Computing the gravitational self-force on a compact object plunging into a Schwarzschild black hole

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We compute the gravitational self-force (or "radiation reaction" force) acting on a particle falling radially into a Schwarzschild black hole. Our calculation is based on the "mode-sum" method, in which one first calculates the individual ℓ-multipole contributions to the self-force (by numerically integrating the decoupled perturbation equations) and then regularizes the sum over modes by applying a certain analytic procedure. We demonstrate the equivalence of this method with the $\zeta-$function scheme. The convergence rate of the mode-sum series is considerably improved here (thus reducing computational requirements) by employing an analytic approximation at large ℓ.

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The space-based gravitational wave detector LISA (Laser Interferometer Space Antenna), scheduled for launch around 2011 [1], will open up a window for the low frequency band below 1Hz, allowing access to a variety of black hole sources. A main target for LISA would be the outburst of gravitational radiation emitted during the capture of a compact star by a supermassive black hole—a 10^{5-7} solar masses black hole of the kind now believed to reside in the cores of many galaxies, including our own [2]. By LISA's launch time, a sufficient theoretical understanding of the orbital evolution of such systems, including radiation reaction effects, must be at hand, to allow design of accurate templates necessary for detection and interpretation of the gravitational waveforms. Due to the extreme mass-ratio typical to the binary systems of interest, the entire problem can be conveniently treated in the context of perturbation theory, being a relatively mature branch of gravitational physics: the compact stellar object is thus modeled by a point-like particle, and its field treated as a perturbation over the fixed Kerr geometry of the large black hole. To leading order in the mass ratio, such a particle then traces a geodesic of the background spacetime, and one asks about radiation-reaction induced corrections to this geodesic.

The "traditional" approach to this problem relies on energy-momentum balance considerations [3]: By calculating the fluxes of energy and angular momentum to infinity and across the event horizon, one attempts to infer the temporal rate of change of the particle's "constants" of motion. This technique is applicable only in adiabatic scenarios, in which the time scale for radiation reaction effect is much larger than the dynamic time-scale of the system; it is not clear yet whether this approximation is valid for the entire range of relevant LISA parameters [3]. Moreover, this approach seems insufficient for tackling generic orbits in Kerr spacetime (i.e., ones both eccentric and inclined) even under the adiabatic approximation. This has led many researches, particularly over the last five years, to turn to the useful notion of the *local self-force* (SF).

Consider a point-like particle of mass μ moving freely around a black hole with mass $M \gg \mu$; and treat the particle's gravitational field, $h_{\alpha\beta} \propto O(\mu)$, as a linear perturbation over the background metric $g_{\alpha\beta}$. In the "SF picture", this particle's equation of motion is written as

$$\mu u^{\alpha}{}_{;\beta} u^{\beta} = F^{\alpha}_{\rm self}, \qquad (1)$$

where u^{α} is the particle's four-velocity, a semicolon denotes covariant differentiation with respect to $g_{\alpha\beta}$, and $F^{\alpha}_{\rm self} \propto O(\mu^2)$ describes the leading-order SF effect. Since the perturbation $h_{\alpha\beta}$ obviously diverges at the particle's location, the problem of obtaining $F^{\alpha}_{\rm self}$ involves the introduction of a reliable regularization scheme. A well established formal prescription for constructing $F^{\alpha}_{\rm self}$, relying on a physically consistent regularization method, became available recently with the work of Mino, Sasaki and Tanaka (MST) [4]. The same formal prescription was introduced independently by Quinn and Wald (QW) [5], based on an axiomatic approach. To allow a practical implementation of the MST/QW formal prescription in actual calculations, a "mode-sum" scheme was later devised in Refs. [6, 7, 8], based on the MST/QW result. An alternative regularization approach, based on the $\zeta-$function technique, was introduced in Ref. [9].

The main objective of this paper is to report on a first actual calculation of the gravitational SF, based on the mode-sum prescription. Focusing, as a test-case, on radial trajectories in a Schwarzschild background, we demonstrate the applicability of this approach, and push forward some analytic and numerical techniques which may later be applied to more general orbits. Among the new results presented here: (i) Two different derivations of the "regularization parameters", independent of each other and of the derivation of [8]; (ii) Consistency of the MST/QW regularization with the ζ-function method; (iii) A first explicit example of the gauge-invariance feature of the regularization parameters predicted in Ref. [10]; (iv) An improved numerical method for integrating the decoupled field equations to fourth order accuracy; (v) An analytic approximation developed for improving the convergence rate of the mode-sum series (see below). Full details of our analysis shall be provided elsewhere [11].

Throughout this paper we use "geometrized" units $G = c = 1$, metric signature $-+++$, and the standard Schwarzschild coordinates t, r, θ, φ. We consider a particle falling radially into a Schwarzschild black hole with mass M, starting at rest at $r = r_0$, and let r_p denote the value of r at the SF evaluation point. In the lack of SF, the particle traces a geodesic characterized by the (conserved) specific energy $E \equiv (1 - 2M/r_0)^{1/2}$. By virtue of the symmetry of the above setup, we obviously have $F^\theta_{\rm self} = F^\varphi_{\rm self} = 0$; we hereafter thus focus on the r, t components of the SF.

Let us start by briefly reviewing the mode-sum method for constructing the gravitational SF: First, one has to calculate the multipole modes of the metric perturbation $h_{\alpha\beta}$ in the harmonic gauge, denoted here by $h^\ell_{\alpha\beta}$ (this refers to the quantity obtained by summing over all azimuthal numbers m and over all ten tensor harmonics for a given multipole number ℓ). This calculation is done through a numerical integration of the decoupled linearized Einstein equations. Then, one constructs the ℓ-mode contribution to the "full" force, denoted here by $F^{\alpha\ell}$, through a certain operation involving 1st-order derivatives of $h^\ell_{\alpha\beta}$ [see Eq. (15) of Ref. [8]]. In the radial motion case, this operation reduces to

$$F^{\alpha\ell}_\pm = \mu \, k^{\alpha\beta\gamma\delta} \bar{h}^\ell_{\beta\gamma;\delta} \tag{2}$$

(evaluated at the particle's location), where $\bar{h}^\ell_{\alpha\beta} \equiv h^\ell_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} h^\ell_{\mu\nu}$, $k^{\alpha\beta\gamma\delta} \equiv u^\beta u^\gamma g^{\alpha\delta}/2 + g^{\beta\gamma} g^{\alpha\delta}/4 + u^\alpha g^{\beta\gamma} u^\delta/4 - g^{\alpha\beta} u^\gamma u^\delta - u^\alpha u^\beta u^\gamma u^\delta/2$, and the $\pm$ sign corresponds to taking the derivative from $r \to r_p^\pm$, respectively. (Note that these force-modes satisfy the normalization condition $u_\alpha F^{\alpha\ell}_\pm = 0$.) While the perturbation itself diverges at the particle's location, the individual modes $h^\ell_{\alpha\beta}$ are continuous everywhere [9]—an important benefit of the mode-sum approach (this holds in the harmonic gauge or in any other gauge related to it by a regular gauge transformation). Typically, however, the derivatives of $h^\ell_{\alpha\beta}$ are found to have a finite discontinuity through the particle's location, yielding two different finite values $F^{\alpha\ell}_\pm$. According to the mode-sum method, the gravitational SF is then constructed through [7]

$$F^\alpha_{\rm self} = \sum_{\ell=0}^{\infty} \left[F^{\alpha\ell}_\pm - A^\alpha_\pm L - B^\alpha - C^\alpha/L \right] - D^\alpha, \tag{3}$$

where $L \equiv \ell + 1/2$ and the (ℓ-independent) quantities A^α, B^α, C^α, and D^α are the so-called "regularization parameters", whose values depend on the orbit under consideration. Roughly speaking, the expression $A^\alpha_\pm L + B^\alpha + C^\alpha/L$ reflects the asymptotic form of $F^{\alpha\ell}_\pm$ at large ℓ [ensuring convergence of the sum in Eq. (3)], while the parameter D^α is a certain residual quantity that arises in the summation over ℓ. (See [7, 8] for an exact definition of these parameters.)

Incorporating a systematic perturbation expansion of the $\ell-$mode Green's function associated with the perturbation equations in the harmonic gauge—an implementation of the technique developed in [7]—we have obtained for radial trajectories [11],

$$A^r_\pm = \mp \frac{\mu^2}{r_p^2} E, \qquad A^t_\pm = \mp \frac{\mu^2}{r_p^2} \frac{\dot{r}_p}{f}, \tag{4a}$$

$$B^r = -\frac{\mu^2}{2r_p^2} E^2, \qquad B^t = -\frac{\mu^2}{2r_p^2} \frac{E\dot{r}_p}{f}, \tag{4b}$$

$$C^\alpha = D^\alpha = 0, \tag{4c}$$

where $f \equiv 1 - 2M/r_p$ and $\dot{r_p} = -(E^2 - f)^{1/2}$. These values agree with those derived (for generic orbits) in Ref. [8] using a different method. Note that whereas the values of $A^\alpha_\pm$, B^α, and C^α can be verified numerically by examining the behavior of the modes $F^{\alpha\ell}_\pm$ at large ℓ (see below), the value of D^α cannot be so verified; hence the importance of our independent derivation of D^α.

The above prescription requires one to tackle the perturbation equations in the harmonic gauge. These equations are separable with respect to ℓ, m [7], but it is not clear how, or whether at all, one could avoid, the coupling occurring between different elements of the tensor-harmonic basis. A more practical derivation of $h^\ell_{\alpha\beta}$ (and, consequently, of $F^{\alpha\ \ell}_\pm$) is possible in the *Regge-Wheeler* (RW) gauge [12, 13]: Here, a well developed formalism [15] allows one to derive all $h^\ell_{\alpha\beta}$ components from two scalar generating functions, by mere differentiation [9, 11]. These two waveforms satisfy a scalar-like wave equation which is easily accessible to numerical treatment. Now, it has been shown [10] that the mode-sum formula (3) is valid, *with the same parameter values*, for any gauge related to the harmonic gauge through a regular gauge transformation. The RW gauge belongs to this regular family of gauges so long as radial trajectories are considered [10]—as in our current work. This shall allow us to work here entirely within the convenient RW gauge.

Using a variant of the Green's function expansion technique mentioned above—this time applied to the perturbation equations in the RW gauge—we have been able to directly obtain the values of A_α, B_α, and C_α associated with the RW-gauge modes $F^{\ell\pm}_\alpha$. (The details of this derivation shall be given in [11].) The RW-gauge parameters thus obtained were found to have, in the head-on case considered here, *precisely the same values* as in the harmonic gauge. This serves as a first explicit demonstration of the regularization parameters' gauge-invariance property predicted in [10].

It should be noted that the mode-sum prescription (3), stemming from the standard MST/QW regularization scheme, completely conforms with the ζ-function regularization approach introduced in Ref. [9]: In the latter too, the SF is brought to the form (3), with the parameter D_α shown [9, 14] to be $\propto \zeta(0, 1/2) = 0$ (where ζ is Riemann's generalized zeta function).

Incorporating the parameter values (4) in Eq. (3) (noticing that $A^\alpha_+ = -A^\alpha_-$), we next write the mode-sum formula in the compact form

$$F^\alpha_{\rm self} = \sum_{\ell=0}^{\infty}\left(\bar{F}^{\alpha\ell} - B^\alpha\right) \equiv \sum_{\ell=0}^{\infty} F^{\alpha\ell}_{\rm reg}, \qquad (5)$$

where $\bar{F}^{\alpha\ell} \equiv (F^{\alpha\ell}_+ + F^{\alpha\ell}_-)/2$, and $F^{\alpha\ell}_{\rm reg}[\propto O(\ell^{-2})]$ are the "regularized" modes. Note that $B^\alpha(r_p)$ describes the asymptotic form of $\bar{F}^{\alpha\ell}$ at $\ell \to \infty$.

We now turn to the actual implementation of the prescription (5), beginning with the numerical calculation of $\bar{F}^{\alpha\ell}$. As already mentioned, our calculation was carried out within the RW gauge. All tensorial components of the ℓ-mode metric perturbations in the RW gauge (in fact, only the even-parity part of $h_{\alpha\beta}$ plays role in our head-on case) are conveniently constructed from a single scalar generating function—Moncrief's gauge-invariant waveform ψ^ℓ [15]. This construction, prescribed in [9], involves twice differentiating ψ^ℓ. Then, the desired modes $F^{\alpha\ell}$ are obtained using Eq. (2). Thus, the numerical task reduces to integrating the (inhomogeneous) hyperbolic wave equation satisfied by ψ^ℓ [16], with the appropriate source term associated with the point-like particle, and with a proper choice of initial data. This numerical problem has been formalized and worked out previously in Ref. [16]. We have developed an improved version of the above numerical scheme, which ensures fourth-order numerical convergence. (This is essential for our purpose, as the construction of $F^{\alpha\ell}$ involves *three* derivatives of the numerical-integration variable ψ^ℓ.)

Typical results from applying the above numerical prescription are presented in Fig. 1, showing the first few (averaged) modes $\bar{F}^{r\ell}$ as a function of r_p for the sample value $r_0 = 14M$ ($E \cong 0.926$), and demonstrating the anticipated $\propto \ell^{-2}$ behavior of $F^{r\ell}_{\rm reg}$. The above construction of $\bar{F}^{\alpha\ell}$ is only applicable to the "radiative" modes $\ell \geq 2$. The modes $l = 0, 1$, although merely reflecting a residual gauge freedom [12], must also be taken into account in the mode sum (5). As shown in [13], the $l = 1$ even-parity perturbation is completely removable by a gauge transformation (interpreted as a translation to the center of mass system), and we may take $\bar{F}^{l=1}_\alpha = 0$. The $l = 0$ perturbation mode (interpreted as a variation in the mass M) is constructed analytically in [11]—the resultant contribution $\bar{F}^{l=0}_\alpha$ is also plotted in Fig. 1. [17]

Since $F^{\alpha\ell}_{\rm reg} \propto O(\ell^{-2})$, the mode sum in (5) admits the slow convergence rate $\propto 1/\ell$. This means that achieving even a modest accuracy requires one to sum over many

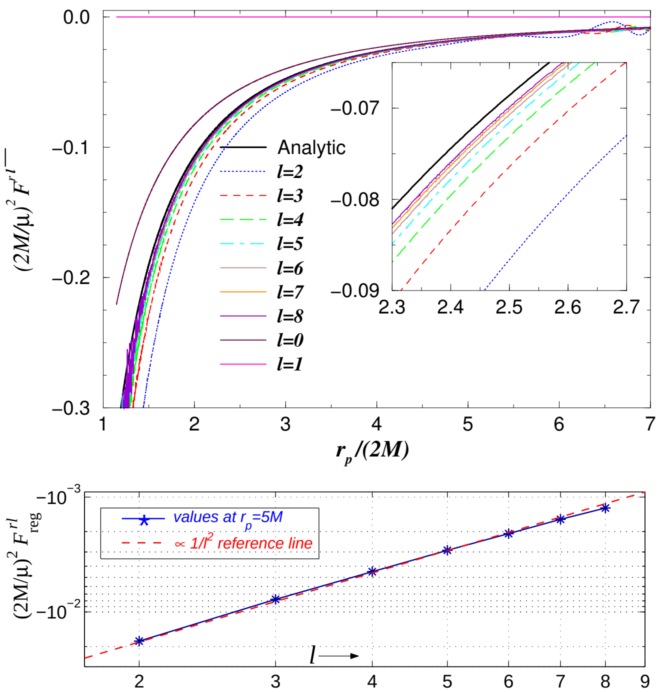

FIG. 1: Upper figure: The "full" modes $\bar{F}^{r\ell}(r_p)$ at $\ell = 0, \ldots, 8$, for a particle released from rest at $r_0 = 14M$. Note how a limiting curve [given analytically by $B^r(r)$] is approached at large ℓ. The wavy feature near $r_p = 14M$ is due to the radiation content of the initial data (chosen here as conformally flat). This "spureous" feature damps down by the time the particle reaches $r_p \sim 10M$, exposing the inherent self force effect. The bottom figure demonstrates the anticipated $\propto \ell^{-2}$ convergence of the difference $F^{r\ell}_{\rm reg} \equiv \bar{F}^{r\ell} - B^r$.

modes, which is numerically very demanding (numerical integration of the decoupled field equations becomes increasingly difficult at growing ℓ). To improve the convergence rate of the mode-sum series, we have obtained [11] an analytic approximation for $F^{\alpha\ell}_{\rm reg}$ at large ℓ: By extending the local analysis of the $\ell-$mode Green's function one order beyond the calculation of the three parameters A^α, B^α, and C^α, we have obtained an analytic expression for the $O(L^{-2})$ term of $F^{\alpha\ell}$. This significantly improves the mode-sum convergence, especially by virtue of the fact that the next, $O(L^{-3})$ term in the mode-sum is expected to vanish (this can be shown for any positive odd power of $1/L$ in the mode sum using straightforward parity arguments [14], and is further supported by our numerical results—see Fig. 2). We have found [11]

$$F^{r\ell}_{\rm reg} = -\frac{15}{16}\mu^2\frac{E^2}{r_p^2}\left(E^2 + \frac{4M}{r_p} - 1\right)L^{-2} + O(L^{-4}), \quad (6a)$$

$$F^{\ell}_{t\ \rm reg} = -\frac{15}{16}\mu^2 E\frac{d}{d\tau}\left(\frac{\dot{r}_p^2}{r_p}\right)L^{-2} + O(L^{-4}), \qquad (6b)$$

where τ is the particle's proper time (the two expressions are negative definite for all $r < r_0$). These expressions are in perfect agreement with the numerical results, as demonstrated in Fig. 2. The contribution of the $O(L^{-2})$ expansion term to the overall SF is now easily obtained analytically, using $\sum_{l=2}^{\infty} L^{-2} \cong 0.49$. The remainder of the mode sum now converges as $\propto \ell^{-3}$. By calculating numerically only the first 10 modes (say), one now obtains the SF to within a mere relative error of $\sim 10^{-3}$ (compare this with a $\sim 10^{-1}$ error when not using the analytic approximation).

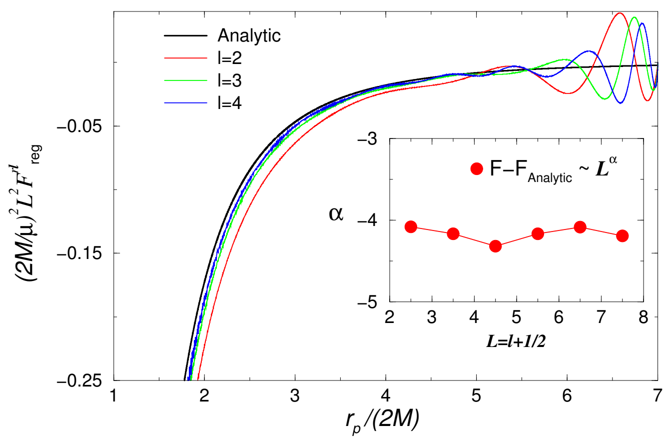

FIG. 2: Analytic approximation *vs.* numerical results: The plot shows the numerically calculated (regularized) modes $F^{r\ell}_{\rm reg}$ along with their large-ℓ analytic approximation, $F^{r\ell}_{\rm analytic}$, as given by the $O(L^{-2})$ term in Eq. (6a). Shown are the modes $\ell = 0, 1, 2, 3, 4$ for $r_0 = 14M$ [times L^2 for the sake of comparison]. The inset shows the reminder $F^{r\ell}_{\rm reg} - F^{r\ell}_{\rm analytic}$ at $r_p = 6M$, demonstrating its anticipated $\propto \ell^{-4}$ behavior. The wavy feature at the onset of the plunge is associated with the "spurious" radiation content of the initial data; the inherent SF is exposed only after these waves are dissipated away.

Figure 3 shows both r and t components of the overall SF resulting from summing up all individual mode contributions. The modes $\ell = 2, \ldots, 8$ were obtained numerically, while for $\ell > 8$ we used the analytic approximation (6) (for $\ell = 0, 1$ we used the exact solutions mentioned above). The radial component of the SF is found to point *inward* (i.e., toward the black hole) throughout the entire plunge. This seems to be a universal feature which does not depend on the starting point r_0. Consequently, the work done by the SF on the particle is positive, resulting in that the energy parameter E *increases* throughout the plunge. The instantaneous rate of change of E is given by [18] $\mu(dE/d\tau) = -F_t$ (≥ 0), and the total change of E is obtained by integrating this expression along the worldline from $\tau(r_0)$ to $\tau(r = 2M)$. It is important to stress, however, that this result will be attached to our specific choice of gauge (as opposed to the energy flux at infinity, which is gauge invariant) [19].

In summary, the mode-sum approach for calculating

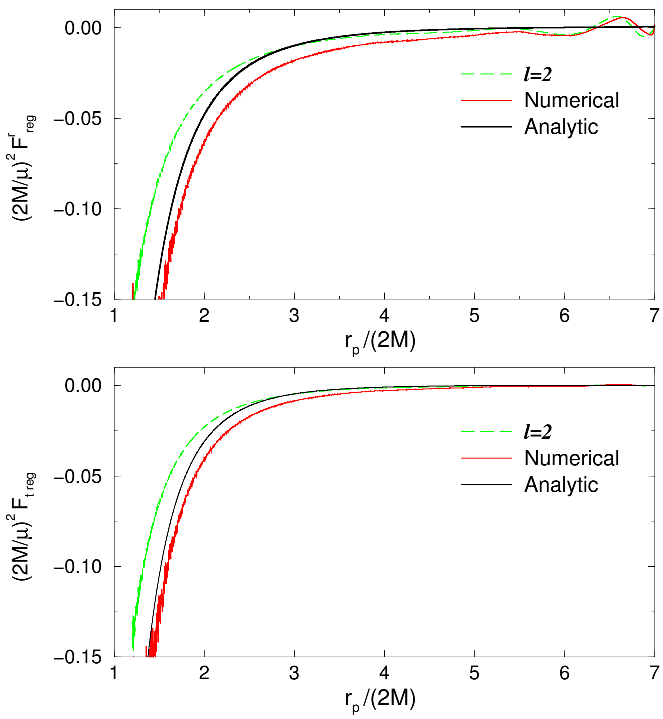

FIG. 3: The upper and bottom panels show, respectively, the r and t components of the overall SF on a particle starting at rest at $r_0 = 14M$. The plots labeled as "numerical" are produced by summing up the numerically calculated (regularized) modes up to $\ell = 8$, and then incorporating our analytic approximation at $\ell > 8$ (these higher modes contribute up to $\sim 20\%$ of the total force). Also given, for comparison, is a curve based entirely on the analytic approximation (6) (summed over $\ell = 2, \ldots, \infty$ plus the exact solutions for $l = 0, 1$); and a curve showing the mere $\ell = 2$ contribution. The latter serves to illustrate the importance of higher ℓ contributions. All curves reach a finite value at the horizon.

the gravitational SF was successfully applied here in the test case of radial motion in Schwarzschild spacetime. We have also demonstrated the feasibility of applying an analytic approximation for improving the mode-sum convergence and even providing a rough estimate to the SF. This marks a significant milestone in our (still long) way toward being able to compute the orbital evolution of generic orbits in Kerr spacetime. The next step along this way, already being considered, is the implementation of the mode-sum prescription to more general orbits in Schwarzschild background. This will provide a first opportunity for validating the SF approach against available calculations based on the standard energy-momentum balance approach.

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[1] Up to date information concerning the joint NASA-ESA project LISA can be found in the project's webpage at http://lisa.jpl.nasa.gov/

[2] A. Ghez, M. Morris, E. E. Becklin, T. Kremenek, and A. Tanner, Nature **407**, 349 (2000).

[3] S. A. Hughes, Phys. Rev. D **61**, 084004 (2000); and references therein.

[4] Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).

[5] T. C. Quinn and R.M. Wald, Phys. Rev. D **56**, 3381 (1997).

[6] The mode-sum scheme was first introduced for the scalar SF in L. Barack and A. Ori, Phys. Rev. D **61**, 061502(R) (2000); L. Barack, *ibid.* **62**, 084027 (2000). For an implementation of this method in the *scalar* case, see, e.g., L. M. Burko, Phys. Rev. Lett. **84**, 4529 (2000); L. Barack and L. M. Burko, Phys. Rev. D**62**, 084040 (2000); L. M. Burko and Y.-T. Liu, *ibid.* **64**, 024006 (2001).

[7] L. Barack, Phys. Rev. D **64**, 084021 (2001).

[8] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).

[9] C. O. Lousto, Phys. Rev. Lett. **84**, 5251 (2000).

[10] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).

[11] L. Barack and C. O. Lousto, in preparation.

[12] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[13] F.J. Zerilli, Phys. Rev. D **2**, 2141 (1970).

[14] C. O. Lousto, Class. Quant. Grav. **18**, 3989 (2001).

[15] V. Moncrief, Annals of Physics (NY), **88**, 323 (1974).

[16] C. O. Lousto and R.H. Price, Phys. Rev. D **55**, 2124 (1997); *ibid.* **56**, 6439 (1997); *ibid.* **57**, 1073 (1998).

[17] The "standard" $l = 0$ solution constructed by Zerilli [13] is pathological at the event horizon. Here we adopt a different, physically consistent, monopole solution having $K^\ell = 0, H_0^\ell = H_1^\ell = H_2^\ell$, in the RW notation [12]. We shall discuss this issue in more detail in [11].

[18] A. Ori, Phys. Lett. A **202**, 347 (1995).

[19] T. C. Quinn and R. M. Wald, Phys. Rev. D**60**, 064009 (1999).