Regularization parameters for the self-force in Schwarzschild spacetime: Scalar case

Leor Barack
Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, D-14476 Golm, Germany

Amos Ori
Department of Physics, Technion–Israel Institute of Technology, Haifa, 32000, Israel
(Received 10 June 2002; published 29 October 2002)

We derive the explicit values of all regularization parameters (RP) for a scalar particle in an arbitrary geodesic orbit around a Schwarzschild black hole. These RP are required within the previously introduced mode-sum method for calculating the local self-force acting on the particle. In this method, one first calculates the (finite) contribution to the self-force due to each individual multipole mode of the particle's field, and then applies a certain regularization procedure to the mode sum, involving the RP. The explicit values of the RP were presented in a recent paper [L. Barack *et al.*, Phys. Rev. Lett. **88**, 091101 (2002)]. Here we give the full details of the RP derivation in the scalar case. The calculation of the RP in the electromagnetic and gravitational cases will be discussed in an accompanying paper.

DOI: 10.1103/PhysRevD.66.084022 PACS number(s): 04.70.Bw, 04.25.Nx

I. INTRODUCTION

The space-based gravitational wave detector LISA (Laser Interferometer Space Antenna), scheduled for launch around 2011 [1], will open up a window for the low-frequency band below 1 Hz, allowing access to a variety of black hole sources. As one of its main targets, LISA is expected to detect the outburst of gravitational radiation emitted during the capture of a compact star by a supermassive black hole—a 10^5-10^7 solar masses black hole of the kind now believed to reside in the cores of many galaxies, including our own [2]. Designing accurate gravitational waveform templates for this type of astrophysical event requires an accurate knowledge of the orbital evolution, including the effect of radiation reaction. The evolution of such extreme mass-ratio systems can be modeled by considering a pointlike test particle moving in the fixed gravitational field of a black hole. One then addresses the question of the local *self-force* acting on this particle. (In special cases, one may study the orbital evolution under radiation reaction using global energy-momentum balance techniques [3]. However, such techniques appear insufficient when dealing with the astrophysically realistic case of nonequatorial eccentric orbits in Kerr spacetime.)

There exists a well established formal framework for calculating self-forces in curve spacetime: DeWitt and Brehme [4] first obtained a formal expression for the *electromagnetic* self-force. More recently, Mino, Sasaki, and Tanaka (MST) [5] have worked out the case of the *gravitational* self-force. [The same results, in both the electromagnetic and gravitational cases, were obtained by Quinn and Wald (QW) [6] using a different method.] The case of the scalar self-force was then analyzed by Quinn [7]. Recently, the two groups of Barack and Ori (BO) and Mino, Nakano, and Sasaki (MNS) have reported [8] on a practical method for implementing the above formal results, allowing actual calculations of the self-force for any geodesic orbit in Schwarzschild spacetime. The purpose of the present paper (together with the one accompanying it [9]) is to provide a full account of the method and results reported in [8].

The notion of self-forces is briefly described as follows. Consider a pointlike particle carrying a charge q, which may represent here a scalar charge, an electric charge, or a mass. The particle is assumed to move freely in the curved background of a black hole with mass $M \gg q$. In the limit $q \to 0$, such a particle is known to move along a geodesic of the background geometry. However, when endowed with a finite charge (or mass), the particle no longer traces a background geodesic, as a result of interaction with its own field. The finite-charge (or finite-mass) correction to the particle's motion is then described in terms of a "self-force": Treating the particle's field as a linear perturbation on the fixed black hole background, the particle's equation of motion is written as

$$\mu a_\alpha = F_\alpha^{\rm self}, \tag{1}$$

where μ is the particle's mass, a_α denotes its (covariant) four-acceleration, and $F_\alpha^{\rm self} \propto O(q^2)$ describes the leading-order self-force effect. (In the gravitational case, the four-acceleration, as well as the self-force, may be defined through a mapping of the particle's worldline into a trajectory in the background spacetime—see Ref. [10].) The formal construction of $F_\alpha^{\rm self}$ is described in [5,6] for the gravitational case, in [4,6] for the electromagnetic case, and in [7] for the scalar case. In all cases, the self-force is constructed through

$$F_\alpha^{\rm self} = \lim_{x\to z} F_\alpha^{\rm tail}(x) + \text{trivial local terms}, \tag{2}$$

where z represents a point on the particle's worldline where the self-force is being evaluated, x is a point in the neighborhood of z, and the local terms are given explicitly in [4–7] (they include the Abraham-Lorentz-Dirac force in the scalar and electromegnetic cases). The quantity $F_\alpha^{\rm tail}(x)$, the "tail" force, is a nonlocal contribution to the self-force, whose occurrence reflects the essential *nonlocal* nature of the radiation reaction effect in curved spacetime: waves emitted by the particle may backscatter off spacetime curvature and later interact back with their emitter. The tail force may formally be constructed through a worldline integral as [4,6,7]

$$F_\alpha^{\rm tail}(x) = \lim_{\epsilon\to 0^+} q\int_{-\infty}^{\tau_0-\epsilon} \hat{\nabla}_\alpha G[x,z(\tau)]d\tau, \tag{3}$$

where τ is the proper time along the particle's worldline, τ_0 is the value of τ at the intersection of the worldline with the past light cone of x, G symbolizes a Green's function for the particle's field, and $\hat{\nabla}_\alpha$ is a certain first-order differential operator acting on G [the explicit form of $\hat{\nabla}_\alpha$, as well as the type of the Green's function (whether a biscalar, a bivector, or a bitensor) depend on the case considered—see [4–7] for details]. Notably, when geodesics in vacuum spacetime are considered (which is often the situation, especially in the gravitational case), the tail force constitutes the *sole* contribution to the self-force. It is the actual evaluation of the tail part that has rendered practical calculations of the self-force most challenging.

It is instructive (and later useful) to write Eq. (3) in the form

$$F_\alpha^{\rm tail}(x) = F_\alpha^{\rm full}(x) - F_\alpha^{\rm dir}(x), \tag{4}$$

where $F_\alpha^{\rm full}(x)$ and $F_\alpha^{\rm dir}(x)$, the ''full'' and ''direct'' forces, are the quantities constructed by replacing the integral in Eq. (3) with $\int_{-\infty}^{\tau_0+\epsilon}$ and $\int_{\tau_0-\epsilon}^{\tau_0+\epsilon}$, respectively. The ''full'' force $F_\alpha^{\rm full}$ is directly obtained from the particle's ''full'' field by acting with $q\hat{\nabla}_\alpha$ [for the scalar case, e.g., see Eq. (15) below]. The ''direct'' force $F_\alpha^{\rm dir}$ is the ''divergent piece'' to be removed, which is associated with the instantaneous effect of waves propagating directly along the particle's light cone. Note that the ''tail'' force is hence attributed to waves scattered *inside* the particle's past light-cone.

A direct implementation of the MST and QW scheme for calculating the self-force in a weak field was introduced recently by Pfenning and Poisson [11]. To allow a practical implementation of this formal scheme for strong-field orbits, BO devised a multipole-mode decomposition method, relying directly on MST and QW's formal result (2). BO's *mode-sum method* was formulated first for the scalar self-force [12], and later for the gravitational self-force [13]. This method has been tested and fully implemented for calculating the scalar self-force in several cases [14,15]. The mode sum scheme (which we review in the next section) is based on decomposing the tail force into individual multipole-mode contributions, relating these contributions to the ''full force'' modes—which are accessible to standard numerical analysis—and then summing over the mode contributions, subject to a certain regularization procedure. This procedure requires knowledge of certain analytic parameters, the ''regularization parameters'' (RP), whose values depend on the orbit under consideration. The RP values were derived previously for a few special orbits in Schwarzschild spacetime: for radial and circular orbits in the scalar case [12] and for radial trajectories in the gravitational case [13,16]. These (rather cumbersome) calculations were carried out through a special local perturbative expansion of the Green's-function's multipole modes, relying directly on the integral formula (3).

In this paper, we present a different approach for the calculation of the RP, based on a direct multipole decomposition of the ''direct'' piece of force. This new approach (already outlined in [8]) allowed a rather convenient calculation of all RP values for a *general* geodesic orbit in Schwarzschild spacetime, as we describe in this paper. In particular, it provided an independent verification for the RP values in the special cases considered previously (using the l-mode Green's-function analysis as mentioned above). Two variants of the new calculation method were worked out independently by the two groups of BO and MNS, yielding the same RP values [8]. The calculation by MNS has been reported in [18]. This paper presents full details of the RP derivation by BO [17].

In its basis, the calculation method presented here is applicable to all three sorts of self-forces: scalar, electromagnetic, and gravitational. We find it most instructive to concentrate first on the scalar case, as a toy model. This model captures the essential parts of the calculation technique, while avoiding several complexities and delicate issues that show up in the gravitational and electromagnetic cases. In this paper, we thus focus on the scalar model, leaving the treatment of the gravitational and electromagnetic cases to an accompanying paper.

It should be commented that other approaches for the calculation of the self-force, not directly relying on the MST and QW formal scheme, were also suggested recently. Lousto [19] introduced an approach also based on a multipole decomposition but employing a proposed zeta-function regularization scheme. Other methods were proposed by Nakano and Sasaki [20] and Detweiler [21]. Most recently, Detweiler and Whiting [22] presented an alternative formulation of the self-force problem in curved spacetime, which was shown to yield the same result for the self-force as the previous MST-QW formulation. This new formulation provides an elegant physical interpretation of the self-force as the force applied by the ''radiative'' part of the particle's self-field.

The paper is arranged as follows. In Sec. II, we review the mode-sum method, and define the regularization parameters. The scalar toy model to be considered in this paper is introduced in Sec. III. The expression for the ''direct'' part of the scalar force is introduced and processed in Sec. IV, and is being formally decomposed into modes in Sec. V. We then prepare for the calculation of the RP by introducing a useful coordinate system, in Sec. VI. The main part of our calculation is contained in Sec. VII, where, through an investigation of the direct force's multipole modes, we obtain all RP values for a general trajectory in Schwarzschild spacetime. Section VIII summarizes the RP values, and Sec. IX provides some concluding remarks.

Throughout this paper we use geometrized units (with $G = c = 1$), and metric signature $-+++$.

II. REVIEWING THE MODE-SUM APPROACH

The mode-sum method was introduced in Ref. [12] for the scalar self-force, and in Ref. [13] for the gravitational self-force. Here we review it using a slightly different perspective (and notation).

In the mode-sum scheme, one first formally expands all three quantities $F_\alpha^{\rm tail}(x)$, $F_\alpha^{\rm full}(x)$, and $F_\alpha^{\rm dir}(x)$ appearing in Eq. (4) into multipole l modes as

$$F_\alpha^{\rm tail}(x)=\sum_{l=0}^{\infty} F_\alpha^{({\rm tail})l}(x),$$

$$F_\alpha^{\rm full}(x)=\sum_{l=0}^{\infty} F_\alpha^{({\rm full})l}(x), \tag{5}$$

$$F_\alpha^{\rm dir}(x)=\sum_{l=0}^{\infty} F_\alpha^{({\rm dir})l}(x)$$

(where, recall, x represents an off-worldline point in the neighborhood of the self-force evaluation point z). Here, $F_\alpha^{({\rm tail})l}$, $F_\alpha^{({\rm full})l}$, and $F_\alpha^{({\rm dir})l}$ are the quantities obtained by summing over all azimuthal numbers m (and, in the gravitational case, also over all ten tensor harmonics), for a given multipole number l. An important benefit of the multipole decomposition is the fact that, whereas $F_\alpha^{\rm full}$ and $F_\alpha^{\rm dir}$ both diverge at $x\to z$, their individual modes attain finite values even at the particle's location (though they are usually found to be discontinuous there). Applying the multipole decomposition to Eq. (4), we obtain

$$F_\alpha^{({\rm tail})l}(x)=F_\alpha^{({\rm full})l}(x)-F_\alpha^{({\rm dir})l}(x). \tag{6}$$

Considering now MST and QW's expression for the self-force, Eq. (2), we have

$$F_\alpha^{\rm self}=F_\alpha^{\rm tail}(x=z)=\sum_l F_\alpha^{({\rm tail})l}(x=z) \tag{7}$$

(hereafter we ignore the trivial local terms and focus on the tail contribution). Note that since the tail force $F_\alpha^{\rm tail}(x)$ is regular at the particle's location z [5,6], one gets $F_\alpha^{\rm self}$ by just evaluating the tail force at $x=z$. We can then write, using Eq. (6),

$$F_\alpha^{\rm self}=\sum_l \lim_{x\to z}F_\alpha^{({\rm tail})l}(x)=\sum_l \left[\lim_{x\to z}F_\alpha^{({\rm full})l}(x)-\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)\right], \tag{8}$$

where the direction of the limit $x\to z$ is considered as *prescribed*. It is important to note here that each of the two limits $\lim_{x\to z}F_\alpha^{({\rm full})l}(x)$ and $\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)$ is, in general, directional dependent. This, however, does not pose a problem (and the third equality in the above chain of equalities is valid) if the direction of the limit is prescribed: one then only has to make sure that the two limits of the full and direct forces are taken in a consistent manner (i.e., from the same direction).

In the last expression of Eq. (8), the sum over l modes is guaranteed to converge [as $F_\alpha^{\rm tail}(x)$ is a regular function]. However, the individual sums over the full-force modes and over the direct-force modes usually diverge. Suppose now that one could construct a function h_α^l that would make the sum $\Sigma_l[\lim_{x\to z}F_\alpha^{({\rm full})l}(x)-h_\alpha^l]$ convergent. Then, we would have [continuing the chain of equalities (8)]

$$F_\alpha^{\rm self}=\sum_l \left[\left(\lim_{x\to z}F_\alpha^{({\rm full})l}(x)-h_\alpha^l\right)-\left(\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)-h_\alpha^l\right)\right]$$
$$=\sum_l \left(\lim_{x\to z}F_\alpha^{({\rm full})l}(x)-h_\alpha^l\right)-\sum_l \left(\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)-h_\alpha^l\right). \tag{9}$$

In principle, the "regularization function" h_α^l is to be obtained by exploring the behavior of the full-force modes at large l. However, this function can also be deduced by analyzing the large-l behavior of the local quantity $F_\alpha^{({\rm dir})l}$—a task accessible to analytic treatment. In all cases considered so far, the function h_α^l was found to have the general form

$$h_\alpha^l=A_\alpha L+B_\alpha+C_\alpha/L, \tag{10}$$

with $L\equiv l+1/2$, and where A_α, B_α, and C_α are l-independent coefficients whose values depend on the details of the trajectory under consideration. Defining

$$D_\alpha\equiv\sum_{l=0}^{\infty}\left[\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)-A_\alpha L-B_\alpha-C_\alpha/L\right], \tag{11}$$

we finally get from Eq. (9)

$$F_\alpha^{\rm self}=\sum_{l=0}^{\infty}\left[\lim_{x\to z}F_\alpha^{({\rm full})l}(x)-A_\alpha L-B_\alpha-C_\alpha/L\right]-D_\alpha. \tag{12}$$

Equation (12) constitutes the basic formula for constructing the self-force through the mode-sum method. The four quantities A_α, B_α, C_α, and D_α are called the "regularization parameters" (RP). The full modes $F_{\alpha l}^{\rm full}$, recall, are directly obtained from the "full" field modes [see Eq. (15) below for the construction of the full force in the scalar case], which, in turn, are calculated using standard numerical techniques. Equation (12) thus describes a practical scheme for constructing the self force, given the values of the RP.

In this paper (dealing with the scalar self-force) and in the accompanying paper (dealing with the gravitational and electromagnetic self-forces) we derive the values of all RP needed for implementing Eq. (12) for any geodesic orbit in Schwarzschild spacetime.

III. SCALAR TOY MODEL

We consider a particle of a scalar charge q, moving freely in the vacuum exterior of a Schwarzschild black hole with mass $M\gg q$. In the lack of self-force, the particle moves along a geodesic $z^\mu(\tau)$ with specific energy and angular momentum parameters $\mathcal{E}$ and $\mathcal{L}$, respectively. We shall consider the self-force acting on the particle at a point along its worldline which we denote by $z\equiv(t_0,r_0,\theta_0,\varphi_0)$ (where t,r,θ,φ

are the standard Schwarzschild coordinates). Let also $x \equiv (t,r,\theta,\varphi)$ denote a point in the close neighborhood of z.

The particle induces a scalar field $\Phi^{\rm full}(x)$, which we shall treat as a linear perturbation over the fixed Schwarzschild background. In our model, the field $\Phi^{\rm full}(x)$ is assumed to satisfy the (minimally coupled) Klein-Gordon equation

$$\Box \Phi^{\rm full} \equiv \Phi^{{\rm full};\alpha}_{;\alpha} = -4\pi\rho, \tag{13}$$

where a semicolon denotes covariant differentiation with respect to the background geometry, and the scalar charge density is given by

$$\rho(x) = q\int_{-\infty}^{\infty} \delta^4[x - z(\tau)](-g)^{-1/2} d\tau \tag{14}$$

(g being the metric determinant). We now define the "full force" as the vector field

$$F_\alpha^{\rm full}(x) \equiv q\Phi^{\rm full}_{,\alpha}. \tag{15}$$

Note that both the full field $\Phi^{\rm full}(x)$ and the full force $F_\alpha^{\rm full}(x)$ obviously diverge on the worldline, but are otherwise well defined.

The force definition (15) complies with Quinn's definition [7]. It differs from the expression used by MNS, which involves a spatial projection of the scalar force [see Eq. (1.3) in [18]]. We prefer to adopt here the force definition (15) for several reasons. (i) It is a simpler definition, which nevertheless serves as an effective toy model for the realistic gravitational case. (ii) It avoids the need to consider an off-worldline extension of the four-velocity, as necessary for defining the spatially projected force. (iii) The force model (15) is naturally derived from a Lagrangian formalism, and is hence consistent with global stress-energy conservation—unlike the spatially projected force [7,23].

Finally, we introduce the notions of the "direct" field $\Phi^{\rm dir}$ and the "tail" field $\Phi^{\rm tail} = \Phi^{\rm full} - \Phi^{\rm dir}$ (see [7,18]), from which the direct and tail forces are derived by

$$F_\alpha^{\rm dir}(x) = q\Phi^{\rm dir}_{,\alpha}, \quad F_\alpha^{\rm tail}(x) = q\Phi^{\rm tail}_{,\alpha}. \tag{16}$$

Recall that the "direct" field is the part of the scalar field propagated directly along the particle's light cone, while the "tail" part is associated with reflections of the field *inside* the light cone.

IV. DIRECT FORCE: PRELIMINARIES

The form of the direct scalar field $\Phi^{\rm dir}$ was worked out by MNS [18] (see also some preliminary results in [24]), by studying the Hadamard expansion of the field equation. Let $\epsilon(x)$ denote the spatial geodesic distance from the point x to the geodesic $z(\tau)$ (i.e., the length of the short geodesic section connecting x to the worldline and normal to it), and let $\delta x^\mu \equiv x^\mu - z^\mu$. Then, the direct scalar field obtained by MNS can be written in the form

$$\Phi^{\rm dir}(x) = \frac{q\hat{f}(\delta x)}{\epsilon} + {\rm const}, \tag{17}$$

where $\hat{f}$ is a regular function of δx (and z) satisfying

$$\hat{f} = 1 + O(\delta x^2) \tag{18}$$

(the explicit form of f will not be needed in the analysis below). Introducing the squared geodesic distance $S(\delta x) \equiv \epsilon^2$, the direct scalar force is then given by

$$F_\alpha^{\rm dir}(x) = q\Phi^{\rm dir}_{,\alpha} = q^2[\hat{f}_{,\alpha} S^{-1/2} - (\hat{f}/2) S^{-3/2} S_{,\alpha}]. \tag{19}$$

Consider now the Taylor expansion of the function $S(\delta x)$ about $\delta x = 0$. We write this expansion as

$$S = S_0 + S_1 + S_2 + \cdots, \tag{20}$$

where $S_0, S_1, \ldots$ represent terms of homogeneous orders $\delta x^2, \delta x^3, \ldots$, respectively. Note that this decomposition of S is no longer covariant, and the individual terms S_n will depend on the choice of coordinate system. Below we shall need only the two leading terms, S_0 and S_1, which we obtain in Appendix A. We find

$$S_0 = (g^0_{\mu\nu} + u_\mu u_\nu)\,\delta x^\mu \delta x^\nu, \tag{21a}$$

$$S_1 = (u_\lambda u_\gamma \Gamma^{\lambda 0}_{\alpha\beta} + g^0_{\alpha\beta,\gamma}/2)\,\delta x^\alpha \delta x^\beta \delta x^\gamma, \tag{21b}$$

where $u^\alpha \equiv dx^\alpha/d\tau$ is the four-velocity at z, and $\Gamma^{\lambda 0}_{\alpha\beta}$ and $g^0_{\alpha\beta}$ denote, respectively, the connection coefficients and metric functions evaluated at $\delta x = 0$ (namely, at $x = z$). Substituting Eqs. (18) and (21) in Eq. (19), we now obtain a Taylor expansion for the direct force, which we may express as

$$F_\alpha^{\rm dir}(x) = q^2[\epsilon_0^{-3} P_\alpha^{(1)} + \epsilon_0^{-5} P_\alpha^{(4)} + \epsilon_0^{-7} P_\alpha^{(7)} + \cdots]. \tag{22}$$

Here, $\epsilon_0 \equiv S_0^{1/2}$, and $P_\alpha^{(n)}$ denote terms of homogeneous order $O(\delta x^n)$. Note that the term of the form $\propto \epsilon^{-1}\delta x$ in Eq. (19) can be written as $\propto \epsilon^{-7}(\epsilon^6 \delta x) \propto \epsilon^{-7}\delta x^7$ and then be absorbed in the term $\epsilon_0^{-7} P_\alpha^{(7)}$. Similarly, terms of the form $\propto \epsilon^{-3}\delta x^2$ may be expressed as $\propto \epsilon^{-5}\delta x^4$ and be absorbed in $\epsilon_0^{-5} P_\alpha^{(4)}$, and so on. Note also that the three terms presented in Eq. (22) are of orders δx^{-2}, δx^{-1}, and δx^0, respectively. The three dots $(\cdots)$ in that equation represent terms that vanish in the limit $\delta x \to 0$ [such as, e.g., $\epsilon_0^{-9} P_\alpha^{10} \propto O(\delta x)$]. In the following analysis, we shall need the explicit values of only $P_\alpha^{(1)}$ and $P_\alpha^{(4)}$, which are given by

$$P_\alpha^{(1)} = -\frac{1}{2} S_{0,\alpha}, \tag{23a}$$

$$P_\alpha^{(4)} = -\frac{1}{2} S_0 S_{1,\alpha} + \frac{3}{4} S_{0,\alpha} S_1. \tag{23b}$$

Now, in constructing the self-force, one is merely concerned with the behavior of the direct force at $x \to z$—see, e.g., Eq. (8). Thus, the terms represented by the three dots $(\cdots)$ in Eq. (22), which vanish in the limit $x \to z$, are irrelevant for calculating the self-force, and may be ignored in our analysis. We hence introduce a "revised" version of the direct force by omitting these terms (retaining, though, the notation $F_\alpha^{\rm dir}$),

$$F_\alpha^{\rm dir}=q^2[F_\alpha^{(A)}+F_\alpha^{(B)}+F_\alpha^{(C)}], \tag{24}$$

where

$$F_\alpha^{(A)}\equiv\epsilon_0^{-3}P_\alpha^{(1)},\quad F_\alpha^{(B)}\equiv\epsilon_0^{-5}P_\alpha^{(4)},\quad F_\alpha^{(C)}\equiv\epsilon_0^{-7}P_\alpha^{(7)}. \tag{25}$$

Note that this splitting of $F_\alpha^{\rm dir}$ holds for any choice of coordinates x^μ which are sufficiently regular in the neighborhood of z (though the coefficients $P_\alpha^{(n)}$ will depend on the choice of coordinates).

V. MULTIPOLE DECOMPOSITION

Next, we consider the multipole decomposition of $F_\alpha^{\rm dir}$. Let

$$F_\alpha^{\rm dir}(x)=\sum_{lm} F_\alpha^{lm}(r,t)Y^{lm}(\theta,\varphi), \tag{26}$$

where $Y^{lm}(\theta,\varphi)$ are spherical harmonics. We denote by $F_{\pm\alpha}^{l}$ the total l-mode contribution to the direct force at z,

$$F_{\pm\alpha}^{l}\equiv\lim_{\delta r\to 0^\pm}\sum_m F_\alpha^{lm}(r,t_0)Y^{lm}(\theta_0,\varphi_0). \tag{27}$$

Note that $F_{\pm\alpha}^{l}=\lim_{x\to z}F_\alpha^{({\rm dir})l}(x)$ [as in Eq. (8), e.g.], where the direction of the limit is explicitly specified such that x approaches z ''from the radial direction.'' The $\pm$ sign corresponds to the two possible radial limits, $r\to r_0^+$ or rather $r\to r_0^-$. This choice of taking the radial limit appears most convenient in our multipole-mode scheme. In particular, it is most easily implemented in the (numerical) calculation of the full force modes (recall that the limit $x\to z$ of both the direct and full forces must be taken from the same direction).

Equation (27) is invariant under rotation in the subspace of angular coordinates θ,φ. We take advantage of this property, and redefine the angular coordinates such that z is located at the pole, i.e., $\theta_0=0$. Due to angular-momentum conservation, the particle is now confined to move on a plane of constant φ, which we take as $\varphi=0,\pi$ (the value of the φ coordinate is fixed along the particle's trajectory, apart from a ''jump'' at the two poles $\theta=0,\pi$). The particle's four-velocity now satisfies $u^\varphi=0$.

The above setup is beneficial in that the l mode $F_{\pm\alpha}^{l}$ is now composed of only the axially symmetric $m=0$ harmonic: Recall that Y^{lm} vanishes at $\theta=0$ for any $m\neq0$, and $Y^{l,m=0}(\theta=0)=[L/(2\pi)]^{1/2}P_l(1)$, where $P_l(\cos\theta)$ is the Legendre polynomial and $P(1)=1$. Consequently, we find from Eq. (27)

$$F_{\pm\alpha}^{l}=\lim_{\delta r\to 0^\pm}[L/(2\pi)]^{1/2}F_\alpha^{l,m=0}(r,t_0). \tag{28}$$

The mode $F_\alpha^{l,m=0}$ is given by the integral

$$F_\alpha^{l,m=0}(r,t)=\int F_\alpha^{\rm dir}(r,t,\theta,\varphi)[Y^{l,m=0}]^*d\Omega$$
$$=[L/(2\pi)]^{1/2}\int F_\alpha^{\rm dir}(r,t,\theta,\varphi)P_l(\cos\theta)d\Omega, \tag{29}$$

where $d\Omega\equiv d\cos\theta d\varphi$ and the asterisk denotes complex conjugation. Combining Eqs. (28) and (29), we finally obtain the following integral expression for the total l-mode direct force:

$$F_{\pm\alpha}^{l}=\lim_{\delta r\to 0^\pm}\frac{L}{2\pi}\int F_\alpha^{\rm dir}(r,t_0,\theta,\varphi)P_l(\cos\theta)d\Omega. \tag{30}$$

VI. REGULAR COORDINATE SYSTEM

The coordinate system (t,r,θ,φ) is singular at $\theta=\theta_0=0$. This singularity makes the expansion (20), (21) inapplicable in these coordinates. To overcome this difficulty, we introduce the two ''locally Cartesian angular coordinates''

$$x=\rho(\theta)\cos\varphi,\quad y=\rho(\theta)\sin\varphi, \tag{31}$$

where $\rho(\theta)$ is a sufficiently regular, odd function of θ, admitting the expansion

$$\rho(\theta)=\theta+\rho_1\theta^3+\rho_2\theta^5+\cdots. \tag{32}$$

For later convenience we shall also demand that $\rho(\theta)$ grows monotonously within the entire domain $0\leqslant\theta<\pi$, such that $\rho(\theta)$ is invertible. [An obvious natural choice would be $\rho=\theta$; however, later we shall make the specific choice $\rho(\theta)=2\sin(\theta/2)$ which will simplify our calculations.]

Using the relations $d\rho/d\theta=1+\rho^2h_1(\rho^2)$ and $\rho^2/\sin^2\theta=1+\rho^2h_2(\rho^2)$ [easily followed from the above definition of $\rho(\theta)$], where h_1 and h_2 are both regular functions of ρ^2, one finds that the contravariant components of the metric tensor now take the form

$$g^{xx}=r^{-2}(1+x^2h_1+y^2h_2),$$
$$g^{yy}=r^{-2}(1+y^2h_1+x^2h_2),$$
$$g^{xy}=r^{-2}(h_1-h_2)xy. \tag{33}$$

The point z is located at $x=y=0$. The above tensor $g^{\alpha\beta}$ is perfectly regular in the neighborhood of this point—and so is the covariant metric $g_{\alpha\beta}$. In the particle's location itself, $x=y=0$, the line element takes the simple form

$$g_{xx}^0=g_{yy}^0=r_0^2,\quad g_{xy}^0=0 \tag{34}$$

[along with $g_{tt}^0=-(1-2M/r_0)$ and $g_{rr}^0=(1-2M/r_0)^{-1}$].

Note that the particle's geodesic is confined to $y=0$ and correspondingly $u^y=0$. Also, since z is located at $x=y=0$, we have $\delta x^x=x$, $\delta x^y=y$. Finally, we comment that the particle's angular momentum is given as $\mathcal{L}=u_x$ *evaluated at* z (but note that u_x is *not* conserved along the geodesic).

VII. INVESTIGATING THE *l* MODE OF THE DIRECT FORCE

A. Is the $x \to z$ limit interchangeable with the Legendre integral?

We now explore in more detail the *l*-mode direct force $F^l_{\pm\alpha}$, based on the integral formula (30). Recalling that the direct force itself is composed of three terms, Eq. (24), we write

$$F^l_{\pm\alpha} = q^2[F^{(A)l}_\alpha + F^{(B)l}_\alpha + F^{(C)l}_\alpha], \tag{35}$$

where $F^{(W)l}_\alpha$ (W standing for A, B, or C) denotes the contribution to $F^l_{\pm\alpha}$ [through Eq. (30)] due to the term $F^{(W)}_\alpha$ of the direct force,

$$F^{(W)l}_{\pm\alpha} = \lim_{\delta r\to 0^\pm} \frac{L}{2\pi}\int F^{(W)}_\alpha(r,t_0,\theta,\varphi)P_l(\cos\theta)d\Omega. \tag{36}$$

Recall that the various terms $F^{(W)}_\alpha$ are given in Eq. (25).

The task of evaluating the various contributions $F^{(W)l}_{\pm\alpha}$ would be much simplified if we could interchange the limit $\delta r \to 0^\pm$ and the integration in Eq. (36). Is such an interchange allowed? In Appendix B we address this question, and show that interchanging the $\delta r \to 0^\pm$ limit and the Legendre integral is indeed allowed for $W=B$ and $W=C$; however, as evident from the explicit calculation below, such an interchange is not valid for $W=A$. Here we present a heuristic argument suggesting why the interchange is valid for $W=B,C$, and why it might fail for $W=A$. A sketch of a mathematical proof is provided in Appendix B.

For a given small separation $\delta x = x - z$, assume that all components of δx^α scale as δr (we assume $\delta r \neq 0$). Since ϵ_0 then scales like δr too, we find that the various terms $F^{(W)}_\alpha$ scale as

$$F^{(A)}_\alpha = \epsilon_0^{-3}P^{(1)}_\alpha \propto \delta r^{-2},$$

$$F^{(B)}_\alpha = \epsilon_0^{-5}P^{(4)}_\alpha \propto \delta r^{-1}, \tag{37}$$

$$F^{(C)}_\alpha = \epsilon_0^{-7}P^{(7)}_\alpha \propto \delta r^{0},$$

where the proportion coefficients only depend on the "direction" of δx^α (i.e., on the ratios between its various components). To consider the interchangeability of the limit and integral in Eq. (36), one is mainly concerned with the contribution to the integral from small x,y values (i.e., from the immediate neighborhood of the integrand's singular point z). To find out how this small piece of integral scales with δr, we consider the small integration area around z, in the xy plane, defined by $\rho = (x^2+y^2)^{1/2} < \delta r$ (for a given $\delta r \neq 0$). Observing that this integration area scales like δr^2 and relying on the scale relations (37), one finds that this small-δr contribution to the integral scales like δr^0 for $F^{l(A)}_{\pm\alpha}$, like δr^1 for $F^{l(B)}_{\pm\alpha}$, and like δr^2 for $F^{l(C)}_{\pm\alpha}$. Namely, upon taking the limit $\delta r \to 0$, the small-δr piece of integration vanishes for $W=B,C$, but not for $W=A$. This suggests that we may interchange the limit and integration for $W=B,C$, but not for $W=A$. A more rigorous mathematical treatment implies that this is indeed the case (see Appendix B).

We are thus allowed to write

$$F^{l(B,C)}_\alpha = \frac{L}{2\pi}\int F^{(B,C)}_\alpha(r_0,t_0,x,y)P_l(\cos\theta)d\Omega. \tag{38}$$

However, for $W=A$ we must use the original expression,

$$F^{l(A)}_{\pm\alpha} = \lim_{\delta r\to 0^\pm} \frac{L}{2\pi}\int F^{(A)}_\alpha(r,t_0,x,y)P_l(\cos\theta)d\Omega. \tag{39}$$

For later convenience we give here explicitly the form of $F^{(B,C)}_\alpha$ for $r=r_0, t=t_0$. We have

$$F^{(B)}_\alpha = \hat{\epsilon}_0^{-5}P^{(4)}_\alpha(x,y), \quad F^{(C)}_\alpha = \hat{\epsilon}_0^{-7}P^{(7)}_\alpha(x,y), \tag{40}$$

where $P^{(n)}_\alpha(x,y)$ is a polynomial of homogeneous order n in x and y, and $\hat{\epsilon}_0$ is the reduction of ϵ_0 to $\delta r = \delta t = 0$: We find, recalling $\rho^2 = x^2+y^2$ and $u_y = 0$,

$$\hat{\epsilon}_0 = (r_0^2\rho^2 + u_x^2x^2)^{1/2}. \tag{41}$$

Note that $\hat{\epsilon}_0$ is an even function of both x and y—a fact that will play a crucial role in the analysis below.

B. Calculating $F^{l(C)}_\alpha$

Let us first evaluate the term $F^{l(C)}_\alpha$. We observe that the integrand in Eq. (38) is composed of three factors: $\hat{\epsilon}_0^{-7} \times P^{(7)}_\alpha(x,y) \times P_l(\cos\theta)$. Since $\hat{\epsilon}_0$ and $\cos\theta(\rho)$ are even function of both x and y, then so are the factors $\hat{\epsilon}_0^{-7}$ and $P_l(\cos\theta)$. However, each of the eight terms of $P^{(7)}_\alpha(x,y)$ (proportional to $x^7y^0, x^6y, \ldots, x^0y^7$) is of odd power in either x or y. Hence, the overall integrand in Eq. (38) is composed only of terms which are odd in either x or y. As a consequence, the integral is found to vanish identically, yielding

$$F^{l(C)}_\alpha = 0. \tag{42}$$

C. Calculating $F^{l(B)}_\alpha$

We next turn to consider $F^{l(B)}_\alpha$. The integrand in Eq. (38) now takes the form $\hat{\epsilon}_0^{-5} \times P^{(4)}_\alpha(x,y) \times P_l(\cos\theta)$. Note that the polynomial $P^{(4)}_\alpha(x,y)$ may now contain terms which are even in both x and y, yielding, in general, a nonvanishing contribution to the integral. To proceed, one thus has to be provided with the explicit form of $P^{(4)}_\alpha$.

The explicit form of the polynomial $P^{(4)}_\alpha(x,y)$ is obtained by substituting for S_0 and S_1 (and their gradients) from Eqs. (21) in Eq. (23b), taking $\delta r = \delta t = 0$ and recalling $\delta x = x$, $\delta y = y$, and $u_y = 0$. One thereby obtains

$$P^{(4)}_\alpha(x,y) = P^{(x)}_\alpha x^4 + P^{(xy)}_\alpha x^2y^2 + P^{(y)}_\alpha y^4 \quad \text{for } \alpha = t,r,x, \tag{43a}$$

$$P^{(4)}_y(x,y) = P^{(x)}_y x^3y + P^{(y)}_y xy^3, \tag{43b}$$

where the various coefficients are explicitly given by

$$P_r^{(x)}=-\frac{1}{2}[f^{-1}\dot r^2 r_0(2u_x^2-r_0^2)+r_0^{-1}(2u_x^4+3u_x^2r_0^2+r_0^4)],$$

$$P_r^{(xy)}=-\frac{1}{2}r_0[3u_x^2+2r_0^2+2f^{-1}\dot r^2(u_x^2-r_0^2)],$$

$$P_r^{(y)}=-\frac{1}{2}r_0^3(1-f^{-1}\dot r^2), \tag{44}$$

$$P_t^{(x)}=-r_0u_t\dot r(u_x^2-r_0^2/2), \quad P_t^{(xy)}=-r_0u_t\dot r(u_x^2-r_0^2),$$

$$P_t^{(y)}=\frac{1}{2}r_0^3u_t\dot r, \tag{45}$$

$$P_x^{(x)}=0, \quad P_x^{(xy)}=\frac{1}{2}r_0u_x\dot r(r_0^2-2u_x^2), \quad P_x^{(y)}=\frac{1}{2}r_0^3u_x\dot r, \tag{46}$$

$$P_y^{(x)}=r_0u_x\dot r(u_x^2-r_0^2/2), \quad P_y^{(y)}=-\frac{1}{2}r_0^3u_x\dot r. \tag{47}$$

In these expressions $f\equiv(1-2M/r_0)$, $\dot r\equiv u^r$, and all four-velocity components are evaluated at z. Note that the components $P_t^{(4)}$, $P_r^{(4)}$, and $P_x^{(4)}$ consist of only terms which are *even* in both x and y. On the other hand, the y component $P_y^{(4)}$ contains only terms which are *odd* in both coordinates.

Consider first the y component: Both terms $\propto x^3y$ and $\propto xy^3$ of the polynomial $P_y^{(4)}$ yield, upon integrating, no contribution to $F_y^{l(B)}$, and one immediately obtains

$$F_y^{l(B)}=0. \tag{48}$$

The other components of $F_\alpha^{l(B)}$ do not similarly vanish: Recalling $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$, and expressing $\hat\epsilon_0$ in the form $\hat\epsilon_0=r_0\rho(\theta)(1+r_0^{-2}u_x^2\cos^2\varphi)^{1/2}$, we may write the double integral in Eq. (38) in the factorized form

$$F_\alpha^{l(B)}=r_0^{-5}I^\theta I_\alpha^\varphi, \tag{49}$$

where

$$I^\theta\equiv\frac{L}{2\pi}\int_{-1}^{1}\frac{P_l(\cos\theta)}{\rho(\theta)}d\cos\theta, \tag{50a}$$

$$I_\alpha^\varphi\equiv\int_0^{2\pi}\frac{P_\alpha^{(x)}\cos^4\varphi+P_\alpha^{(xy)}\cos^2\varphi\sin^2\varphi+P_\alpha^{(y)}\sin^4\varphi}{(1+r_0^{-2}u_x^2\cos^2\varphi)^{5/2}}d\varphi. \tag{50b}$$

We now take advantage of the freedom we still have in specifying the function $\rho(\theta)$, and make the convenient choice

$$\rho=2\sin(\theta/2). \tag{51}$$

With this choice, the integral I^θ becomes a standard one, reading simply

$$I_\theta=(2\pi)^{-1} \tag{52}$$

[see, e.g., Eq. (7.225-3) of [25]]. The integral I_α^φ, in turn, is a linear combination of standard elliptic integrals. It can be expressed as

$$I_\alpha^\varphi=a_\alpha\hat K(w)+b_\alpha\hat E(w), \tag{53}$$

where $\hat K(w)$ and $\hat E(w)$ are two complete elliptic integrals of the first and second kinds, respectively, the argument w is given by

$$w\equiv\frac{u_x^2}{r_0^2+u_x^2}, \tag{54}$$

and the coefficients a_α and b_α read

$$a_\alpha=-\frac{4}{3}(r_0/u_x)^3w^{-1/2}[a^{(x)}P_\alpha^{(x)}+a^{(xy)}P_\alpha^{(xy)}+a^{(y)}P_\alpha^{(y)}], \tag{55}$$

$$b_\alpha=-\frac{4}{3}(r_0/u_x)w^{-3/2}[b^{(x)}P_\alpha^{(x)}+b^{(xy)}P_\alpha^{(xy)}+b^{(y)}P_\alpha^{(y)}],$$

with

$$\begin{aligned} a^{(x)}&=(w+2)(w-1),\\ a^{(xy)}&=-2(w-1),\\ a^{(y)}&=(3w-2),\\ b^{(x)}&=2(w-1)^2(w+1),\\ b^{(xy)}&=-(w-2)(w-1),\\ b^{(y)}&=2(1-2w). \end{aligned} \tag{56}$$

The explicit form of the desired contribution $F_\alpha^{l(B)}$ (for $\alpha=r,t,x$) is finally obtained by inserting the values of $P_\alpha^{(x)}$, $P_\alpha^{(xy)}$, and $P_\alpha^{(y)}$ [given in Eqs. (44)–(47)] in the above expressions for a_α and b_α, constructing I_α^φ through Eq. (53), and substituting in Eq. (49). This yields

$$F_r^{l(B)}=\frac{1}{r_0^2}\frac{(\dot r^2-2u_t^2)\hat K(w)+(\dot r^2+u_t^2)\hat E(w)}{\pi fV^{3/2}}, \tag{57a}$$

$$F_t^{l(B)}=\frac{1}{r_0^2}\frac{u_t\dot r[\hat K(w)-2\hat E(w)]}{\pi V^{3/2}}, \tag{57b}$$

$$F_x^{l(B)}=\frac{1}{r_0}\frac{\dot r[\hat K(w)-\hat E(w)]}{\pi(u_x/r_0)V^{1/2}}, \tag{57c}$$

where $V\equiv1+u_x^2/r_0^2$.

Note the remarkable fact that the contribution $F_\alpha^{l(B)}$ is *independent of l*.

D. Calculating $F_{\pm\alpha}^{l(A)}$

Finally, let us evaluate $F_{\pm\alpha}^{l(A)}$. Recalling $F_{\alpha}^{(A)}=\epsilon_0^{-3}P_{\alpha}^{(1)}$ and using Eqs. (23a) and (21a), Eq. (39) becomes

$$F_{\pm\alpha}^{l(A)}=-[L/(2\pi)](g^0_{\alpha\beta}+u_\alpha u_\beta)\tilde{F}_{\pm}^{l\beta}, \tag{58}$$

where

$$\tilde{F}_{\pm}^{l\beta}\equiv \lim_{\delta r\to 0^{\pm}}\int \delta x^{\beta}\epsilon_0^{-3}P_l(\cos\theta)d\Omega. \tag{59}$$

Note that we have already taken here the limit $\delta t\to 0$, hence the integrand ($\propto \delta x^{\beta}$) vanishes identically for $\beta=t$. Also, since $\cos\theta(\rho)$ and ϵ_0, given explicitly by

$$\epsilon_0=[r_0^2\rho^2+g^0_{rr}\delta r^2+(u_r\delta r+u_x x)^2]^{1/2}, \tag{60}$$

are both even functions of y, the integral in Eq. (59) obviously vanishes for $\beta=y$. Hence,

$$\tilde{F}_{\pm}^{lt}=\tilde{F}_{\pm}^{ly}=0. \tag{61}$$

Consider now Eq. (59) for the two remaining components, $\beta=r,x$. First, we change the integration variables to x,y. Since the Jacobian is $\partial(\theta,\varphi)/\partial(x,y)=(\rho\rho')^{-1}$ (where $\rho'=d\rho/d\theta$), Eq. (59) becomes

$$\tilde{F}_{\pm}^{l\beta}\equiv \lim_{\delta r\to 0^{\pm}}\int \delta x^{\beta}\epsilon_0^{-3}H(\rho)dxdy, \tag{62}$$

where $H(\rho)\equiv P_l(\cos\theta)\sin\theta(\rho\rho')^{-1}$. The function $H(\rho)$ is a regular, even function of θ (and of ρ), with $H(0)=1$. We thus write it as $H(\rho)=1+\rho^2\hat{H}(\rho)$, where the function $\hat{H}(\rho)$ admits a regular (even) Taylor expansion at $\rho=0$. Accordingly, we divide $\tilde{F}_{\pm}^{l\beta}$ into two contributions,

$$\tilde{F}_{\pm}^{l\beta}=\lim_{\delta r\to 0^{\pm}}(I_1^{\beta}+I_2^{\beta}), \tag{63}$$

where

$$I_1^{\beta}\equiv\int \delta x^{\beta}\epsilon_0^{-3}dxdy,$$
$$I_2^{\beta}\equiv\int \delta x^{\beta}\epsilon_0^{-3}\rho^2\hat{H}(\rho)dxdy. \tag{64}$$

Consider first the contribution I_2^{β}: Near $\rho=0$, the integrand in this term scales like δr^0, thus the integrated singular contribution scales like δr^2. Hence, based on precisely the same argument applied in Appendix B with regard to the term $F_{\alpha}^{(C)}$, we find that the integral I_2^{β} is sufficiently regular to allow us to interchange the orders of the $\delta r\to 0$ limit and integration,

$$\lim_{\delta r\to 0^{\pm}} I_2^{\beta}=I_2^{\beta}(\delta r=0). \tag{65}$$

Doing so, we find that the contribution from I_2^{β} to $\tilde{F}_{\pm}^{l\beta}$ vanishes for either $\beta=r$ or $\beta=x$: For $\beta=r$, the integrand vanishes identically; for $\beta=x$, the integrand, evaluated at $\delta r=0$, becomes an odd function of x [see Eq. (41)], which vanishes upon integrating.

To calculate the remaining contribution I_1^{β}, we divide the domain of integration in Eq. (64) into two regions: Let $H^{\rm in}$ denote the square $-h<x,y<h$, for some particular $0<h<1$ (say, $h=1/10$), and let $H^{\rm out}$ denote the remaining integration area over the sphere, outside $H^{\rm in}$. Correspondingly, we divide the integral I_1^{β} into two contributions, as $I_1^{\beta}=I_1^{\beta{\rm in}}+I_1^{\beta{\rm out}}$. Now, since the integrand of $I_1^{\beta{\rm out}}$ contains no singularity (the only singularity on the sphere occurs at $x=y=0$, which is located in $H^{\rm in}$), in evaluating $\lim_{\delta r\to 0^{\pm}}I_1^{\beta{\rm out}}(\delta r)$ we are allowed to interchange the limit and integration,

$$\lim_{\delta r\to 0^{\pm}} I_1^{\beta{\rm out}}=I_1^{\beta{\rm out}}(\delta r=0). \tag{66}$$

Precisely as in the case of the integral I_2^{β} considered above, this contribution is then found to vanish for either $\beta=r$ or $\beta=x$. We are thus left with $\tilde{F}_{\pm}^{l\beta}=\lim_{\delta r\to 0^{\pm}}I_1^{\beta{\rm in}}$, namely,

$$\tilde{F}_{\pm}^{l\beta}=\lim_{\delta r\to 0^{\pm}}\int_{-h}^{h}\int_{-h}^{h}\delta x^{\beta}\epsilon_0^{-3}dxdy. \tag{67}$$

We proceed by considering separately the two components $\beta=r$ and $\beta=x$. Let us begin with the r component: Rescaling the integration variables as $X\equiv x/\delta r$ and $Y\equiv y/\delta r$, we find

$$\tilde{F}_{\pm}^{lr}=\lim_{\delta r\to 0_{\pm}}\int_{-h/\delta r}^{h/\delta r}\int_{-h/\delta r}^{h/\delta r}[\tilde{\epsilon}_{\pm}(X,Y)]^{-3}dXdY, \tag{68}$$

where

$$\tilde{\epsilon}_{\pm}\equiv\epsilon_0/\delta r=\pm[g^0_{rr}+r_0^2(X^2+Y^2)+(u_r+u_xX)^2]^{1/2}, \tag{69}$$

and the $\pm$ sign refers to the sign of δr. Note that $\tilde{\epsilon}_{\pm}$ (and hence the entire integrand) is independent of δr, such that the $\delta r\to 0_{\pm}$ limit becomes trivial:

$$\tilde{F}_{\pm}^{lr}=\pm\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[g^0_{rr}+r_0^2(X^2+Y^2)+(u_r+u_xX)^2]^{-3/2}dXdY \equiv\pm U. \tag{70}$$

This is an elementary integral [see, e.g., Eq. (3.252-3) of [25], in conjunction with Eq. (3.252-2) therein], yielding

$$U=(2\pi/r_0)[r_0^2u_r^2+g^0_{rr}(r_0^2+u_x^2)]^{-1/2}. \tag{71}$$

Note the relation

$$U=-\frac{2\pi f}{r_0^2u_t}, \tag{72}$$

which stems directly from the "radial" geodesic equation of motion, $(u^r)^2=u_t^2-(1-u_x^2/r_0^2)f$.

Consider next the case $\beta=x$. It is not possible to treat this case the same as the case $\beta=r$, by changing the integration variables to X,Y: doing so, the integrand becomes $X\tilde{\epsilon}_\pm^{-3}$, and the double integral does not strictly converge at infinity. We therefore apply here a different method to evaluate the limit $\delta r\to 0^\pm$ in Eq. (67). First, we express $\tilde{F}_\pm^{lx}$ as

$$\tilde{F}_\pm^{lx}=\lim_{\delta r\to 0^\pm}\int_{-h}^{h}\int_{-h}^{h}xS_0^{-3/2}dxdy, \tag{73}$$

where, recall, $S_0=\epsilon_0^2=r_0^2(x^2+y^2)+g_{rr}^0\delta r^2+(u_r\delta r+u_x x)^2$ (with the limit $t\to t_0$ already taken). Now, S_0 is quadratic in δx^α, and its derivative with respect to x is a linear combination of both x and δr. One easily obtains the relation

$$x=\alpha S_{0,x}+\beta\delta r, \tag{74}$$

where the coefficients α and β are given by

$$\alpha=\frac{1}{2(r_0^2+u_x^2)},\quad \beta=-\frac{u_x u_r}{r_0^2+u_x^2}. \tag{75}$$

Substituting x from Eq. (74) in Eq. (73), we express $\tilde{F}_\pm^{lx}$ as the sum of two integrals:

$$\tilde{F}_\pm^{lx}=\lim_{\delta r\to 0^\pm}\left[\alpha\int S_{0,x}S_0^{-3/2}dxdy+\beta\int \delta r\epsilon_0^{-3}dxdy\right] \equiv \alpha I_i+\beta I_{ii}. \tag{76}$$

In what follows we show that I_i vanishes, leaving us with only the contribution from I_{ii}, which is just proportional to the r component $\tilde{F}_\pm^{lr}$ calculated above.

Considering first I_i, we carry out the trivial integration over x, obtaining

$$I_i=\lim_{\delta r\to 0^\pm}\int_{-h}^{h}[-2S_0^{-1/2}]_{x=-h}^{x=h}dy. \tag{77}$$

The integration over y is then a standard one, but one does not need to carry it out explicitly: Observing that the integrand is now a regular function of y and δr throughout the entire range of integration, we are allowed to interchange the limit and integration. Noticing then $S_0(\delta r=0,x=+h)=S_0(\delta r=0,x=-h)$, we immediately conclude

$$I_i=0. \tag{78}$$

Consider next I_{ii}. Comparing with Eq. (67) (for the r components) we find simply $I_{ii}=\tilde{F}_\pm^{lr}$, hence

$$\tilde{F}_\pm^{lx}=\beta\tilde{F}_\pm^{lr}=\pm\beta U. \tag{79}$$

Having calculated all components of $\tilde{F}_\pm^{l\beta}$, we may now construct $F_{\pm\alpha}^{l(A)}$ through Eq. (58). We obtain

$$F_{\pm t}^{l(A)}=\mp[L/(2\pi)]u_t(u_r+\beta u_x)U=\pm\frac{Lfu_r}{r_0^2+u_x^2}, \tag{80a}$$

$$F_{\pm r}^{l(A)}=\mp[L/(2\pi)][f^{-1}+u_r(u_r+\beta u_x)]U=\pm\frac{Lf^{-1}u_t}{r_0^2+u_x^2}, \tag{80b}$$

$$F_{\pm x}^{l(A)}=\mp[L/(2\pi)][\beta(r_0^2+u_x^2)+u_xu_r]U=0, \tag{80c}$$

$$F_{\pm y}^{l(A)}=0, \tag{80d}$$

where we have substituted for U and β from Eqs. (72) and (75), respectively.

Note the remarkable fact that the contribution $F_{\pm\alpha}^{l(A)}$ is *precisely* proportional to L.

VIII. VALUES OF THE REGULARIZATION PARAMETERS

In conclusion of the calculation carried out in the previous section, we have found that the l-mode direct force $F_{\pm\alpha}^{l}$ is composed of only two contributions: one—completely described by $F_{\pm\alpha}^{l(A)}$—is precisely proportional to L, and the other—completely described by $F_{\pm\alpha}^{l(B)}$—is independent of L. No other powers of L are present. Recalling the definition of the RP in Sec. II, we then conclude that the term $F_{\pm\alpha}^{l(A)}$ contributes only to the parameter A_α and that the term $F_{\pm\alpha}^{l(B)}$ contributes only to B_α. Recalling Eq. (35), we identify the RP as

$$LA_{\pm\alpha}=q^2F_{\pm\alpha}^{l(A)},\quad B_\alpha=q^2F_{\pm\alpha}^{l(B)},\quad C_\alpha=0. \tag{81}$$

Furthermore, from Eq. (11) we immediately get $D_\alpha=0$. The explicit values of $A_{\pm\alpha}$ and B_α are then obtained by substituting the expressions derived above for the quantities $F_{\pm\alpha}^{l(A,B)}$—Eqs. (48), (57), and (80).

To give a useful summary of the RP values thus obtained, we shall transform the angular coordinates x,y back to the *standard* θ,φ coordinates, in which the orbit is equatorial (i.e., confined to $\theta=\pi/2$). The quantities $F_r^{\rm dir}(x)$ and $F_t^{\rm dir}(x)$ are unaffected by this transformation, therefore the r and t components of all RP are unchanged. However, $F_x^{\rm dir}$ and $F_y^{\rm dir}$ transform to $F_\theta^{\rm dir}$ and $F_\varphi^{\rm dir}$ in a manner which is not completely trivial, and we need to find the corresponding θ and φ components of the RP. Note that *a priori* there is no guarantee that the RP will transform like vectors at the evaluation point, because the RP depend on $F_\alpha^{\rm dir}$ in the neighborhood of z; and the transformation $(x,y)\to(\theta,\varphi)$ involves nontrivial functions of the angular coordinates, which may affect the mode decomposition. However, in Appendix C we show that in fact all the RP do transform (in this particular coordinate transformation) like four-vectors at z. It is trivial to show that at the evaluation point

$$x_{,\theta}=y_{,\varphi}=0\quad x_{,\varphi}=-y_{,\theta}=1. \tag{82}$$

[For concreteness we consider here the transformation described by a $\pi/2$ rotation about the horizontal axis $\varphi=0$, which takes z from the pole to the point (θ_0,φ_0)

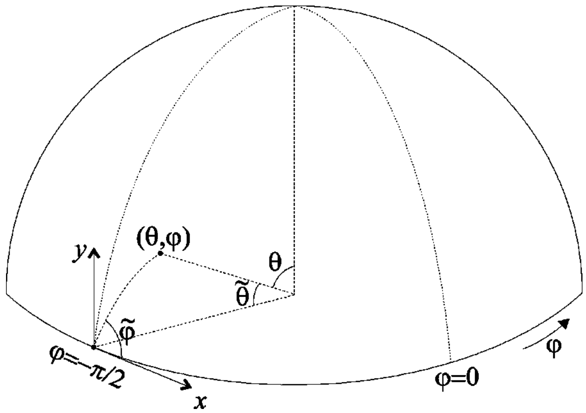

FIG. 1. A sketch showing the various quantities involved in constructing the coordinate transformation $(x,y)\rightarrow(\theta,\varphi)$. Shown is the "northern" hemisphere $t=r=\text{const}$, $\theta\geqslant 0$. The particle moves along an equatorial orbit, and is momentarily located at $(\theta,\varphi)=(\pi/2,-\pi/2)$. x,y is a pair of "locally Cartesian angular coordinates" at the particle's location, as described in the text.

$=(\pi/2,-\pi/2)$—see Fig. 1 and Appendix C for more details. Note, however, that the RP in the θ,φ coordinates do not depend on φ_0, due to the symmetry of rotations in φ.] Therefore, $B_\varphi=B_x$ and $B_\theta=-B_y$ (and the same for A_φ,A_θ). Note that $u_x(z)$ now becomes $u_\varphi\equiv\mathcal{L}$, the conserved azimuthal angular momentum. In the standard θ,φ coordinates, the RP are then given by $A_\theta=B_\theta=0$,

$$A_{\pm r}=\mp\frac{q^2}{r_0^2}\frac{\mathcal{E}}{fV},\quad A_{\pm t}=\pm\frac{q^2}{r_0^2}\frac{\dot{r}}{V},\quad A_\varphi=0, \tag{83a}$$

$$B_r=\frac{q^2}{r_0^2}\frac{(\dot{r}^2-2\mathcal{E}^2)\hat{K}(w)+(\dot{r}^2+E^2)\hat{E}(w)}{\pi f V^{3/2}}, \tag{83b}$$

$$B_t=\frac{q^2}{r_0^2}\frac{\mathcal{E}\dot{r}[\hat{K}(w)-2\hat{E}(w)]}{\pi V^{3/2}}, \tag{83c}$$

$$B_\varphi=\frac{q^2}{r_0}\frac{\dot{r}[\hat{K}(w)-\hat{E}(w)]}{\pi(\mathcal{L}/r_0)V^{1/2}}, \tag{83d}$$

$$C_\alpha=D_\alpha=0, \tag{83e}$$

where

$$w\equiv\frac{\mathcal{L}^2}{\mathcal{L}^2+r_0^2},\quad V\equiv 1+\mathcal{L}^2/r_0^2, \tag{84}$$

$f\equiv(1-2M/r_0)$, and $\dot{r}^2\equiv(u^r)^2=\mathcal{E}^2-fV$. Recall $\mathcal{E}=-u_t$ and $\mathcal{L}=u_\varphi$ are the (conserved) specific energy and angular momentum parameters.

We comment that the parameter $A_{\pm\alpha}$ is normal to the four-velocity: $A_{\pm\alpha}u^\alpha=0$. However, the parameter B_α (as the self-force itself in our model) is, in general, *not* normal to u^α: An explicit calculation yields

$$B_\alpha u^\alpha=-2\frac{q^2}{r_0^2}\frac{\dot{r}\hat{E}(w)}{\pi V^{1/2}}. \tag{85}$$

Finally, we give here the RP values for the special case of a radial geodesic, i.e., $\mathcal{L}=0$: Noting, in this case, $w=0$, $V=1$, and $\dot{r}^2=\mathcal{E}^2-f$, and recalling $\hat{K}(0)=\hat{E}(0)=\pi/2$, the nonvanishing components in Eqs. (83) reduce to

$$A_{\pm r}^{\text{radial}}=\mp\frac{q^2}{r_0^2}(\mathcal{E}/f),\quad A_{\pm t}^{\text{radial}}=\pm\frac{q^2}{r_0^2}\dot{r}, \tag{86a}$$

$$B_r^{\text{radial}}=\frac{q^2}{2r_0^2}f^{-1}(\mathcal{E}^2-2f),\quad B_t^{\text{radial}}=-\frac{q^2}{2r_0^2}\mathcal{E}\dot{r}. \tag{86b}$$

[The vanishing of $B_\varphi^{\text{radial}}$ is obvious from symmetry considerations. Note that B_φ vanishes at the limit $w\rightarrow 0$ despite the factor $\mathcal{L}$ in the denominator, because $\hat{K}(w)-\hat{E}(w)=O(w)=O(\mathcal{L}^2)$.] These values are in agreement with the ones derived in [12] using the *l*-mode Green's-function expansion method.

IX. CONCLUDING REMARKS

The mode-sum scheme described by Eq. (12), with the explicit RP values calculated in this paper, Eqs. (83), provides one with a practical means—yet one based on a physically well-established regularization scheme—for calculating the scalar self-force for any geodesic orbit around a Schwarzschild black hole. Recall that the full modes $F_\alpha^{(\text{full})l}$ needed for fully implementing this mode-sum scheme are to be obtained from the *l* modes of the scalar field Φ, which, in turn, are to be calculated using standard numerical techniques.

The RP values derived here, Eqs. (83), were obtained independently by MNS [18] using a different approach. In their analysis, MNS decomposed the direct field using the "standard" θ,φ coordinates (in which the motion is equatorial), in which case the $m\neq 0$ modes contribute as well. MNS then derived an analytic expression for the contribution of each l,m mode of the direct force, expanded in powers of M/r. By explicitly summing up this expansion (and summing over m), MNS were able to recover all RP values.

The RP values (83) reduce, in the special cases of radial motion ($\mathcal{L}=0$) or circular motion ($\dot{r}=0$), to the values derived previously [12] using a completely independent analytic approach (namely, by locally analyzing the *l*-mode Green's function, as we briefly mention in the Introduction). In these cases, the values of the parameters A_α, B_α, and C_α have also been confirmed numerically, by calculating the full-force modes [15].

The RP calculation method presented in this paper is directly applicable to the more realistic case of the *gravitational* self-force acting on a mass particle, as well as to the case of the *electromagnetic* self-force acting on an electrically charged particle. Both cases shall be treated in an accompanying paper [9], where we obtain the gravitational and electromagnetic RP for general orbits in Schwarzschild spacetime (the results in the gravitational case were provided in [8]). The extension of our scalar field analysis to the gravitational and electromagnetic cases involves several complexities which require special care. In particular, one has to tackle the technical issue of extending the four-velocity vector off the worldline [9]. A more fundamental issue concerns the gauge dependence of the gravitational self-force [10].

ACKNOWLEDGMENTS

We are grateful to Lior Burko, Yasushi Mino, Hiroyuki Nakano, and Misao Sasaki for interesting discussions and stimulating interaction. L.B. was supported by the Marie Curie Fund through the European Community program IHP-MCIF-99-1 under contract number HPMF-CT-2000-00851.

APPENDIX A: DERIVATION OF S_0 AND S_1

In this appendix, we calculate the two leading terms in the expansion of $S\equiv S(x,z)$ [the square of the geodesic distance from the point x to the geodesic $z(\tau)$] in powers of $\delta x^\mu \equiv x^\mu - z^\mu$. This expansion takes the form

$$S=S_0+S_1+S_2+\cdots, \tag{A1}$$

in which the term S_n is of homogeneous order δx^{n+2}, and we wish to calculate S_0 and S_1.

In flat space, using Cartesian coordinates y^α (with $y^\alpha=0$ at z), we obviously have $S=(\eta_{\alpha\beta}+u_\alpha u_\beta)y^\alpha y^\beta \equiv S_0$, where $\eta_{\alpha\beta}$ is the flat space metric. In curved space (or in curvilinear coordinates), each of the terms S_n (like S itself) will be a certain function of $g_{\alpha\beta}$ and its derivatives. From simple dimensionality considerations, it is clear that S_0 may not include any derivatives of $g_{\alpha\beta}$, and that S_1 may include only first-order derivatives of the latter (in addition to $g_{\alpha\beta}$ itself).

Let y^α be locally Cartesian coordinates at the evaluation point z, with $y^\alpha=0$ at z. Namely, at $x=z$, the metric functions in the coordinates y^α are just $\eta_{\mu\nu}$, and their first-order derivatives vanish. Since no second- or higher-order derivatives appear in S up to the desired order, we must have

$$S=(\eta_{\alpha\beta}+u'_\alpha u'_\beta)y^\alpha y^\beta+O(y^4), \tag{A2}$$

where a prime denotes vectorial components in the y^α coordinate system. We now transform from y^α back to our original coordinates x^μ. Recall that S is a biscalar, and is hence invariant under this transformation. Writing the Taylor expansion of $y^\alpha(\delta x^\mu)$,

$$y^\alpha=\frac{\partial y^\alpha}{\partial x^\lambda}\delta x^\lambda+\frac{1}{2}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\delta x^\mu \delta x^\nu+O(\delta x^3) \tag{A3}$$

(in which all coefficients are evaluated at z), and substituting it in the right-hand side of Eq. (A2), we find

$$S=\left[(\eta_{\alpha\beta}+u'_\alpha u'_\beta)\frac{\partial y^\alpha}{\partial x^\lambda}\frac{\partial y^\beta}{\partial x^\epsilon}\right]\delta x^\lambda \delta x^\epsilon$$
$$+\left[(\eta_{\alpha\beta}+u'_\alpha u'_\beta)\frac{\partial y^\alpha}{\partial x^\lambda}\frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu}\right]\delta x^\lambda \delta x^\mu \delta x^\nu$$
$$+O(\delta x^4).$$

Comparing this to Eq. (A1), we identify the first and second terms on the right-hand side with S_0 and S_1, respectively. Using the obvious tensorial transformation rule, we find

$$S_0=(g_{\lambda\epsilon}+u_\lambda u_\epsilon)\delta x^\lambda \delta x^\varepsilon. \tag{A4}$$

To calculate S_1 we need the second-order transformation coefficients, which are given by

$$\frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu}=\Gamma^\epsilon_{\mu\nu}\frac{\partial y^\beta}{\partial x^\epsilon}$$

[see, e.g., Eq. (3.2.11) of [26]]. Therefore,

$$S_1=\left[(\eta_{\alpha\beta}+u'_\alpha u'_\beta)\frac{\partial y^\alpha}{\partial x^\lambda}\frac{\partial y^\beta}{\partial x^\epsilon}\Gamma^\epsilon_{\mu\nu}\right]\delta x^\lambda \delta x^\mu \delta x^\nu$$
$$=(g_{\lambda\epsilon}+u_\lambda u_\epsilon)\Gamma^\epsilon_{\mu\nu}\delta x^\lambda \delta x^\mu \delta x^\nu.$$

Recalling that

$$g_{\lambda\epsilon}\Gamma^\epsilon_{\mu\nu}\delta x^\lambda \delta x^\mu \delta x^\nu=\frac{1}{2}g_{\mu\nu,\lambda}\delta x^\lambda \delta x^\mu \delta x^\nu,$$

we finally obtain

$$S_1=(u_\lambda u_\gamma \Gamma^{\lambda 0}_{\alpha\beta}+g^0_{\alpha\beta,\gamma}/2)\delta x^\alpha \delta x^\beta \delta x^\gamma. \tag{A5}$$

APPENDIX B: INTERCHANGEABILITY OF THE $r\rightarrow r_0$ LIMIT AND THE LEGENDRE INTEGRAL

In this appendix, we explore the interchangeability of the limit and integration in Eq. (36)—an issue crucial for the calculation carried out in Sec. VII. For convenience, let us write $F^{(W)l}_{\pm\alpha}=[L/(2\pi)]\hat{F}^{(W)l}_{\pm\alpha}$, where

$$\hat{F}^{(W)l}_{\pm\alpha}\equiv \lim_{\delta r\rightarrow 0^\pm}\int_0^\pi d\theta \int_0^{2\pi} d\varphi F^{(W)}_\alpha(r,t_0,\theta,\varphi)$$
$$\times P_l(\cos\theta)\sin\theta. \tag{B1}$$

Here, recall, W stands for A, B, or C, with

$$F^{(A)}_\alpha \equiv \epsilon_0^{-3}P^{(1)}_\alpha, \quad F^{(B)}_\alpha \equiv \epsilon_0^{-5}P^{(4)}_\alpha, \quad F^{(C)}_\alpha \equiv \epsilon_0^{-7}P^{(7)}_\alpha, \tag{B2}$$

where ϵ_0 is given explicitly in Eq. (60), and $P^{(n)}_\alpha$ represents a polynomial of homogeneous order n in $\delta x^\mu \equiv x^\mu - z^\mu$. We shall show that interchanging the limit and integration in Eq. (B1) is valid for $W=B,C$, and explain why our proof fails in the case $W=A$.

We begin by considering the case $W=B$. Let $R\equiv|\delta r|^s$ for some $0<s<1$. Consider the range of small δr (such that $R<1$). We split the θ integral in Eq. (B1) into three domains: (i) $\theta<R$, (ii) $R<\theta<1$, and (iii) $1<\theta<\pi$. Correspondingly, the above double integral can be expressed as $\mathcal{I}_i+\mathcal{I}_{ii}+\mathcal{I}_{iii}$, hence

$$\hat{F}^{(B)l}_{\pm\alpha}=\lim_{\delta r\to 0^\pm}\mathcal{I}_i(\delta r)+\lim_{\delta r\to 0^\pm}\mathcal{I}_{ii}(\delta r)+\lim_{\delta r\to 0_\pm}\mathcal{I}_{iii}(\delta r). \tag{B3}$$

Consider first the internal integral $\mathcal{I}_i$:

$$\mathcal{I}_i(\delta r)=\int_0^R d\theta\int_0^{2\pi}d\varphi F^{(B)}_\alpha(\delta r,\theta,\varphi)\,P^l(\cos\theta)\sin\theta. \tag{B4}$$

Since at the relevant limit $R\to 0$, we have $\theta\ll 1$ throughout the range of integration, and we may approximate this integral [using $\sin\theta d\theta\simeq\rho d\rho$, as well as $P^l(\cos\theta)\simeq 1$ and $\rho(R)\simeq R$] as

$$\mathcal{I}_i(\delta r)\simeq\int_0^R\rho d\rho\int_0^{2\pi}d\varphi F^{(B)}_\alpha(\delta r,\rho,\varphi). \tag{B5}$$

Now, $F^{(B)}_\alpha=\epsilon_0^{-5}P^{(4)}_\alpha$. We may bound ϵ_0 and $P^{(4)}_\alpha$ as $\epsilon_0>c_1|\delta r|$ and $|P^{(4)}_\alpha|<c_2\rho^4$, where hereafter c_n are some positive constants. Consequently, $F^{(B)}_\alpha$ can be bounded as $|F^{(B)}_\alpha|<c_3|\delta r|^{-5}\rho^4<c_3|\delta r|^{-5}R^4$. Since the "integration area" in Eq. (B5) is πR^2, we then obtain the upper bound

$$|\mathcal{I}_i|<\pi c_3|\delta r|^{-5}R^6=c_4|\delta r|^{6s-5}. \tag{B6}$$

Taking, e.g., $s=0.9$, we find that $|\mathcal{I}_i|<c_4|\delta r|^{0.4}$ and hence it vanishes at the limit $\delta r\to 0^\pm$:

$$\lim_{\delta r\to 0_\pm}\mathcal{I}_i(\delta r)=0. \tag{B7}$$

Consider next $\mathcal{I}_{ii}$, which is defined by the double integral

$$\mathcal{I}_{ii}(\delta r)=\int_R^1 d\theta\int_0^{2\pi}d\varphi F^{(B)}_\alpha(\delta r,\theta,\varphi)\,P^l(\cos\theta)\sin\theta. \tag{B8}$$

Let us introduce the quantity $\Delta F^{(B)}_\alpha\equiv F^{(B)}_\alpha(\delta r)-F^{(B)}_\alpha(\delta r=0)$, and the corresponding integral

$$\Delta\mathcal{I}_{ii}(\delta r)\equiv\int_R^1 d\theta\int_0^{2\pi}d\varphi\Delta F^{(B)}_\alpha(\delta r,\theta,\varphi)P^l(\cos\theta)\sin\theta, \tag{B9}$$

such that

$$\mathcal{I}_{ii}(\delta r)=\Delta\mathcal{I}_{ii}(\delta r)+\int_R^1 d\theta\int_0^{2\pi}d\varphi F^{(B)}_\alpha(\delta r=0,\theta,\varphi)\times P^l(\cos\theta)\sin\theta. \tag{B10}$$

Since $s<1$, in the entire range (ii) we have $|\delta r|\ll R\leqslant\theta$ (for $|\delta r|\ll 1$), and thus $|\delta r|\ll\rho$. We may approximate $\Delta F^{(B)}_\alpha$ at $|\delta r|\ll 1$ as

$$\Delta F^{(B)}_\alpha\simeq d(\epsilon_0^{-5}P^{(4)}_\alpha)/d(\delta r)|_{\delta r=0}\times\delta r. \tag{B11}$$

From Eq. (60) (and recalling $|x|\leqslant\rho$) we observe that, at $\delta r=0$, ϵ_0^{-1} and $d(\epsilon_0^{-1})/d(\delta r)$ are bounded from above by $\propto\rho^{-1}$ and $\propto$const, respectively. At $\delta r=0$, we may also upper bound $P^{(4)}_\alpha$ and $dP^{(4)}_\alpha/d(\delta r)$ by $\propto\rho^4$ and $\propto\rho^3$, respectively. Applying these bounds in Eq. (B11) we then obtain

$$|\Delta F^{(B)}_\alpha|<c_5|\delta r|\rho^{-2}. \tag{B12}$$

[Note that in Eq. (B11) we have assumed that either S or $P^{(4)}_\alpha$ (or both) include terms linear in δr. In special cases where only terms quadratic in δr are present in both quantities, we shall instead arrive at the bound $|\Delta F^{(B)}_\alpha|<c_6\delta r^2\rho^{-3}$. Since $|\delta r|/\rho\ll 1$, this is even smaller than the bound (B12), and so the entire derivation below remains valid.]

Using Eq. (B12) and recalling $|P^l(\cos\theta)|\leqslant 1$, we can now bound $\Delta I_{ii}(\delta r)$ by

$$|\Delta\mathcal{I}_{ii}(\delta r)|<c_5\int_R^1 d\theta\int_0^{2\pi}d\varphi\sin\theta|\delta r|\rho^{-2}=2\pi c_5\int_R^1 d\theta\sin\theta|\delta r|\rho^{-2}. \tag{B13}$$

Since $\sin\theta/\rho$ and $d\theta/d\rho$ are bounded in domain (ii), the last integral can now be bounded as

$$|\Delta\mathcal{I}_{ii}(\delta r)|<c_6|\delta r|\int_{\rho(R)}^{\rho(1)}\rho^{-1}d\rho. \tag{B14}$$

[We point out here that in the way we have chosen $\rho(\theta)$ in Sec. VII—see Eq. (51)—$d\theta/d\rho$ is unbounded at $\theta\to\pi$. It is this divergence that forced us to terminate domain (ii) at $\theta=1$, and to introduce domain (iii).] Upon integration, we find

$$|\Delta\mathcal{I}_{ii}(\delta r)|<c_6|\delta r|[\ln\rho(1)-\ln\rho(R)]. \tag{B15}$$

For small R we have $\rho(R)\simeq R=|\delta r|^s$; therefore

$$|\Delta\mathcal{I}_{ii}(\delta r)|<c_6\ln\rho(1)|\delta r|-sc_6|\delta r|\ln|\delta r|. \tag{B16}$$

Clearly, both terms vanish as $\delta r\to 0^\pm$, hence

$$\lim_{\delta r\to 0^\pm}\Delta\mathcal{I}_{ii}(\delta r)=0. \tag{B17}$$

From Eq. (B10), we then have

$$\lim_{\delta r\to 0^{\pm}} \mathcal{I}_{ii}(\delta r) = \lim_{\delta r\to 0^{\pm}} \int_{R(\delta r)}^{1} d\theta \int_0^{2\pi} d\varphi\, F_{\alpha}^{(B)}(\delta r=0,\theta,\varphi) \times P^l(\cos\theta)\sin\theta = \int_0^1 d\theta \int_0^{2\pi} d\varphi\, F_{\alpha}^{(B)}(\delta r=0,\theta,\varphi) \times P^l(\cos\theta)\sin\theta. \tag{B18}$$

Finally, consider the third contribution, $\mathcal{I}_{iii}$, defined by

$$\mathcal{I}_{iii}(\delta r) = \int_1^{\pi} d\theta \int_0^{2\pi} d\varphi F_{\alpha}^{(B)}(\delta r,\theta,\varphi) P^l(\cos\theta)\sin\theta. \tag{B19}$$

In full analogy with the analysis above, we define

$$\Delta\mathcal{I}_{iii}(\delta r) = \int_1^{\pi} d\theta \int_0^{2\pi} d\varphi \Delta F_{\alpha}^{(B)}(\delta r,\theta,\varphi) P^l(\cos\theta)\sin\theta. \tag{B20}$$

The above bound, $|\Delta F_{\alpha}^{(B)}|<c_5|\delta r|\rho^{-2}$, is valid in range (iii) too. Since in this range ρ is bounded from below, we may now write $|\Delta F_{\alpha}^{(B)}|<c_7|\delta r|$; and since $P^l(\cos\theta)$ and $\sin\theta$ are both bounded by unity, the entire integrand is then bounded by $c_7|\delta r|$:

$$|\Delta\mathcal{I}_{iii}(\delta r)| < c_7|\delta r| \int_1^{\pi} d\theta \int_0^{2\pi} d\varphi = 2\pi^2 c_7 |\delta r|. \tag{B21}$$

Again, this quantity vanishes at the limit $\delta r\to 0$; hence, by the same considerations used above for range (ii) [see the chain of Eqs. (B18)], we obtain

$$\lim_{\delta r\to 0_{\pm}} \mathcal{I}_{iii}(\delta r) = \int_1^{\pi} d\theta \int_0^{2\pi} d\varphi\, F_{\alpha}^{(B)}(\delta r=0,\theta,\varphi) \times P^l(\cos\theta)\sin\theta. \tag{B22}$$

Substituting Eqs. (B7), (B18), and (B22) in Eq. (B3), we obtain

$$\hat{F}_{\alpha}^{l(B)} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi F_{\alpha}^{(B)}(\delta r=0,\theta,\varphi) P^l(\cos\theta)\sin\theta. \tag{B23}$$

Namely, in the calculation of $\hat{F}_{\pm\alpha}^{l(B)}$—and thus also $F_{\pm\alpha}^{l(B)}$—we are allowed to interchange the limit $\delta r\to 0$ and the integration. Note that since $\hat{F}_{\alpha}^{(B)}$ admits a well-defined limit at $\delta r\to 0$ (except at $\theta=0$—which, however, was shown not to affect the integral), we have omitted its $\pm$ label.

The same proof can immediately be applied to $\hat{F}_{\pm\alpha}^{l(C)}$. Evaluating $\mathcal{I}_i$, we find this time that $|F_{\alpha}^{(C)}|$ is bounded from above by $c_8|\delta r|^{-7}\rho^7 < c_8|\delta r|^{-7}R^7$, hence (taking again $s=0.9$)

$$|\mathcal{I}_i| < c_9|\delta r|^{-7}R^9 = c_9|\delta r|^{9s-7} = c_9|\delta r|^{1.1}\to 0 \tag{B24}$$

as $\delta r\to 0$. Evaluating next $\Delta\mathcal{I}_{ii}$, we obtain this time [in analogy with Eq. (B12)] $|\Delta F_{\alpha}^{(C)}|<c_{10}|\delta r|/\rho$, and hence

$$|\Delta\mathcal{I}_{ii}(\delta r)| < c_{11}|\delta r| \int_{\rho(R)}^{\rho(1)} \rho^0 d\rho < c_{12}|\delta r|, \tag{B25}$$

which again vanishes as $\delta r\to 0$. The calculation of $\Delta\mathcal{I}_{iii}$ proceeds exactly as for $W=B$ (the only difference is that now $\Delta F_{\alpha}^{(C)}\propto|\delta r|/\rho$ instead of $|\delta r|/\rho^2$, but this does not affect the above evaluation of $\Delta\mathcal{I}_{iii}$ in any way). Again we find $|\Delta\mathcal{I}_{iii}(\delta r)|<2\pi^2 c_{13}|\delta r|$, which vanishes at the limit $\delta r\to 0$. We conclude that the limit and integration may be interchanged for $W=C$ as well,

$$\hat{F}_{\alpha}^{l(C)} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi F_{\alpha}^{(C)}(\delta r=0,\theta,\varphi) P^l(\cos\theta)\sin\theta. \tag{B26}$$

Finally, it is instructive to see how the above type of arguments fails for $W=A$. Since $F_{\alpha}^{(A)}=\epsilon_0^{-3}P_{\alpha}^{(1)}$, in evaluating $\mathcal{I}_i$ one obtains $|\mathcal{I}_i|<c_{14}|\delta r|^{-3}R^3=c_{14}|\delta r|^{3(s-1)}$. Then, evaluating $\Delta\mathcal{I}_{ii}$, one obtains

$$|\Delta\mathcal{I}_{ii}(\delta r)| < c_{15}|\delta r| \int_{\rho(R)}^{\rho(1)} \rho^{-2} d\rho, \tag{B27}$$

which at the limit of small R yields $|\Delta\mathcal{I}_{ii}(\delta r)|<c_{15}|\delta r|/R=c_{15}|\delta r|^{1-s}$. Obviously, for any $s\leqslant 1$ the bound for $\mathcal{I}_i$ will fail to vanish as $\delta r\to 0$, and for any $s\geqslant 1$ the bound for $\Delta\mathcal{I}_{ii}$ will fail to vanish at this limit. [Note also that in the case $s\geqslant 1$ the inequality $|\delta r|\ll\rho$, used above in evaluating $\Delta\mathcal{I}_{ii}$, is no longer valid throughout range (ii).] In fact, it becomes evident from the explicit calculation in Sec. VII that for $W=A$ the limit $\delta r\to 0$ *cannot* be interchanged with the integration.

APPENDIX C: TRANSFORMING TO EQUATORIAL ORBIT

In Sec. VII, we analyzed the multipole decomposition of the direct force in a Schwarzschild coordinate system in which the particle is momentarily at the pole. We then transformed to locally Cartesian angular coordinates x,y and calculated the RP in the system $x_{\rm po}^{\mu}\equiv(t,r,x,y)$. Usually (e.g., in numerical calculations) one adopts a more natural pair of angular coordinates, in which the particle's orbit is confined to the equatorial plain ($\theta=\pi/2$)—throughout this appendix we shall denote this system by $x_{\rm eq}^{\mu}\equiv(t,r,\theta,\varphi)$. The goal of this appendix is, given the RP values in the system $x_{\rm po}^{\mu}$, to obtain the corresponding values in the system $x_{\rm eq}^{\mu}$.

In the system $x_{\rm eq}^{\mu}$ we have $u^{\theta}=0$ and $u_{\varphi}=\mathcal{L}$, where, recall, $\mathcal{L}$ is the conserved specific angular momentum. Let the particle be momentarily at $(\theta,\varphi)=(\theta_0,\varphi_0)=(\pi/2,-\pi/2)$.[1] Consider the set of spherical coordinates $\tilde{\theta},\tilde{\varphi}$, defined such that the particle is located at their pole, $\tilde{\theta}=0$, and $\tilde{\varphi}=0,\pi$ coincides with $\theta=\pi/2$—see Fig. 1. (These are, in fact, the same spherical coordinates used throughout the paper; here we merely use a different notation, as the symbols θ,φ are reserved for the angular coordinates of the $x^{\mu}_{\rm eq}$ system.) The ''locally Cartesian'' coordinates at z are then given by [see Eq. (31)]

$$x=\rho(\tilde{\theta})\cos\tilde{\varphi}=2\sin(\tilde{\theta}/2)\cos\tilde{\varphi},$$
$$y=\rho(\tilde{\theta})\sin\tilde{\varphi}=2\sin(\tilde{\theta}/2)\sin\tilde{\varphi}. \tag{C1}$$

Relating the spherical coordinates $\tilde{\theta},\tilde{\varphi}$ to the standard pair θ,φ is a straightforward geometrical problem, and one finds

$$\cos\tilde{\theta}=-\sin\theta\sin\varphi,\quad \cot\tilde{\varphi}=\tan\theta\cos\varphi. \tag{C2}$$

This allows us to express x,y directly as functions of θ,φ. We obtain

$$x(\theta,\varphi)=2^{1/2}\frac{\sin\theta\cos\varphi}{\sqrt{1-\sin\theta\sin\varphi}},$$
$$y(\theta,\varphi)=2^{1/2}\frac{\cos\theta}{\sqrt{1-\sin\theta\sin\varphi}}, \tag{C3}$$

which describes explicitly the transformation between $x^{\mu}_{\rm eq}$ and $x^{\mu}_{\rm po}$. Note that this transformation is regular on the entire sphere [except at the point $(\theta,\varphi)=(\pi/2,\pi/2)$, which, however, is irrelevant for our analysis].

[1] We consider here a specific value of φ_0 in order to simplify the following expressions, and to make the correspondence between the $\tilde{x},\tilde{y}$ and x,y coordinates (see below) easily apparent. Note, however, that our final result—the RP values in the system $x^{\mu}_{\rm eq}$—does not depend on the choice of φ_0, as the l-mode decomposition of the direct force in that system is invariant under rotations about the polar axis.

Given the above transformation rule, the desired components $F^{\rm dir}_{\theta}$ and $F^{\rm dir}_{\varphi}$ are constructed as

$$F^{\rm dir}_{\theta}=x_{,\theta}F^{\rm dir}_{x}+y_{,\theta}F^{\rm dir}_{y},$$
$$F^{\rm dir}_{\varphi}=x_{,\varphi}F^{\rm dir}_{x}+y_{,\varphi}F^{\rm dir}_{y}. \tag{C4}$$

It is now useful to consider the Taylor expansion of the various partial derivatives about the particle's location: Introducing $\Delta\theta\equiv\theta-\pi/2$ and $\Delta\varphi\equiv\varphi-(-\pi/2)$, we find

$$x_{,\theta}=-\frac{3}{4}\Delta\theta\Delta\varphi+O(\delta x^4), \tag{C5a}$$

$$x_{,\varphi}=1-\frac{1}{8}\Delta\varphi^2-\frac{3}{8}\Delta\theta^2+O(\delta x^4), \tag{C5b}$$

$$y_{,\theta}=-1-\frac{1}{8}\Delta\varphi^2+\frac{1}{8}\Delta\theta^2+O(\delta x^4), \tag{C5c}$$

$$y_{,\varphi}=-\frac{1}{4}\Delta\theta\Delta\varphi+O(\delta x^4), \tag{C5d}$$

where $O(\delta x^4)$ represents corrections of fourth order in $\Delta\theta$ and $\Delta\varphi$. Substituting these exapnsions in Eqs. (C4), we obtain, near the particle's location,

$$F^{\rm dir}_{\theta}\simeq-F^{\rm dir}_{y},\quad F^{\rm dir}_{\varphi}\simeq F^{\rm dir}_{x}, \tag{C6}$$

where corrections are due to terms of the form $\epsilon_0^{-7}P^{(7)}_{\alpha}$ (recall the notation introduced in Sec. IV) and higher-order terms that vanish at $x\to z$. From the analysis of Sec. VII, it is clear that such correction terms do not contribute to either $F^{(A)}_{\alpha}$ or $F^{(B)}_{\alpha}$, and their contribution to $F^{(C)}_{\alpha}$ vanishes in the multipole decomposition. Hence, none of the RP will be affected by omitting these correction terms and replacing the approximation in Eq. (C6) with an exact equality. Consequently, we find that the RP transform under $x^{\mu}_{\rm po}\to x^{\mu}_{\rm eq}$ as vectors at z, namely

$$R_{\theta}=-R_y,\quad R_{\varphi}=R_x, \tag{C7}$$

where R_{α} stands for any of the RP (obviously, R_t and R_r do not change). Note also the relations $u_{\theta}=-u_y(z)(=0)$ and $u_{\varphi}=u_x(z)$ (u_t and u_r do not change).

[1] Up to date information about the joint NASA-ESA project LISA can be found in the project's webpage at http://lisa.jpl.nasa.gov/

[2] A. Ghez, M. Morris, E.E. Becklin, T. Kremenek, and A. Tanner, Nature (London) **407**, 349 (2000).

[3] See, e.g., the early work by M. Davis, W.H. Press, R.H. Price, and R. Ruffini, Phys. Rev. Lett. **27**, 1466 (1971); or the later work by E. Poisson and M. Sasaki, Phys. Rev. D **51**, 5753 (1995) (and see also references therein).

[4] B.S. DeWitt and R.W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).

[5] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).

[6] T.C. Quinn and R.M. Wald, Phys. Rev. D **56**, 3381 (1997).

[7] T.C. Quinn, Phys. Rev. D **62**, 064029 (2000).

[8] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).

[9] L. Barack and A. Ori, ''Regularization parameters for the self-force in Schwarzschild spacetime: II. gravitational case,'' gr-qc/0209072.

[10] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).

[11] M.J. Pfenning and E. Poisson, Phys. Rev. D **65**, 084001 (2002).

[12] L. Barack and A. Ori, Phys. Rev. D **61**, 061502(R) (2000); L. Barack, *ibid.* **62**, 084027 (2000).

[13] L. Barack, Phys. Rev. D **64**, 084021 (2001).

[14] L.M. Burko, Class. Quantum Grav. **17**, 227 (2000); L.M. Burko, Y.-T. Liu, and Y. Soen, Phys. Rev. D **63**, 024015 (2001); L.M. Burko and Y.-T. Liu, *ibid.* **64**, 024006 (2001).
[15] L.M. Burko, Phys. Rev. Lett. **84**, 4529 (2000); L. Barack and L.M. Burko, Phys. Rev. D **62**, 084040 (2000).
[16] L. Barack and C.O. Lousto, Phys. Rev. D **66**, 061502(R), (2002).
[17] The same approach was used earlier by Burko (unpublished), for the case of radial orbits.
[18] Y. Mino, H. Nakano, and M. Sasaki, gr-qc/0111074.
[19] C.O. Lousto, Phys. Rev. Lett. **84**, 5251 (2000).
[20] H. Nakano and M. Sasaki, Prog. Theor. Phys. **105**, 197 (2001).
[21] S. Detweiler, Phys. Rev. Lett. **86**, 1931 (2001).
[22] S. Detweiler and B.F. Whiting, gr-qc/0202086.
[23] T.C. Quinn and R.M. Wald, Phys. Rev. D **60**, 064009 (1999).
[24] Y. Mino and H. Nakano, Prog. Theor. Phys. **100**, 507 (1998).
[25] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 5th ed. (Academic, Boston, 1994).
[26] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).