Gravitational lensing in spherically symmetric static spacetimes with centrifugal force reversal

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Abstract

In Schwarzschild spacetime the value $r = 3m$ of the radius coordinate is characterized by three different properties: (a) there is a "light sphere", (b) there is "centrifugal force reversal", (c) it is the upper limiting radius for a non-transparent Schwarschild source to act as a gravitational lens that produces infinitely many images. In this paper we prove a theorem to the effect that these three properties are intimately related in *any* spherically symmetric static spacetime. We illustrate the general results with some examples including black-hole spacetimes and Morris-Thorne wormholes.

1 Introduction

In a Schwarzschild spacetime with mass m, the horizon at the value $r = 2m$ of the radius coordinate plays a distinguished role. However, also the value $r = 3m$ is of particular interest. As a matter of fact, this value is characterized by three quite different properties.

First, a geodesic with circular orbit of radius r around the center must be timelike for $r > 3m$, lightlike for $r = 3m$, and spacelike for $r < 3m$. A light ray emitted tangentially to a circle of radius r will go to infinity for $r > 3m$, it will stay on this circle for $r = 3m$, and it will go towards the center for $r < 3m$. For this reason, one often refers to the surface $r = 3m$ as to a "light sphere" or a "photon sphere". For a detailed study of the geodesics in Schwarzschild spacetime the reader may consult Chandrasekhar [1]; the notion of a photon sphere is discussed from a more general perspective in a recent paper by Claudel, Virbhadra and Ellis [2].

Second, an observer moving on a (non-geodesic) circular orbit of radius r feels a centrifugal force that is pointing in the direction of increasing r, as in Newtonian physics, for $r > 3m$. However, for $r < 3m$ the centrifugal force is pointing in the direction of decreasing r. An observer at $r = 3m$ feels no centrifugal force at all. This phenomenon of centrifugal force reversal was discussed in a long series of articles by Marek Abramowicz with various coauthors, see e.g. Abramowicz and Prasanna [3]; there is also a forthcoming book by Abramowicz and Sonego [4].

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Third, we may fix a constant $r_1 > 2m$ and ask about the number of future-pointing lightlike geodesics in the spacetime region $r > r_1$ that start on a given t-line γ and end at a given spacetime point p. If we exclude the exceptional case that p and γ lie on a common spatial axis through the center, this number is finite if $r_1 > 3m$. However, this number is infinite if $r_1 \leq 3m$. Viewing γ as the worldline of a light source and p as an observation event, this result admits the following interpretation. A non-transparent Schwarzschild source of radius $r_1 > 3m$ acts as a gravitational lens that produces finitely many images whereas the number of images is infinite in the case $r_1 \leq 3m$. A discussion of lensing in the Schwarzschild spacetime, including results on image positions on the observer's sky and on relative magnitudes of the images, can be found in Virbhadra and Ellis [5], also see Frittelli, Kling and Newman [6].

It is an interesting question to ask whether these three properties coincidentally come together in the Schwarzschild spacetime. It is the purpose of this paper to demonstrate that this is not the case. More precisely, we are going to prove a theorem to the effect that, in any spherically symmetric and static spacetime, the presence of a light sphere, the occurence of centrifugal force reversal, and multiple imaging with infinitely many images are closely related phenomena.

To work this out, we consider (3+1)-dimensional spacetimes with metrics of the form

$$g = -A(\rho)^2 dt^2 + B(\rho)^2 d\rho^2 + C(\rho)^2 \big(d\vartheta^2 + \sin^2\vartheta d\varphi^2 \big) \tag{1}$$

with some strictly positive C^2 functions A, B and C. Here we assume that ϑ and φ have their usual range as standard coordinates on the 2-sphere S^2 whereas t is assumed to range over $\mathbb{R}$ and ρ is assumed to range over some open interval $]\rho_{\min}, \rho_{\max}[$ with $-\infty \leq \rho_{\min} < \rho_{\max} \leq \infty$. So the topology of the (3+1)-dimensional spacetime manifold is $\mathbb{R}^2 \times S^2$.

This is the general form of a spherically symmetric and static spacetime. The assumptions on A, B, and C to be strictly positive make sure that the Killing vector field ∂_t is timelike and that the metric has Lorentzian signature. The meaning of the functions A, B and C is the following. $A(\rho)dt$ is the proper-time differential along the t-lines, $B(\rho)d\rho$ is the proper-length differential along the ρ-lines and $4\pi C(\rho)^2$ is the area of the sphere at constant ρ and constant t. If the derivative $C'(\rho)$ is different from zero for all $\rho \in \,]\rho_{\min}, \rho_{\max}[$, we may transform to a "Schwarzschild-like radius coordinate" r via $\rho \longmapsto r = C(\rho)$. However, we also want to include examples, such as wormholes, where the derivative of C does have zeros. Therefore, we stick with the more general radius coordinate ρ.

It is our goal to investigate, for a point p and a t-line γ in such a spacetime, the light rays (i.e., future-pointing lightlike geodesics) that start on γ and terminate at p. For any choice of p and γ we can achieve by a spatial rotation that p and γ are in the hyperplane $\vartheta = \pi/2$. If, in the new coordinate system, the φ-coordinates of p and γ do not differ by a multiple of π, then all light rays from γ to p are confined to this hyperplane. In the exceptional case that the φ-coordinates of p and γ do differ by a multiple of π, i.e., that p and γ lie on a common radial axis P, every light ray from γ to p in the hyperplane $\vartheta = \pi/2$ gives rise to a one-real-parameter family of light rays from γ to p, resulting by applying rotations around the axis P. (Such a one-parameter-family of light rays indicates that the observer at p is seeing an *Einstein ring* of the light source with worldline γ.) In any case, knowledge of the light rays in the hyperplane $\vartheta = \pi/2$ will be sufficient to know all light rays. For that reason we may restrict our consideration to (2+1)-dimensional spacetimes with metrics of the form

$$g = -A(\rho)^2 dt^2 + B(\rho)^2 d\rho^2 + C(\rho)^2 d\varphi^2 \tag{2}$$

with t ranging over $\mathbb{R}$, φ ranging over $\mathbb{R}$ mod 2π, and ρ ranging over $]\rho_{\min}, \rho_{\max}[$ with $-\infty \le \rho_{\min} < \rho_{\max} \le \infty$.

2 Lightlike geodesics

Solving the geodesic equation for the metric (2), which can be done explicitly up to quadratures, is a standard exercise. In this section we summarize, for later convenience, the relevant equations for lightlike geodesics. To that end we first observe that a lightlike geodesic β of the metric (2) admits the following three constants of motion,

$$g(\dot{\beta}, \dot{\beta}) = -A(\rho)^2\, \dot{t}^2 + B(\rho)^2\, \dot{\rho}^2 + C(\rho)^2\, \dot{\varphi}^2 = 0\,, \tag{3}$$

$$-g(\dot{\beta}, \partial_t) = A(\rho)^2\, \dot{t} = E\,, \tag{4}$$

$$g(\dot{\beta}, \partial_\varphi) = C(\rho)^2\, \dot{\varphi} = L\,, \tag{5}$$

where an overdot denotes differentiation with respect to the curve parameter. We restrict to the case $E = 1$, thereby singling out for each geodesic a unique parametrization that is future-pointing with respect to the time coordinate t.

Inserting (4) with $E = 1$ and (5) into (3) results in

$$A(\rho)^2\, B(\rho)^2\, \dot{\rho}^2 + L^2\, V(\rho) \;=\; 1\,, \tag{6}$$

where we have introduced the potential

$$V(\rho) = A(\rho)^2\, C(\rho)^{-2} \tag{7}$$

which will play the central role throughout our analysis. Please note that this potential is unaffected by a conformal change of the metric. We now divide (6) by $V(\rho)$ and differentiate the resulting equation with respect to the curve parameter. After dividing by $\dot{\rho}$ we find

$$\ddot{\rho} + \dot{\rho}^2 \big(C'(\rho) C(\rho)^{-1} + B'(\rho) B(\rho)^{-1} \big) = -\tfrac{1}{2}\, C(\rho)^2\, B(\rho)^{-2}\, A(\rho)^{-4}\, V'(\rho)\,. \tag{8}$$

Although we divided by $\dot{\rho}$, this equation has to hold, by continuity, also at points where $\dot{\rho} = 0$.

Clearly, the constant map $s \longmapsto \rho(s) = \rho_0$ is a solution of the differential equation (8) if and only if $V'(\rho_0) = 0$. This is the necessary and sufficient condition for a lightlike geodesic with circular orbit to exist at radius ρ_0. From (6) we read that, for such a geodesic, the constant of motion L has to satisfy the equation $L^2\, V(\rho_0) = 1$. A stability analysis of (8) shows that a circular light orbit at ρ_0 is stable with respect to perturbations of the initial condition if $V''(\rho_0) > 0$ and unstable if $V''(\rho_0) < 0$.

Also from (6) and (8) we read that along a lightlike geodesic with constant of motion L the radius coordinate has a strict local maximum (or a strict local minimum, respectively) at points where $L^2\,V(\rho) = 1$ and $V'(\rho) > 0$ (or $V'(\rho) < 0$, respectively). Other extrema cannot occur along lightlike geodesics with non-circular orbits.

For later purpose we observe that (5) and (6) imply

$$\dot{\varphi} = \frac{L\,B(\rho)\,C(\rho)^{-1}}{\sqrt{V(\rho)^{-1} - L^2}}\,|\dot{\rho}|. \tag{9}$$

Here we have made use of the fact that, by (5), $\dot{\varphi}$ always has the same sign as L. By integration, (9) yields the orbits of the light rays in the (ρ, φ)-plane.

3 Centrifugal force reversal

In this section we want to discuss the "centrifugal force" felt by an observer in circular motion in the metric (2). This is the only case that is of interest to us in this paper. For possible generalizations to non-circular motions in arbitrary stationary spacetimes we refer the reader, e.g., to Abramowicz, Carter and Lasota [7] and to Bini, Carini and Jantzen [8].

On a (2+1)-dimensional spacetime with metric (2), we introduce the timelike vector field

$$U = \frac{1}{\sqrt{1-v^2}}\Big(A(\rho)^{-1}\,\partial_t \,\pm\, v\,C(\rho)^{-1}\,\partial_\varphi\Big) \tag{10}$$

with some constant $v \in [0, 1[$. The integral curves of this vector field can be interpreted as worldlines of observers that move on circular orbits with constant 3-velocity v (in units of the velocity of light) with respect to the static observers whose worldines are the t-lines. By (10), the vector field U is normalized according to $g(U, U) = -1$, so its integral curves are parametrized by proper time.

In general, the integral curves of U are no geodesics, i.e., the 4-acceleration $\nabla_U U$ does not vanish. With respect to an observer moving along an integral curve of U, the relative acceleration of a freely falling observer with a momentarily tangential worldline is given by $-\nabla_U U$. In correspondence with standard non-relativistic terminology, this quantity could be viewed as "inertial acceleration". We are interested in its radial component which is readily calculated with the help of the identity $-g(\partial_\rho, \nabla_U U) = \frac{1}{2}\big(L_{\partial_\rho} g\big)\big(U\,,\,U\big)$, where $L_{\partial_\rho} g$ denotes the Lie derivative of g with respect to the vector field ∂_ρ . This results in

$$-g(\partial_\rho, \nabla_U U) = -A'(\rho)\,A(\rho)^{-1} - \frac{v^2}{2(1-v^2)}\,C(\rho)^2\,A(\rho)^{-2}\,V'(\rho)\,, \tag{11}$$

with V defined by (7).

On the right-hand side of (11), we interpret the first term as *gravitational acceleration* and the second as *centrifugal acceleration.* (By multiplying each of those accelerations with the observer's mass we get the respective "force".) These names are justified since the first term is independent of v, whereas the second term is proportional to v^2 in lowest order. Hence, for velocities small compared to the velocity of light the centrifugal term has, indeed, the same v-dependence as in Newtonian physics.

From (11) we read that the sign of the centrifugal term is determined by the sign of V'. The centrifugal acceleration is pointing in the direction of increasing ρ at all values of ρ

with $V'(\rho) < 0$, and it is pointing in the direction of decreasing ρ at all values of ρ where $V'(\rho) > 0$. In the following we are interested in the situation that V' changes sign at some radius ρ_0. In this situation we say that there is "centrifugal force reversal" at ρ_0. It is one of our goals to prove that then the gravitational field produces infinitely many images for static light sources and observers at radii close to ρ_0.

By comparison with the preceding section we see that centrifugal force reversal can occur at ρ_0 only if there is a circular light orbit at ρ_0. Note, however, that the occurence of a circular light orbit is not sufficient for centrifugal force reversal; the potential V might have a saddle.

The following observation is also of interest. In accordance with (11), U can be geodesic ($\nabla_U U = 0$) only at those points where the centrifugal acceleration is exactly balanced by the gravitational acceleration. If the gravitational acceleration is pointing in the direction of decreasing ρ, $A'(\rho) > 0$, this is impossible at any radius ρ where $V'(\rho) > 0$. In this sense, validity of the inequality $V'(\rho) > 0$ has the effect that a freely falling object (with subluminal velocity) cannot stay at radius ρ.

4 Multiplicity results for light rays

In the preceding sections we have emphasized the role of the potential V, defined by (7). In particular, we have seen that the zeros of V' indicate circular light orbits and that the sign of V' determines the direction of the centrifugal force. In this section we shall state and discuss a theorem that relates the occurence or non-occurence of extrema (minima, maxima, or saddles) of the potential V to multiple imaging. The proof of this theorem will be given in the subsequent section.

Theorem 4.1. *Consider a $(2+1)$-dimensional Lorentzian manifold (M, g) with metric of the form (2), where the coordinate ranges are $t \in \mathbb{R}$, $\varphi \in \mathbb{R}$ mod 2π, and $\rho \in \,]\rho_{\min}, \rho_{\max}[$ with $-\infty \le \rho_{\min} < \rho_{\max} \le \infty$, hence $M \simeq \mathbb{R}^2 \times S^1$. Fix a point p in M and an integral curve γ of ∂_t and denote the radius coordinates of γ and p by ρ_1 and ρ_2, respectively. Let $N(p, \gamma)$ be the number of future-pointing lightlike geodesics in M that start on γ and terminate at p, with two geodesics being identified if one is a reparametrization of the other. Then the following is true for the potential V defined by (7).*

(a) *If V' has no zeros on the whole interval $]\rho_{\min}, \rho_{\max}[$, then $N(p, \gamma)$ is finite.*

(b) *If there is a $\rho_0 \in \,]\rho_{\min}, \rho_{\max}[$ such that $V(\rho) \le V(\rho_0)$ for all $\rho \in \,]\rho_{\min}, \rho_{\max}[$, then $N(p, \gamma)$ is infinite.*

(c) *Assume that there is a $\rho_0 \in \,]\rho_{\min}, \rho_{\max}[$ with $V'(\rho_0) = 0$ such that $V'(\rho) < 0$ for all $\rho \in \,]\rho_{\min}, \rho_0[$ and $V'(\rho) > 0$ for all $\rho \in \,]\rho_0, \rho_{\max}[$. Assume, in addition, that $\lim_{\rho \to \rho_{\min}} V(\rho) = \lim_{\rho \to \rho_{\max}} V(\rho)$. Then $N(p, \gamma)$ is either zero or infinite. By keeping γ fixed and moving p an appropriate distance along the circle $\rho = \rho_2$ one can always achieve that $N(p, \gamma)$ is infinite. Moreover, by an arbitrarily small perturbation of the metric coefficients A, B, C one can always achieve that $N(p, \gamma)$ is infinite for any choice of p and γ.*

(d) *Assume there is a $\rho_0 \in \,]\rho_{\min}, \rho_{\max}[$ with $V'(\rho_0) = 0$ such that $V'(\rho) < 0$ for all $\rho \in \,]\rho_{\min}, \rho_{\max}[$ with $\rho \neq \rho_0$. If $\rho_1 < \rho_0$ or $\rho_2 < \rho_0$, then $N(p, \gamma)$ is finite. If both $\rho_1 \ge \rho_0$ and*

$\rho_2 \geq \rho_0$*, then* $N(p, \gamma)$ *is infinite.*
Part (d) *implies an analogous result for the case that the inequality* $V'(\rho) > 0$*, instead of* $V'(\rho) < 0$*, holds for all* $\rho \in \,]\rho_{\min}, \rho_{\max}[$ *with* $\rho \neq \rho_0$*, simply by a coordinate transformation* $\rho \longmapsto -\rho$*.*

Every future-pointing lightlike geodesic that starts on γ and terminates at p may be interpreted as giving an image of a light source with worldline γ for an observer at p. Hence, in the case of part (b), part (c) and the second half of part (d) of the theorem there are infinitely many such images. The proof of the theorem will demonstrate that in all these cases the result can be strengthened in the following way. For every integer $n_0 \in \mathbb{N}$ there are such geodesics with winding numbers $n > n_0$ and $n < -n_0$. Here the *winding number* of a curve $\beta : [s_1, s_2] \longrightarrow M$ is defined as the biggest integer $n \in \mathbb{Z}$ such that

$$2\,\pi\, n \;\leq\; \int_{s_1}^{s_2} \dot{\varphi}(s) ds \tag{12}$$

where $\dot{\varphi}(s)$ is the usual shorthand notation for $\frac{d}{ds}\varphi\big(\beta(s)\big)$. Hence, in the case of part (b), part (c) and the second half of part (d) of the theorem there are infinitely many images that correspond to light rays winding in the positive φ-direction ($n \geq 0$) and infinitely many images that correspond to light rays winding in the negative φ-direction ($n < 0$).

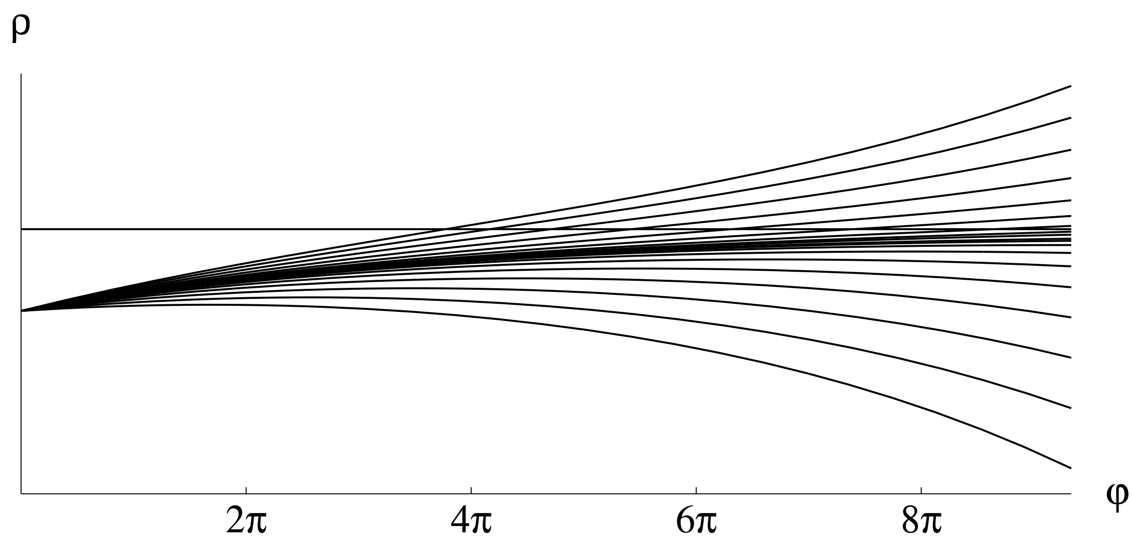

Figure 1: This figure shows in a φ–ρ–diagram the behavior of light rays near an unstable circular light orbit, or, what is the same, near a local maximum of the potential V. We have drawn light rays issuing from a point near the circular light orbit, the latter being indicated by the horizontal line. For producing the picture we have chosen $V(\rho) = (a(\rho - \rho_0)^2 + b)^{-1}$ and $B(\rho)\, C(\rho)^{-1} = c$ with some constants $a > 0$, b and c. The qualitative features, however, are the same near any unstable circular light orbit. As the φ–coordinate is 2π–periodic, the horizontal axis should be thought of as rolled into a circle. Keeping this in mind, the picture clearly illustrates part (b) of Theorem 4.1.

A major value of this theorem is in the fact that any of its four parts can be applied to arbitrarily small intervals $]\rho_{\min}, \rho_{\max}[$. In particular, parts (b), (c) and (d) of this theorem characterize multiple imaging behavior near local maxima, strict local minima and saddles of the potential V. Figures 1, 2 and 3 show the qualitative behavior of light rays near extrema of V and may serve as illustrations of parts (b), (c), and (d), respectively, of Theorem 4.1. The discussion of light rays near a minimum of V is more subtle than near a maximum or near a saddle for the following reason. There is a class of spherically symmetric static spacetimes in which, for the constant of motion L varying over some interval, all light rays have periodic orbits with the same period, see the lower half of Figure 2. These spacetimes could be viewed as lightlike analogues of the *Bertrand spacetimes* discussed by Perlick [9] which are characterized by periodic orbits of *timelike* geodesics. In such a "lightlike Bertrand spacetime" there are pairs of source and observer which cannot be connected by any light ray, whereas other pairs can be connected by infinitely many light rays.

In a nutshell, Theorem 4.1 says that, in a metric of the form (2), multiple imaging with infinitely many images occurs if and only if this metric admits a circular light orbit. The latter may be a local minimum, a local maximum or a saddle of the potential V. If we exclude saddles, then the occurence of a circular light orbit is equivalent to centrifugal force reversal. As saddles are non-generic in the sense that they can be destroyed by an arbitrarily small perturbation of the metric functions, it is thus justified to summarize Theorem 4.1 in the following way. In a generic metric of the form (2), centrifugal force reversal is necessary and sufficient for the occurence of multiple imaging with infinitely many images.

Finally, we want to add some words of caution to prevent the reader from possible misinterpretations. In Theorem 4.1 we identify two lightlike geodesics if one is a reparametrization of the other. This, however, does *not* imply that images are identified if they are situated at the same spot on the observer's sky. Running through a periodic light orbit arbitrarily often in positive φ-direction (or in negative φ-direction, respectively) gives infinitely many images for any pair of source and observer on this orbit, provided that both the source and the observer are "transparent" in the sense that they do not block light rays; however, all these infinitely many images are situated at the same spot on the observer's sky, one behind the other. So the observer will actually *see* only two images, one corresponding to geodesics winding in positive φ-direction and one corresponding to geodesics winding in negative φ-direction. This peculiar situation occurs only if light source and observer are on a periodic light orbit, so it is non-generic. In a generic situation with infinitely many images the images will always be situated at infinitely many different positions on the observer's sky.

Also, it is worthwile to remark that, in the case of an infinite sequence of images, the apparent brightness of these images necessarily goes to zero. Physically, this follows from the fact that a light source cannot emit an infinite amount of energy. Since any detector has a finite sensitivity, it will register only finitely many images. There is a second reason why it is impossible to actually observe an infinite sequence of images. As infinitely many points on the celestial sphere must have an accumulation point, the limited resolution of any detector implies that it is impossible to resolve all of them. Therefore, a mathematical statement that there are infinitely many images at infinitely many different positions on the observer's sky physically only means that the observer can see arbitrarily many images by

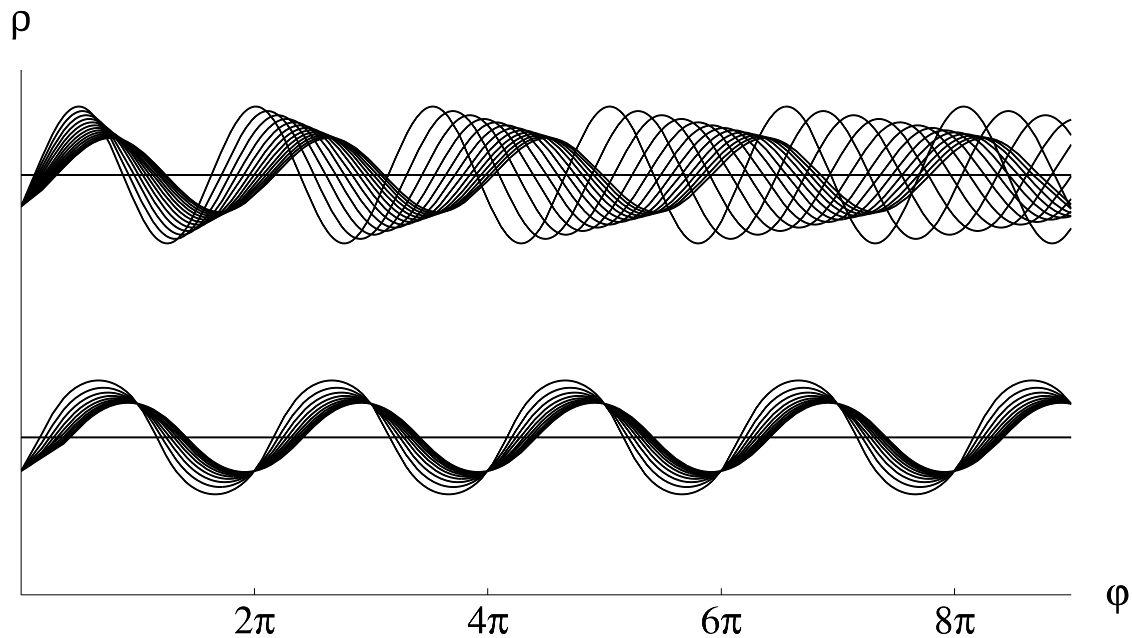

Figure 2: This figure shows in a φ–ρ–diagram the behavior of light rays near a stable circular light orbit, i.e., near a local minimum of the potential V, for two different choices of V. In both cases we have drawn light rays issuing from a point near the circular light orbit, the latter being indicated by a horizontal line. The upper diagram is valid for $V(\rho) = (-a(\rho - \rho_0)^2 + b)^{-1}$, the lower diagram for $V(\rho) = a \cosh^2\big((\rho - \rho_0)/b\big)$, and in both cases we have chosen $B(\rho)\, C(\rho)^{-1} = \, c\,$, with constants $a > 0$, b and c. Along any light ray close to a stable circular light orbit, ρ is a periodic function of φ. This is true independent of the special form of V. However, the period may depend on the constant of motion L, as in our first example, or it may be constant, as in our second example where the period is equal to $2\pi b/c$. In the first case, the light rays issuing from a particular point cover each point in a neighborhood of the circular light orbit infinitely often. In the second case this is true only if the constant period is an irrational multiple of 2π. If the period is a rational multiple of 2π (as in our picture where we have chosen $b = c$), then the light rays issuing from a particular point cover some points in a neighborhood of the circular light orbit infinitely often whereas other points are not met at all. This is obviously a highly non-generic situation. It can be destroyed by an arbitrarily small perturbation of the metric coefficients in such a way that the period becomes L-dependent (or, as an alternative, in such a way that the period becomes a constant but irrational multiple of 2π). In this sense, light rays near a *generic* stable circular light orbit qualitatively behave like in the upper diagram. Some more insight can be gained from studying the proof of part (c) of Theorem 4.1, see Section 5 below.

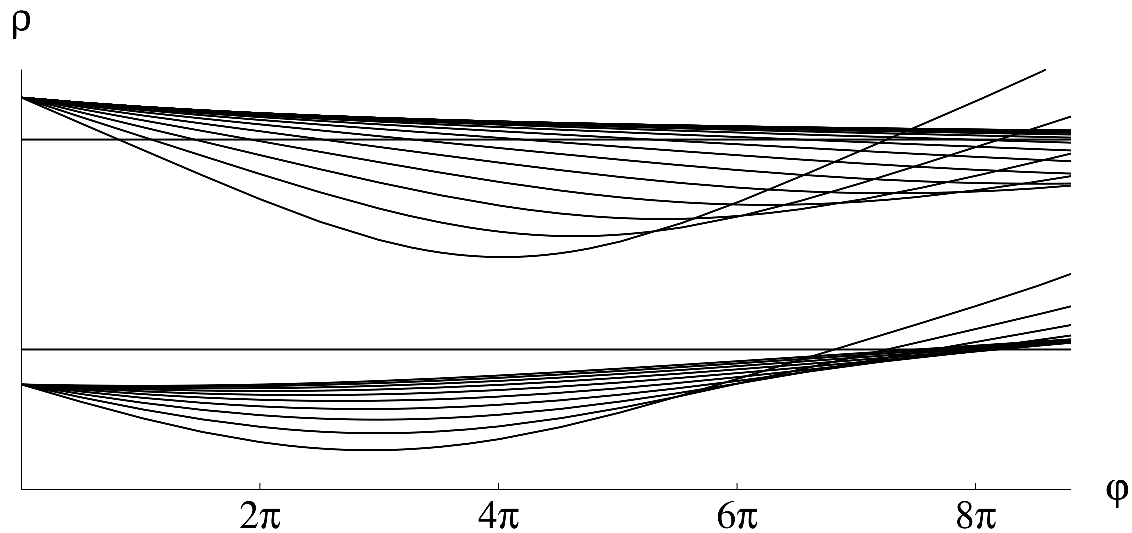

Figure 3: This figure shows in a φ–ρ–diagram the behavior of light rays near a half-stable circular light orbit, i.e., near a saddle of the potential V. For producing the picture we have chosen $V(\rho) = (a(\rho - \rho_0)^3 + b)^{-1}$ and $B(\rho)\,C(\rho)^{-1} = c$ with constants $a > 0$, b and c. The upper diagram shows light rays issuing from a point in the region $\rho > \rho_0$, the lower diagram from a point in the region $\rho < \rho_0$. The qualitative features are the same near any saddle of V.

choosing a detector of sufficiently high sensitivity and resolution (as long as the light source can be treated as pointlike and quantum-theoretical limits on the measuring process play no role).

5 Proof of Theorem 4.1

We make a transformation $]\rho_{\min}, \rho_{\max}[\longrightarrow]u_{\min}, u_{\max}[$, $\rho \longmapsto u$ of the radius coordinate such that

$$du = B(\rho)\,C(\rho)^{-1}\,d\rho\,, \tag{13}$$

and we write

$$f(u) = C(\rho)^2\,A(\rho)^{-2} = V(\rho)^{-1}\,. \tag{14}$$

This puts the orbit equation (9) into the form

$$\dot{\varphi} = \frac{L|\dot{u}|}{\sqrt{f(u) - L^2}}\,. \tag{15}$$

For the evaluation of this equation we shall use the following elementary lemma.

Lemma 5.1. *Let f be a C^2 function $]u_{\min}, u_{\max}[\longrightarrow \mathbb{R}^+$. Fix two real numbers u_a and u_b such that $u_{\min} < u_a < u_b < u_{\max}$ and let K be a constant such that $K^2 \le f(u)$ for all $u \in [u_a, u_b]$.*
(a) *If there is a parameter value $u_0 \in [u_a, u_b]$ with $f(u_0) = K^2$ and $f'(u_0) = 0$, then*

$$\int_{u_a}^{u_b} \frac{du}{\sqrt{f(u) - K^2}} = \infty \,. \tag{16}$$

(b) *If such a parameter value u_0 does not exist, then the integral on the left-hand side of* (16) *is finite.*

Proof. To prove part (a) we choose a small but non-zero real number ε such that

$$\int_{u_a}^{u_b} \frac{du}{\sqrt{f(u) - K^2}} \ge \Big| \int_{u_0}^{u_0+\varepsilon} \frac{du}{\sqrt{f(u) - K^2}} \Big| \,. \tag{17}$$

This is possible because the integrand is positive. (If $u_0 = u_a$ we have to choose $\varepsilon > 0$; if $u_0 = u_b$ we have to choose $\varepsilon < 0$. In any other case we may choose ε either positive or negative.) As, by assumption, $f(u_0) = K^2$ and $f'(u_0) = 0$, Taylor's theorem yields $f(u) = K^2 + h(u)(u - u_0)^2$ with some continuous function h. By the mean value theorem, there is a constant $B_0(\varepsilon) \in \mathbb{R}$ such that

$$\Big| \int_{u_0}^{u_0+\varepsilon} \frac{du}{\sqrt{f(u) - K^2}} \Big| = \Big| \frac{1}{B_0(\varepsilon)} \int_{u_0}^{u_0+\varepsilon} \frac{du}{u - u_0} \Big| = \infty \,. \tag{18}$$

(18) and (17) demonstrate that part (a) of the lemma is true. – Under the assumptions of part(b) of the lemma, the equation $f(u_0) = K^2$ can hold only for $u_0 = u_a$ with $f'(u_a) > 0$ or for $u_0 = u_b$ with $f'(u_b) < 0$ (or for both). We choose a small positive ε and write

$$\int_{u_a}^{u_b} \frac{du}{\sqrt{f(u) - K^2}} = \Big(\int_{u_a}^{u_a+\varepsilon} + \int_{u_a+\varepsilon}^{u_b-\varepsilon} + \int_{u_b-\varepsilon}^{u_b} \Big) \frac{du}{\sqrt{f(u) - K^2}} \,. \tag{19}$$

If $f(u_a) \neq K^2$, the first integral on the right-hand side of (19) is certainly finite. If $f(u_a) = K^2$, Taylor's theorem yields $f(u) = K^2 + C_0(u - u_a) + h(u)(u - u_a)^2$ with a constant $C_0 > 0$ and a continuous function h. Then the mean value theorem guarantees the existence of a constant $A_0(\varepsilon) \in \mathbb{R}$ such that

$$\int_{u_a}^{u_a+\varepsilon} \frac{du}{\sqrt{f(u) - K^2}} = \frac{A_0(\varepsilon)}{\sqrt{C_0}} \int_{u_a}^{u_a+\varepsilon} \frac{du}{\sqrt{u - u_a}} = \frac{2\, A_0(\varepsilon)\sqrt{\varepsilon}}{\sqrt{C_0}} \,. \tag{20}$$

So the first integral on the right-hand side of (19) is always finite. By a completely analogous calculation one shows that the last integral on the right-hand side of (19) is always finite. As the middle integral is obviously finite, this completes the proof of part (b) of the lemma. □

We are now ready to prove the theorem. Having replaced ρ with u, we denote the coordinates of γ and p by (u_1, φ_1) and (u_2, φ_2), respectively. From equation (15) we read that the orbit of a lightlike geodesic remains an orbit of a lightlike geodesic if it is run through in the opposite direction. Therefore, it is no loss of generality if we assume that $u_1 \le u_2$. Moreover, we may assume $0 \le \varphi_2 - \varphi_1 < 2\pi$.

Proof of part (a) *of Theorem* 4.1: By (13) and (14), our assumption of V' having no zeros is equivalent to f' having no zeros. We shall give the proof for the case that $V' < 0$ which is equivalent to $f' > 0$. The case that $V' > 0$ and, thus, $f' < 0$ is then covered as well, because we are always free to change ρ into $-\rho$ and, thereby, u into $-u$. If $V' < 0$, we read from (8) that along any lightlike geodesic the coordinate ρ cannot have other extrema than strict local minima. By (13), the same is true for the coordinate u. Hence, there are two classes of lightlike geodesics from (u_1, φ_1) to (u_2, φ_2): (i) those along which u is a strictly monotonous function, and (ii) those along which u has exactly one extremum, namely a local minimum. What we have to prove is that both classes contain only finitely many members. For a lightlike geodesic of class (i), integration of the orbit equation (15) yields

$$\varphi_2 - \varphi_1 + 2\pi n \,=\, L \int_{u_1}^{u_2} \frac{du}{\sqrt{f(u) - L^2}} \tag{21}$$

where $n \in \mathbb{Z}$ is the winding number. The possible values of L are restricted according to $-K < L < K$, where $K = \sqrt{f(u_a(L))}$. Now part (b) of Lemma 5.1 implies that the right-hand side of (21) is bounded, i.e., (21) can hold only for finitely many integers $n \in \mathbb{Z}$. As the right-hand side of (21) is a strictly monotonous function of L, this proves that there are only finitely many values of L possible for a lightlike geodesic of class (i) from (u_1, φ_1) to (u_2, φ_2). Clearly, the initial condition u_1, φ_1 together with the value of L fixes a solution of (6) and (5) (to be expressed in the new coordinates (u, φ)) uniquely up to extension. As along a geodesic of class (i) u cannot take the value u_2 more than once, this concludes the proof that class (i) contains only finitely many geodesics. – For a lightlike geodesic of class (ii), integration of the orbit equation (15) yields

$$\varphi_2 - \varphi_1 + 2\pi n \,=\, L\Big(\int_{u_a(L)}^{u_1} + \int_{u_a(L)}^{u_2} \Big) \frac{du}{\sqrt{f(u) - L^2}} \tag{22}$$

where $u_a(L)$ is the minimum value of u along the geodesic. $u_a(L)$ is related to L by the equation $f(u_a(L)) = L^2$. Again, the possible values of L are restricted according to $-\sqrt{f(u_1)} < L < \sqrt{f(u_1)}$. By part (b) of Lemma 5.1, both integrals on the right-hand side of (22) remain bounded, even if L^2 approaches its maximal value. Thus, (22) can hold only for finitely many integers n. As in the case of class (i), this leads to the conclusion that there are only finitely many values of L possible for a lightlike geodesic from (u_1, φ_1) to (u_2, φ_2). As along a geodesic of class (ii) u can take the value u_2 at most twice, this demonstrates that class (ii) contains only finitely many geodesics. □

Proof of part (b) *of Theorem* 4.1: We assume that the coordinate transformation $\rho \longmapsto u$ maps ρ_0 to the value u_0 and we distinguish two cases: (A) $u_0 \in [u_1, u_2]$, (B) $u_0 \notin [u_1, u_2]$. In case (A) it is our goal to demonstrate that there is a solution of (15) from (u_1, φ_1) to (u_2, φ_2) with winding number n, for any $n \in \mathbb{Z}$. We first observe that this is obviously true if $u_1 = u_2$ since, by assumption of case (A), this implies that (u_1, φ_1) and (u_2, φ_2) are two points on a circular light orbit with radius u_0; so we can construct the desired light rays simply by running through this light orbit as often as necessary. Therefore, it is no loss of generality if we assume for the proof in case (A) that $u_1 < u_2$. For a geodesic along which the radius coordinate u increases monotonically from u_1 to u_2 integration of the orbit equation (15) yields (21). If we set $K = \sqrt{f(u_0)}$, the allowed values for L are restricted by $-K < L < K$. Evidently, the right-hand side of (21) is a strictly increasing function of L. By Lemma 5.1, the integral on the right-hand side of (21) goes to infinity for $|L| \to K$. As the integrand is positive, we have thus shown that the right-hand side of (21) varies monotonically from $-\infty$ to ∞ if L varies from $-K$ to K. Hence, for any $n \in \mathbb{Z}$ there is an allowed value of L such that (21) is satisfied. – In case (B) we may assume that $u_0 < u_1 \leq u_2$ because we are free to make a coordinate transformation $u \longmapsto -u$. Moreover, we may assume that $K^2 < f(u)$ for all $u \in \,]u_0, u_1[$, where again $K = \sqrt{f(u_0)}$. This can be achieved by replacing, if necessary, the original u_0 with the maximal value at which the condition was violated. (Please note that the theorem does not assume uniqueness of ρ_0 or, what is the same, of u_0.) With these assumptions, we consider light rays along which the radius coordinate monotonically decreases from u_1 to some value $u_a(L) \in [u_0, u_1]$ and then monotonically increases to u_2, where $f(u_a(L)) = L^2$. For such a light ray integration of the orbit equation (15) yields (22). Here the allowed values for L are restricted by $K^2 \leq L^2 \leq (K + \varepsilon)^2$ with some positive ε. For $|L| \to K$, which implies $u_a(L) \to u_0$, both integrals in (22) go to infinity, owing to Lemma 5.1. Thus, if L varies over all allowed positive (or negative, respectively) values, the left-hand side of (22) varies from some positive value to $+\infty$ (or from some negative value to $-\infty$, respectively). This implies that (22) can be satisfied for all $n \in \mathbb{Z}$ with $|n|$ bigger than some $n_0 \in \mathbb{N}$. □

Proof of part (c) *of Theorem* 4.1: By assumption, there is a parameter value $u_0 \in \,]u_{\min}, u_{\max}[$ such that $f'(u) > 0$ for $u \in \,]u_{\min}, u_0[$ and $f'(u) < 0$ for $u \in \,]u_0, u_{\max}[$. As (13) defines u only up to an additive constant, we may assume that $u_0 = 0$. With $\sqrt{f(0)} = K$, the orbit equation (15) says that L is restricted by $L^2 \leq K^2$. For $L = \pm K$ we get the circular light orbit at $u = u_0 = 0$; along light rays with $L^2 < K^2$, if sufficiently extended, u oscillates between a minimum value $u_a(L)$ and a maximum value $u_b(L)$ with $f(u_a(L)) = f(u_b(L)) = L^2$. We are interested in light rays passing through the points (u_1, φ_1) and (u_2, φ_2), so we must have $u_a(L) \leq u_1 \leq u_2 \leq u_b(L)$. This restricts the possible values of L according to $(L_0 - \delta)^2 < L^2 \leq L_0^2$ where $L_0 \leq K$ and δ are some positive constants. For any light ray along which u starts at u_1, increases monotonically to $u_b(L)$, oscillates k times to $u_a(L)$ and back to $u_b(L)$ and finally decreases monotonically to u_2, integration of the orbit equation (15) yields

$$(\Delta\varphi)(k, L) = \Phi(L) + k\Psi(L) \tag{23}$$

where

$$\Phi(L) = L\Big(\int_{u_1}^{u_b(L)} + \int_{u_2}^{u_b(L)}\Big)\frac{du}{\sqrt{f(u) - L^2}} \; . \tag{24}$$

and

$$\Psi(L) = 2L \int_{u_a(L)}^{u_b(L)} \frac{du}{\sqrt{f(u) - L^2}} \; . \tag{25}$$

It is our goal to prove that for any $n \in \mathbb{Z}$ with $|n|$ sufficiently large we can choose k and L such that

$$(\Delta\varphi)(k, L) = \varphi_2 - \varphi_1 + 2n\pi \,. \tag{26}$$

To that end, fix some k and let L run over all allowed positive (or negative, respectively) values. Then $(\Delta\varphi)(k, L)$ ranges over an interval of length $\alpha + k\beta$, where α and β are independent of k and β is zero if and only if the function Ψ is constant on the interval $]L_0 - \delta, L_0]$ (or on the interval $[-L_0, -L_0 + \delta[$, respectively). If $\beta \neq 0$, we can secure the overlapping of intervals pertaining to neighboring values of k by choosing k sufficiently large, $k > \big(\Psi(L_0) - \alpha\big)/\beta$, so we can satisfy equation (26) for any $n \in \mathbb{Z}$ with $|n|$ sufficiently large. Now let us assume that $\beta = 0$, i.e., that $\Psi(L)$ takes a constant value P_0 (necessarily $P_0 > 0$) for all allowed values of L. If $2\pi/P_0$ is irrational, the numbers $\{(\Delta\varphi)(k, L)|k \in \mathbb{N}\}$ modulo 2π are dense in the circle $\mathbb{R}$ modulo 2π, for any allowed value of L. Hence, if L varies over an arbitrarily small interval around a positive (or negative, respectively) allowed value, these numbers cover the circle infinitely often, i.e., (26) can be satisfied for infinitely many positive (or negative, respectively) values of n. If $2\pi/P_0$ is rational, the numbers $\{(\Delta\varphi)(k, L)|k \in \mathbb{N}\}$ modulo 2π meet only finitely many points of the circle $\mathbb{R}$ modulo 2π, for any allowed value of L. Hence, if L varies over a small interval around a positive (or negative, respectively) allowed value, (26) is satisfied either for infinitely many positive (or negative, respectively) values of k or for no such value at all, depending on $\varphi_2 - \varphi_1$. In the latter case one can obviously achieve that $N(p, \gamma)$ is infinite by moving p some distance along the circle $\rho = \rho_2$ (and leaving γ fixed). The case $\beta = 0$, i.e., the case that Ψ is constant on a whole interval, is indeed possible. E. g., if $f(u) = K^2 \cosh^{-2}(Pu)$, we find $\Psi(L) = 2\pi/P$ for all $L^2 < K^2$. It is also possible to construct (non-analytic but arbitrarily often differentiable) examples where Ψ is constant on some interval but not everywhere constant. The property of Ψ being constant on a whole interval can always be destroyed by an arbitrarily small perturbation of f (i.e., of the metric coefficients) as can be seen from (25). □

Proof of part (d) *of Theorem* 4.1: By assumption, the function f has a saddle at some value u_0. If $u_1 = u_2 = u_0$, then there are infinitely many light rays from (u_1, φ_1) to (u_2, φ_2) because we can run through the circular light orbit at u_0 as often as we like. Therefore we may exclude the case $u_1 = u_2 = u_0$ for the rest of the proof. Then the assumptions imply that, along any light ray from (u_1, φ_1) to (u_2, φ_2), u cannot have other extrema than strict local minima. Hence there are two classes of such light rays, as in the proof of part (a): (i) Those along which u is monotonous such that (21) holds, and (ii) those along which u passes through exactly one local minimum at some value $u_a(L)$ such that (22) holds. If $u_1 < u_0$ or $u_2 < u_0$, then L is restricted by $L^2 \leq L_0^2$ with some $L_0^2 < f(u_0)$, so the integrals on the right-hand side of (21) and (22) are bounded by Lemma 5.1. As in the proof of part (a), this implies that $N(p, \gamma)$ is finite. If $u_1 \geq u_0$ and $u_2 \geq u_0$, L^2 is allowed to vary over an interval $[K^2 - \delta, K^2]$ with $K = \sqrt{f(u_0)}$. By Lemma 5.1, the right-hand side of (22) correspondingly varies from some positive value to $+\infty$ for $L > 0$, and it varies from some negative value to $-\infty$ for $L < 0$. Hence, (22) can be satisfied for any $n \in \mathbb{Z}$ with $|n|$ sufficiently large. □

6 Examples

6.1 Black-hole spacetimes

We consider spherically symmetric and static (3+1)-dimensional spacetimes of the form

$$g = -A(r)^2 dt^2 + B(r)^2 dr^2 + r^2 \big(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \big) \tag{27}$$

where ϑ and φ are standard coordinates on the 2-sphere, t ranges over $\mathbb{R}$ and r ranges over an interval $]r_H, \infty[$ with $0 < r_H < \infty$. Restricting to the hyperplane $\vartheta = \pi/2$ gives a (2+1)-dimensional spacetime of the kind considered in the preceding sections, with the Schwarzschild-like coordinate r replacing our general radial coordinate ρ.

In this situation the potential (7) takes the form $V(r) = r^{-2} A(r)^2$. We assume that the Killing vector field ∂_t becomes lightlike, i.e., $A(r) \to 0$, in the limit $r \to r_H$. This indicates that there is a (Killing) horizon at $r = r_H$. In addition, we assume that $A(r)$ remains bounded for $r \to \infty$ which is true if the metric (27) is asymptotically flat. These assumptions imply that the potential $V(r)$ goes to zero both for $r \to r_H$ and for $r \to \infty$, so the strictly positive function V must attain its absolute maximum somewhere on the interval $]r_H, \infty[$. Thus, part (b) of Theorem 4.1 implies that every observer sees infinitely many images of any static light source in this spacetime. This result should be compared with Theorem 4.3 in Claudel, Virbhadra and Ellis [2] which says that, under some mathematical conditions different from ours, every spherically symmetric and static black hole must be surrounded by at least one “photon sphere”. In our version, there is a photon sphere at the maximum of the potential V. Please note that V may have several extrema, so there may be additional photon spheres. Correspondingly, the centrifugal force in such a spacetime is pointing in the direction of increasing r near infinity and it is pointing in the direction of decreasing r near the horizon; in between, it may change its direction several times.

As a particular example we consider the Reissner–Nordström spacetime which is the unique spherically symmetric and static black-hole solution of the Einstein–Maxwell equations. The above result may also be illustrated with black-hole solutions of the Einstein–Yang-Mills–Higgs... equations. The Reissner–Nordström metric is of the form (27) with

$$A(r)^2 = B(r)^{-2} = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \, , \tag{28}$$

cf., e.g., Hawking and Ellis [10], p. 156, so the potential $V(r)$ takes the form

$$V(r) = \frac{1}{r^2} - \frac{2m}{r^3} + \frac{e^2}{r^4} \, . \tag{29}$$

We restrict to the case that the constants m and e satisfy $0 \le |e| < m$, and we let the radius coordinate r range over the interval $]r_H, \infty[$ with $r_H = m + \sqrt{m^2 - e^2}$. Then the Reissner–Nordström metric gives the spacetime around a non-rotating object with mass m and charge e that has undergone gravitational collapse. Clearly, $V(r) \to 0$ both for $r \to r_H$ and for $r \to \infty$, so the above result implies that in the Reissner–Nordström spacetime every observer sees infinitely many images of every static light source. However, with the metric explicitly given, we can strengthen this general result in the following way. From (29) we calculate that V has exactly one extremum, namely a maximum at

$$r_0 = \frac{3m}{2} + \sqrt{\frac{9m^2}{4} - 2e^2} \, . \tag{30}$$

Hence, part (b) of Theorem 4.1 implies that inside any shell $r_{\text{min}} < r < r_{\text{max}}$ that contains the radius r_0 every event can be reached from every t-line by infinitely many future-pointing lightlike geodesics that are completely contained in this shell. On the other hand, there are only finitely many such geodesics, by part (a) of Theorem 4.1, if the shell does not contain the radius r_0. In the Schwarzschild case $e = 0$ equation (30) reduces to $r_0 = 3m$ and we find the features discussed already in the introduction. – For a more detailed discussion of light rays in the Reissner-Nordström metric the reader may consult Chandrasekhar [1], Chapter 5, or Kristiansson, Sonego and Abramowicz [13].

6.2 Wormhole spacetimes

Morris and Thorne [11], also see Morris, Thorne and Yurtsever [12], consider wormhole spacetimes where the metric has the form

$$g = -e^{2\Phi(\ell)} dt^2 + d\ell^2 + r(\ell)^2 \big(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2 \big) \, . \tag{31}$$

Here ϑ and φ are standard coordinates on the 2-sphere, t ranges over $\mathbb{R}$ and ℓ ranges over all of $\mathbb{R}$ as well. Restricting to the hyperplane $\vartheta = \pi/2$ gives a (2+1)-dimensional spacetime of the kind considered in the preceding sections, with the proper-length coordinate ℓ replacing our general radial coordinate ρ. Morris and Thorne assume that the metric (31) is asymptotically flat for $\ell \to \infty$ as well as for $\ell \to -\infty$ which means to require that $r(\ell)^2 \to \infty$ whereas $\Phi(\ell)$ remains bounded for $\ell \to \pm\infty$. As a consequence, the strictly positive potential

$$V(\ell) = r(\ell)^{-2} e^{2\Phi(\ell)} \tag{32}$$

goes to zero for $\ell \to \pm\infty$, so it must attain its absolute maximum on $\mathbb{R}$. The (not necessarily unique) value ℓ_0 of the radius coordinate where this takes place indicates an unstable circular light orbit, similar to the black-hole case. According to our general terminology, there is "centrifugal force reversal" at ℓ_0. However, we admit that in this special example our terminology might be viewed as a bit misleading because neither the direction of increasing ℓ nor the direction of decreasing ℓ could be interpreted properly as "away from the center" everywhere. Notwithstanding this semantic problem, the observation that V attains its absolute maximum on $\mathbb{R}$ makes part (b) of Theorem 4.1 applicable. Hence, every t-line γ can be joined to every point p by infinitely many lightlike geodesics, i.e., every Morris-Thorne wormhole acts as a gravitational lens that produces infinitely many images. Incidentally, this result remains true if the two asymptotically flat regions are glued together (as in the lower part of Fig.1 in Morris and Thorne [11]); after this identification, however, the spacetime does not fit into our general framework because spherical symmetry is lost.

More specific results are possible if we consider the special case that the potential (32) is monotonously increasing on $]-\infty, 0[$ and monotonously decreasing on $]0, \infty[$, with a local maximum at $\ell = 0$. Then inside any shell $\ell_{\min} < \ell < \ell_{\max}$ with $\ell_{\min} < 0$ and $\ell_{\max} > 0$ every event can be reached from every t-line by infinitely many future-pointing lightlike geodesics that are completely contained in this shell.

6.3 Interior Schwarzschild solution

As another illustration of our results we want to consider light rays in an interior Schwarzschild solution, i.e., inside a spherically symmetric and static material body. This is, of course, physically meaningful only in the case that the body is transparent. The reader might think of our interior solution as a (rough) model for a globular cluster.

As in subsection 6.1 we consider a spherically symmetric and static spacetime of the form (27), but this time we assume that the Schwarzschild-like radius coordinate r ranges over $]\,0\,,\,r_1\,[$ with some positive constant r_1. We assume that this spacetime metric (i) solves the Einstein field equation for a perfect fluid, (ii) has a regular center, and (iii) can be continuously joined to the Schwarzschild solution

$$g = -\Big(1 - \frac{2m}{r}\Big)\, dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2 \big(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \big) \tag{33}$$

at the radius r_1, with the pressure p going to zero for $r \to r_1$. Condition (ii) requires, in particular, that the metric coefficient $A(r)$ remains finite for $r \to 0$, so the potential $V(r) = A(r)^2 r^{-2}$ goes to infinity for $r \to 0$. Condition (iii), together with condition (i), requires $A'(r)$ and, thus, the derivative $V'(r)$ of the potential to be continuous at r_1 (see, e.g., Kramer, Stephani, MacCallum, Herlt [14], eq. (14.2b)), i.e.,

$$V'(r) \to \frac{2(3m - r_1)}{r_1^4} \qquad \text{for } \ r \to r_1 \, . \tag{34}$$

If $2m < r_1 < 3m$, these conditions on V imply that V' has to change sign, i.e., that there is centrifugal force reversal, somewhere between 0 and r_1. By Theorem 4.1, this implies that the gravitational field produces infinitely many images for light sources and observers placed in an appropriate shell $r_{\min} < r < r_{\max}$. (Either V has a strict local minimum at some r_0 such that part (c) of Theorem 4.1 applies to some neighborhood of r_0, or V is constant on some interval such that part (b) of Theorem 4.1 applies to that interval.)

Already in the introduction we have discussed the known fact that a Schwarzschild source of radius $r_1 \in \,]2m, 3m[$ produces infinitely many images for any light source and any observer outside the body. The analysis in this subsection demonstrates that the same is true for appropriately placed light sources and observers inside the body. – For the existence of circular light orbits in an interior Schwarzschild solution the reader may also consult Example 6 in Claudel, Virbhadra and Ellis [2].

7 Outlook

It is interesting to remark that some of the multiple imaging results presented in Theorem 4.1 can be proven, as an alternative, with the help of Morse theory. Relevant background material can be found in a book by Masiello [15], see, in particular, Theorem 6.5.6 in this book. This theorem says that, in regions of stationary spacetimes whose boundaries satisfy a certain "light convexity" assumption, any observer sees infinitely many images of any light source. It is easy to check that, if we specialize to circular shells in (2+1)-dimensional spacetimes with metrics of the form (2), this light convexity assumption can be expressed in terms of the potential (7) in the following way. The shell $\rho_1 < \rho < \rho_2$ has a light convex boundary if and only if $V'(\rho_1) > 0$ and $V'(\rho_2) < 0$, i.e., there must be centrifugal force reversal somewhere in the shell. This demonstrates that part (b) of Theorem 4.1 can be proven, as an alternative, with the Morse theoretical techniques detailed in Masiello's book. In this paper we were able to give elementary proofs of all results, using the fact that for the class of spacetimes considered the geodesic equation can be explicitly integrated up to quadratures; so there was no need to use "sophisticated" methods such as Morse theory. However, Morse theory could be an appropriate tool for generalizing our results to spacetimes which are not spherically symmetric and static such that an explicit analysis of the geodesic equation is not possible. As long as the spacetimes are stationary, the above-mentioned results of Masiello [15] could be used as a basis; for Morse theory on non-stationary spacetimes we refer to Giannoni, Masiello and Piccione [16].

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