-(K-c), K-c]$, the function G is analytic there. While $\bar{\tau} \rightarrow \infty$ as $u \rightarrow c$ and $\bar{\tau} \rightarrow -\infty$ as $u \rightarrow 2K - c$ by (98), we have now $\tau \rightarrow \pm \frac{2\Theta_*}{\beta}$, whence $\bar{\tau} \rightarrow \pm\infty$, as $x \rightarrow \pm(K-c)$. It follows from the ODE satisfied by $\bar{\tau}$ that we can solve the equation above for $x(\tau)$ with $\tau \in]-\frac{2\Theta_*}{\beta}, \frac{2\Theta_*}{\beta}[$. A direct calculation shows that $G'(x, k) > 0$ at $x = \pm(K-c)$. This implies that $x(\tau)$ extends as an analytic function into an open neighbourhood of $[-\frac{2\Theta_*}{\beta}, \frac{2\Theta_*}{\beta}]$.

Inserting now $x(\tau)$ into the equation $t = sn(K-x, k)$, we finally get

$$\frac{\bar{r}_*}{\bar{r}} = 1 - \frac{2(1-k^2)k^2}{2k^2-1} \frac{sn^2(x(\tau))}{1-k^2 sn^2(x(\tau))}. \quad (100)$$

In the limit when $r \rightarrow r_+$ we have $k \rightarrow 1$ and thus $sn(u, k) \rightarrow \tanh u$. The formula (100) thus suggests that the solution $\bar{r}(\tau, r)$ of (100) approaches the constant solution $\bar{r} = \hat{r}$ in that limit, as we know already by general arguments. However, since $K \rightarrow \infty$ while $c \rightarrow 1/2 \log((\sqrt{2}+1)/(\sqrt{2}-1))$ as $k \rightarrow 1$, the precise behaviour of the solution in that limit requires a quite careful discussion involving also the behaviour of $G(x, k)$.

The right hand side of (100) is an analytic function of τ in an open interval containing $[-\tau_{scri}(r), \tau_{scri}(r)]$, with $\tau_{scri}(r) \equiv \frac{2\Theta_*}{\beta} = \frac{r}{(r+\frac{m}{2})(r-\frac{m}{2})}$. It vanishes precisely at the points $\tau = \pm\tau_{scri}$ at which the conformal factor (88) vanishes. Furthermore, it depends analytically on r for $r_+ < r < \infty$. It follows that

$$\bar{r}(r, \tau) \Theta(r, \tau) \quad \text{is positive and analytic for} \quad r \in]r_+, \infty[, \quad \tau \in [-\tau_{scri}(r), \tau_{scri}(r)]. \quad (101)$$

Using the coordinate $z = \frac{1}{\bar{r}}$ in the first of the line elements (78) and rescaling with the conformal factor $\Omega = z$ gives the smooth conformal representation

$$\Omega^2 \tilde{g} = z^2 (1 - 2mz) dw^2 - 2dw dz - d\sigma^2, \quad (102)$$

of the Schwarzschild metric which extends analytically through future null infinity, given here by $\mathcal{J}^+ = \{z = 0\}$.

Integrating (95) with \bar{r} given by (100) and writing the solution in terms of z , we obtain our conformal geodesics in the form

$$\tau \rightarrow (w(\tau, r), z(\tau, r)). \quad (103)$$

From the discussion above it is clear that the first of the functions on the right hand side is an analytic function of both variables for $r \in]r_+, \infty[$, $\tau \in [-\tau_{scri}(r), \tau_{scri}(r)]$. We show that this holds true also for the second function. It is clear that the function w is analytic near \tilde{S} . Parametrizing w in terms of z , we obtain from (94), (95)

$$\frac{dw}{dz} = - \left(\beta^2 (1 - \bar{r}_* z) (1 - \alpha^2 z^2) + (\beta + \gamma z) \sqrt{\beta^2 (1 - \bar{r}_* z) (1 - \alpha^2 z^2)} \right)^{-1}.$$

The assertion now follows because the function on the right hand side is analytic for z in an open interval of the form $] -\epsilon, \frac{1}{\bar{r}_*}[$ with some $\epsilon > 0$.

It follows in particular that there exists a smooth function $\hat{w}(r)$, with $r \in]r_+, \infty[$, such that $w(\tau, r) \rightarrow \hat{w}(r)$ as $\tau \rightarrow \tau_{scri}$ (or, for symmetry reasons, as $\tau \rightarrow -\tau_{scri}$). We show that

$$\hat{w}(r) \rightarrow \infty \quad \text{as } r \rightarrow r_+, \quad \hat{w}(r) \rightarrow -\infty \quad \text{as } r \rightarrow \infty.$$

The first assertion follows from the observation that

$$w = -\hat{r} - 2m \log(\hat{r} - 2m) + \frac{25}{4} m \log \left(\frac{2\Theta_* + \beta\tau}{2\Theta_* - \beta\tau} \right),$$

along the conformal geodesic with $\bar{r} = \hat{r}$ and the fact that the solutions are jointly smooth in the initial data and the parameter. The second assertion follows from a comparison of w with the solution to

$$\frac{du}{dz} = - \left(\beta^2 (1 - \bar{r}_* z) \left(1 - \left(\frac{\alpha}{\bar{r}_*}\right)^2\right) + \frac{1}{2} \sqrt{(1 - \bar{r}_* z) \left(1 - \left(\frac{\alpha}{\bar{r}_*}\right)^2\right)} \right)^{-1},$$

which satisfies $u = w$ at $z = \frac{1}{\bar{r}_*}$. Since $0 \geq \frac{dw}{dz} \geq \frac{du}{dz}$ for $z \in [0, \frac{1}{\bar{r}_*}]$, it follows that the value $\hat{u}(r)$ of u at $z = 0$ gives an upper estimate for $\hat{w}(r)$. Since the direct integration gives $\hat{u}(r)$ with $\hat{u}(r) \rightarrow -\infty$ as $r \rightarrow \infty$, the assertion follows.

4.2.3 Conformal geodesics on which \bar{r} is decreasing

It is sufficient to discuss the case $\frac{m}{2} < r < r_-$. Now (94) must hold with the “-” sign. The function α in the factorization (97) then* satisfies $\bar{r}_* < \alpha$ and $\alpha \rightarrow \infty$ as $\bar{r}_* \rightarrow \frac{m}{2}$ while $\alpha \rightarrow \hat{r}$ as $\bar{r}_* \rightarrow \hat{r}$. If we set

$$t \equiv \left(\frac{1}{2} \left(1 + \frac{\alpha}{\bar{r}}\right) \right)^{\frac{1}{2}}, \quad k \equiv \left(\frac{1}{2} \left(1 + \frac{\alpha}{\bar{r}_*}\right) \right)^{-\frac{1}{2}}, \quad e \equiv \sqrt{2},$$

such that $e > 1$, $0 < k < 1$, the integration of (94) yields

$$\beta \bar{r} = \sqrt{2(2 - k^2)} \int_{\frac{1}{k}}^{\frac{1}{e} \sqrt{1 + \frac{\alpha}{\bar{r}}}} \frac{dt}{(t^2 e^2 - 1) \sqrt{(t^2 k^2 - 1)(t^2 - 1)}}.$$

The substitution $k \operatorname{sn}(u, k) = 1/t$ and $c \in]0, K[$ such that $\operatorname{sn}(c, k) = 1/e$ give

$$\beta \bar{\tau} = k^2 \sqrt{\frac{2-k^2}{2}} \int_u^K \frac{\operatorname{sn}^2 v}{1 - \frac{k^2}{2} \operatorname{sn}^2 v} dv = -2 Z(c) (u - K) + \log \frac{\theta_4\left(\frac{\pi(u+c)}{2K}\right)}{\theta_4\left(\frac{\pi(u-c)}{2K}\right)}.$$

Setting $x = K - u$ and observing (99), we finally obtain

$$\tau = F(x, k) \equiv \frac{2\Theta_*}{\beta} \tanh \left(k^2 \sqrt{\frac{2-k^2}{2}} \int_0^x \frac{1 - \operatorname{sn}^2 v}{2 - k^2(1 + \operatorname{sn}^2 v)} dv \right). \quad (104)$$

The function $F(x)$ is real analytic on \mathbb{R} with $F'(x) \geq 0$ and $F'(x) = 0$ iff $x = (2m+1)K$ with $m \in \mathbb{Z}$. Thus (104) can be solved to obtain an analytic function $x(\tau)$ which maps the interval $] -\tau_s, \tau_s[$, with $2\Theta_*/\beta < \tau_s \equiv G(K, k)$, diffeomorphically onto $] -K, K[$. The solution can then be written

$$\bar{r} = \bar{r}_* \frac{(2-k^2)(1 - \operatorname{sn}^2(x(\tau)))}{2 - k^2(1 + \operatorname{sn}^2(x(\tau)))}. \quad (105)$$

From the discussion above it follows that $\bar{r}(\tau) \rightarrow 0$ and $d\bar{r}/d\tau \rightarrow \mp\infty$ as $\tau \rightarrow \pm\tau_s$. Thus in this case there does not exist a smooth (though a continuous) extension of $\bar{r}(\tau)$ beyond its physical domain. The limit $r \rightarrow r_+$ implies $k \rightarrow 1$. In particular, it follows from (104), (105) that the right hand side of (105) goes in this limit to \hat{r} for constant τ and $\tau_s \rightarrow \tau_i$.

The limit $r \rightarrow \frac{m}{2}$ implies $k \rightarrow 0$. Since $\beta = 0$ at $r = \frac{m}{2}$, the expression for τ_s appears to give a nonsensical result at that point. However, we have $2\Theta_* k^2/\beta \rightarrow 2/m$. Expanding the right hand side of (104) and observing that $K(0) = \pi/2$ thus gives $\tau_s \rightarrow \pi/4m$ as $r \rightarrow m/2$, consistent with the fact the conformal geodesics approach in the limit metric geodesics with length $\bar{\tau} = \frac{\tau_s}{\Theta_*(\bar{r}_*=2m)} = m\pi$.

To follow the conformal geodesics through the horizon, one has to integrate for given $\bar{r}(\tau)$ the equation

$$v' = \frac{\gamma + \beta \bar{r} - \sqrt{(\gamma + \beta \bar{r})^2 - F(\bar{r})}}{F(\bar{r})}, \quad (106)$$

whose right hand side defines a positive analytic function of \bar{r} for $\bar{r} > 0$. The conformal geodesics are then obtained in the form

$$\tau \rightarrow x(\tau, r) = (v(\tau, r), \bar{r}(\tau, r)), \quad (107)$$

where the function on the right hand side are analytic for (τ, r) with $r \in]\frac{m}{2}, r_+[$, $\tau \in [0, \tau_s(r)[$.

4.3 Analytic coordinates covering the Schwarzschild-Kruskal space-time and its null infinity

If we set now $y^1 = r$, $y^2 = \phi$, $y^3 = \theta$ on \tilde{S} , drag these coordinates along with the congruence of conformal geodesics constructed in the previous section, and

set $y^0 = \tau$, we obtain a smooth coordinate system near \tilde{S} . The purpose of the following discussion is to establish the following result (we ignore here the coordinate singularity arising from the use of coordinates on S^2 which can easily be removed):

The conformal Gauss system y^ν defines a smooth global coordinate system on the Schwarzschild-Kruskal space-time which extends smoothly to null infinity.

It remains to be shown that the conformal geodesics of the congruence do not develop caustics. Because of the symmetry of the functions (79) and of our family of curves under $s \rightarrow -s$, $\rho \rightarrow -\rho$, it is sufficient, to control the behaviour of the congruence on the subset $\{s \geq 0, \rho \geq 0\}$ of the Schwarzschild-Kruskal space-time. Since the coordinates y^μ are regular near \tilde{S} , they are known to be regular in particular near the set $\{s = 0, \rho = 0\}$. Therefore we can work with the coordinates w, \bar{r} or z, \bar{r} or v, \bar{r} respectively.

Using the functions on the right hand sides of (103) and (107), we can in principle express the corresponding line elements in terms of the coordinates y^μ . Though the resulting metric coefficients will be smooth where (103) and (107) are analytic, the Jacobian of the transformation may drop rank at various places and we may end up with a degenerate representation of the metric. Having available the explicit solution, we could try to check this by a direct calculation. Since the explicit calculation will be tedious and, in particular, because this method will not apply to more general cases, we prefer to employ an argument which is similar to the analysis of the behaviour of metric geodesics congruences in terms of the Jacobi equation.

Consider the transformation provided by (107). We need to show that the tangent vector field $\dot{x} = \Theta^{-1} \dot{x}' = \Theta^{-1} X$ and the connecting vector field $x_{,r} = \bar{x}_{,r} - \dot{x} \tau_{,r} = Z - \dot{x} \tau_{,r}$ are linearly independent on their domain of definition. In terms of the 2-form (71) (with $h = F dv^2 - 2dv d\bar{r}$, $x^0 = v$, $x^1 = \bar{r}$) this can be expressed as the requirement that the invariant $\Theta^{-1} \epsilon_h(X, Z)$ does not vanish. In the case of (103) the nondegeneracy up to and in fact beyond null infinity would follow if it could be shown that

$$-(\bar{r} \Theta)^2 (\dot{w} z_{,r} - \dot{z} w_{,r}) \neq 0 \quad \text{for } r_* \in]r_+, \infty[, \tau \in [0, \tau_{scri}(r)].$$

If z is replaced by \bar{r} again, this condition translates in terms of $x(\tau, r) = (v(\tau, r), \bar{r}(\tau, r))$ and (71) (with $h = F dw^2 + 2dw d\bar{r}$, $x^0 = w$, $x^1 = \bar{r}$) into the requirement that

$$0 \neq \epsilon_{\Theta^2 h}(\dot{x}, x_{,r}) = \Theta \epsilon_h(X, Z),$$

in the domain given above. Here the factor Θ is of course most significant because it vanishes at τ_{scri} .

The proof that these requirements are met by our transformations relies on a differential equation satisfied by $\epsilon_h(X, Z)$. To make use of (75), we observe that the various representations of the Schwarzschild metric with warping function

$f = \bar{r}$ and second metric $k = -d\sigma^2$ give the value $c = \frac{1}{2} f^2 D_A D^A f = -m$ for the constant in (69). It follows

$$\begin{aligned} D_X \epsilon_h(X, Z) &= \epsilon_h(D_X X, Z) + \epsilon_h(X, D_X Z) = \epsilon_h(-\beta \epsilon(X, \cdot)^\sharp, Z) + \epsilon_h(X, D_X Z) \\ &= -\beta h(X, Z) + \epsilon_h(X, D_X Z), \end{aligned}$$

and similarly

$$\begin{aligned} D_X D_X \epsilon_h(X, Z) &= \\ \beta^2 \epsilon_h(X, Z) - 2\beta h(X, D_X Z) + \epsilon_h(X, D_X D_X Z) &= \\ = (\beta^2 + \frac{2m}{\bar{r}^3}) \epsilon_h(X, Z) + D_Z \beta, & \end{aligned}$$

where in the second equation $D_X Z = D_Z X$, $h(X, X) = 1$, and (75) have been used. The invariant $\epsilon_h(X, Z)$ thus satisfies the ODE

$$D_X D_X \epsilon_h(X, Z) - (\beta^2 + k) \epsilon_h(X, Z) = D_Z \beta,$$

with $k = \frac{2m}{\bar{r}^3}$, and on \tilde{S} the initial conditions

$$\epsilon_h(X, Z)_* = \frac{(r + \frac{m}{2})^2}{r^2} > 1, \quad (D_X \epsilon_h(X, Z))_* = 0 \quad \text{for } r \geq \frac{m}{2}.$$

The quantity $D_Z \beta$, which is constant along the conformal geodesics, is given by

$$D_Z \beta = \beta_{,r} = -2 \frac{r^2 - 2mr + \frac{m^2}{4}}{(r + \frac{m}{2})^4}.$$

It vanishes at $\check{r}_\pm \equiv \frac{2 \pm \sqrt{3}}{2} m$, where $\bar{r}(\check{r}_\pm) = 3m > \hat{r}$, and satisfies

$$D_Z \beta > 0 \quad \text{for } \frac{m}{2} \leq r < \check{r}_+, \quad D_Z \beta < 0 \quad \text{for } \check{r}_+ < r.$$

A lower estimate for $\epsilon_h(X, Z)$ can be obtained as follows. Denote by u and v the solutions to the ODE problems

$$w'' - (\beta^2 + k)w = f, \quad w(0) = 1, \quad w'(0) = 0,$$

where $f = 0$ in the case of u and $f = -1$ in the case of v . Then u is strictly increasing with $u \geq \cosh(\beta \bar{\tau})$ for $\bar{\tau} \geq 0$, and $\epsilon_h(X, Z)$ can be given in the form

$$\epsilon_h(X, Z) = u \left(\epsilon_h(X, Z)_* + \left(1 - \frac{v}{u}\right) \beta_{,x} \right). \quad (108)$$

Since $(u - v)'' - (\beta^2 + k)(u - v) = 1$ and the function $u - v$ has vanishing initial conditions at $\bar{\tau} = 0$, it follows that $u \geq v$ for $\bar{\tau} \geq 0$. Since v changes its sign, a

further estimate is needed. We derive a representation of v in terms of u . Since $u > 0$ there exists a function f with $v = f u$. The ODE's satisfied by u and v imply for f the equation

$$f'' = -2 \frac{u'}{u} f' - \frac{1}{u},$$

which has because of the initial conditions for u and v the solution

$$f = 1 - \int_0^{\bar{\tau}} \left(\frac{1}{u^2} \int_0^{\tau'} u d\tau'' \right) d\tau'.$$

Since u is strictly increasing, it follows

$$\begin{aligned} 0 \leq 1 - \frac{v}{u} &= \int_0^{\bar{\tau}} \left(\frac{1}{u^2} \int_0^{\tau'} u d\tau'' \right) d\tau' \leq \int_0^{\bar{\tau}} \frac{1}{u^2} u \tau' d\tau' \leq \int_0^{\bar{\tau}} \frac{\tau'}{\cosh(\beta \tau')} d\tau' \\ &\leq 2 \int_0^{\bar{\tau}} \tau' e^{-\beta \tau'} d\tau' = \frac{2}{\beta^2} (1 - (\beta \bar{\tau} + 1) e^{-\beta \bar{\tau}}) \quad \text{for } \bar{\tau} \geq 0, \end{aligned}$$

which implies

$$0 \leq 1 - \frac{v}{u} \leq \frac{2}{\beta^2}. \quad (109)$$

A direct calculation gives $-1 < 2\beta^{-2} D_Z \beta < 0$ for $r > \check{r}_+$. It follows that

$$\epsilon_h(X, Z)_* + \left(1 - \frac{v}{u}\right) \beta_{,X} \geq \epsilon_h(X, Z)_* \quad \text{for } \frac{m}{2} < r \leq \check{r}_+,$$

$$\epsilon_h(X, Z)_* + \left(1 - \frac{v}{u}\right) \beta_{,X} \geq \epsilon_h(X, Z)_* - 1 = \frac{m r_* + \frac{m^2}{4}}{r_*^2} \quad \text{for } \check{r}_+ < r.$$

Since (62) gives under our assumptions

$$\Theta = \frac{\Theta_*}{\cosh^2\left(\frac{\beta}{2} \bar{\tau}\right)}, \quad (110)$$

(108) implies that for given $r > \frac{m}{2}$ there is a constant $c > 0$ such that

$$\Theta \epsilon_h(X, Z) \geq \Theta_* c \frac{\cosh(\beta \bar{\tau})}{\cosh^2\left(\frac{\beta}{2} \bar{\tau}\right)} \geq \Theta_* c.$$

In the region covered by (107) it suffices of course to get a lower estimate for $\epsilon_h(X, Z)$, because Θ is positive where the conformal geodesics approach the singularity. On the curves with $\bar{\tau} = \hat{r}$, which separate the domains where (103) and (107) are valid, $c^2 \equiv \beta^2 + k$ with $c = \text{const.} > 0$ and thus

$$\begin{aligned} \Theta \epsilon_h(X, Z) &= \Theta (\epsilon_h(X, Z)_* \cosh(c \bar{\tau}) + c^{-2} D_Z \beta (\cosh(c \bar{\tau}) - 1)) \\ &\geq \Theta_* \epsilon_h(X, Z)_* > 0. \end{aligned}$$

Since $\beta = 0$ but $k \geq k_* = (2m)^{-2}$, $D_Z \beta = m^{-2}$ on the curves starting with $r = \frac{m}{2}$, it follows that $\Theta = \Theta_* = (2m)^{-2}$ and a similar result is obtained.

4.4 Global numerical evolution for a class of standard Cauchy data

The methods described above offer a possibility to study in detail the complete numerical evolution of three-dimensional space-times without symmetries which are determined by certain asymptotically flat Cauchy data. In [1] has been shown the existence of smooth, time-symmetric, asymptotically flat solutions of the vacuum constraint which coincide with certain given time-symmetric initial data on compact sets and with Schwarzschild data in a neighbourhood of space-like infinity. If these data can be constructed numerically, it is easy to determine numerically hyperboloidal initial data implied by the Cauchy data.

Consider the time symmetric initial data set (82). The set \tilde{S} can be embedded by the transformation $r = \tan \frac{\chi}{2}$ into the 3-dimensional standard unit sphere $S = S^3$. With the conformal factor $\Theta_*^S = 2(1+r^2)^{-1}(1+\frac{m}{2r})^{-2} = 2 \cos^2 \frac{\chi}{2} (1 + \frac{m}{2} \cot \frac{\chi}{2})^{-2}$, the rescaled metric $(\Theta_*^S)^2 \tilde{h} = -(d\chi^2 + \sin^2 \chi d\sigma^2)$ coincides with the restriction of the standard metric on S^3 to the set $S \setminus \{i_0, i_\pi\}$, where $i_0 = \{\chi = 0\}$, $i_\pi = \{\chi = \pi\}$. If we set $m = 0$, we get Minkowski data and conformally compactified Minkowski data respectively.

For given χ_0 with $\frac{\pi}{2} < \chi_0 < \pi$ let ξ be a smooth function on \mathbb{R} such that $\xi(x) = 0$ for $x \leq \frac{\pi}{2}$, $\xi(x) = 1$ for $x \geq \chi_0$ and such that $\xi' \geq 0$. We define $\psi \in C^\infty(-\infty, \pi]$ by setting it equal to 1 for $x \leq \frac{\pi}{2}$ and equal to $1 + \xi(x) (\cot \frac{x}{2} - 1)$ for $\frac{\pi}{2} < x < \pi$. Then $\psi' \leq 0$ and $\frac{1}{2} \leq M \equiv \sup_{x < \pi} \frac{d\psi}{dx}(x) < \infty$.

Suppose we are given time symmetric initial data on \mathbb{R}^3 which agree with Schwarzschild data of mass $m < \frac{1}{M}$ near space-like infinity and are such that the metric \tilde{h} , suitably written in terms of the coordinates ϕ, θ, χ on S , satisfies

$$\tilde{h} = -\frac{(1 + \frac{m}{2} \cot \frac{\chi}{2})^4}{4 \cos^4 \frac{\chi}{2}} (d\chi^2 + \sin^2 \chi d\sigma^2) \quad \text{for } \chi > \chi_0. \quad (111)$$

The conformal factor

$$\Theta_* = \frac{2 \cos^2 \frac{\chi}{2}}{(1 + \frac{m}{2} \psi(\chi))^2}, \quad (112)$$

is smooth on $S \setminus \{i_\pi\}$, coincides for $\chi \geq \chi_0$ with Θ_*^S , has non-vanishing gradient in $S \setminus \{i_0, i_\pi\}$ and goes to zero at i_π . It defines a conformal compactification of the data such that $h = \Theta_*^2 \tilde{h}$ coincides with the standard metric of the 3-sphere for $\chi > \chi_0$. We choose initial data b_* for the 1-form field which annihilate the normals of the initial hypersurface and satisfy $b_* = \Theta_*^{-1} d\Theta_*$ in $S \setminus \{i_\pi\}$.

Since the time evolution of the data will be Schwarzschild near i_π , it can be determined there by the methods described above. It is clear that we can construct a smooth hyperboloidal hypersurface, which coincides with S on the set $\{\chi \leq \chi_0\}$ and extends to the future null infinity of the Schwarzschild part. It should not be difficult to determine the corresponding initial data for the conformal field equations, possibly by a numerical integration (as shown in [5], this reduces to solving a system of ODE's). Since there are codes available

to evolve such data numerically (cf. [2], [8] and the references given there) we could in principle calculate their evolution in time.

If such data, satisfying (111) for a fixed χ_0 , can be constructed for sufficiently small mass m they will be close to corresponding Minkowskian data and the result of [3] suggests that they will evolve into solutions possessing complete past and future null infinities and regular points i^\pm at future and past time-like infinity. If we use a gauge in which b_* picks up on $S \setminus \{i_\pi\}$ a suitable component in the direction of the tangent vectors of the conformal geodesics, it will be possible to construct a gauge of the type considered above which smoothly covers the complete future of the initial hypersurface as well as $\mathcal{J}^+ \cup \{i^+\}$. The work in [8] suggests that such solutions can be calculated numerically in their entirety.

The numerical calculation of such space-time is, of course, not an end in itself. In fact, the solution will have somewhat curious features. It follows from [6] that they will have vanishing Newman-Penrose constants. If the solution admit regular points i^+ , it then follows from [7] that the rescaled conformal Weyl tensor will vanish at those points. This situation, which is more special than the one considered in [8], suggests that the method of gluing a Schwarzschild end to a given solution of the constraints produces data of rather restricted radiation content. However, such calculations will allow us to perform detailed tests of the code under completely controlled assumptions and to study the robustness of the code and of the gauge conditions by choosing data with an increasing value of the mass which eventually yields the time evolution of a collapsing gravitational field.

5 Concluding remarks

We have described in detail the construction of a global system of conformal Gauss coordinates on the Schwarzschild-Kruskal solution which extend smoothly and without degeneracy through null infinity. Furthermore, we have shown that the conformal factor naturally associated with this system defines a smooth conformal extension of the Schwarzschild-Kruskal space-time which gives to null infinity a finite location in the new coordinates which is determined by our choice of initial data.

We did not try to work out in detail the behaviour of the fields in the conformal extension constructed here. An analysis of the fields near space-like infinity can be found in [5]. The behaviour of the fields near time-like infinity, which is of considerable interest for the numerical calculation of space-times, has still to be investigated.

There is a property of the Gauss system which we only indicated but which may turn out to be quite important. While the regularity of a conformal Gauss systems is essentially decided by its underlying conformal geodesics (considered here as point sets), there always exists a huge class of different time slicings based on one and the same underlying congruence of conformal geodesics. The consequences of this freedom still have to be explored.

Of course, there are many more such coordinate systems. It would be interesting to see whether the initial data for congruences of conformal geodesics which lead to such coordinates can be characterized in a general way.

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