

Fuchsian analysis of $S^2 \times S^1$ and S^3 Gowdy spacetimes

Fredrik Ståhl¹

Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Golm, Germany

E-mail: Fredrik.Stahl@mh.se

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Abstract

The Gowdy spacetimes are vacuum solutions of the Einstein equations with two commuting Killing vectors having compact spacelike orbits with T^3 , $S^2 \times S^1$ or S^3 topology. In the case of T^3 topology, Kichenassamy and Rendall have found a family of singular solutions which are asymptotically velocity dominated by construction. In the case when the velocity is between 0 and 1, the solutions depend on the maximal number of free functions. We consider the similar case with $S^2 \times S^1$ or S^3 topology, where the main complication is the presence of symmetry axes. The results for T^3 may be applied locally except at the axes, where one of the Killing vectors degenerates. We use Fuchsian techniques to show the existence of singular solutions similar to the T^3 case. We first solve the analytic case and then generalize to the smooth case by approximating smooth data with a sequence of analytic data. However, for the metric to be smooth at the axes, the velocity must be -1 or 3 there, which is outside the range where the constructed solutions depend on the full number of free functions. A plausible explanation is that in general a spiky feature may develop at the axis, a situation which is unsuitable for a direct treatment by Fuchsian methods.

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1. Introduction

The singularity theorems by Hawking and Penrose show that singularities will be present in a spacetime under some quite general conditions [1]. Apart from existence, these results give very little information on the nature of these singularities. There is, however, a general proposal for the structure of the generic cosmological singularity, put forth by Belinskii, Khalatnikov and Lifshitz (BKL) [2]. The argument is based on formal calculations which originally were without rigorous mathematical proofs. In essence, the BKL picture consists of two predictions: firstly, the evolution of a spatial point near the singularity is unaffected by the

¹ Present address: TFM, Mitthögskolan, 83125 Östersund, Sweden.

evolution of nearby points. Thus, the dynamics of an inhomogeneous model can be described by a homogeneous model at each spatial point, with the parameters of the homogeneous model depending on the spatial coordinates. Secondly, the dynamics is expected to be oscillatory, similar to the situation in the homogeneous Bianchi IX models. Loosely speaking, the evolution of a spatial point near a cosmological singularity should appear as an infinite sequence of epochs, each similar to a different Kasner spacetime.

Some cosmological models exhibit a simpler behaviour without oscillations, termed ‘asymptotically velocity dominated’ (AVD) [3, 4]. For these models there is no oscillation and the evolution of a given spatial point approaches a particular Kasner solution. This is exhibited in the evolution equations by the spatial derivative terms becoming negligible in relation to time derivatives.

The behaviour of general spacetimes is still out of reach of the mathematical techniques available, so it makes sense to study particular subclasses of solutions, subject to some particular symmetry and/or a particular choice of matter. Recently, there has been considerable progress in the study of spatially homogeneous spacetimes [5–9]. There is also an interesting result which establishes the AVD behaviour for the general Einstein equations coupled to a scalar field, without any symmetry assumptions [10].

It is of obvious interest to relax the symmetry assumptions from homogeneous to inhomogeneous spacetimes. For simplicity, we restrict our attention to the spatially compact case in vacuum. The largest symmetry group that is compatible with inhomogeneities is then $U(1) \times U(1)$, which in the spatially compact case leads to the Gowdy models [11, 12]. The possible spatial topologies are then essentially \mathbf{T}^3 , $\mathbf{S}^2 \times \mathbf{S}^1$ or \mathbf{S}^3 . (In the more general case of *local* $U(1) \times U(1)$ symmetry, other topologies are possible, see [13].)

For Gowdy spacetimes it is possible to define a quantity, the ‘velocity’, which describes the dynamics near the singularity. The terminology is slightly misleading since the velocity is only uniquely defined up to a sign. While a sign can be introduced, it depends on the particular parametrization of the metric. If AVD holds, the velocity has a (space-dependent) limit at the singularity.

For the subclass of polarized Gowdy models, which describe universes with polarized gravitational waves, the Einstein evolution equations reduce to a single linear ordinary differential equation, which can be solved formally in terms of a series of Legendre polynomials. The polarized solutions have been shown to be AVD [4, 14, 15].

In the case of the full Gowdy models with \mathbf{T}^3 spatial topology, numerical simulations show that the velocity has a limit between 0 and 1 for almost all spatial points in generic situations [16–19]. The exceptions are isolated points where the velocity remains outside this range, which results in a discontinuous behaviour, ‘spikes’, in the limit. A family of singular solutions has been constructed analytically by Kichenassamy and Rendall using the Fuchsian methods [20], based on previous work on formal asymptotic expansions by Grubisic and Moncrief [21]. These solutions have a prescribed asymptotic behaviour. In the case when the asymptotic velocity is between 0 and 1, the solutions depend on the full number of free functions, which indicates that they correspond to an open set of initial data. Recently, Ringström found a condition on regular Cauchy data which produces the same asymptotics as the solutions found by Kichenassamy and Rendall [22], extending a result by Chruściel for small data close to that of a Kasner $(2/3, 2/3, -1/3)$ spacetime [23]. The initial data satisfying these conditions form an open set. Families of solutions with spikes have also been constructed by Rendall and Weaver, showing that the spikes are the real features of the model and not numerical artefacts [24].

Much less is known about the unpolarized Gowdy spacetimes with $\mathbf{S}^2 \times \mathbf{S}^1$ or \mathbf{S}^3 topology. Garfinkle has done numerical simulations of Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ spacetimes, which show similar

behaviour as in the T^3 case. It is the purpose of this paper to provide a Fuchsian analysis of Gowdy $S^2 \times S^1$ and S^3 spacetimes similar to that in [20] for T^3 spacetimes.

The Fuchsian algorithm [20, 25] has been proved to be a valuable tool in showing the existence of singular solutions to nonlinear wave equations. Variants of it have been used to show the existence of Cauchy horizons with certain properties [26], to analyse the properties of isotropic singularities [27–32], to construct families of singular Gowdy or plane symmetric spacetimes [20, 33–36] and for the general result on scalar field spacetimes [10].

The original version of the Fuchsian technique only applies to the analytic case, since the proof uses the Cauchy formula in the complex domain to estimate derivatives in terms of the functions themselves. There are a number of results extending the Fuchsian algorithm to give smooth solutions [27–29, 33, 37, 38] in various contexts. In section 5 we will modify the argument in [38] to apply it in our case.

Finally, we note that there is an additional motivation for studying Gowdy spacetimes with $S^2 \times S^1$ topology apart from cosmology, since axisymmetric and stationary black-hole interiors are, in fact, Gowdy $S^2 \times S^1$ spacetimes. These include the region of Kerr spacetime between the inner and outer horizons [39] and the interiors of the ‘distorted black holes’ studied by Geroch and Hartle [40].

The outline of this paper is as follows. We first give a short outline of some of the results for Gowdy T^3 spacetimes in section 2. In section 3, we give an introduction to the Gowdy $S^2 \times S^1$ and S^3 spacetimes, introduce coordinates and a parametrization of the metric, and rewrite the field equations in a suitable form. We also discuss the restrictions needed for the metric to be smooth at the axes and the properties of some previously known solutions. In section 4, we give a short review of the Fuchsian technique of [20] and construct analytic solutions of the field equations. This is then extended to smooth solutions in section 5 by means of a generalization of the technique in [38]. Finally, we discuss the shortcomings and implications of the results in section 6.

2. Gowdy T^3 spacetimes

Since our discussion will, to some extent, be parallel to the case with T^3 spatial topology, we outline some of the results for this case here. The line element for Gowdy T^3 spacetimes may be written as

$$ds^2 = e^A(-dt^2 + d\theta^2) + t[e^P(d\phi + Q d\chi)^2 + e^{-P} d\chi^2], \quad (1)$$

where θ , ϕ and χ are suitably chosen coordinates on T^3 and A , P and Q are functions of t and θ only. There is an initial singularity at $t = 0$. The velocity v is defined up to a sign by

$$v^2 := (t\partial_t P)^2 + e^{2P}(t\partial_t Q)^2.$$

We may also set the sign of v to equal the sign of $\partial_t P$, but note that this is parametrization dependent. As noted in the introduction, for the AVD models the velocity has a limit at the initial singularity at $t = 0$. We will call this limit the asymptotic velocity $\kappa(\theta)$.

In [20], Kichenassamy and Rendall use Fuchsian methods to show that there exist solutions which depend on the full number of free analytic functions in the case when $0 < \kappa < 1$. The solutions have a controlled asymptotic behaviour, namely,

$$P(t, \theta) = \kappa(\theta) \ln t + \varphi(\theta) + t^\epsilon u(t, \theta), \quad (2a)$$

$$Q(t, \theta) = Q_0(\theta) + t^{2\kappa(\theta)}[\psi(\theta) + v(t, \theta)], \quad (2b)$$

near $t = 0$ with $u, v \rightarrow 0$ as $t \rightarrow 0^+$ and ϵ a small positive constant. Similar results can be obtained for κ outside this range, but then with only three free functions. Recently, a

condition on regular initial data which give rise to the asymptotics (2) has been found by Ringström [22].

Numerical studies indicate that the asymptotic velocity is between 0 and 1 in general, except for the isolated spatial points where other values are allowed. These discontinuous features in the asymptotic velocity are usually referred to as ‘spikes’. For the values of θ near such a discontinuity, the spacetime is not AVD and evolves in an inhomogeneous way. Families of solutions with spikes have been constructed analytically by Rendall and Weaver [24], and we will now outline their construction since it will be relevant for our discussion of the other topologies below.

The result in [24] relies on the observation that the form of the metric and the evolution equations are invariant under two transformations. The first transformation, called inversion, corresponds to an interchange of the two Killing vectors. Under inversion, P and Q transform to two new functions P_1 and Q_1 , which satisfy the original equations. Now, if P and Q are solutions with asymptotics given by (2), and Q_0 has an isolated zero at a point $\theta = \theta_0$, it turns out that the coefficient of the leading order term in P_1 will be discontinuous and Q_1 will have a singularity at $t = 0, \theta = \theta_0$. Since this behaviour is absent in the P and Q variables, these discontinuities have no geometric meaning, the cause being simply a bad choice of parametrization of the metric. Such features are called ‘false spikes’.

The other transformation is called a Gowdy-to-Ernst transformation in [24]. Applying this transformation to P_1 and Q_1 we obtain new solutions P_2 and Q_2 . If Q_0 has an isolated zero at $\theta = \theta_0$, so that P_1 and Q_1 have a false spike, there will be a discontinuity in the leading order term in P_2 , but Q_2 will be regular. The divergence of the Kretschmann curvature invariant is of a different order in t along the curve $\theta = \theta_0$ than along the curves with constant θ in a punctured neighbourhood of θ_0 . Therefore, these phenomena have a real geometric meaning, and they are called ‘true spikes’.

The above discussion of parametrizations, transformations and spikes will be relevant to our treatment of the $\mathbf{S}^2 \times \mathbf{S}^1$ and \mathbf{S}^3 cases in the next section for several reasons. Firstly, the same phenomena will occur in the other topologies since the \mathbf{T}^3 results apply as long as we stay away from the axes. Secondly, it is quite possible that similar phenomena can occur at the axes as well. Thirdly, we will make use of both an inversion and a Gowdy-to-Ernst transformation to construct a well-defined evolution system in a neighbourhood of one of the symmetry axes.

3. Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ and \mathbf{S}^3 spacetimes

We now turn to the Gowdy spacetimes with $\mathbf{S}^2 \times \mathbf{S}^1$ or \mathbf{S}^3 spatial topology. First, some comments on the derivation of the Gowdy models are in order. We assume that (M, g) is a $U(1) \times U(1)$ -symmetric spatially compact spacetime with spacelike group orbits. More precisely, let $M = I \times \Sigma$ where $I \subset \mathbf{R}$ and Σ is a connected orientable compact three-manifold, and assume that the $U(1) \times U(1)$ group acts effectively on Σ . It then follows [12] that the topology of Σ is that of either \mathbf{T}^3 , $\mathbf{S}^2 \times \mathbf{S}^1$, \mathbf{S}^3 or a lens space $\mathbf{L}(p, q)$. The \mathbf{T}^3 case has been commented upon in the previous section, and $\mathbf{L}(p, q)$ may be covered by \mathbf{S}^3 , so we only consider $\Sigma \simeq \mathbf{S}^2 \times \mathbf{S}^1$ or \mathbf{S}^3 here. (More precisely, the papers cited in section 2 concern the \mathbf{T}^3 case with vanishing twist constants. See [36] for a similar treatment with non-vanishing twist but restricted to the polarized case.)

We introduce coordinates on Σ as follows. Let θ label the orbits of $U(1) \times U(1)$ in Σ and let (ϕ, χ) be the coordinates on each orbit induced by the standard coordinates on $U(1) \times U(1)$. For $\Sigma \simeq \mathbf{S}^2 \times \mathbf{S}^1$, we choose χ to be a cyclic coordinate on the \mathbf{S}^1 part, $\chi \in [0, 2\pi] \bmod 2\pi$, and (θ, ϕ) to be spherical coordinates on the \mathbf{S}^2 part, $\theta \in [0, \pi]$ and

$\phi \in [0, 2\pi] \bmod 2\pi$. This fixes (ϕ, χ) up to translation. For $\Sigma \simeq S^3$, the fix points to the first $U(1)$ factor of $U(1) \times U(1)$ form a circle S_1 , and the set of fix points to the second factor is a similar circle S_2 (this is a nontrivial consequence of the effective action, see, e.g., [41]). The parametrization is chosen such that S_1 and S_2 correspond to $\theta = 0$ and $\theta = \pi$, respectively. This fixes (ϕ, χ) up to translation in this case as well. Note that the two cases are similar in a neighbourhood of one of the axes.

If (M, g) is assumed to be a maximal globally hyperbolic development of generic data on Σ , then M contains the set $(0, \pi) \times \Sigma$, on which the line element of the metric g may be parametrized as

$$ds^2 = e^A(-dt^2 + d\theta^2) + d\sigma^2, \tag{3}$$

where $d\sigma^2$ is a metric on the orbits with determinant $R = c \sin t \sin \theta$ for some positive constant c [12]. We set $c = 1$ from now on, since a constant conformal factor does not affect the qualitative aspects of the model. Here ‘generic’ refers to an open and dense subset of all Cauchy data; see [12] for the exact definition. We will call this model the Gowdy model over Σ [11].

In the T^3 case the metric can also be written in the form (3), but with the orbit metric determinant t . It follows that for $S^2 \times S^1$ or S^3 , the metric can be brought into the T^3 form by a change of coordinates whenever $\theta \neq 0$ and $\theta \neq \pi$. Thus the result of [20] and [38] may be applied locally to any subset of M where θ is bounded away from 0 and π , and what remains is to study a neighbourhood of one of the symmetry axes. (Strictly speaking, in the smooth case it is necessary to show that local solutions can be patched together. This may be done using the domain of dependence result of theorem 5.1 below.) Also, by the symmetry of the problem, we may restrict our attention to a neighbourhood of $\theta = 0$ and $t = 0$. When we write ‘the axis’ below, we will mean the axis at $\theta = 0$, although all arguments apply to the other axis at $\theta = \pi$ as well.

3.1. Parametrizations

The orbit metric $d\sigma^2$ may be parametrized in several ways. If we use a parametrization similar to that for the T^3 case (1),

$$ds^2 = e^A(-dt^2 + d\theta^2) + R[e^P(d\phi + Q d\chi)^2 + e^{-P} d\chi^2], \tag{4}$$

the Einstein equations decouple as two evolution equations involving P and Q and two constraint equations involving P, Q and A [12]. Let $(t, \theta) \in M \subset \mathbf{R}^2$ with the Minkowski metric $\eta = -dt^2 + d\theta^2$ and $(P, Q) \in N \subset \mathbf{R}^2$ with the hyperbolic metric $h = dP^2 + e^{2P} dQ^2$. Given the map $\Psi: M \ni (t, \theta) \mapsto (P, Q) \in N$, we define a section $d\Psi$ of $T^*M \otimes \Psi^*(TN)$ by

$$d\Psi = dP \otimes \Psi^* \frac{\partial}{\partial P} + dQ \otimes \Psi^* \frac{\partial}{\partial Q}. \tag{5}$$

Then the evolution equations may be written as the wave-map-type equation

$$\text{tr}_\eta[\mathcal{D}(h, \eta)(R d\Psi)] = 0, \tag{6}$$

where $\mathcal{D}(h, \eta)$ is a connection on $T^*M \otimes \Psi^*(TN)$ defined in terms of the connections of (M, η) and (N, h) (see [12] for details).

The ‘velocity’ $v(t, \theta_0)$ is defined up to a sign as the h -velocity of the solution (P, Q) of (6) along a curve $\theta = \theta_0$, rescaled by an appropriate factor. In the T^3 case the factor is usually taken to be t , but in our case it is more convenient to choose $\tan t$. So we define the velocity v by

$$v^2 = (DP)^2 + e^{2P} (DQ)^2, \tag{7}$$

where $D = \tan t \partial / \partial t$.

An ‘asymptotically velocity dominated’ (AVD) solution is a solution which asymptotically approaches a solution to the ‘AVD equations’, obtained by dropping pure spatial derivative terms in the evolution equations [3, 4]. For an AVD solution, it follows immediately from the AVD equations that the velocity v tends to a finite limit as $t \rightarrow 0$. We will refer to this limit as the asymptotic velocity. It is also possible to fix the sign of the velocity to be the same as the sign of DP , but the sign will then depend upon the parametrization as we will see below.

Note that P cannot be regular at the axis. The regular parametrization used by Garfinkle in his numerical work [42] is obtained from (4) by setting

$$P = \hat{P} + \ln \sin \theta \quad \text{and} \quad A = \hat{P} + \gamma + \ln \sin t, \quad (8)$$

with γ , \hat{P} and Q smooth functions of t and θ . Unfortunately, the corresponding evolution equations contain coefficients that are singular at the axes. This is not a major obstacle for the numerics since the full terms appearing in the equations are well behaved, but it does prevent a direct analytical treatment.

Another parametrization may be obtained by interchanging the roles of the Killing vectors $\partial/\partial\phi$ and $\partial/\partial\chi$. This corresponds to an inversion in the hyperbolic plane (N, h) , so the field equations are invariant under this reparametrization. Explicitly,

$$ds^2 = e^A(-dt^2 + d\theta^2) + R[e^Y(d\chi + X d\phi)^2 + e^{-Y} d\phi^2], \quad (9)$$

where

$$Y = P + \ln(e^{-2P} + Q^2) \quad \text{and} \quad X = \frac{Q}{e^{-2P} + Q^2}. \quad (10)$$

Note that there is a subtlety here. If $P \rightarrow \infty$ as $t \rightarrow 0$, the rate of blow-up of Y will be dramatically different depending on whether Q vanishes or not. This is a consequence of the parametrization, and must be distinguished from the real geometric features of the model.

The parametrization (9) is again singular at the axis, but this may be resolved by substituting $Y = Z - \ln R$. We also set $A = \lambda - Z + 2 \ln \sin t$. The metric then becomes

$$ds^2 = e^{\lambda-Z} \sin^2 t (-dt^2 + d\theta^2) + e^Z (d\chi + X d\phi)^2 + R^2 e^{-Z} d\phi^2. \quad (11)$$

From lemma 5.1 of [12], (11) defines a smooth metric with the desired symmetry if and only if λ , Z and X are smooth functions of θ on \mathbf{S}^2 , with X and λ vanishing at $\theta = 0$ and $\theta = \pi$. Note that a function is smooth in a neighbourhood of $\theta = 0$ or $\theta = \pi$ in \mathbf{S}^2 if and only if it is a smooth function of $\sin^2 \theta$ there.

Because of the invariance of h under inversion, the velocity is given by

$$v^2 = (DY)^2 + e^{2Y}(DX)^2 = (1 - DZ)^2 + e^{2Z} R^{-2}(DX)^2. \quad (12)$$

As mentioned above, it is possible to assign a sign to v by setting it equal to the sign of DP . Assume, for illustrative purposes, that Q is independent of t and that $DP \rightarrow \kappa(\theta)$ and $P \rightarrow \infty$ as $t \rightarrow 0$, where $\kappa(\theta) < 0$. It then follows from (10) that $DY = DZ - 1 \rightarrow \kappa$ as $t \rightarrow 0$ wherever $Q \neq 0$, but $DY = DZ - 1 \rightarrow -\kappa$ for those values of θ where Q vanishes. So the sign defined with respect to DP cannot be expressed in terms of DZ alone. If κ is smooth, the limit of DZ has discontinuities and the limit of X has singularities at points where Q has isolated zeros (note that for any smooth metric of form (4), Q must vanish at the axes). Since DP and Q have regular asymptotic behaviour in this case, these discontinuities are artefacts of the parametrization. This is an example of false spikes as mentioned in section 2. The same argument may, of course, also be applied in the other direction, giving discontinuities in the asymptotic behaviour of DP from a well-behaved DZ . For the rest of this paper, we fix the sign of the velocity to be equal to the sign of $1 - DZ$.

3.2. Einstein's equations

In the last section we obtained a parametrization which is regular at the axis. We will now modify the parametrization to obtain evolution equations with regular coefficients as well.

We will denote the partial derivatives by subscripts, e.g., $f_\theta := \partial_\theta f = \partial f / \partial \theta$. As mentioned above, Einstein's equations decompose into the wave map equation (6) for Z and X and two constraint equations for λ , Z and X . If we denote the covariant derivative of the Minkowski metric η on N by \mathcal{D} , (6) can be written as

$$R^{-1} \mathcal{D}_A (R \mathcal{D}^A Z) = R^{-2} e^{2Z} \mathcal{D}_A X \mathcal{D}^A X, \tag{13a}$$

$$R \mathcal{D}_A (R^{-1} e^{2Z} \mathcal{D}^A X) = 0, \tag{13b}$$

and the constraint equations are

$$2(\lambda_\pm \pm \cot t) R_\pm = 4(R_\theta)_\pm + R (Z_\pm^2 + e^{2Z} R^{-2} X_\pm^2), \tag{14}$$

where we have used the notation $f_\pm := f_\theta \pm f_t$ [12].

The integrability of constraints (14) is ensured by the evolution equations (13). When R_+ and R_- are nonzero, the constraints determine λ up to a constant, and the constant is fixed by the smoothness requirement, i.e., that $\lambda = 0$ at $\theta = 0$. Since $R = \sin t \sin \theta$, R_+ and R_- vanish when $\theta = t$ and $\theta = \pi - t$, respectively. So this happens for two values of θ in almost all Cauchy surfaces. At these points, the constraints become ‘matching conditions’ for Z and X [11, 13]. If the matching conditions hold on an initial Cauchy surface, they are preserved by the evolution equations [42]. In what follows we will assume without further comment that the solutions are chosen such that the matching conditions hold.

To obtain a system with smooth coefficients, we reparametrize the metric using an Ernst potential as in [12], a procedure which is similar to the Kramer–Neugebauer transformation in stationary and axisymmetric spacetimes [43]. Let

$$\Omega := -R^{-1} e^{2Z} (X_\theta dt + X_t d\theta). \tag{15}$$

It then follows from (13b) that Ω is a closed form. Thus we may write $\Omega = d\omega$ for some function ω , defined up to a constant, on any simply connected open set containing $\theta = 0$. Inverting relation (15) shows that if ω is a smooth function of θ on a neighbourhood of $\theta = 0$ on S^2 , so is X . This determines X up to a constant, which is fixed by the requirement that $X = 0$ at $\theta = 0$ so that the metric is smooth at the axis. Following [24], we will call the transformation $(Y, X) \mapsto (Z, \omega)$ a ‘Gowdy-to-Ernst’ transformation.

Expressing the evolution equation (13a) in terms of Z and ω gives

$$R^{-1} \mathcal{D}_A (R \mathcal{D}^A Z) = -e^{-2Z} \mathcal{D}_A \omega \mathcal{D}^A \omega, \tag{16a}$$

and the identity $\mathcal{D}_{[A} \mathcal{D}_{B]} X = 0$ together with (15) gives a second evolution equation

$$\mathcal{D}_A (R e^{-2Z} \mathcal{D}^A \omega) = 0. \tag{16b}$$

Expanding (16) we get

$$Z_{tt} + \cot t Z_t - Z_{\theta\theta} - \cot \theta Z_\theta = -e^{-2Z} (\omega_t^2 - \omega_\theta^2) \tag{17a}$$

$$\omega_{tt} + \cot t \omega_t - \omega_{\theta\theta} - \cot \theta \omega_\theta = 2(Z_t \omega_t - Z_\theta \omega_\theta). \tag{17b}$$

We will now rewrite the equations in a form which makes the regularity of the coefficients at the axis explicit, and which is more suited for the Fuchsian techniques. Put $\tau := \sin t$ and $D := \tan t \partial_t = \tau \partial_\tau$ as before, and let $\Delta := \partial_\theta^2 + \cot \theta \partial_\theta + (\sin \theta)^{-2} \partial_\phi^2$, e.g., the Laplacian on S^2 with respect to the metric $d\theta^2 + \sin^2 \theta d\phi^2$ induced by the natural embedding in

Euclidean \mathbf{R}^3 . We also write $\nabla f \cdot \nabla g := f_\theta g_\theta + (\sin \theta)^{-2} f_\phi g_\phi$ and $(\nabla f)^2 := \nabla f \cdot \nabla f$. Since Z and ω are independent of ϕ , (17) can be written as

$$(1 - \tau^2)D^2Z - \tau^2DZ = \tau^2\Delta Z - e^{-2Z}[(1 - \tau^2)(D\omega)^2 - \tau^2(\nabla\omega)^2] \quad (18a)$$

$$(1 - \tau^2)D^2\omega - \tau^2D\omega = \tau^2\Delta\omega + 2[(1 - \tau^2)DZD\omega - \tau^2\nabla Z \cdot \nabla\omega]. \quad (18b)$$

Constraints (14) expressed in terms of Z and ω are

$$D\lambda + \tan^2 t \cot \theta \lambda_\theta = -2(1 + \tan^2 t) + \frac{1}{2}[(DZ)^2 + \tan^2 t Z_\theta^2 + e^{-2Z}((D\omega)^2 + \tan^2 t (\omega_\theta^2))] \quad (19a)$$

$$D\lambda + \tan \theta \lambda_\theta = \tan \theta (Z_\theta DZ + e^{-2Z} \omega_\theta D\omega). \quad (19b)$$

In the analytic case, the Fuchsian techniques can, in fact, be applied directly to (18) in the variables τ and θ . For the smooth case, however, we have to cast the equations into a symmetric hyperbolic form, which cannot be done using the θ variable alone because the axial symmetry will lead to coefficients singular in θ at $\theta = 0$. We, therefore, introduce an approximately Cartesian coordinate system on a neighbourhood of $\theta = 0$ on \mathbf{S}^2 . Let $x := \sin \theta \cos \phi$ and $y := \sin \theta \sin \phi$. In the variables (x, y) , the metric on \mathbf{S}^2 is

$$d\theta^2 + \sin^2 \theta d\phi^2 = (1 - x^2 - y^2)^{-1}[(1 - y^2) dx^2 + 2xy dx dy + (1 - x^2) dy^2], \quad (20)$$

hence

$$\Delta = (1 - x^2)\partial_x^2 - 2xy\partial_x\partial_y + (1 - y^2)\partial_y^2 - 2x\partial_x - 2y\partial_y \quad (21)$$

and

$$\nabla f \cdot \nabla g = (1 - x^2)f_x g_x - xy(f_x g_y + f_y g_x) + (1 - y^2)f_y g_y. \quad (22)$$

Inserting these expressions into (18) then gives a system in the coordinates t, x and y with coefficients regular in x and y at the axis.

3.3. Restrictions at the axes

As mentioned in the introduction, asymptotic velocity dominance for Gowdy spacetimes may be loosely formulated as that the metric tends to a Kasner metric when $t \rightarrow 0$ for a fixed point in Σ . The presence of symmetry axes imposes additional restrictions on the asymptotic behaviour along the axis.

To make this precise we calculate the generalized Kasner exponents. These are defined as the eigenvalues q_i of the renormalized second fundamental form $(\text{tr } k)^{-1}k_{ij}$, expressed in an orthonormal frame on Σ . From the expression for metric (11) we find that

$$q_1 = (D\lambda - DZ + 2)(D\lambda - DZ + 4)^{-1} \quad (23a)$$

$$q_{2,3} = (1 \pm \nu)(D\lambda - DZ + 4)^{-1}, \quad (23b)$$

where ν is the velocity as given in (12).

Because of the rotational symmetry, the only possible values of the Kasner exponents at the axis are $(0, 0, 1)$ and $(2/3, 2/3, -1/3)$ since the two eigenvalues corresponding to eigenvectors tangent to Σ must agree. From the smoothness requirements $D\lambda$ and DZ vanish at the axis, and using our sign convention we have $\nu = 1 - DZ$ there. It then follows immediately from (23) that for the $(0, 0, 1)$ case, $DZ \rightarrow 2$ and $\nu \rightarrow -1$ as $t \rightarrow 0$ along the axis, while for the $(2/3, 2/3, -1/3)$ case, $DZ \rightarrow -2$ and $\nu \rightarrow 3$.

As noted at the beginning of this section, the \mathbf{T}^3 results of [20] apply to sets not containing the axis at $\theta = 0$. In that case, we obtain solutions with the full number of free functions only

in the case when $0 < \nu < 1$ (and $DP < 0$) in the limit $t \rightarrow 0$. Also, numerical simulations have shown that the velocity is indeed driven into this range, except at isolated spatial points [16] (see also [24] for a heuristic argument). So we cannot hope to show the existence of velocity-dominated solutions depending on the full number of free functions since ν is either -1 or 3 on the axis. (Actually, in our case DZ will play the role of ν , but the same conclusion can be drawn since $\nu = 1 - DZ$ on the axis.)

3.4. Some known solutions

3.4.1. Polarized solutions. The polarized solutions form the subclass with $Q \equiv 0$, so called because they describe spacetimes with polarized gravitational waves. In this case the evolution equations (18) reduce to a linear ordinary differential equation for P , the Euler–Poisson–Darboux equation. The solutions may be expressed explicitly in terms of Legendre polynomials [15]. Also, rigorous asymptotic expansions in a neighbourhood of the singularity have been found [4, 14]. In our notation,

$$Z(t, \theta) = k(\theta) \ln \sin t + \varphi(\theta) + u(t, \theta) \quad (24a)$$

$$\lambda(t, \theta) = \frac{1}{2}(k(\theta)^2 - 4) \ln \sin t + \psi(\theta) + v(t, \theta), \quad (24b)$$

with $|D^m \partial_\theta^n u| \leq C \sin^2 t |\ln \sin t|$ and $|D^m \partial_\theta^n v| \leq C \sin^2 t |\ln \sin t|^2$ for $m = 0, 1$, all n and a constant C . The functions k and φ may be chosen freely as long as they satisfy the constraints, and the solutions are all asymptotically velocity dominated. Since λ must vanish at the axis for the metric to be smooth, it follows immediately from the expansions that $k = \pm 2$ there, and the velocity $\nu = 1 - k$ must tend to -1 or 3 along the axis. The Kretschmann curvature scalar is unbounded as $t \rightarrow 0$ unless $\nu = 1 - k = \pm 1$, $k_\theta = 0$ and $k_{\theta\theta} = 0$.

There is a way in which one may obtain unpolarized solutions from a given polarized solution by means of an Ehlers transformation. This is used to obtain the ‘reference solution’ used to validate the numerical code in [42] (see also [44]). The technique works because a combination of an Ehlers transformation with Gowdy-to-Ernst transformations gives an isometry of the hyperbolic plane (N, h) . The velocity of the transformed solution has the same asymptotic behaviour as for the original polarized solution.

3.4.2. Black holes. The region of Kerr spacetime between the inner and outer horizons may be written as a Gowdy spacetime over $S^2 \times S^1$, by choosing as time coordinate a rescaling of the usual radial coordinate [39]. In our parametrization (11), the velocity ν tends to -1 along the axis and to $+1$ elsewhere. The discontinuous behaviour is an artefact of the parametrization, since in the parametrization (4), $DP \rightarrow -1$ everywhere. This provides an example of Gowdy spacetimes on $S^2 \times S^1$ without curvature singularities. The extremal case corresponding to the interior of the Schwarzschild black hole is a polarized Gowdy solution with $\nu \rightarrow -1$ at the horizon and $\nu \rightarrow 3$ at the curvature singularity.

The same idea can be applied to the ‘distorted black-hole’ spacetimes studied by the Geroch and Hartle. These are constructed by the analytic continuation of certain Weyl solutions, essentially corresponding to perturbations on an exterior Schwarzschild background, through the horizon. The solutions are static, but they can be different from Schwarzschild spacetime by violating asymptotic flatness or by allowing matter in the exterior region. The interior is a polarized Gowdy spacetime with $S^2 \times S^1$ topology. If we let the Schwarzschild solution be given in Gowdy coordinates with $P = P_S$, the Geroch–Hartle solution is given by $P = P_S + U$, where U is regular on the horizon. It is possible to show that U must be regular at the other singularity as well. Thus, the asymptotic velocity is the same as for the interior Schwarzschild solution.

Note that for polarized Gowdy spacetimes, analyticity is a necessary and sufficient condition for the existence of an extension with a compact Cauchy horizon [14]. The necessity follows from the fact that on the extension, the evolution equation becomes the Laplace equation in suitably chosen coordinates.

3.4.3. Numerical results. Garfinkle has done numerical simulations of Gowdy spacetimes on $\mathbf{S}^2 \times \mathbf{S}^1$, with similar results as in the \mathbf{T}^3 case [42]. All solutions have velocity -1 at the axis, without spiky features, which probably can be explained by the particular choice of initial data. We should emphasize, however, that Garfinkle's calculations are carried out in the parametrization (8), and rewriting his results in our parametrization (11) gives solutions with 'false spikes' at the axis. Since we cannot generate solutions with spikes by the method in this paper, our results cannot be compared directly with those in [42].

4. The analytic case

4.1. Fuchsian systems

Consider a system of partial differential equations on \mathbf{R}^{n+1} , whose solutions are expected to have a singularity as $t \rightarrow 0$. The Fuchsian algorithm is based on the following idea: decompose the unknown into a prescribed singular part, depending on a number of arbitrary functions, and a regular part w . If the system can be rewritten as a Fuchsian system of the form

$$t\partial_t w + N(x)w = tf(t, x, w, w_x), \quad (25)$$

where w_x denotes the collection of spatial derivatives of w , the Fuchsian method applies. In the analytic case we have the following theorem.

Theorem 4.1 (Kichenassamy and Rendall [20]). *Assume that N is an analytic matrix near $x = 0$ such that there is a constant C with $\|\sigma^N\| \leq C$ for $0 < \sigma < 1$, where σ^N is the matrix exponential of $N \ln \sigma$. Also, suppose that f is a locally Lipschitz function of w and w_x which preserves analyticity in x and continuity in t . Then the Fuchsian system (25) has a unique solution in a neighbourhood of $x = 0$ and $t = 0$ which is analytic in x and continuous in t , and tends to 0 as $t \rightarrow 0$.*

Note that theorem 4.1 also applies to the case

$$t\partial_t w + N(x)w = t^\alpha f(t, x, w, w_x), \quad (26)$$

since changing the t variable to t^α transforms the system into the form (25).

4.2. Low velocity, analytic case

Let k, φ, ω_0 and ψ be real analytic functions of θ on a neighbourhood of $\theta = 0$ in \mathbf{S}^2 , with $k \in (0, 1)$. We introduce new unknowns u and v such that

$$Z(\tau, \theta) = k(\theta) \ln \tau + \varphi(\theta) + \tau^\epsilon u(\tau, \theta) \quad (27a)$$

$$\omega(\tau, \theta) = \omega_0(\theta) + \tau^{2k(\theta)}(\psi(\theta) + v(\tau, \theta)), \quad (27b)$$

where $\epsilon > 0$ is a small constant which we will fix later. The idea is to use theorem 4.1 to show that u, v, Du and Dv tend to 0 as $t \rightarrow 0$, so that (27) provides an asymptotic expansion of Z and ω . The term 'low velocity' is then justified by the fact that the velocity v tends to $1 - k \in (0, 1)$ as $t \rightarrow 0$, as is easily verified from definition (12). For this reason we will

refer to $1 - k$ as the ‘asymptotic velocity’. However, remember that from the discussion in section 3.3, the velocity should be -1 or 3 at the axis, so the solutions constructed in this section are *not* regular at the axis.

In terms of X , expansion (27b) of ω corresponds to

$$X = X_0(\theta) + \tau^{2(1-k(\theta))}(\tilde{\psi}(\theta) + \tilde{v}(\tau, \theta)), \tag{28}$$

for some functions $X_0, \tilde{\psi}$ and \tilde{v} , which can be compared directly with the \mathbf{T}^3 results [20].

Inserting expansion (27) into system (18) gives

$$\begin{aligned} (1 - \tau^2)D^2u &= -(1 - \tau^2)[2\epsilon Du + \epsilon^2u] + \tau^{2-\epsilon}[k + (\ln \tau)\Delta k + \Delta\varphi] \\ &\quad + \tau^2[(D + \epsilon)u + \Delta u] - e^{-2\varphi-2\tau^\epsilon u}\{(1 - \tau^2)\tau^{2k-\epsilon}[(D + 2k)(\psi + v)]^2 \\ &\quad - \tau^{2-2k-\epsilon}(\nabla\omega_0)^2 - 2\tau^{2-\epsilon}\nabla\omega_0 \cdot [\nabla(\psi + v) + 2(\ln \tau)(\psi + v)\nabla k] \\ &\quad - \tau^{2+2k-\epsilon}[\nabla(\psi + v) + 2(\ln \tau)(\psi + v)\nabla k]^2\} \end{aligned} \tag{29a}$$

$$\begin{aligned} (1 - \tau^2)D^2v &= -2k(1 - \tau^2)Dv + 2(1 - \tau^2)\tau^\epsilon(D + \epsilon)u(D + 2k)(\psi + v) \\ &\quad + \tau^2[2(\ln \tau)\nabla k \cdot \nabla(\psi + v) + 2(\ln \tau)(\psi + v)\Delta k + (D + 2k)(\psi + v) \\ &\quad + \Delta(\psi + v)] + \tau^{2-2k}\Delta\omega_0 - 2\tau^{2-2k}\nabla\omega_0 \cdot [(\ln \tau)\nabla k + \nabla\varphi + \tau^\epsilon\nabla u] \\ &\quad - 2\tau^2[\nabla\varphi + \tau^\epsilon\nabla u][2(\ln \tau)(\psi + v)\nabla k + \nabla(\psi + v)]. \end{aligned} \tag{29b}$$

Next, we introduce the variables

$$w := (U_0, U_1, U_2, U_3, V_0, V_1, V_2, V_3) = (u, Du, \tau u_x, \tau u_y, v, Dv, \tau v_x, \tau v_y), \tag{30}$$

and also write \bar{U} and \bar{V} as shorthands for the vectors (U_2, U_3) and (V_2, V_3) . Using (21) and (22) system (29) may then be written as

$$DU_0 = U_1 \tag{31a}$$

$$\begin{aligned} (1 - \tau^2)DU_1 &= -(1 - \tau^2)[2\epsilon U_1 + \epsilon^2U_0] + \tau^{2-\epsilon}[k + (\ln \tau)\Delta k + \Delta\varphi] + \tau^2(U_1 + \epsilon U_0) \\ &\quad + \tau[(1 - x^2)U_{2x} - xy(U_{2y} + U_{3x}) + (1 - y^2)U_{3y} - 2xU_2 - 2yU_3] \\ &\quad - e^{-2\varphi-2\tau^\epsilon U_0}\{(1 - \tau^2)\tau^{2k-\epsilon}[V_1 + 2k(\psi + V_0)]^2 - \tau^{2-2k-\epsilon}(\nabla\omega_0)^2 \\ &\quad - 2\tau^{1-\epsilon}\nabla\omega_0 \cdot [\bar{V} + \tau\nabla\psi + 2\tau(\ln \tau)(\psi + V_0)\nabla k] \\ &\quad - \tau^{2k-\epsilon}[\bar{V} + \tau\nabla\psi + 2\tau(\ln \tau)(\psi + V_0)\nabla k]^2\} \end{aligned} \tag{31b}$$

$$DU_2 = \tau(U_{0x} + U_{1x}) \tag{31c}$$

$$DU_3 = \tau(U_{0y} + U_{1y}) \tag{31d}$$

$$DV_0 = V_1 \tag{31e}$$

$$\begin{aligned} (1 - \tau^2)DV_1 &= -2k(1 - \tau^2)V_1 + 2(1 - \tau^2)\tau^\epsilon(U_1 + \epsilon U_0)(V_1 + 2k(\psi + V_0)) \\ &\quad + \tau^2[V_1 + 2k(\psi + V_0) + 2(\ln \tau)(\psi + V_0)\Delta k + 2(\ln \tau)\nabla k \cdot \nabla\psi + \Delta\psi] \\ &\quad + 2\tau(\ln \tau)\nabla k \cdot \bar{V} + \tau^{2-2k}\Delta\omega_0 + \tau[(1 - x^2)V_{2x} - xy(V_{2y} + V_{3x}) \\ &\quad + (1 - y^2)V_{3y} - 2xV_2 - 2yV_3] - 2\tau^{2-2k}\nabla\omega_0 \cdot [(\ln \tau)\nabla k + \nabla\varphi \\ &\quad + \tau^\epsilon\nabla U_0] - 2\tau[\tau\nabla\varphi + \tau^\epsilon\bar{U}] \cdot [2(\ln \tau)(\psi + V_0)\nabla k + \nabla(\psi + V_0)] \end{aligned} \tag{31f}$$

$$DV_2 = \tau(V_{0x} + V_{1x}) \tag{31g}$$

$$DV_3 = \tau(V_{0y} + V_{1y}). \tag{31h}$$

If we choose ϵ such that $0 < \epsilon < \min\{2k, 2 - 2k\}$, system (31) is of the form (26) for some α (after dividing (31b) and (31f) by $1 - \tau^2$), and a rescaling of τ gives a Fuchsian system

of the form (25). By the criterion in [10], or by direct computation of σ^N , the conditions of theorem 4.1 are fulfilled. (Note that the essential features of the system, in particular the matrix N and the powers of τ , are the same as in the \mathbf{T}^3 case [20].) We come to the following conclusion.

Theorem 4.2. *Suppose that k, φ, ω_0 and ψ are real analytic functions of θ in a neighbourhood of $\theta = 0$ in \mathbf{S}^2 such that $k \in (0, 1)$ for each θ . Let ϵ be a positive constant less than $\min\{2k, 2 - 2k\}$. There exists a unique solution of the Einstein equations (13) of form (27) in a neighbourhood of $\theta = 0$ and $t = 0$ such that u, v, Du and Dv all tend to 0 as $t \rightarrow 0$.*

4.3. Negative velocity, analytic case

In the case when we only assume k to be positive, allowing values of k greater than 1, we may use the same expansion (27) as in the low velocity case. However, examining the system (31) we see that we must assume that ω_0 is constant in order to avoid negative powers of τ . This is again similar to the \mathbf{T}^3 case, and is to be expected since numerical simulations indicate that spiky features may appear as $t \rightarrow 0$ [20, 42]. Again, using definition (12) of the velocity shows that the asymptotic velocity is $1 - k$, which is negative if $k > 1$. Expansion (27b) of ω implies

$$X = X_0(\theta) + \tau^\epsilon \tilde{v}(\tau, \theta) \quad (32)$$

for some X_0 and \tilde{v} , which is in agreement with [20].

By the same arguments as in the low velocity case we have a similar existence theorem for a smaller set of data, including the regular case with velocity -1 at the axis.

Theorem 4.3. *Suppose that k, φ and ψ are real analytic functions of θ in a neighbourhood of $\theta = 0$ in \mathbf{S}^2 such that $k > 0$ for each θ , and suppose that ω_0 is a constant. Let ϵ be a positive constant less than $2k$. There exists a unique solution of the Einstein equations (13) of form (27) in a neighbourhood of $\theta = 0$ and $t = 0$ such that u, v, Du and Dv all tend to 0 as $t \rightarrow 0$.*

4.4. High velocity, analytic case

If k is negative, we replace expansion (27) with

$$Z(\tau, \theta) = k(\theta) \ln \tau + \varphi(\theta) + \tau^\epsilon u(\tau, \theta) \quad (33a)$$

$$\omega(\tau, \theta) = \omega_0(\theta) + \tau^\epsilon v(\tau, \theta). \quad (33b)$$

Note that we do not get the full number of free functions in this case either. Again, this is because of the possibility of spiky features in the asymptotic behaviour. The calculations are similar to the $k > 0$ case and will not be repeated here. It turns out that the system is valid for $k < 1/2$ (corresponding to an asymptotic velocity greater than $1/2$), by choosing $\max\{0, 2k\} < \epsilon < \min\{2, 2 - 2k\}$. The corresponding expansion of X is

$$X = X_0 + \tau^{2(1-k(\theta))} (\tilde{\psi}(\theta) + \tilde{v}(\tau, \theta)), \quad (34)$$

for some functions $\tilde{\psi}$ and \tilde{v} and an integration constant X_0 (which must vanish for the solution to be smooth at the axis). These solutions include the regular case with velocity 3 at the axis.

Theorem 4.4. *Suppose that k, φ and ω_0 are real analytic functions of θ in a neighbourhood of $\theta = 0$ in \mathbf{S}^2 , such that $k < 1/2$ for each θ . Let ϵ be a positive constant such that $\max\{0, 2k\} < \epsilon < \min\{2, 2 - 2k\}$. There exists a unique solution of the Einstein equations (13) of form (33) in a neighbourhood of $\theta = 0$ and $t = 0$ such that u, v, Du and Dv all tend to 0 as $t \rightarrow 0$.*

5. The smooth case

In the previous section, we established the existence of real analytic solutions with the desired asymptotic behaviour in a number of cases. Here, we will consider the corresponding smooth solutions, using a generalization of the approximation scheme in [38].

The idea is to approximate smooth asymptotic data $(k, \varphi, \omega_0, \psi)$ by a sequence of analytic data $(k_m, \varphi_m, \omega_{0m}, \psi_m)$, and to show convergence of the corresponding analytic solutions w_m to a smooth solution w . Since the argument does not depend on the details of the Gowdy equations, we will study a more general symmetric hyperbolic system with a singular term and sufficiently well-behaved coefficients. Also, since we are only interested in a neighbourhood of one of the axes in $S^2 \times S^1$ or S^3 , we will carry out the argument on a subset of $[0, \infty) \times \mathbf{R}^n$.

5.1. Regularity

We will need the following notion of regularity, taken from [38].

Definition 5.1. *A function $f(t, x)$ from an open subset $\Omega \subset [0, \infty) \times \mathbf{R}^n$ to \mathbf{R}^m is said to be regular if it is C^∞ for all $t > 0$ and if its partial derivatives of any order with respect to $x \in \mathbf{R}^n$ extend continuously to $t = 0$ in $\overline{\Omega}$.*

We can now specify the symmetric hyperbolic system to be studied in the rest of this section.

Definition 5.2. *We say that the system of differential equations*

$$tA^0(t, x)\partial_t w + N(x)w + tA^j(t, x, w)\partial_j w = tf(t, x, w) \quad (35)$$

is regular symmetric hyperbolic if A^0 is uniformly positive definite and symmetric, the A^j are symmetric, and all coefficients are regular.

Note that our terminology differs from the usual one here, since the full system (35) is not regular at $t = 0$ in the usual sense. In [38], the case when A^0 is the identity was considered, so the question at hand is under which circumstances the construction in [38] can be extended to more general A^0 .

5.2. Formal solutions

In our case, the analytic solutions will be obtained from a Fuchsian system

$$t\partial_t w + \tilde{N}(x)w = t\tilde{f}(t, x, w, w_x), \quad (36)$$

with the property that a solution of (36) is also a solution of (35). Such a Fuchsian system cannot always be found. If A^0 is independent of t , we can simply multiply both sides of (35) by $(A^0)^{-1}$ to get $\tilde{N} := (A^0)^{-1}N$ and $\tilde{f} := (A^0)^{-1}(f - A^j\partial_j w)$. If A^0 depends on t as well, this is no longer possible in general since $(A^0)^{-1}N$ will depend on t . However, if there is a decomposition $(A^0)^{-1}(t, x) = B_0(x) + t^\alpha B_1(t, x)$ for some regular matrices B_0 and B_1 and a constant α , the t dependence of $(A^0)^{-1}N$ may be included in \tilde{f} (after rescaling t if $\alpha < 1$). This is indeed the case for the Gowdy equations, e.g., (31).

Note that there is a subtlety here. In order to obtain a Fuchsian system for which theorem 4.1 holds, it may be necessary to redefine the dependent variables w . For example, in (31f), we must replace ∇U_0 with an expression in terms of U_2 and U_3 to get a symmetric hyperbolic system, but then theorem 4.1 would not apply anymore. So the equivalence of (36) and (35) is not a trivial matter, and has to be shown in each case.

Note also that most of the arguments in section 5.3 hold in the more general case when A^0 depends on w as well. However, in that case, it is often not possible to rewrite the symmetric hyperbolic system as a Fuchsian system of form (36).

In proving the existence of smooth solutions to (35) it is important that the matrix N is positive definite. This is not the case in general, so we need to modify the system to fulfil this requirement. We need the following definition of formal solutions from [38].

Definition 5.3. A finite sequence (w_1, w_2, \dots, w_p) of functions defined on an open subset $\Omega \subset [0, \infty) \times \mathbf{R}^n$ is called a formal solution of order p of the differential equation (36) on Ω if

- (i) each w_i is regular and
- (ii) $t\partial_t w_i + \tilde{N}(x)w_i - t\tilde{f}(t, x, w_i, \partial_x w_i) = O(t^i)$ for all i as $t \rightarrow 0$ in Ω .

We also define formal solutions of system (35) in a completely analogous way. The existence of formal solutions of (36) is provided by the following result.

Lemma 5.1 (Rendall [38]). *If \tilde{f} is regular and \tilde{N} is smooth and satisfies $\|\sigma^{\tilde{N}}\| \leq C$ for some constant C and all σ in a neighbourhood of 0, then (36) has a formal solution of any given order which vanishes at $t = 0$.*

To proceed it is necessary to show that a formal solution of (36) of order p is also a formal solution of (35) of order p . Because of the nontrivial relation between the systems, this must also be shown on a case by case basis. We will assume for the time being that this can be done.

Given a formal solution $\{w_1, \dots, w_i\}$, we can obtain a system for $z_i := t^{1-i}(w - w_i)$, where w is the sought solution of the original system (35). The procedure is similar to that in [38], so the details will be omitted. We end up with a system

$$tA^0(t, x)\partial_t z_i + (N(x) + (i-1)A^0(t, x))z_i + tA^j(t, x, w_i + t^{i-1}z_i)\partial_j z_i = tf_i(t, x, z_i), \quad (37)$$

where the regular function f_i is constructed from A^j and f and depends on w_i and $\partial_x w_i$ as well. The point is that since A^0 is uniformly positive definite, the coefficient $N + (i-1)A^0$ of the singular part in (37) is uniformly positive definite for large enough i . We will refer to (37) as the positive definite system and show an existence theorem for smooth solutions of this system in the next section.

Finally, when starting from a Fuchsian system of the form (36), we have to show that the obtained solution of (35) is also a solution of (36). To sum up, we need to verify the following to be able to apply theorem 5.1 below:

- There is a regular symmetric hyperbolic system of the form (35).
- There is a Fuchsian system of the form (36), satisfying the conditions in theorem 4.1.
- Analytic and formal solutions of (36) are also analytic and formal solutions of (35), respectively.
- The solution to (35) obtained from theorem 5.1 is also a solution of (36).

5.3. The existence theorem

We now turn to proving the existence of solutions to (37). Given smooth asymptotic data $S := \{k, \varphi, \omega_0, \psi\}$ on $\mathcal{U} \subset \mathbf{R}^n$, we may construct a sequence of analytic data $S_m := \{k_m, \varphi_m, \omega_{0m}, \psi_m\}$ on \mathcal{U} which converges to S in $C^\infty(\mathcal{U})$, uniformly on compact subsets. If the formal solutions are constructed as in the proof of lemma 5.1 (see [38]), the analytic formal solutions w_{mi} of order i corresponding to the analytic data S_m converge to

a formal solution w_i of order i corresponding to the smooth data S as $m \rightarrow \infty$, and the convergence is uniform on compact subsets. This also holds for spatial derivatives of any order. Hence, spatial derivatives of any order of the coefficients of the positive definite system (37) converge on compact subsets as $m \rightarrow \infty$. It follows that on any compact subset there is an i such that the coefficient involving $N(x)$ is positive definite for all m in our case, so we fix such a value of i and omit the index i from now on.

The global existence theorem for Gowdy spacetimes over $S^2 \times S^1$ and S^3 [12] implies that there are smooth solutions of (37) on a common time interval for all m . Thus our problem can be solved by proving the following theorem.

Theorem 5.1. *Let $z_m(t, x)$ be a sequence of regular solutions on $[0, t_1) \times \mathcal{U} \subset [0, \infty) \times \mathbf{R}^n$, with $z_m(0, x) = 0$, to a sequence of regular symmetric hyperbolic equations*

$$tA_m^0(t, x)\partial_t z_m + N_m(t, x)z_m + tA_m^j(t, x, z_m)\partial_j z_m = tf_m(t, x, z_m). \tag{38}$$

Suppose that N_m is uniformly positive definite for each m and that the coefficients converge uniformly to A_0^0, N_0, A_0^j and f_0 on compact subsets as $m \rightarrow 0$, with the same properties as A_m, N_m, A_m^j and f_m , and that the corresponding spatial derivatives converge uniformly as well. Then z_m converges to a regular solution z_0 of the corresponding system with coefficients A_0^0, N_0, A_0^j and f_0 on $[0, t_0) \times \mathcal{U}$ for some t_0 , with $z_0(0, x) = 0$.

Proof. The idea is to use energy estimates to show that $\{z_m\}$ is a Cauchy sequence. Since the proof is very similar to the case when A^0 is the identity [38], we only give an outline of the important steps here, with emphasis on the differences.

First, we consider the system satisfied by spatial derivatives of z . The collection \mathbf{z} of spatial derivatives up to order s satisfies a system similar to (38),

$$tA_m^0(t, x)\partial_t \mathbf{z}_m + \mathbf{N}_m(t, x)\mathbf{z}_m + tA_m^j(t, x, z_m)\partial_j \mathbf{z}_m = t\mathbf{f}_m(t, x, z_m), \tag{39}$$

obtained by differentiating (38) and substituting the equations for lower order spatial derivatives. The problem is that \mathbf{N}_m is not necessarily positive definite, due to the presence of off-diagonal blocks depending on $(A_m^0)^{-1}$ and spatial derivatives of A_m^0 and N_m . But this can be dealt with by multiplying the spatial derivatives $D^\alpha z_m$ by $K^{|\alpha|}$ for a sufficiently small constant K as in [38] (here α is a multi-index). Let $\mathbf{z}_m := \{K^{|\alpha|} D^\alpha z_m; |\alpha| \leq s\}$ be the collection of weighted spatial derivatives up to order s and assume that $\mathbf{A}_m, \mathbf{N}_m, \mathbf{A}_m^j$ and \mathbf{f}_m are the corresponding weighted coefficient matrices. For example, in one spatial dimension the system for $\mathbf{z}_m = (z_m, \partial_x z_m)$ has coefficients

$$\begin{aligned} \mathbf{A}_m^0 &= \begin{bmatrix} A_m^0 & 0 \\ 0 & A_m^0 \end{bmatrix} & \mathbf{N}_m &= \begin{bmatrix} N_m & 0 \\ K D_m N_m & N_m \end{bmatrix} \\ \mathbf{A}_m^j &= \begin{bmatrix} A_m^j & 0 \\ K D_m A_m^j & A_m^j \end{bmatrix} & \mathbf{f}_m &= \begin{bmatrix} f_m \\ K D_m f_m \end{bmatrix}, \end{aligned} \tag{40}$$

where D_m is the operator $\partial_x - (\partial_x A_m^0)(A_m^0)^{-1}$. The convergence of the coefficients for any fixed s follows from the convergence of the coefficients of the original equation.

We will need appropriately weighted Sobolev norms

$$\|u\|_{H^s} := \left(\sum_{|\alpha| \leq s} K^{2|\alpha|} \langle D^\alpha u, A_m^0 D^\alpha v \rangle_{L^2} \right)^{1/2}. \tag{41}$$

Note that these are equivalent to the usual Sobolev norms. The norms depend *a priori* on m since they include a factor A_m^0 , but since A_m^0 converges uniformly on compact subsets to A^0

which is uniformly positive definite, the equivalence can be taken to be independent of m on compact subsets for sufficiently large m .

Second, a domain of dependence result may be obtained by standard techniques. The argument is the same as that in [38] and will be omitted. Thus we need only consider the problem on a compact subset.

Next, we will show that the sequence $\|z_m\|_{H^s}$ is bounded. Differentiating and using the definitions of \mathbf{z}_m and \mathbf{A}_m^0 we get

$$\partial_t (\|z_m\|_{H^s}^2) = 2 \langle \mathbf{A}_m^0 \partial_t \mathbf{z}_m, \mathbf{z}_m \rangle_{L^2} + \sum_{|\alpha| \leq s} K^{2|\alpha|} \langle D^\alpha z_m, (\partial_t \mathbf{A}_m^0) D^\alpha z_m \rangle_{L^2}, \tag{42}$$

and using (39) to substitute for $\mathbf{A}_m^0 \partial_t \mathbf{z}_m$ gives

$$\partial_t (\|z_m\|_{H^s}^2) = -2t^{-1} \langle \mathbf{N}_m \mathbf{z}_m, \mathbf{z}_m \rangle_{L^2} + R_m, \tag{43}$$

where R_m contains the same terms as in the regular case with $\mathbf{N}_m \equiv 0$. The first term on the right is negative since \mathbf{N}_m is positive definite by construction, and R_m may be estimated as in the regular case, giving

$$R_m \leq C_1 \|z_m\|_{H^s}^2 + C_2 \|z_m\|_{H^s}. \tag{44}$$

Here C_1 and C_2 are polynomials in $\|(A^0)^{-1}\|_{L^\infty}$, $\|D^\alpha A^0\|_{L^\infty}$, $\|D^\alpha A^j\|_{L^\infty}$ and $\|D^\alpha f\|_{L^\infty}$ for $|\alpha| \leq s$, and in $\|z_m\|_{H^k}$ for any given $k > n/2 + 1$ (by the Sobolev embedding theorem). Since all of the coefficients converge and we may choose $k = s$ if $s > n/2 + 1$, $\partial_t (\|z_m\|_{H^s}^2)$ is bounded by a polynomial in $\|z_m\|_{H^s}$ whose coefficients are independent of m . Applying Gronwall's lemma then gives that $\|z_m\|_{H^s}$ is bounded.

We can in fact obtain a stronger estimate. Since $\|z_m\|_{H^s}$ is bounded, it follows that the coefficients C_1 and C_2 are bounded, so they can be chosen to be constants. A second application of Gronwall's lemma then gives that $t^{-1} \|z_m\|_{H^s}$ is bounded if $s > n/2 + 1$.

Finally, we show that $\{z_m\}$ is a Cauchy sequence in the H^s norm. Let the difference between consecutive elements of the sequence be given by $v_m := z_m - z_{m-1}$ and put $\mathbf{v}_m := \mathbf{z}_m - \mathbf{z}_{m-1}$. From (39) we get

$$\begin{aligned} \frac{1}{2} \partial_t (\|v_m\|_{H^s}^2) &= -t^{-1} \langle \mathbf{N}_m \mathbf{v}_m, \mathbf{v}_m \rangle_{L^2} - t^{-1} \langle (\mathbf{N}_m - \mathbf{N}_{m-1}) \mathbf{z}_{m-1}, \mathbf{v}_m \rangle_{L^2} \\ &\quad - \langle (\mathbf{A}_m^0 - \mathbf{A}_{m-1}^0) \partial_t \mathbf{z}_{m-1}, \mathbf{v}_m \rangle_{L^2} - \langle \mathbf{A}_m^j(z_m) \partial_j \mathbf{v}_m, \mathbf{v}_m \rangle_{L^2} + \langle \mathbf{v}_m, \mathbf{g}_m \rangle_{L^2}, \end{aligned} \tag{45}$$

where

$$\mathbf{g}_m := t(\mathbf{f}_m(z_m) - \mathbf{f}_{m-1}(z_{m-1})) - t(\mathbf{A}_m^j(z_m) - \mathbf{A}_{m-1}^j(z_{m-1})) \partial_j \mathbf{z}_{m-1}. \tag{46}$$

The first term on the right-hand side of (45) is negative and can be discarded, and using that $t^{-1} \|z_{m-1}\|_{H^s}$ is bounded, the second term is less than $C \|\mathbf{N}_{m+1} - \mathbf{N}_m\|_{L^\infty} \|v_m\|_{H^s}$. To estimate the third term we need a bound on $\|\partial_t z_{m-1}\|_{H^s}$, but this can be obtained by applying the bound on $t^{-1} \|z_{m-1}\|_{H^{s+1}}$ to (39). The fourth term may be estimated by doing a partial integration as in the regular case. The last term is estimated by inserting some terms and by applying the mean value theorem and the bound on $\|z_{m-1}\|_{H^{s+1}}$. We end up with an estimate of the form

$$\partial_t (\|v_m\|_{H^s}^2) \leq C_1 \|v_m\|_{H^s}^2 + C_2 \|v_m\|_{H^s}, \tag{47}$$

where C_1 is a constant independent of m and C_2 is a polynomial in $\|\mathbf{A}_m^0 - \mathbf{A}_{m-1}^0\|_{L^\infty}$, $\|\mathbf{N}_m - \mathbf{N}_{m-1}\|_{L^\infty}$, $\|\mathbf{A}_m^j - \mathbf{A}_{m-1}^j\|_{L^\infty}$ and $\|\mathbf{f}_m - \mathbf{f}_{m-1}\|_{L^\infty}$ which tends to 0 as $m \rightarrow \infty$. Applying Gronwall's lemma again, we conclude that $\|v_m\|_{H^s} \rightarrow 0$ as $m \rightarrow \infty$.

We have shown that the solutions z_m of the positive definite system (38) converge in the H^s norm to a smooth solution for each s . A potential problem is that the time interval of convergence may depend on s . But by standard techniques for symmetric hyperbolic equations the solutions may be extended to a common time interval, so we can conclude that z_m converge as smooth functions as well. \square

5.4. Low velocity, smooth case

From the discussion in section 5.2, we first need to write the Gowdy equations (31) in symmetric hyperbolic form. We rewrite (31) as

$$DU_0 = U_1 \tag{48a}$$

$$\begin{aligned} (1 - \tau^2)DU_1 = & -(1 - \tau^2)[2\epsilon U_1 + \epsilon^2 U_0] + \tau^{2-\epsilon}[k + (\ln \tau)\Delta k + \Delta\varphi] + \tau^2(U_1 + \epsilon U_0) \\ & + \tau[(1 - x^2)U_{2x} - xy(U_{2y} + U_{3x}) + (1 - y^2)U_{3y} - 2xU_2 - 2yU_3] \\ & - e^{-2\varphi - 2\tau^\epsilon U_0} \{ (1 - \tau^2)\tau^{2k-\epsilon}[V_1 + 2k(\psi + V_0)]^2 - \tau^{2-2k-\epsilon}(\nabla\omega_0)^2 \\ & - 2\tau^{1-\epsilon}\nabla\omega_0 \cdot [\bar{V} + \tau\nabla\psi + 2\tau(\ln \tau)(\psi + V_0)\nabla k] \\ & - \tau^{2k-\epsilon}[\bar{V} + \tau\nabla\psi + 2\tau(\ln \tau)(\psi + V_0)\nabla k]^2 \} \end{aligned} \tag{48b}$$

$$(1 - x^2)DU_2 - xyDU_3 = (1 - x^2)U_2 - xyU_3 + \tau(1 - x^2)U_{1x} - \tau xyU_{1y} \tag{48c}$$

$$(1 - y^2)DU_3 - xyDU_2 = (1 - y^2)U_3 - xyU_2 + \tau(1 - y^2)U_{1y} - \tau xyU_{1x} \tag{48d}$$

$$DV_0 = V_1 \tag{48e}$$

$$\begin{aligned} (1 - \tau^2)DV_1 = & -2k(1 - \tau^2)V_1 + 2(1 - \tau^2)\tau^\epsilon(U_1 + \epsilon U_0)(V_1 + 2k(\psi + V_0)) \\ & + \tau^2[V_1 + 2k(\psi + V_0) + 2(\ln \tau)(\psi + V_0)\Delta k + 2(\ln \tau)\nabla k \cdot \nabla\psi + \Delta\psi] \\ & + 2\tau(\ln \tau)\nabla k \cdot \bar{V} + \tau^{2-2k}\Delta\omega_0 + \tau[(1 - x^2)V_{2x} - xy(V_{2y} + V_{3x}) \\ & + (1 - y^2)V_{3y} - 2xV_2 - 2yV_3] - 2\tau^{2-2k}\nabla\omega_0 \cdot [(\ln \tau)\nabla k + \nabla\varphi] \\ & - 2\tau^{1+\epsilon-2k}\nabla\omega_0 \cdot \bar{U} - 2\tau^\epsilon[\tau^{1-\epsilon}\nabla\varphi + \bar{U}] \cdot [2\tau(\ln \tau)(\psi + V_0)\nabla k + \tau\nabla\psi + \bar{V}] \end{aligned} \tag{48f}$$

$$(1 - x^2)DV_2 - xyDV_3 = (1 - x^2)V_2 - xyV_3 + \tau(1 - x^2)V_{1x} - \tau xyV_{1y} \tag{48g}$$

$$(1 - y^2)DV_3 - xyDV_2 = (1 - y^2)V_3 - xyV_2 + \tau(1 - y^2)V_{1y} - \tau xyV_{1x}. \tag{48h}$$

Because of the factor $\tau^{1+\epsilon-2k}$, we must restrict k to the case $0 < k < 3/4$. After an appropriate rescaling of the time variable, the system is clearly a regular symmetric hyperbolic system of the form (35).

Next, we need to address the question if solutions of (31) are solutions of (48) as well. System (48) is obtained from (31) by multiplying with a regular positive definite matrix A^0 and replacing $\tau\nabla U_0$ with \bar{U} . For analytic solutions, the evolution equations (31a), (31c) and (31d) imply $D(\tau\nabla U_0 - \bar{U}) = 0$. But $\tau\nabla U_0 - \bar{U} = 0$ at $\tau = 0$, so $\tau\nabla U_0 = \bar{U}$ on the whole development. Thus an analytic solution of (31) is an analytic solution of (48). Repeating the argument for formal solutions of order i of (31), we obtain formal solutions of (48) of order i .

We can now form the corresponding positive definite system, where $i = 3$ suffices to make the coefficient of the singular term positive definite. Applying theorem 5.1 then gives the existence of a smooth solution to (48).

It remains to show that the obtained solution is a solution of the Fuchsian system (31). The problem is again to verify the substitution $\bar{U} = \tau\nabla U_0$. From (48a), (48c) and (48d) it follows that $D(\bar{U} - \tau\nabla U_0) = \bar{U} - \tau\nabla U_0$, and since \bar{U} and U_0 vanish at $\tau = 0$ we must have $\bar{U} - \tau\nabla U_0 = c(x)\tau$ for some function $c(x)$. There seems to be no direct way of showing that $c(x) \equiv 0$ follows from (48). But our solution is constructed as a limit of analytic solutions with $c(x) \equiv 0$, so this must hold for the smooth solution as well. To see this, let the analytic functions converging to \bar{U} and U_0 be \bar{U}_m and U_{0m} . We have $\bar{U}_m - \tau\nabla U_{0m} = 0$ for all m , so $\|c(x)\|\tau = \|(\bar{U} - \tau\nabla U_0) - (\bar{U}_m - \tau\nabla U_{0m})\| \leq \|\bar{U} - \bar{U}_m\| + \tau\|\nabla U_0 - \nabla U_{0m}\|$, (49)

where the norm is the supremum norm on any compact subset. The right-hand side tends to zero for all τ as $m \rightarrow \infty$, which shows that $\bar{U} = \tau \nabla U_0$. Therefore our solution of (48) must also be a solution of (31).

We summarize the results of this section in the following theorem.

Theorem 5.2. *If k, φ, ω_0 and ψ are smooth functions of θ and $0 < k < 3/4$ for all θ , there exists a solution of Einstein's equations (13) in a neighbourhood \mathcal{U} of $\theta = 0$ of the form (27) where $2k - 1 < \epsilon < \min\{2k, 2 - 2k\}$ and u and v are regular and tend to 0 as $t \rightarrow 0$. Given the form of the expansion and a choice of ϵ , the solution is unique.*

5.5. Negative velocity, smooth case

Since exponents of τ involving $-2k$ only appear in terms containing $\nabla \omega_0$, the argument in the previous section applies immediately to the negative velocity case with $k > 0$ and constant ω_0 .

Theorem 5.3. *If k, φ and ψ are smooth functions of θ such that $k > 0$ for all θ and ω_0 is a constant, there exists a solution of the Einstein equations (13) in a neighbourhood \mathcal{U} of $\theta = 0$ of the form (27) where $\epsilon < 2k$ and u and v are regular and tend to 0 as $t \rightarrow 0$. Given the form of the expansion and a choice of ϵ , the solution is unique.*

5.6. High velocity, smooth case

The calculations for the high velocity case are similar to the low velocity case and will be omitted. We have the following existence result.

Theorem 5.4. *If k, φ and ω_0 are smooth functions of θ and $k < 1/2$ for all θ , there exists a solution of the Einstein equations (13) in a neighbourhood \mathcal{U} of $\theta = 0$ of the form (33) where $\max\{0, 2k\} < \epsilon < \min\{2, 2 - 2k\}$ and u and v are regular and tend to 0 as $t \rightarrow 0$. Given the form of the expansion and a choice of ϵ , the solution is unique.*

5.7. Intermediate velocity, smooth case

It remains to treat the case when $3/4 \leq k < 1$. The idea is to include one more term in the expansion (27a) of Z ,

$$Z(\tau, \theta) = k(\theta) \ln \tau + \varphi(\theta) + \alpha(\theta) \tau^{2-2k} + \tau^{2-2k+\epsilon} u(\tau, \theta), \quad (50)$$

where α is to be chosen so as to eliminate the leading order terms in the equations. We keep the original expansion for ω as in (27b).

The resulting system is similar to (48), but with DU_1 and DU_2 given by

$$\begin{aligned} (1 - \tau^2)DU_1 = & -(1 - \tau^2)[(4 - 4k + 2\epsilon)U_1 + (2 - 2k + \epsilon)^2U_0] - 4\tau(\ln \tau)\nabla k \cdot \bar{U} \\ & + \tau^{2k-\epsilon}[k + (\ln \tau)\Delta k + \Delta\varphi] + \tau^{2-\epsilon}(4k^2 - 10k + 6)\alpha \\ & + \tau^{2-\epsilon}[\Delta\alpha - 2(\ln \tau)(\alpha\Delta k + 2\nabla\alpha \cdot \nabla k) + 4(\ln \tau)^2\alpha(\nabla k)^2] \\ & + \tau^2[U_1 + (2 - 2k + \epsilon)U_0 - 2(\ln \tau)U_0\Delta k + 4(\ln \tau)^2U_0(\nabla k)^2] \\ & + \tau[(1 - x^2)U_{2x} - xy(U_{2y} + U_{3x}) + (1 - y^2)U_{3y} - 2xU_2 - 2yU_3] \\ & - \exp(-2\varphi - 2\alpha\tau^{2-2k} - 2\tau^{2-2k+\epsilon}U_0)\{(1 - \tau^2)\tau^{4k-2-\epsilon}[V_1 + 2k(\psi + V_0)]^2 \\ & - 2\tau^{2k-1-\epsilon}\nabla\omega_0 \cdot [\bar{V} + \tau\nabla\psi + 2\tau(\ln \tau)(\psi + V_0)\nabla k] - \tau^{4k-2-\epsilon}[\bar{V} + \tau\nabla\psi \\ & + 2\tau(\ln \tau)(\psi + V_0)\nabla k]^2\} + \tau^{-\epsilon}[\exp(-2\varphi - 2\alpha\tau^{2-2k} \\ & - 2\tau^{2-2k+\epsilon}U_0)(\nabla\omega_0)^2 - (2 - 2k)^2\alpha] \end{aligned} \quad (51a)$$

$$\begin{aligned}
 (1 - \tau^2)DV_1 = & -2k(1 - \tau^2)V_1 + 2\tau(\ln \tau)\nabla k \cdot \bar{V} + 2(1 - \tau^2)\tau^{2-2k+\epsilon}(U_1 + (2 - 2k \\
 & + \epsilon)U_0)(V_1 + 2k(\psi + V_0)) + \tau^2[V_1 + 2k(\psi + V_0) + 2(\ln \tau)(\psi + V_0)\Delta k \\
 & + 2(\ln \tau)\nabla k \cdot \nabla \psi + \Delta \psi] + \tau^{2-2k}[\Delta \omega_0 + 2(1 - \tau^2)(2 - 2k) \\
 & \times \alpha(V_1 + 2k(\psi + V_0))] + \tau[(1 - x^2)V_{2x} - xy(V_{2y} + V_{3x}) + (1 - y^2)V_{3y} \\
 & - 2xV_2 - 2yV_3] - 2\tau^{2-2k}\nabla \omega_0 \cdot [(\ln \tau)\nabla k + \nabla \varphi] - 2\tau^{3-4k+\epsilon}\nabla \omega_0 \cdot \bar{U} \\
 & - 2\tau^{4-4k}\nabla \omega_0 \cdot [\nabla \alpha - 2(\ln \tau)(\alpha + \tau^\epsilon U_0)\nabla k] - 2\tau^{2-2k}[\tau^{2k-1}\nabla \varphi + \tau \nabla \alpha \\
 & - 2\tau(\ln \tau)(\alpha + \tau^\epsilon U_0)\nabla k + \bar{U}] \cdot [2\tau(\ln \tau)(\psi + V_0)\nabla k + \tau \nabla \psi + \bar{V}], \quad (51b)
 \end{aligned}$$

We choose $\alpha := (2 - 2k)^{-2}(\nabla \omega_0)^2 \exp(-2\varphi)$ to cancel the $\tau^{-\epsilon}$ factor in the last term on the right-hand side of (51a). That term will then be of order $\tau^{2-2k-\epsilon}$, so the order of the $(\nabla \omega_0)^2$ term is unchanged from the previous case (48b).

The problematic exponents of τ are now $2 - 2k - \epsilon$, $3 - 4k + \epsilon$ and $2k - 1 - \epsilon$. We have to choose the number $\epsilon > 0$ such that $4k - 3 < \epsilon < 2k - 1$, but we also need to ensure that $2 - 2k - \epsilon > 0$, which is compatible with $4k - 3 < \epsilon$ if and only if $1/2 < k < 5/6$. In that case, we may apply the arguments of sections 5.2 and 5.3 to show that a regular solution exists.

Contrary to what was claimed in [38], it is not possible to cover the whole range $1/2 < k < 1$ since we can only choose α such that the last term in (51a) vanishes to first order in τ . In equation (27) of [38] the corresponding higher-order terms have been left out.

We can, however, cover small intervals of k closer to 1 by repeating the method above. Replacing expansion (27a) by (50) is equivalent to performing the transformation $u \mapsto \alpha \tau^{2-2k-\epsilon} + \tau^{2-2k}u$. Repeating this transformation i times, where each α is chosen such that the leading order coefficient of $(\nabla \omega_0)^2$ is cancelled at each stage, gives a system with positive powers of τ if and only if

$$1 - \frac{1}{2i} < k < 1 - \frac{1}{2i + 4} \tag{52}$$

and

$$1 - 2(i + 1)(1 - k) < \epsilon < \min\{2 - 2k, 1 - 2i(1 - k)\}. \tag{53}$$

The interval $(1/2, 1)$ is covered by the infinite sequence of intervals $(1 - 1/2i, 1 - 1/(2i + 4))$. The expansion of Z for a given i is

$$Z(\tau, \theta) = k(\theta) \ln \tau + \varphi(\theta) + \sum_{j=1}^i \alpha_j(\theta) \tau^{(2-2k)j} + \tau^{(2-2k)i+\epsilon} u(\tau, \theta). \tag{54}$$

Note that as k tends to 1, we have to include an increasing number of terms in the expansion (54) of Z . This might seem to be contradictory since we keep the original expansion of ω , and the higher-order terms of ω should affect Z at some finite order. But, for any given k , the number of terms in (54) is finite since i is bounded above by (52). Also, the exponent of τ in the terms containing α_j is always between 0 and 1, so higher-order contributions are still encoded in u .

The above discussion motivates the following theorem.

Theorem 5.5. *If k, φ, ω_0 and ψ are smooth functions of θ and k satisfies (52) for all θ and some natural number i , there exists a solution of the Einstein equations (13) in a neighbourhood \mathcal{U} of $\theta = 0$ of the form (54) and (27b) where ϵ satisfies (53) and u and v are regular and tend to 0 as $t \rightarrow 0$. Given the form of the expansion and a choice of ϵ , the solution is unique.*

6. Discussion

We have constructed families of solutions to the evolution equations for Gowdy spacetimes with $\mathbf{S}^2 \times \mathbf{S}^1$ or \mathbf{S}^3 spatial topology. When the asymptotic velocity is between 0 and 1 ($0 < k < 1$) we obtain solutions depending on four free functions, which is the same number as in the general solution, while outside that range the solutions include only three free functions. The solutions are asymptotically velocity dominated by construction. Unfortunately, for regular asymptotically velocity-dominated solutions of the full Einstein equations the asymptotic velocity must be -1 or 3 ($k = \pm 2$) at the axes. This can also be checked directly by inserting the asymptotic expansions of the solutions into the constraints (19). Since both analytical and numerical arguments indicate that the velocity is between 0 and 1 for a large set of initial data in \mathbf{T}^3 Gowdy, and these results may be applied for the other topologies on sets not containing the axes, it seems that sharp features will develop at the axes as $t \rightarrow 0$ in general situations.

There are two possible interpretations. Firstly, there is the possibility of false spikes, which result from an inappropriate choice of parametrization of the metric. This is also what makes our solutions different from the numerical solutions [42]. The problem may be traced to different choices of parametrizations of the hyperbolic plane. It might be possible to choose this parametrization differently so as to avoid coordinate singularities, for example, using the ball model coordinates [45]. This is of equal importance in the study of the dynamics of \mathbf{T}^3 Gowdy, in particular when trying to verify the numerical observation that the velocity is eventually driven below 1 in a mathematically rigorous way.

Secondly, there is the possibility of true spikes at the axes. In [24], \mathbf{T}^3 Gowdy solutions with spikes are constructed by applying suitable transformations to a given solution with velocity between 0 and 1. At first sight, it seems possible to do something similar for the other topologies. As outlined in section 2, the transformations used in [24] are an inversion which interchanges the Killing vectors and a Gowdy-to-Ernst transformation. A combination of these transformations preserves the evolution equations in the $\mathbf{S}^2 \times \mathbf{S}^1$ and \mathbf{S}^3 cases as well. In fact, the equations for Z and ω are the same as the equations for $-P$ and Q (in the notation of section 3.1). But P is singular at the axes, not just at $t = 0$, so this method of constructing solutions with spikes does not work in our case. This does, of course, not rule out the possibility that there will be true spikes at the axes in general.

Finally, some remarks on the asymptotic behaviour of the curvature are in order. Using the asymptotic expansions for our solutions, (27) or (33), it is straightforward to calculate that a necessary condition for the Kretschmann scalar $R_{ijkl}R^{ijkl}$ to be bounded is that the asymptotic velocity is ± 1 ($k = 0$ or $k = 2$), which is in agreement with the result for the polarized models described in section 3.4.1. All of the black-hole solutions mentioned in section 3.4.2 are, of course, extendible through the horizon, and the asymptotic velocity is indeed ± 1 there. While such non-generic solutions can be ignored from the cosmological point of view, they might be interesting by analogue to black-hole interiors. One interesting question that remains to be answered is under which circumstances general Gowdy models can be extended through a compact Cauchy horizon. In particular, is analyticity necessary as in the polarized case [14]?

To conclude, it seems that for more progress on Gowdy $\mathbf{S}^2 \times \mathbf{S}^1$ and \mathbf{S}^3 spacetimes to be made, a better understanding and handling of the spikes are needed. If this can be done for \mathbf{T}^3 Gowdy, it should be possible to adapt the techniques to the other topologies using some of the arguments of this paper.

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