

# Global Prescribed Mean Curvature foliations in cosmological spacetimes with matter

## Part I

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### Abstract

This work investigates some global questions about cosmological spacetimes with two dimensional spherical, plane and hyperbolic symmetry containing matter. The result is, that these spacetimes admit a global foliation by prescribed mean curvature surfaces, which extends at least towards a crushing singularity. The time function of the foliation is geometrically defined and unique up to the choice of an initial Cauchy surface.

This work generalizes a similar analysis on constant mean curvature foliations and avoids the topological obstructions arising from the existence problem.

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# 1 Introduction

The flavour of General Relativity viewed as an initial value problem for the field equations is based on the geometrical nature of the equations, which implies the diffeomorphism invariance and the absence of a metric background structure. This has consequences for spacelike foliations, in that the time function of a foliation is not canonical, but arbitrary unless it is tied to some geometrical quantity. The latter would turn the analysis of the global structure of spacetimes into an investigation of the asymptotic behaviour of such a foliation.

In spacetimes with certain symmetries some global foliations, defined by time coordinates tied to the symmetries are known, e.g. [Re], [An].

A geometrically defined foliation which does not depend on the symmetries are the well known constant mean curvature (CMC) foliations, where the time coordinate is given by the mean curvature of the leaves, which varies continuously from leaf to leaf, see [R1] for a survey of this topic and [R4], [R6] for foliations of spacetimes with symmetries.

Unfortunately the CMC constructions suffer from the existence problem, which is unsolved in general. To overcome this difficulty, a foliation with leaves of prescribed mean curvature (PMC) has been constructed at least locally in time in [H2] for cosmological spacetimes. The prescription is given implicitly, letting the mean curvature vary continuously along the normal vector field of the foliation relative to a given Cauchy surface. The time function of the foliation is geometrically defined and turns out to be intrinsic, in that coupled to Einstein's field equations adapted to the leaves, one obtains Cauchy data for spacetimes foliated by PMC leaves.

The aim of the present paper is to globalize this result for certain spacetimes. Motivated by the method in [R3], where satisfying results about cosmic censorship have been obtained for spatially homogeneous models, I consider here cosmological spacetimes with two dimensional spherical, plane and hyperbolic symmetry. This choice is the first step of successively lowering the degree of symmetry to obtain more and more general results. I focus on the following questions:

How large is the maximal interval for the time function and does this maximal foliation cover the whole spacetime?

One guideline taken from [R3] is, that the foliation may be extendible as long as the mean curvature of the leaves remains finite. We will see, that this principle stays true with some modifications.

Another important aspect for the construction of global foliations is the choice of an appropriate matter model. The key requirements for the matter fields are their regularity in a regular geometric background as well as some energy conditions. These requirements are not trivially satisfied, since the first one rules out matter such as the perfect fluid, which is known to develop singularities (shocks) in a regular geometric background. For dust it has been shown in [R5], [IR], that there is no way to construct a global CMC foliation, which covers the whole dust-filled cosmological spacetime. These counterexamples emphasize dramatically the importance of choosing appropriate matter models. Furthermore demanding energy conditions seems natural and obvious, but we will see, that the non-negative pressures condition plays a special role in the improvements in section 3.5.

Section 2 fixes notation and states some basic definitions. We attack the main questions in section 3, following closely the treatment in [R4]. In the final section the results will be discussed.

## 2 Basic definitions and formulas

### 2.1 Spacetimes and foliations

A spacetime is a pair  $(M, g)$ , where  $M$  denotes a four dimensional smooth and orientable Lorentz manifold with metric  $g$  and signature  $(-+++)$ . The metric induces the Levi-Civita connection  ${}^4\nabla$  on  $M$ . If  $\{X_\alpha\}$  denotes a local basis of vector fields on  $M$ , we define the connection coefficients  ${}^4C_{\alpha\beta}^\gamma$  relative to this basis by  ${}^4\nabla_{X_\alpha} X_\beta = {}^4C_{\alpha\beta}^\gamma X_\gamma$  and we get the Christoffel symbols  ${}^4\Gamma$  and Ricci rotation coefficients  ${}^4\gamma$  by specializing to a coordinate basis or an orthonormal frame, respectively. The sign convention for the curvature is fixed by the definition  ${}^4R(X, Y)Z := {}^4\nabla_X {}^4\nabla_Y Z - {}^4\nabla_Y {}^4\nabla_X Z - {}^4\nabla_{[X, Y]} Z$ , where  $X, Y, Z$  are vector fields. The curvature tensor is then defined as  ${}^4R(W, Z, X, Y) := g(W, {}^4R(X, Y)Z)$  with Ricci tensor  ${}^4R_{\alpha\beta} = {}^4R_{\alpha\mu\beta}^\mu$  and scalar curvature  ${}^4R = {}^4R_\mu^\mu$ , written in abstract index notation of the Ricci calculus.

Then the Einstein tensor reads  $G_{\alpha\beta} = {}^4R_{\alpha\beta} - \frac{1}{2}{}^4Rg_{\alpha\beta}$  and the field equations are  $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$  or equivalently  ${}^4R_{\alpha\beta} = 8\pi (T_{\alpha\beta} - \frac{1}{2}(\text{tr } T)g_{\alpha\beta})$ , where  $T_{\alpha\beta}$  denotes the energy momentum tensor of the matter fields. The matter quantities are the energy density  $\rho := T_{\mu\nu}n^\mu n^\nu$ , the momentum density  $j_\beta := -T_{\mu\nu}n^\mu h_\beta^\nu$  and the stress tensor  $S_{\alpha\beta} := T_{\mu\nu}h_\alpha^\mu h_\beta^\nu$  with respect to an observer, represented by a unit timelike vector  $n$ , where  $h_{\alpha\beta} := g_{\alpha\beta} + n_\alpha n_\beta$  denotes the orthogonal projector on  $\{n\}^\perp$  in covariant notation.

Einstein's field equations in vacuum ( $T_{\alpha\beta} = 0$ ) have a well posed Cauchy problem in harmonic coordinates, thus one obtains spacetimes as solutions of Einstein's field equations with matter, whenever the equations describing the matter fields and the energy momentum tensor couple to the field equations in harmonic coordinates, such that the Cauchy problem remains well posed. We will see examples in subsection 2.2 (compare [W],[FR] for an introduction/analysis of the Cauchy problem for Einstein's equations).

In this work we confine ourselves to *cosmological* solutions  $(M, g)$  of Einstein's field equations. Due to [Ba] these are globally hyperbolic and spatially compact spacetimes, where the Ricci tensor contracted twice with any timelike vector is non-negative (timelike convergence condition). This last condition can be reexpressed in terms of the matter variables as  $\rho + \text{tr } S \geq 0$  for any observer, which is the strong energy condition.

Now let us pay attention to an additional structure. A foliation  $\{S_t\}$ ,  $t \in I \subset \mathbb{R}$  ( $I$  interval containing zero) of (a part of)  $(M, g)$  by spacelike hypersurfaces induces on each leaf the unit normal vector field  $n$ , the metric  $h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$ , which also serves as orthogonal projection and the second fundamental form  $k_{\alpha\beta} := -h_\alpha^\mu h_\beta^\nu {}^4\nabla_\mu n_\nu$  (the definition of  $k_{\alpha\beta}$  fixes the sign conventions used in this work). The second fundamental form is a symmetric tensor, intrinsic to the leaves of the foliation, and can also be written as the Lie derivative of the 3-metric  $h$  with respect to the normal vector field,  $k_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n h_{\alpha\beta}$ . The 3-metric determines further geometrical objects on the leaves, such as the Levi-Civita connection  $\nabla$ , the Christoffel symbols  $\Gamma$ , the Ricci rotation coefficients  $\gamma$  and the curvature tensor  $R(\cdot)$ . Tensors intrinsic to the leaves of the foliation will carry Latin indices in the abstract index notation.

The parameter  $t$  of the foliation has timelike gradient and thus can be regarded as (coordinate-) time. Given local coordinates  $(x^i)$  on  $S_0$ , we can Lie-transport them to neighbouring leaves along an arbitrary family of transversal curves, parametrized by  $x$ . We will express equations containing coordinate components with respect to the adapted coordinates  $(t, x)$ .

The lapse function  $N$  and the shift vector  $\nu \perp n$  on the leaves are defined by  $\partial_t = Nn + \nu$ , thus  $N = -g(\partial_t, n)$  and  $\nu = \partial_t - Nn$ . Then we have  $1 = dt(\partial_t) = N dt(n)$ . Further,  $dt$

is (co-)orthogonal to the leaves and if we denote the conormal of the leaves by  $\sigma$  we see that  $dt = -N^{-1}\sigma$  or  $\sigma = -Ndt$ , thus  $N^{-1}$  measures the length of  $dt$ .

The most common example involving the lapse function is the event horizon in Schwarzschild spacetime, where the coordinate time explodes along the worldline of infalling observers  $n$ , thus  $N^{-1}$  explodes or  $N \rightarrow 0$ . In this work we will be faced with a complementary scenario, where we have to ensure, that the lapse function does not explode, corresponding to the phenomenon of recollapse, where coordinate time freezes, as  $dt \rightarrow 0$ . If this occurs, one could try to reparametrize the foliation by setting  $\tilde{t} = f(t)$  for some monotone function  $f$  and one gets  $d\tilde{t} = f'(t) dt$ ,

$$\tilde{N} = \frac{1}{f'(t)}N = \frac{dt}{d\tilde{t}}N \quad ,$$

and the same relation holds for shift.

The 3 + 1 split of the spacetime geometry by means of lapse and shift ends up in the 3 + 1 form of the field equations. The constraint equations are

$$(1a) \quad R + H^2 - |k|^2 = 16\pi\rho \quad (\text{Hamiltonian constraint})$$

$$(1b) \quad \nabla^j k_{ij} - \nabla_i H = 8\pi j_i \quad (\text{momentum constraint}) \quad ,$$

with  $|k|^2 = k_{\alpha\beta}k^{\alpha\beta}$  and  $H = \text{tr } k$  denotes the mean curvature of the leaves. The ADM equations read

$$(2a) \quad \partial_t h_{ij} = -2Nk_{ij} + \nabla_i \nu_j + \nabla_j \nu_i$$

$$(2b) \quad \partial_t k_{ij} = -\nabla_i \nabla_j N + N \left( R_{ij} + Hk_{ij} - 2k_i^r k_{rj} - 8\pi(S_{ij} + \frac{1}{2}(\rho - \text{tr } S)h_{ij}) \right) \\ + \nu^r \nabla_r k_{ij} + k_{rj} \nabla_i \nu^r + k_{ir} \nabla_j \nu^r \quad .$$

Taking the trace of the second equation and eliminating the scalar curvature  $R$  by the Hamiltonian constraint we obtain the lapse equation

$$(3) \quad \Delta N + N \left( |k|^2 + 4\pi(\rho + \text{tr } S) \right) = (\partial_t - \nu)H \quad ,$$

which serves as a constraint of the foliation. Note, that in cosmological spacetimes, the term in brackets is always non-negative. If it turns out to be non-vanishing, then the left-hand side of (3) can be shown to be an isomorphic mapping of  $N$ , considered as an element of some Sobolev space  $H^s$  into  $H^{s-2}$ . This observation motivates the following

### 2.1. Definition.

A *Prescribed Mean Curvature (PMC) foliation* is defined to be a foliation  $\{S_t\}$  satisfying (3), with

$$(4) \quad (\partial_t - \nu)H = Nn(H) := 1 \quad ,$$

thus the mean curvature of the leaves is forced to vary uniformly along the normals of the leaves.

In [H2] I proved the following local in time result:

## 2.2. Theorem.

Let  $(M, g)$  be a smooth, globally hyperbolic spacetime, obeying the strong energy condition, with compact Cauchy surface  $\Sigma$  and

$$(5) \quad \lambda = |k|^2 + 4\pi(\rho + \text{tr } S)$$

does not vanish identically on  $\Sigma$ .

Then there exists a  $T > 0$  and a unique smooth PMC foliation  $\{S_t\}$ ,  $t \in [-T, T]$  in  $(M, g)$ , with  $\Sigma = S_0$ .

Note, that the setting here is quite general, no symmetry assumptions have to be made and essentially the strong energy condition turns out to be sufficient for the local in time existence of a unique PMC foliation up to the choice of an initial Cauchy surface.

The aim of the present paper is to globalize the result. Here are two problems involved: How large is the interval of values taken by the time coordinate and does the global foliation then cover the whole spacetime? To answer these questions in general there seem to be no techniques available up to now. One strategy to obtain global results is, to study first spacetimes with some spatial symmetry, taking advantage of the simplifications of the equations. Then the hope is, that the techniques developed in these cases give insight into the nature of more general classes of spacetimes, by successively lowering the degree of symmetry. Here we will focus on spacetimes with two dimensional spacelike orbits of symmetry and three (local) Killing vector fields. In the second part of this paper we will consider spacetimes with two commuting (local) Killing vector fields. These cases are indeed the first steps of this program, since the initial analysis for locally spatially homogeneous spacetimes has already been done and lead to very strong results (see [R3] for the exact analysis or [H1] for an overview about the results in the present context).

## 2.2 Matter models

Before getting deeper into the analysis just motivated I introduce some matter models and their coupling to the Einstein equations, with special emphasis on the Cauchy problem. Since energy conditions will play an important role in the estimates we perform later on, I assemble the relevant ones for the present work first.

- The dominant energy condition. Its statement is, that for all orthonormal frames  $\{{}^4e_\alpha\}$  with  ${}^4e_0$  timelike

$$T({}^4e_0, {}^4e_0) \geq |T({}^4e_\alpha, {}^4e_\beta)| \quad .$$

Another formulation of this condition is, that for any observer, the local energy density  $\rho$  is non-negative and the momentum density  $j$  is non-spacelike, thus 'matter cannot travel faster than light', a statement, that can be proved rigorously (see for example section 4.3 of [HE]), leading to the result, that if the energy momentum tensor obeys the dominant energy condition and vanishes on a set  $S$ , then it vanishes on the whole Cauchy development  $D(S)$  of the set, thus  $D(S)$  is a vacuum spacetime.

- The strong energy condition states that for all timelike vectors  $v$  the inequality

$$T(v, v) \geq \frac{1}{2}(\text{tr } T)g(v, v)$$

holds, or equivalently for any observer  $\rho + \text{tr} S \geq 0$ , thus the stresses do not become too negative. Another formulation is  ${}^4R(v, v) \geq 0$ , also known as the timelike convergence condition, which contributes to the expansion of timelike geodesic congruences a negative term, thus shifting the balance towards contraction to a final singularity of the congruence. Hawking's famous singularity theorems guarantees then geodesic incompleteness in the past provided further the existence of a Cauchy surface with uniform negative mean curvature.

We already used the strong energy condition as an integral part in the definition of cosmological spacetimes and its meaning for foliations.

- The non-negative pressures condition demands the stress tensor  $S$  to be positive definite. This condition ensures in some sense, that the pressures contribute more to attraction than to repulsion, leading to a finite lifetime of the spacetime under certain circumstances. This somewhat unexpected behaviour is a true relativistic effect, which does not have a Newtonian counterpart.

### 2.2.1 The Maxwell equations

The Maxwell field is described by a two form  $F$ , subject to the Maxwell equations

$$(6a) \quad dF = 0 \quad \iff \quad {}^4\nabla_{[\kappa} F_{\mu\nu]} = 0$$

$$(6b) \quad d \star F = 4\pi \star J \iff {}^4\nabla_{\mu} F^{\alpha\mu} = 4\pi J^{\alpha} \quad ,$$

where  $J$  denotes the electromagnetic charge current density. Alternatively the covariant derivatives can be replaced by ordinary derivatives.

It is well known, that the Maxwell equations admit (locally) a reformulation in terms of the vector potential  $A$ , a one form with  $dA = F$ . Then the remaining inhomogeneous equation reads  ${}^4\nabla^{\mu} {}^4\nabla_{\alpha} A_{\mu} - {}^4\nabla^{\mu} {}^4\nabla_{\mu} A_{\alpha} = 4\pi J_{\alpha}$ . Fixing the gauge invariance of the equation by the Lorentz gauge, we obtain the system

$$(7a) \quad {}^4\nabla^{\mu} A_{\mu} = 0 \quad (\text{Lorentz-gauge})$$

$$(7b) \quad \square A_{\alpha} = 4\pi J_{\alpha} - R_{\alpha\mu} A^{\mu} \quad ,$$

with  $\square = -{}^4\nabla^{\mu} {}^4\nabla_{\mu}$  and the curvature term arises as a consequence of some commutations of derivative operators. In this formulation the second equation is a wave equation, hence can be brought into first order symmetric hyperbolic form, if the source term is appropriate. Thus we get local existence and uniqueness for this equation in a given spacetime. One can further show, that then the Lorentz gauge propagates and we get indeed a unique local solution of the Maxwell equations. Spatially global solution can be obtained by the usual patching argument, arising from localizing the equation with respect to an appropriate partition of unity of the initial data on some Cauchy surface.

Given an electromagnetic field  $F$  we can form the associated energy momentum tensor  $E$ , defined as

$$(8) \quad E_{\alpha\beta} = \frac{1}{4\pi} \left( F_{\alpha\mu} F_{\beta}^{\mu} - \frac{1}{4} |F|^2 g_{\alpha\beta} \right) \quad ,$$

where  $|F|^2 = F^{\mu\nu} F_{\mu\nu}$ .

$E$  is tracefree, satisfies the relation  ${}^4\nabla_{\mu} E^{\alpha\mu} = -F^{\alpha}_{\mu} J^{\mu}$  and the dominant and strong energy condition hold.

Since the Maxwell equations turn out to be symmetric hyperbolic at least locally, we can couple them to the Einstein equation to get a symmetric hyperbolic system of equations in harmonic coordinates. Thus we end up with a well-posed Cauchy problem for the Einstein-Maxwell system, as long as the electromagnetic charge current density is in appropriate form.

### 2.2.2 The Vlasov equation

The Vlasov equation is a model for a collisionless gas. It describes the motion of a huge number of structureless particles in spacetime. We need only the case, where the particles have unit mass, where the equation is composed of a non negative function  $f$ , defined on the mass shell of particles of mass one  $P := \{v \in TM \mid g(v, v) = -1, \text{ future pointing}\}$ , representing the particle distribution, and a geodesic spray  $X$  on  $P$ . The equation reads

$$(9a) \quad X(f) = 0 \quad ,$$

with

$$(9b) \quad X = p^\mu \partial_\mu - {}^4\Gamma_{\mu\nu}^k p^\mu p^\nu \frac{\partial}{\partial p^k} = v^\alpha e_\alpha - {}^4\gamma_{\mu\nu}^k v^\mu v^\nu \frac{\partial}{\partial v^k} \quad ,$$

where  $p^\mu$  denotes the components of the momentum of the particles with respect to the given coordinates and  $v^\alpha$  are the components of the momentum with respect to an orthonormal frame. They are related by  $p^\mu = ({}^4e_\alpha)^\mu v^\alpha$  and on the mass shell  $p^0$  is determined by the components  $p^i$  and  $v^0 = \sqrt{1 + \delta_{ij} v^i v^j}$ .

Inserting the definition  ${}^4e_\alpha = ({}^4e_\alpha)^\mu \partial_\mu$  and doing a 3+1 decomposition (with the exception, that we do not write down the explicit expression for  ${}^4\gamma_{0j}^k$ ) we can reformulate the Vlasov equation as

$$(9') \quad 0 = \partial_t f + (N \frac{v^j}{v^0} (e_j)^i - \nu^i) \partial_i f - \left( e_i (N) v^0 + N (-k_{rs} (e_i)^r (e_j)^s + \delta_{ik} {}^4\gamma_{0j}^k) v^j + N \delta_{ik} \gamma_{rs}^k \frac{v^r v^s}{v^0} \right) \frac{\partial}{\partial v^i} f \quad .$$

The Cauchy problem for the Vlasov equation in a given spacetime is easy, since it is a linear, scalar equation with characteristic vector field  $(Y, Q)$  satisfying

$$\begin{aligned} \dot{Y}^\alpha &= Q^\alpha \\ \dot{Q}^\alpha &= -{}^4\Gamma_{\kappa\lambda}^\alpha Q^\kappa Q^\lambda \quad , \end{aligned}$$

and since  $X(f) = 0$ ,  $f$  is constant along the characteristics.

From the particle distribution  $f$  one can construct other physically meaningful quantities such as the energy momentum tensor by integration over the tangent spaces. We denote the part of the mass shell in the fibre over  $x \in M$  by  $P_x := P \cap T_x M$ . Then we define the energy-momentum tensor  $T$  by

$$(10) \quad T_{\alpha\beta}(x) := - \int_{P_x} f p_\alpha p_\beta \sqrt{|g|} / p_0 dp^1 dp^2 dp^3 \quad .$$

$T$  is divergence free, satisfies the dominant and strong energy condition and the non-negative pressures condition.

To obtain the matter quantities  $\rho$ ,  $j$  and  $S$  one has to calculate the components of the

energy momentum tensor with respect to an orthonormal frame. These components have the following representation:

$$(10') \quad T({}^4e_\alpha, {}^4e_\beta)(x) = \int_{P_x} f v_\alpha v_\beta / v^0 dv^1 dv^2 dv^3$$

Now let us consider the Einstein-Vlasov system. The energy momentum tensor automatically satisfies  $\text{div } T = 0$  as mentioned above, thus there arise no additional equations from the Bianchi identities. But unfortunately the coupled system of equations is not symmetric hyperbolic in any sense, due to the fact, that it is a system of integro-differential equations. Nevertheless, the local existence proof applies similar techniques as in the case of quasilinear symmetric hyperbolic systems where the peculiarity in the construction for the Einstein-Vlasov system consists in bounding the support of  $f$  in the tangent space: There is no a priori bound on the velocities of the particles, and no localization argument available as for the spacetime coordinates. In order to estimate the matter quantities appearing in the coupled system, one has to control the maximal velocity uniformly during the construction. This has been done for example in [R2], establishing a well-posed Cauchy problem for the Einstein-Vlasov system.

With this result at hand, it is easy to extend this result to the Einstein-Vlasov-Maxwell system, since the Einstein-Maxwell equations are symmetric hyperbolic and when coupling the Vlasov equation to them nothing new appears.

### 2.3 Further conventions

For the convenience of the reader I cite the well known Gronwall estimate, which is central to the analysis of partial differential equations and will be used here in this work. I adopt the convention of denoting any generic constant by  $C$ .

#### 2.3. Proposition (Gronwall's inequality).

Let  $I \subset \mathbb{R}$  be an interval,  $t_0 \in I$  and  $\alpha, \beta, u \in C(I, \mathbb{R}_+)$ , with

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \beta(s) u(s) ds \right|$$

for all  $t \in I$ .

Then

$$u(t) \leq \alpha(t) + \left| \int_{t_0}^t \alpha(s) \beta(s) e^{|\int_s^t \beta(r) dr|} ds \right|$$

holds for all  $t \in I$ .

The proof of Gronwall's inequality in this particular form can be found in [A].

### 3 Spacetimes with two dimensional spherical, plane and hyperbolic symmetry

#### 3.1 The geometry of surface symmetric spacetimes

Let  $(M, g)$  be a smooth, globally hyperbolic spacetime which is topologically of the form  $\mathbb{R} \times S^1 \times F$ , with  $F$  a compact, orientable surface. The submanifolds  $\{\tau\} \times S^1 \times F$  are assumed to be Cauchy surfaces of  $M$ . The universal covering  $\hat{F}$  of  $F$  induces a spacetime  $(\hat{M}, \hat{g})$  by  $\hat{M} = \mathbb{R} \times S^1 \times \hat{F}$  and  $\hat{g} = p^*g$ ,  $p : \hat{M} \rightarrow M$  the canonical projection. Moreover, there is a group  $G$  of isometries acting on  $(\hat{M}, \hat{g})$ .

Then  $(M, g)$

- is called *spherically symmetric*, if  $F = S^2$  and  $G = SO(3)$  acts isometrically and without fixed points on  $S^1 \times S^2$
- is *plane symmetric*, if  $F = T^2$  and  $G = E_2$  (Euclidian group) acts isometrically on  $\hat{F} = \mathbb{R}^2$
- has *hyperbolic symmetry*, if  $F$  has genus greater than one and the connected component of the symmetry group  $G$  of the hyperbolic plane,  $H^2$ , acts isometrically on  $\hat{F} = H^2$  (thus  $F = H^2/\Gamma$ , with  $\Gamma$  a discrete group of isometries of  $H^2$ )

and the matter quantities remain invariant under the isometries.

To collect these cases, each such spacetime is called *surface symmetric*, the diffeomorphic images of  $F$  in the product decomposition of  $M$  *surfaces of symmetry* and each surface in  $M$  diffeomorphic to  $S^1 \times F$  will be called *symmetric*.

In expressions involving indices, lower case Greek indices range from 0 to 3, lower case Latin indices (preferably taken from the middle of the alphabet) range from 1 to 3 and upper case Latin indices (from the beginning of the alphabet) take the values 2 or 3.

The isometric action forces the curvature of the surfaces of symmetry up to rescaling to be  $\epsilon = 1, 0, -1$  in the spherical, plane and hyperbolic case, respectively. Therefore they can be coordinatized by the well known angles  $(\vartheta, \varphi)$  which cast the metric  $\tilde{g}$  of the surfaces of symmetry (considered for a moment as abstract manifolds) into the form

$$(11) \quad \tilde{g} = d\vartheta^2 + \epsilon_{\vartheta}^2 d\varphi^2 \quad , \quad \epsilon_{\vartheta} := \begin{cases} \sin \vartheta & \epsilon = 1 \\ 1 & \epsilon = 0 \\ \sinh \vartheta & \epsilon = -1 \end{cases} .$$

Define the area radius function  $r$  on a surface of symmetry  $F$  (embedded in  $M$ ) to be

$$(12) \quad r = \sqrt{\frac{1}{4\pi} \text{Vol}(F)} \quad ,$$

then  $r$  is independent of  $(\vartheta, \varphi)$  and the metric of  $F$  reads

$$(13) \quad \bar{g} = r^2 \tilde{g} \quad .$$

With respect to any symmetric Cauchy surface  $S$  we have the timelike unit normal vector  $n$  of  $S$  in  $M$ . Regarding  $n$  as a normal vector to  $F$  in  $M$ , we can define a second unit normal  $m$  of  $F$  by the conditions, that  $m$  is tangent to  $S$  and the system  $(n, m, \bar{e}_2, \bar{e}_3)$

is positively oriented, where we set  $\bar{e}_A := r^{-1}\tilde{e}_A$ ,  $\tilde{e}_2 := \partial_2$ ,  $\tilde{e}_3 := \epsilon_\vartheta^{-1}\partial_3$ . We denote the associated second fundamental forms of  $(F, \bar{g})$  in  $(M, g)$  by  $\kappa$  and  $\lambda$ , with

$$(14a) \quad \lambda_{AB} = -\frac{1}{2}m(\bar{g}_{AB}) = -\frac{1}{2}m(r^2)\tilde{g}_{AB} = \frac{1}{2}(\text{tr } \lambda)\bar{g}_{AB} \quad , \quad \text{tr } \lambda = -\frac{2}{r}m(r)$$

$$(14b) \quad \kappa_{AB} = -\frac{1}{2}n(\bar{g}_{AB}) = -\frac{1}{2}n(r^2)\tilde{g}_{AB} = \frac{1}{2}(\text{tr } \kappa)\bar{g}_{AB} \quad , \quad \text{tr } \kappa = -\frac{2}{r}n(r)$$

Consider now a Gaussian coordinate neighbourhood  $(x', \vartheta, \varphi)$  of a surface of symmetry  $F$ , covering (a part of) a symmetric Cauchy surface  $S$ . The metric  $h$  of  $S$  then takes the form  $h = dx'^2 + \sqrt{|h|}\tilde{g}$ . The projection of geodesics starting in  $(\hat{M}, \hat{g})$  orthogonal at  $F$  remain orthogonal to all surfaces of symmetry. Following them until their projection meets  $F$  again, the symmetry allows only two possibilities: The point of return is the same as the starting point or an antipodal point, in which case we force the geodesic to turn a second time around the circle. Let  $L$  denote the length of the geodesic. Setting  $a = 2\pi \left( \int_0^L |h(z)|^{-1/4} dz \right)^{-1}$ , we define a new coordinate  $x(x')$  by  $a \int_0^{x'} |h(z)|^{-1/4} dz$ . In the coordinates  $(x^i) = (x, \vartheta, \varphi)$  the metric has the representation

$$(15) \quad h = A^2(dx^2 + a^2\tilde{g}) = A^2 dx^2 + \bar{g} \quad ,$$

with  $A(x) = a^{-1}|h(x)|^{1/4}$  defined on  $S^1$ . Comparing this with equation (13) shows  $r = Aa$ . The corresponding Laplacian  $\Delta$  acting on a function  $\psi$  on  $S$  can now be calculated to

$$\begin{aligned} \Delta\psi &= -h^{ij}\nabla_i\nabla_j\psi = -h^{11}\nabla_1\nabla_1\psi - h^{AB}\nabla_A\nabla_B\psi \\ &= -h^{11}\nabla_1\nabla_1\psi + h^{AB}\Gamma_{AB}^1\psi' + \bar{\Delta}\psi \\ &= -A^{-2}(\psi'' + A^{-1}A'\psi') + \bar{\Delta}\psi \quad , \end{aligned}$$

where  $\bar{\Delta}$  denotes the Laplacian of  $\bar{g}$ , and the prime differentiation with respect to  $x$ , a convention, which we adopt for the rest of this work.

Furthermore, the symmetry and the given coordinate representations permits the second fundamental form  $k$  to have the form

$$(16) \quad k = A^2K dx^2 + \frac{1}{2}(\text{tr } \kappa)\bar{g} \quad ,$$

where the coefficients are functions on  $S^1$ . Taking the trace yields the mean curvature  $H = \text{tr } k$  and we get the relation

$$(17) \quad H - K = \text{tr } \kappa \quad .$$

So far we know the intrinsic and extrinsic geometry for a symmetric Cauchy surface  $\Sigma$  in  $M$ . Let us turn now to the 3+1-geometry. Theorem 2.2 states the conditions to guarantee local in time existence for a PMC foliation  $\{S_t\}$  of a neighbourhood of  $\Sigma$ . We need, that  $\lambda$  defined by equation (5) is non-negative and does not vanish identically on  $\Sigma$ . Again  $n$  is the unit normal on  $\Sigma$  and  $T$  the energy-momentum tensor of the spacetime. Therefore we get a sufficient condition by the assumption that  $M$  satisfies the strong energy condition (hence the matter term in (5) is non-negative) and that there exists at least one point in  $\Sigma$ , with  $\lambda > 0$ . The strictness of the inequality is not a restriction, since one always can perform a slight deformation of  $\Sigma$ , such that the second fundamental form is not identically zero, which does the job. The same reasoning works, of course, if  $M$  satisfies the dominant energy condition and the non-negative-pressures condition (then both parts of the matter term in (5) are non-negative). Thus, in surface symmetric spacetimes, some energy conditions are sufficient for the existence of a local in time PMC foliation.

Having constructed a local in time PMC foliation, one can ask, if the leaves of the foliation are symmetric, when  $\Sigma$  has been chosen to be symmetric (Note, that due to symmetry  $\lambda = \lambda(x)$  is a function on  $S^1$  only). As described in [H2] the PMC foliation is given as a limit  $(w, N)$  of functions  $(w^j, N^j)$  on  $\Sigma$  in some Sobolev space. Here  $w^j$  describes a family of spacelike hypersurfaces in  $M$  and  $N^j$  converges towards the lapse function of  $w$ .  $(w^j, N^j)$  are defined as solutions of the sequence of symmetric-hyperbolic elliptic systems

$$\begin{aligned} \partial_t w^j + A^i(w^{j-1}, N^{j-1}) \partial_i w^j + B(w^{j-1}, N^{j-1}, DN^{j-1}) &= 0 \\ \Delta(w^{j-1})N^j + \lambda(w^{j-1}, Dw^{j-1})N^j &= 1 \quad , \end{aligned}$$

with  $w^0$  representing  $\Sigma$  and  $N^0 = 1$ , hence respecting the symmetries. The underlying metric structure of the system is the sequence of first fundamental forms  $h_{j-1}$  of the surfaces  $w^{j-1}$ . If all the  $(w^j, N^j)$  respect the symmetries, then the PMC foliation also, hence it suffices to show, that  $(w^j, N^j)$  respects the symmetries, if  $w^{j-1}, N^{j-1}$  does so.

This is clear for  $w^j$ , because the symmetric hyperbolic equation can be localized, and the pullback to  $\hat{M}$  is invariant under the action of the isometry group.

Suppressing the index  $j - 1$  from some quantities determined by the metric  $h_{j-1}$ , the Laplacian reads

$$\Delta(w^{j-1})N^j = -A^{-2}((N^j)'' + A^{-1}A'(N^j)') + \bar{\Delta}(w^{j-1})N^j \quad .$$

The elliptic equation then becomes

$$-A^{-2}((N^j)'' + A^{-1}A'(N^j)') + \lambda(w^{j-1}, Dw^{j-1})N^j + \bar{\Delta}(w^{j-1})N^j = 1 \quad .$$

If  $\bar{\Omega} = \sqrt{|\bar{g}|} = r^2 \epsilon_\vartheta$  denotes the volume form of the surfaces of symmetry  $F$  in  $M$ , one finds  $\text{Vol}(F) = \int_{[0, \pi] \times [0, 2\pi]} \bar{\Omega} = 4\pi r^2$ , as desired. Setting  $L := -A^{-2}(\frac{d^2}{dx^2} + A^{-1}A' \frac{d}{dx})$  and  $\tilde{N}^j(x) := \frac{1}{\text{Vol}(F)} \int_{[0, \pi] \times [0, 2\pi]} N^j \bar{\Omega} = \frac{1}{4\pi} \int_{[0, \pi] \times [0, 2\pi]} N^j \epsilon_\vartheta$ , then integration of the elliptic equation,  $\int_{[0, \pi] \times [0, 2\pi]} \epsilon_\vartheta (L + \lambda + \bar{\Delta})N^j = \int_{[0, \pi] \times [0, 2\pi]} \epsilon_\vartheta = 4\pi$ , yields

$$(L + \lambda + \bar{\Delta})\tilde{N}^j = 1 \quad ,$$

since on the one hand  $\int_{[0, \pi] \times [0, 2\pi]} (L + \lambda)N^j \epsilon_\vartheta = 4\pi(L + \lambda)\tilde{N}^j$  and on the other hand  $\int_{[0, \pi] \times [0, 2\pi]} (\bar{\Delta}N^j) \epsilon_\vartheta = r^{-2} \int_{[0, \pi] \times [0, 2\pi]} (\bar{\Delta}N^j) \bar{\Omega} = r^{-2} \int_F \bar{\Delta}N^j$  vanishes as well as  $\bar{\Delta}\tilde{N}^j$ .

Thus  $\tilde{N}^j$  is a solution of the elliptic equation. Uniqueness then gives us  $N^j = \tilde{N}^j$ , thus  $N^j = N^j(x)$ , which expresses the symmetry of  $N^j$ , as desired.

Thus we have shown, given a symmetric Cauchy surface  $\Sigma$  in  $M$  admitting a local in time PMC foliation, that all leaves of the foliation are symmetric, too and coordinatizing them in the way described above yields:

### 3.1. Proposition.

*Let  $(M, g)$  be a surface symmetric spacetime obeying the strong energy condition. Then there are coordinates  $(x^\mu) = (t, x, \vartheta, \varphi)$  adapted to a local in time PMC foliation  $\{S_t\}$  of a neighbourhood  $U = ]t_1, t_2[ \times \Sigma$  of  $\Sigma = S_0$  in  $M$ , which cast the metric into the form*

$$(18) \quad g = -N^2 dt^2 + A^2 ((dx + \nu dt)^2 + a^2 \tilde{g}) \quad .$$

*All coefficients except  $a$  are functions on  $]t_1, t_2[ \times S^1$ , whereas  $a$  depends only on the time function  $t$ .  $A$  and  $a$  are everywhere positive,  $N$  denotes the lapse function of the foliation and  $\nu$  the non-vanishing component of the shift vector, uniquely fixed by the condition  $\nu(t, 0) = 0$ .*

Supplementary to the notation already introduced, let an overdot denote differentiation with respect to  $t$ , while keeping the prime as a marker for differentiation with respect to the coordinate  $x$ .

Finally we find for the orthonormal frame  $\{{}^4e_\mu\}$  and its dual  $\{{}^4\sigma^\mu\}$  canonically induced by the 3 + 1-split:

$$\begin{aligned} {}^4e_0 &= n = N^{-1}(\partial_0 - \nu\partial_1) & {}^4\sigma^0 &= Ndt \\ {}^4e_1 &= m = A^{-1}\partial_1 & {}^4\sigma^1 &= A(\nu dt + dx^1) \\ {}^4e_2 &= (Aa)^{-1}\partial_2 & {}^4\sigma^2 &= Aa dx^2 \\ {}^4e_3 &= (Aa\epsilon_\vartheta)^{-1}\partial_3 & {}^4\sigma^3 &= Aa\epsilon_\vartheta dx^3 \end{aligned} \quad .$$

### 3.2 The field equations

Given the PMC foliation, we can write down the field equations in the 3+1-representation. The symmetries suggest to represent the matter quantities completely with respect to an orthonormal frame, thus we define  $j := -T(n, m) = A^{-1}j_1$  and  $S_{ij} = T({}^4e_i, {}^4e_j)$ .

Then one calculates the constraint equations (1) to be

$$\begin{aligned} (19) \quad (A^{1/2})'' &= \frac{1}{8}A^{5/2}\left(H^2 - \frac{1}{2}(H - K)^2 - K^2 - 16\pi\rho\right) + \frac{1}{4}A^{1/2}a^{-2}\epsilon \\ (20) \quad K' &= -3A^{-1}A'K + A^{-1}A'H + H' + 8\pi Aj \end{aligned}$$

The foliation is fixed by the lapse equation (3) and the PMC condition (4)

$$\begin{aligned} (21) \quad N'' &= -A^{-1}A'N' + A^2N\left(\frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S)\right) - A^2 \\ (22) \quad \dot{H} &= 1 + \nu H' \end{aligned}$$

and the evolution equations (2) read

$$\begin{aligned} (23) \quad \dot{A} &= -NAK + A\nu' + A'\nu \\ (24) \quad \dot{a} &= -\frac{1}{2}Na(H - 3K) - a\nu' \\ (25) \quad \dot{K} &= \nu K' - A^{-2}(N'' - A^{-1}A'N') \\ &\quad + N\left(-4A^{-5/2}(A^{1/2})'' + (A^{-2}A')^2 + HK - 8\pi(A^{-2}S_{11} + \frac{1}{2}(\rho - \text{tr } S))\right) \end{aligned}$$

Integrating the equation for  $a$  over the circle yields  $\dot{a} = \mu a$ , with  $\mu = -\frac{1}{2}\int_{S^1} N(H - 3K)$ , since  $\int_{S^1} \nu'$  vanishes. Inserting this back gives an equation for shift:

$$(26) \quad \nu' = -\frac{1}{2}N(H - 3K) + \frac{1}{2}\int_{S^1} N(H - 3K)$$

Differentiation of the equation for  $H$  with respect to  $x$  yields an equation for  $H'$ :

$$(27) \quad (\partial_t - \nu\partial_x)H' = \nu'H'$$

In summary we have equations for space and time derivatives of the fundamental forms  $h$  and  $k$ . Moreover there are equations for the spacelike derivatives of lapse and shift, but unfortunately there is no information about their time derivatives.

So far we started with a given surface symmetric spacetime, admitting a local in time PMC foliation, to which we adapted the field equations. This raises the opposite question: Given the equations (19)-(27) and appropriate data, does there exist a solution, and how unique is it? To answer this question we first need to state more precisely the term 'appropriate data':

### 3.2. Definition.

A *symmetric initial data set* is a smooth collection  $(\Sigma, h, k)$  consisting of a 3-manifold  $\Sigma$  diffeomorphic to  $S^1 \times F$  with metric  $h$  and a symmetric tensor field  $k$  on  $\Sigma$ , where  $\Sigma$  admits coordinates, such that  $h$  and  $k$  could be written in the form shown in section 3.1.

If there are matter fields present, then it is assumed, that there is also smooth symmetric matter data and equations, leading to a well posed Cauchy problem of the reduced field equations in harmonic coordinates.

The smoothness of the quantities appearing in the definition is required, since the transformation to harmonic coordinates involves derivatives.

### 3.3. Proposition.

Let  $(\Sigma, h, k)$  be a symmetric initial data set, with matter obeying the strong energy condition and (compare equation (5) for definition)  $\lambda > 0$  somewhere on  $\Sigma$ . Further, let  $t_0$  denote an arbitrary real number.

Then there exists a  $\delta > 0$  and a PMC foliated surface symmetric spacetime  $(\bar{M}, \bar{g})$  diffeomorphic to  $]t_0 - \delta, t_0 + \delta[ \times \Sigma$  with an embedding  $\iota : \Sigma \rightarrow \bar{M}$ , satisfying  $\iota(\Sigma) = S_{t_0}$  and  $\iota_* h, \iota_* k$  are the first and second fundamental form of  $S_{t_0}$  in  $(\bar{M}, \bar{g})$ .  $(\bar{M}, \bar{g})$  obeys the strong energy condition and  $\bar{g}$  can be written in the form described in proposition 3.1. This construction is unique up to the choice of  $t_0$  and  $\delta$ .

*Proof.*

On the induced manifold  $\hat{\Sigma}$  diffeomorphic to  $S^1 \times \hat{F}$  the induced data is invariant under the group action, hence the Cauchy developments also and we get a surface symmetric Cauchy development of the data, admitting a symmetric local in time PMC foliation near  $\Sigma$  on some time interval  $]t_0 - \delta, t_0 + \delta[$  and allowing a set of coordinates stated in proposition 3.1. The uniqueness property stated above follows from the geometric uniqueness of solutions of Einstein's equations associated with the uniqueness of the PMC foliation, once the remaining degree of freedom has been fixed by the requirement  $\iota(\Sigma) = S_{t_0}$ . Note, that lapse is fixed by the ellipticity of the respective PMC equation and shift is fixed by equation (26) and the condition  $\nu(t, 0) = 0$ . ■

In surface symmetric spacetimes, there is another way to express the constraint equations (19) and (20) in terms of 'optical scalars', see [GM] for an enlightening presentation. The geodesic null congruences determined by  $k_{\pm} = m \pm n$  give rise to the null expansions

$$(28) \quad \vartheta_{\pm} := -(\text{tr } \lambda \pm \text{tr } \kappa) = \frac{2}{r}(m(r) \pm n(r)) = \frac{2}{r}k_{\pm}(r) = 2A^{-2}A' \mp (H - K) \quad ,$$

where (14), with (17) has been used, together with the relation  $r = Aa$  for the area radius defined in (12). The formula illustrates the definition of  $r$  as a volume measure, whose variation along  $k_{\pm}$  is described by the (negative of the) trace of the second fundamental form associated to this direction.

Now we can write the constraint equations for an arbitrary symmetric Cauchy surface  $S$  symmetrically as

$$\begin{aligned} 8\pi(\rho + j) &= -m(\vartheta_-) - \frac{3}{4}\vartheta_-^2 + \vartheta_-H + r^{-2}\epsilon \\ 8\pi(\rho - j) &= -m(\vartheta_+) - \frac{3}{4}\vartheta_+^2 - \vartheta_+H + r^{-2}\epsilon \end{aligned}$$

and taking  $\omega_{\pm} := r\vartheta_{\pm}$  as the fundamental variable we get (compare [GM])

$$(29) \quad m(\omega_{\pm}) = -8\pi r(\rho \mp j) \mp \omega_{\pm}H + \frac{1}{4r}(\omega_+\omega_- - 2\omega_{\pm}^2 + 4\epsilon) \quad .$$

Furthermore, the area radius serves as a warping function in the warped product  $M = B \times_r F$  of the two dimensional spacetime  $(B, \underline{g})$  with  $(F, \bar{g})$ , where  $B = \mathbb{R} \times S^1$  is the quotient  $M/G$  and  $\underline{g} = g|_B$ . We adopt the convention that lower case Latin letters from the beginning of the alphabet range from 0 to 1, and objects intrinsic or orthogonally projected to  $(B, \underline{g})$  will be marked by an underbar. Einsteins equations in this framework can be considered as equations in  $(B, \underline{g})$  for the field  $r : B \rightarrow \mathbb{R}$ :

$$(30) \quad \underline{\nabla}_a \underline{\nabla}_b r = \frac{M}{r^2} \underline{g}_{ab} - 4\pi r (\underline{T}_{ab} - \text{tr} \underline{T} \underline{g}_{ab}) \quad ,$$

where  $\underline{T}$  denotes the projected energy-momentum tensor into the spacetime  $(B, \underline{g})$  and the mass function  $M$  is defined as

$$(31) \quad M := \frac{1}{2}r(\epsilon - \underline{\nabla}^a r \underline{\nabla}_a r) = \frac{1}{2}r(\epsilon - {}^4\nabla^\alpha r {}^4\nabla_\alpha r) = \frac{1}{2}r(\epsilon - \frac{1}{4}r\vartheta_+ r\vartheta_-) \quad ,$$

since  $\vartheta_+\vartheta_- = 4/r^2 {}^4\nabla^\alpha r {}^4\nabla_\alpha r$ .  $M$  turns out to be the Hawking mass  $m_H(F)$  (up to a factor  $-\frac{1}{2}\chi$  for  $\text{genus}(F) \geq 3$ ):

$$(32) \quad m_H(F) := \frac{\text{Vol}(F)^{1/2}}{(4\pi)^{3/2}} \left( \pi\chi(F) - 1/8 \int_F \vartheta_+\vartheta_- \right)$$

and differentiation of (31) yields the mass flux equation

$$(33) \quad \underline{\nabla}_a M = 4\pi r^2 (\underline{T}_{ab} - \text{tr} \underline{T} \underline{g}_{ab}) \underline{\nabla}^b r \quad .$$

### 3.2.1 Expanding and recollapsing models

Let us first consider the definition of mass in (31). In the spherically symmetric case we see, that  $\text{grad} r$  is spacelike as long as  $2M < r$  holds, a condition we are familiar with in connection with the Schwarzschild spacetime.

In the plane and hyperbolic case the picture is quite different, since  $\text{grad} r$  turns out to be timelike, as long as  $2M/r > \epsilon$ . In fact, the lemmas 2.3, 2.4 and 2.5 in [R4] prove for spacetimes with hyperbolic symmetry and in non-flat plane symmetric spacetimes  $\text{grad} r$  is timelike, provided the dominant energy condition is fulfilled. Thus we cannot think about  $r$  as some radial, spacelike coordinate any longer and this fact will play a central role in our further analysis:

We can choose the time orientation in those spacetimes to find  $\text{grad} r$  past pointing, *throughout the whole spacetime*. Then we define the time orientation on the cotangent bundle by metric transport from the tangent bundle, such that  $dr$  turns out to be future pointing. Therefore  $r$  increases with time, which means, that the area of the surfaces of symmetry increases with time, and the spacetime expands in this sense. Thus we will

call these spacetimes expanding. We get an equivalent characterisation by the relation between  $\text{grad } r$  and the null expansions:  $\vartheta_+$  and  $\vartheta_-$  have fixed and opposite signs (in particular  $\vartheta_+ > 0$ ,  $\vartheta_- < 0$ ), since  $\vartheta_{\pm} = (2/r)k_{\pm}(r) = (2/r)dr(k_{\pm})$  is the contraction of a timelike with a null vector.

Therefore we can decide, given a symmetric initial data set  $(\Sigma, h, k)$  with non-flat plane or hyperbolic symmetry, with matter obeying the dominant energy condition, which direction is expanding or contracting. This is important, since we expect some singularity towards the contracting direction and therefore we will pay attention to the past development  $D^-(\Sigma)$  which represents the contracting direction with respect to our conventions.

Note, that we obtained this information without referring to the mean curvature  $H = \text{tr } k$ . But it turns out, that the mean curvature of an arbitrary Cauchy surface  $S$  in expanding spacetimes is also restricted in some way:

First remember, that  $H = \text{tr } k = -\text{div } n$  measures the convergence of the geodesic congruence, future pointing and orthogonal to  $S$ . Thus,  $H < 0$  everywhere on  $S$  corresponds to the notion of expansion and Hawking's theorem proves the existence of a singularity in the past. In an expanding spacetime in our sense,  $H$  is not necessarily confined to be negative everywhere. But we will see, that it is impossible for  $H$  to become non-negative everywhere on  $S$ : The explicit formula (28) for  $\vartheta_{\pm}$  shows in connection with our sign conventions, that  $H < K$ . If  $H$  were non-negative, then  $|H| < |K|$ , thus  $H^2 - K^2 < 0$  and integration of the Hamiltonian constraint (19) over  $S^1$  yields a contradiction.

With the dominant energy condition we can show more. Writing the Hamiltonian constraint as  $R + H^2 - |k|^2 = 16\pi\rho$  we get  $R + H^2 \geq 0$ . Assuming  $H \equiv 0$  on  $S$ , we would get  $R \geq 0$ . But symmetric surfaces  $S$  in spacetimes with plane and hyperbolic symmetry obey the topological conditions (i), respectively (ii), of theorem 5.2. in [SY], which imply, that  $S$  cannot have positive scalar curvature and must be flat in case of non-negative scalar curvature. Thus  $S$  must be flat and the scalar curvature vanishes. This in turn forces  $\rho = 0$  and  $k = 0$  by the Hamiltonian constraint. In the plane symmetric case then the spacetime is flat by  $k = 0$ ,  $\rho = 0$  and the dominant energy condition, contradiction. In the hyperbolic symmetric case integration of equation (19) yields a contradiction, too.

Putting all this together we conclude without loss of generality, that surface symmetric spacetimes, obeying the dominant energy condition, which are plane symmetric and not flat or have hyperbolic symmetry are everywhere expanding, with  $dr$  timelike future pointing,  $\vartheta_+ > 0$ ,  $\vartheta_- < 0$  and any symmetric Cauchy surface  $S$  is not maximal with mean curvature not everywhere positive on  $S$ .

Of course, these arguments do not work in the spherically symmetric case, where  $\text{grad } r$  is spacelike in  $\{2M < r\}$  and no fixed sign of the expansions  $\vartheta_{\pm}$  can be expected, fitting into the the general belief in the closed universe recollapse conjecture, which precludes expansion of the whole spacetime. In particular we expect the existence of a maximal hypersurface in  $M$ .

### 3.3 A priori estimates for the field equations

Our aim now is to get sufficient estimates, that allow the construction of a global PMC foliation. The 'size' of the foliation is measured by the mean curvature  $H$ , thus *we are looking for uniform estimates of the geometric and matter quantities in terms of  $H$ .*

So let  $(M, g)$  be a surface symmetric spacetime, satisfying the dominant and strong energy condition. Let us assume, that there is a Cauchy surface  $\Sigma$  in  $M$  with mean curvature  $H < 0$ . In particular we get a local in time PMC foliation  $\{S_t\}$ ,  $t \in ]t_1, t_2[$  with  $\Sigma = S_0$  by

theorem 2.2. If the spacetime possesses plane or hyperbolic symmetry, we choose the time orientation in correspondence to the conventions introduced in 3.2.1, thus  $H$  decreases with decreasing PMC time. If  $(M, g)$  is spherically symmetric we choose the time orientation, that  $H$  decreases with PMC time, too. Then in either case we expect to find a singularity at least in the past  $D^-(\Sigma)$  of  $\Sigma$ .

In  $D^-(\Sigma)$  the mean curvature is bounded from above by  $H \leq \bar{H} < 0$  and  $H = \bar{H}$  only on  $\Sigma$ . Thus  $|H|$  is bounded from below and we find the following estimates for the field equations in  $D^-(\Sigma)$  as long as  $H$  remains finite:

- At first we consider the constraint equation (29) on a fixed leaf. At the critical points of  $\omega_{\pm}$  we find together with the dominant energy condition the important inequality

$$(34) \quad |r\vartheta_{\pm}| \leq 2 \left( |Hr| + \sqrt{(Hr)^2 + \epsilon} \right)$$

as shown in [R4].

For plane and hyperbolic symmetry this inequality (34) can be strengthened to

$$|\vartheta_{\pm}| \leq 4|H| \leq C \quad \implies \quad |A^{-2}A'| \leq C \quad \text{and} \quad |K| \leq C$$

by the definition (28) of the null expansions.

In the spherically symmetric case the argument is more complicated: First, the work [B] of Burnett shows, that under the additional assumption of the non-negative-pressure condition we have  $r \leq C$  and  $0 < C \leq M$ . Using the upper bound for  $r$  on the right-hand side of (34) we get with (31)

$$|r\vartheta_{\pm}| \leq C \quad \implies \quad \frac{M}{r} \leq C$$

Thus we get  $r^{-1} \leq CM^{-1} \leq C$ , hence  $r$  is bounded above and below away from zero. Inserting this in the estimate (34) leads to  $|\vartheta_{\pm}| \leq Cr^{-1} \leq C$  and we are in the same situation as in the plane/hyperbolic case, hence  $|A^{-2}A'| \leq C$  and  $|K| \leq C$ .

- Now consider the lapse equation (21) on a fixed leaf. At the point, where  $N$  attains its maximum  $\bar{N}$ , we have

$$\bar{N} \leq \left( \frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right)^{-1} \leq C/H^2 \leq C \quad ,$$

due to the strong energy condition. Hence we have a bound for  $|N|$ .

- Next, there is a bound for shift. Examination of formula (26) shows, that all quantities on the right hand side are bounded, so we get  $|\nu'| \leq C$ , and therefore, using  $\nu(t, 0) = 0$ :  $|\nu(t, x)| \leq |\nu(t, 0)| + \int_{S^1} |\nu'| \leq C$ .
- By the way, the equation (27) for  $H'$  provides a bound for  $|H'|$ , since the coefficient on the right-hand side is bounded and applying Gronwall gives the desired estimate.
- With the information about  $\nu'$  examination of (24),  $\dot{a} = a \left( -\frac{1}{2}N(H - 3K) - \nu' \right)$ , shows, that the factor in brackets is already bounded and we get an inequality of the form  $|\partial_t \ln|a|| \leq C$ , which leads to a bound for  $|a|$  and  $|a^{-1}|$ .
- The same line of argumentation works for  $A$ . Equation (23) can be written as  $\dot{A} - \nu A' = A(-NK + \nu')$ , with the factor on the right-hand side bounded. So we get bounds for  $|A|$  and  $|A^{-1}|$ , too.  
Since we have already bounded the null expansions  $\vartheta_{\pm} = 2A^{-2}A' \pm (H - K)$ , one sees easily, that even  $|A'| \leq C$ .

- Integration of equation (19) over the circle yields an inequality

$$\begin{aligned} \frac{1}{8} \int_{S^1} A^{5/2} 16\pi\rho &= \frac{1}{8} \int_{S^1} A^{5/2} \left( H^2 - K^2 - \frac{1}{2}(H - K)^2 \right) + \frac{1}{4} \int_{S^1} A^{1/2} a^{-2} \epsilon \\ &\leq \frac{1}{8} \int_{S^1} A^{5/2} H^2 + \frac{1}{4} \int_{S^1} A^{1/2} a^{-2} \quad . \end{aligned}$$

From this one concludes the boundedness of  $\int_{S^1} \rho$ , and by the dominant energy condition the boundedness of  $\int_{S^1} |j|$  and  $\int_{S^1} |S|$ . Integrating now equation (21) starting at a point, where  $N' = 0$ , we get

$$\begin{aligned} N' &= - \int A^{-1} A' N' + \int A^2 N \left( \frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right) - \int A^2 \\ |N'| &\leq C + \int |A^{-1} A' N'| \quad . \end{aligned}$$

The bound for  $|N'|$  then follows from Gronwall's inequality.

- Furthermore, the bounds for  $A, A^{-1}$  together with the basic estimate for  $\vartheta_{\pm}$  and the boundedness of  $\int_{S^1} \rho$  are enough to apply the proof of the lemma in [R5], which ends up with  $|N^{-1}| \leq C$ .

Collecting all these estimates we get the

### 3.4. Proposition.

*Let  $(M, g)$  be a surface symmetric spacetime, obeying the dominant and strong energy condition and in the spherically symmetric case the non-negative-pressure condition, too. Assume the existence of a symmetric Cauchy surface  $\Sigma$  with strictly negative mean curvature. In particular we get from proposition 3.1 a PMC time coordinate  $t$ , ranging in  $]t_1, t_2[$  with  $\Sigma = \{t = 0\}$  and  $H$  decreases with decreasing  $t$ .*

*Then we have uniformly on  $]t_1, 0]$*

$$|A|, |A^{-1}|, |A'|, |a|, |a^{-1}|, |H|, |H'|, |K|, |N|, |N^{-1}|, |N'|, |\nu|, |\nu'| \leq C \quad .$$

To put this result into some framework, we establish some formalism, to have useful abbreviations at hand as well as to make clear the dependence between estimates of geometric quantities and matter variables.

### 3.5. Definition.

$$\mathcal{F} := (A, a, N, \nu, H, K)$$

collects the quantities describing the geometry of the foliation and

$$\Phi := (\rho, j, S)$$

abbreviates the matter quantities.

We have already estimated the quantities  $\mathcal{F}$ ,  $A^{-1}, a^{-1}, N^{-1}$ , as well as  $A', N', \nu', H'$  and  $\dot{A}, \dot{a}, \dot{H}$  (by inspection of the field equations). The idea is now, to bound all quantities  $\mathcal{F}, \Phi$  together with all of their derivatives uniformly on the time interval  $]t_1, 0]$ . Then there exists a smooth extension to the closure of the interval, which serves as a new symmetric initial data set for the field equations in the sense described in definition 3.2. Note, that the

bounds for  $A^{-1}, a^{-1}, N, N^{-1}$  ensure, that the geometry remains regular at the boundary of the time interval and the  $C^\infty$ -bounds for lapse and shift turn out to be necessary to obtain  $C^\infty$ -bounds for the fundamental forms.

Proposition 3.3 then sets us in the position to extend the foliation at least in the past direction, where  $H < 0$  holds: Construct the solution stated in proposition 3.3 and embed  $M$  into the maximal Cauchy development of the data.

To carry out this program, we need some knowledge about the matter quantities. First, the matter has to obey the dominant and strong energy conditions. Second, assume that the regularity of the geometry guarantees the regularity of the matter in a certain way: For all non-negative integers  $m$  and  $n$  we have

$$(35) \quad \begin{aligned} |\partial_t^m \partial_x^n \mathcal{F}| \leq C &\implies |\partial_t^m \partial_x^n \Phi| \leq C \\ |\partial_t^m \partial_x^{n+1} \mathcal{F}| \leq C &\implies |\partial_t^{m+1} \partial_x^n \Phi| \leq C \quad , \end{aligned}$$

then the following lemmas hold.

### 3.6. Lemma.

Assume, that the matter fulfills (35). Then for arbitrary non-negative integers  $m, n$

$$\begin{aligned} \forall_{k < m} \forall_l |\partial_t^k \partial_x^l \mathcal{F}| \leq C &\implies |\partial_t^m \partial_x^{n+1} \mathcal{F}| \leq C \\ \forall_{l \leq n} |\partial_t^m \partial_x^l \mathcal{F}| \leq C & \end{aligned}$$

holds.

### 3.7. Lemma.

Assume, that the matter fulfills (35). Then for arbitrary non-negative integers  $m, n$

$$\begin{aligned} \forall_{k < m} \forall_l |\partial_t^k \partial_x^l \mathcal{F}| \leq C &\implies |\partial_t^{m+1} \partial_x^n \mathcal{F}| \leq C \\ \forall_{l \leq n+1} |\partial_t^m \partial_x^l \mathcal{F}| \leq C & \end{aligned}$$

holds.

Together with proposition 3.4 the lemmas accomplish the task of bounding all necessary derivatives of the geometric quantities, by first bounding all spatial derivatives  $\partial_x^n \mathcal{F}$  (proposition 3.4 and lemma 3.6) and then successively all derivatives  $\partial_t^k \partial_x^l \mathcal{F}$ , with  $k+l = n$  by alternative applications of the lemmas. Therefore, in view of proposition 3.4 the validity of property (35) will be enough to extend the foliation. We will prove (35) for some matter models in the next section. But first, of course, we have to prove the lemmas.

*Proof of lemma 3.6.*

A: Applying  $\partial_t^m \partial_x^n$  on equation (19) and integrating along  $S^1$  yields a bound for the difference  $|(\partial_t^m \partial_x^n A^{1/2})'(y) - (\partial_t^m \partial_x^n A^{1/2})'(x)|$  and using  $\int_{S^1} (\partial_t^m \partial_x^n A^{1/2})' = 0$  (hence is bounded), gives a bound for  $\partial_t^m \partial_x^{n+1} A^{1/2}$ , hence for  $\partial_t^m \partial_x^{n+1} A$ .

a: trivial

$\nu$ : Apply  $\partial_t^m \partial_x^n$  on (26), then the right-hand side of (26) is bounded by assumption, from which the claim follows immediately.

- N: Again applying  $\partial_t^m \partial_x^n$  on the lapse equation (21) yields an equation of the form (21) for  $\partial_t^m \partial_x^n N''$  plus some already bounded term (by assumption and the already proven bound for  $\partial_t^m \partial_x^n A'$ ). The boundedness of  $\partial_t^m \partial_x^n N'$  follows then from Gronwall's estimate after integrating the equation for  $\partial_t^m \partial_x^n N''$  along  $S^1$  starting at a point with  $\partial_t^m \partial_x^n N' = 0$ .
- H: Differentiation of equation (27) gives an expression of the form  $(\partial_t - \nu \partial_x) \partial_t^m \partial_x^n H' = B_1 \partial_t^m \partial_x^n H' + B_2$ , with  $B_1, B_2$  bounded, where the assumptions and the already obtained bound for  $\partial_t^m \partial_x^{n+1} \nu$  have been used. Thus applying Gronwall's inequality yields a bound for  $\partial_t^k \partial_x^{n+1} H$ .
- K: Apply  $\partial_t^m \partial_x^n$  on equation (20), then the right-hand side is bounded by the previous estimates, hence bounding  $\partial_t^m \partial_x^{n+1} K$ . ■

*Proof of lemma 3.7.*

- A: Applying  $\partial_t^m \partial_x^n$  on (23) yields immediately a bound for  $\partial_t^m \partial_x^n \dot{A}$ , since the right-hand side is already bounded.
- a: The same argument with  $\partial_t^m$  instead of  $\partial_t^m \partial_x^n$  applied to equation (24) works in this case to bound  $\partial_t^m \partial_x^n \dot{a}$ .
- H: And again,  $\partial_t^m \partial_x^n$  on equation (22), yields an estimate for  $\partial_t^m \partial_x^n \dot{H}$ .
- K: Apply  $\partial_t^m \partial_x^n$  on equation (25), then the only terms on the right-hand side, not already known to be bounded are  $\partial_t^m \partial_x^n N''$  and  $\partial_t^m \partial_x^n A''$ . But inserting equations (21), respectively (19) for  $N''$  and  $A''$ , one sees easily, that  $\partial_t^m \partial_x^n N''$  and  $\partial_t^m \partial_x^n A''$  indeed are bounded by the right-hand sides of their equations, hence  $\partial_t^{m+1} \partial_x^n K$  is bounded.
- N: Unfortunately there is no explicit equation for  $\dot{N}$ . So we have to apply the derivative operator  $\partial_t^{m+1} \partial_x^n$  to equation (21), therefore producing already estimated terms involving  $\partial_t^{m+1} \partial_x^n$  applied on the quantities treated above, as well as on  $\Phi$  (bounded by property (35)), but also a term involving  $\partial_t^{m+1} \partial_x^{n+1} A$  on the right-hand side. If this term turns out to be bounded, then the equation for  $\partial_t^{m+1} \partial_x^n N''$  has the form

$$\begin{aligned} \partial_t^{m+1} \partial_x^n N'' &= B - A^{-1} A' \partial_t^{m+1} \partial_x^n N' \\ &\quad + A^2 \partial_t^{m+1} \partial_x^n N \left( \frac{1}{2} (H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right) - A^2 \end{aligned}$$

with  $|B|$  bounded. On  $S^1$ ,  $\partial_t^{m+1} \partial_x^n N$  attains its maximum, from which we can infer

$$\partial_t^{m+1} \partial_x^n N \leq \left\{ (1 - B/A^2) \left( \frac{1}{2} (H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right)^{-1} \right\}_{\max}$$

hence it is bounded from above. Similarly for the minimum:

$$\partial_t^{m+1} \partial_x^n N \geq \left\{ (1 - B/A^2) \left( \frac{1}{2} (H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right)^{-1} \right\}_{\min}$$

which is bounded from below and we get  $|\partial_t^{m+1} \partial_x^n N| \leq C$ .

It remains to show the boundedness of  $\partial_t^{m+1} \partial_x^{n+1} A = \partial_t^m \partial_x^{n+1} \dot{A}$ . Inserting equation (23) for  $\dot{A}$ , we see on the right-hand side beside a bounded term the quantities  $\partial_t^m \partial_x^n A''$  and  $\partial_t^m \partial_x^{n+1} \nu'$ . Inserting equation (19) for  $A''$  and equation (26) for  $\nu'$ , we finally obtain bounds for  $\partial_t^m \partial_x^n A''$  and  $\partial_t^m \partial_x^{n+1} \nu'$ , hence for  $\partial_t^m \partial_x^{n+1} \dot{A}$ , as desired.

$\nu$ : The final estimate is straightforward: Apply the operator  $\partial_t^{m+1}\partial_x^{n-1}$  to equation (26), which yields immediately a bound for  $\partial_t^{m+1}\partial_x^n\nu$  since the right-hand side of the equation is already bounded by the arguments above. ■

### 3.4 Higher order estimates

Here we prove the matter regularity condition (35) for Einstein-Vlasov, Einstein-Maxwell and the Einstein-Vlasov-Maxwell system, achieving the goal of this work.

#### 3.4.1 Collisionless matter

The 3+1-split of the Vlasov equation (9) with respect to the symmetries reads

$$(36) \quad \begin{aligned} 0 &= \partial_t f + \left( NA^{-1} \frac{v^1}{v^0} - \nu \right) \partial_x f \\ &+ \left( -A^{-1} N' v^0 + NK v^1 + NA^{-2} A' \frac{(v^2)^2 + (v^3)^3}{v^0} \right) \frac{\partial}{\partial v^1} f \\ &- N \left( A^{-2} A' \frac{v^1}{v^0} - \frac{1}{2}(H - K) \right) v^B \frac{\partial}{\partial v^B} f \quad , \end{aligned}$$

where  $v^0 := \sqrt{1 + \delta_{ij} v^i v^j}$  and  $0 = \partial_{x^2} f = \partial_{x^3} f = v^3 \frac{\partial}{\partial v^2} f - v^2 \frac{\partial}{\partial v^3} f$  (by symmetry) has been used.

The energy momentum tensor (10) associated with the Vlasov equation leads to the matter quantities

$$\begin{aligned} \rho &= \int f v^0 dv \\ j &= \int f v^1 dv \\ S_{ab} &= \int f v_a v_b / v^0 dv \quad . \end{aligned}$$

Now we investigate the matter regularity property (35). Since  $f$  is constant along the characteristics of the Vlasov equation, given an initial particle distribution  $f_0$  on the mass shell over  $\Sigma$ , we get the matter distribution for each time  $t$  as a function  $f(t, y, w)$  on the mass shell over the leaf  $S_t$ , by  $f(t, y, w) = f_0(Y(t, y, w), V(t, y, w))$ , where  $(Y, V)$  denote the characteristic curve of the Vlasov equation through the point  $(0, y, w) \in P_{(0, y)}$ ,  $(0, y) \in \Sigma$ .

Thus the matter quantities  $\rho, j, S$  are bounded, provided the support of  $f$  remains bounded. Define now

$$\bar{P}_f(t) := \{ \sup |v| \mid v \in \text{supp } f(t, y, w) \forall (0, y, w) \in P_{(0, y)} \} \quad ,$$

then the matter quantities are bounded by  $C(1 + \bar{P}_f(t))^4$ , hence it is sufficient, to control  $\bar{P}_f(t)$  on  $]t_1, 0]$ .

Since the characteristic curves  $(Y, V)$  are integral curves of  $X$  and all coefficients in the components of  $X$  are already bounded by proposition 3.4, the characteristics itself are bounded,  $\bar{P}_f(t) \leq C$  as desired.

Now we need to iterate this procedure. Assume, that all  $\partial_t^k \partial_x^l \mathcal{F}$  and  $\partial_t^k \partial_x^l \Phi$  for  $k+l = m+n$  are bounded. To bound the derivatives of the matter quantities of order  $m+n+1$ ,

differentiate the Vlasov equation  $m + n + 1$  times (with respect to  $t$  and  $x$  only). This yields linear equations for  $\partial_t^{\bar{k}} \partial_x^{\bar{l}} f$ , for all non-negative integers  $\bar{k} + \bar{l} = m + n + 1$ , of the form

$$X(\partial_t^{\bar{k}} \partial_x^{\bar{l}} f) = B \quad ,$$

where  $B$  vanishes outside the support of  $f$ , which bounds the support of  $B$ . Moreover,  $B$  involves derivatives of order  $m + n + 1$  of the quantities  $\mathcal{F}$  and  $A'$ ,  $N'$ . For derivatives of order at most  $m$  in  $t$ , we see the boundedness of  $|B|$ , by applying similar arguments as used in the proof of lemma 3.6. Thus in this case  $\partial_t^{\bar{k}} \partial_x^{\bar{l}} f$  is bounded and  $\bar{P}_{\partial_t^{\bar{k}} \partial_x^{\bar{l}} f}$ , since they have the same characteristics, hence the derivatives of the matter quantities of order  $m + n + 1$ , involving derivatives of order at most  $m$  in  $t$  are bounded. Moreover, due to the simple dependence of the characteristics on the frame variables  $v$ , all derivatives of  $f$  with respect to  $v$  are bounded, too.

In view of this fact, we are able to bound all derivatives of order  $m + n + 1$  of the matter quantities, by considering the Vlasov equation as an equation for  $\partial_t^{m+1} \partial_x^{n+1} f$ , for which the right-hand side is known to be bounded.

Therefore the first part of property (35) holds. For the second part we must show, that the spacelike derivatives of  $\mathcal{F}$  can be redistributed to timelike derivatives of  $\Phi$ . But this has already been done in the proof of the first part of (35), and we get the

### 3.8. Theorem.

*Let  $(M, g, f)$  be a surface symmetric solution of the Einstein-Vlasov system, which possesses a symmetric Cauchy surface  $\Sigma$  with strictly negative mean curvature  $H \leq \bar{H} < 0$  and  $H = \bar{H}$  somewhere on  $\Sigma$ .*

*Then all of the past of  $\Sigma$  admits a PMC foliation  $\{S_t\}$ , where  $t$  takes all values in the interval  $]-\infty, 0]$  and  $H$  takes all values in  $]-\infty, \bar{H}]$ .*

#### 3.4.2 Maxwell field

Let  $F$  denote the electromagnetic field. The symmetry simplifies  $F$ : Due to the symmetries  $F$  can be written relative to the orthonormal coframe  $\{^4\sigma^\mu\}$  introduced in section 3.1 as

$$F = -\hat{e}(t, x) \ ^4\sigma^0 \wedge \ ^4\sigma^1 + \hat{b}(t, x) \ ^4\sigma^2 \wedge \ ^4\sigma^3 \quad ,$$

since all other components are forced to vanish and  $F$  and  $^4\sigma^0 \wedge \ ^4\sigma^1$ ,  $^4\sigma^2 \wedge \ ^4\sigma^3$  remain invariant under the action of the symmetry group.

To obtain an explicit form of the Maxwell equations  $F$  and  $*F$  are represented in the given coordinates by

$$F_{\alpha\beta} \sim \begin{pmatrix} 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix} \quad \begin{aligned} e(t, x) &= NA \hat{e} \\ b(t, x, \vartheta) &= (Aa)^2 \epsilon_\vartheta \hat{b} \end{aligned}$$

$$*F_{\alpha\beta} \sim \begin{pmatrix} 0 & \kappa b & 0 & 0 \\ -\kappa b & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^{-1} e \\ 0 & 0 & -\kappa^{-1} e & 0 \end{pmatrix} \quad \begin{aligned} \sqrt{|g|} &= NA^3 a^2 |\epsilon_\vartheta| \\ \kappa &:= NA^{-1} a^{-2} |\epsilon_\vartheta^{-1}| \end{aligned} .$$

The Maxwell equations (6) in vacuum are given by setting  $J^\alpha = 0$ . We get for the magnetic field

$$\begin{aligned} F'_{23} = 0 & \iff b' = 0 \\ \dot{F}_{23} = 0 & \iff \dot{b} = 0 \end{aligned} .$$

Since  $\epsilon_\vartheta$  does not depend on  $(t, x)$ , we find  $\partial_\alpha((Aa)^2 \hat{b}) = 0$  for  $\alpha = 0, 1$ , hence

$$\begin{aligned} \hat{b} &= C(Aa)^{-2} \\ b &= C\epsilon_\vartheta \end{aligned} .$$

For the electric field we find

$$\begin{aligned} *F'_{23} = 0 & \implies (\kappa^{-1}e)' = 0 \\ *\dot{F}_{23} = 0 & \implies (\kappa^{-1}e)\dot{\phantom{e}} = 0 \end{aligned} .$$

(Note, that  $*F_{01,2}$  vanishes, since  $\kappa b$  contains no  $\epsilon_\vartheta$ ). Again, since  $\epsilon_\vartheta$  does not depend on  $(t, x)$ , we get  $\partial_\alpha((Aa)^2 \hat{e}) = 0$ , for  $\alpha = 0, 1$ , hence

$$\begin{aligned} \hat{e} &= C(Aa)^{-2} \\ e &= CNA^{-1}a^{-2} \end{aligned} .$$

Since  $N, A^{-1}, a^{-1}$  are already bounded, the same is true for the electromagnetic fields  $e$  and  $b$ .

The energy momentum tensor (8) takes the simple form

$$E({}^4e_\alpha, {}^4e_\beta) = \frac{1}{8\pi}(\hat{e}^2 + \hat{b}^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

and we get the matter quantities as

$$\begin{aligned} \rho &= E({}^4e_0, {}^4e_0) = -S_{11} = \text{tr } S = \frac{1}{8\pi}(\hat{e}^2 + \hat{b}^2) = \frac{1}{8\pi}(C(Aa)^{-4} + C(Aa)^{-4}) \\ j &= 0 \end{aligned} .$$

The property (35) posed in section 3.3 is obviously fulfilled, hence we get the

### 3.9. Theorem.

*Let  $(M, g, F)$  be a surface symmetric solution of the Einstein-Maxwell system with plane or hyperbolic symmetry, which possesses a symmetric Cauchy surface  $\Sigma$  with strictly negative mean curvature  $H \leq \bar{H} < 0$  and  $H = \bar{H}$  somewhere on  $\Sigma$ .*

*Then all of the past of  $\Sigma$  admits a PMC foliation  $\{S_t\}$ , where  $t$  takes all values in the interval  $]-\infty, 0]$  and  $H$  takes all values in  $]-\infty, \bar{H}]$ .*

Note that the restriction to plane and hyperbolic symmetry is necessary, since the electromagnetic energy momentum tensor does not obey the non-negative pressures condition, and therefore fails to satisfy the assumptions of proposition 3.4 in the case of spherical symmetry.

### 3.4.3 Charged particles

Now consider the Vlasov-Maxwell system in  $(M, g)$ . We obtain the coupled equations by modifying the uncharged Vlasov equation (36) by adding the term

$$e \left( A^{-1} + A^{-2}A' - N^{-1}\nu(1 + A^{-1}A')\frac{v^1}{v^0} \right) \frac{\partial f}{\partial v^1}$$

on the right-hand side. The matter current  $J$  according to the energy-momentum tensor

$$\Theta = T + E \quad , \quad \begin{aligned} \rho &= \Theta({}^4e_0, {}^4e_0) \\ j &= -\Theta({}^4e_0, {}^4e_1) \end{aligned}$$

is of the form

$$J = \rho^4e_0 + j^4e_1 = N^{-1}\rho\partial_0 + (A^{-1}j - N^{-1}\nu\rho)\partial_1 \quad .$$

The homogeneous Maxwell equations remain unchanged, which yields as before

$$\begin{aligned} \hat{b} &= C(Aa)^{-2} \\ b &= C\epsilon_{\vartheta} \quad . \end{aligned}$$

Since  $*J_{\alpha\beta\gamma} = \sqrt{|g|}\epsilon_{\mu\alpha\beta\gamma}J^\mu$ , and again for  $\alpha = 0, 1$ ,  $\partial_\alpha(\kappa^{-1}e) = |\epsilon_{\vartheta}|\partial_\alpha((Aa)^2\hat{e})$  the inhomogeneous equations read

$$\begin{aligned} (\kappa^{-1}e)' &= 4\pi\sqrt{|g|}J^0 & \iff & \quad \hat{e}' = -2A^{-1}A'\hat{e} + 4\pi A\rho \\ (\kappa^{-1}e)' &= -4\pi\sqrt{|g|}J^1 & & \quad \hat{e}' = -2(Aa)^{-1}(Aa)'\hat{e} + 4\pi(A\nu\rho - Nj) \end{aligned} \quad .$$

To get a bound for  $\hat{e}$ , note that the inhomogeneous Maxwell equations are of the form

$$\partial_\alpha\hat{e} = \varphi_\alpha\hat{e} + \psi_\alpha \cdot \binom{\rho}{j} \quad , \quad \alpha = 0, 1$$

with  $\varphi_\alpha, \psi_\alpha$  bounded by proposition 3.4. Integrating the second equation gives

$$\left| \int_{S^1} \dot{\hat{e}} \right| = \left| \int_{S^1} \varphi_0\hat{e} + \int_{S^1} \psi_0 \cdot \binom{\rho}{j} \right| \leq C \left| \int_{S^1} \hat{e} \right| + C \quad ,$$

since  $\int_{S^1} \rho, \int_{S^1} |j| \leq C$  by integrating equation (19) as shown in the proof of proposition 3.4. Using then the inequality

$$\left| \int_{S^1} \hat{e} \right| (t) \leq \left| \int_{S^1} \hat{e} \right| (0) + \int_0^t \left| \int_{S^1} \dot{\hat{e}} \right|$$

and applying Gronwall's inequality on  $\left| \int_{S^1} \hat{e} \right|$  we obtain

$$\left| \int_{S^1} \hat{e} \right| \leq C \quad .$$

Now using the first equation, we get analogously

$$|\hat{e}(t, y) - \hat{e}(t, x)| = \left| \int_x^y \hat{e}' \right| = \left| \int_{S^1} \varphi_1\hat{e} + \int_{S^1} \psi_1 \cdot \binom{\rho}{j} \right| \leq C \left| \int_{S^1} \hat{e} \right| + C \leq C \quad ,$$

where  $\left| \int_{S^1} \hat{e} \right| \leq C$  has been used.

Thus we have

$$\begin{aligned} |\hat{e}(t, y) - \hat{e}(t, x)| &\leq C \\ \left| \int_{S^1} \hat{e} \right| &\leq C \quad \implies \quad |\hat{e}| \leq C \end{aligned}$$

as desired.

Now property (35) is obtained by bounding appropriate derivatives of  $\hat{e}$ : Application of  $\partial_t^m \partial_x^{n-1}$  on the equation for  $\hat{e}'$  shows the first part of property (35) and application of  $\partial_t^m \partial_x^n$  on the equation for  $\hat{e}$  establishes the second part. This proves:

### 3.10. Theorem.

*Let  $(M, g, f, F)$  be a surface symmetric solution of the Einstein-Vlasov-Maxwell system with plane or hyperbolic symmetry, which possesses a symmetric Cauchy surface  $\Sigma$  with strictly negative mean curvature  $H \leq \bar{H} < 0$  and  $H = \bar{H}$  somewhere on  $\Sigma$ . Then all of the past of  $\Sigma$  admits a PMC foliation  $\{S_t\}$ , where  $t$  takes all values in the interval  $] -\infty, 0]$  and  $H$  takes all values in  $] -\infty, \bar{H}]$ .*

## 3.5 Improving the results

The purpose of this section is to get rid of the requirement of strictly negative mean curvature on the Cauchy surfaces, which seems to be a rather technical restriction. Let us instead assume  $\Sigma$  to be a symmetric Cauchy surface in the surface symmetric spacetime  $(M, g)$  with matter obeying the dominant and strong energy condition as well as the non-negative pressures condition, such as the surface symmetric Einstein-Vlasov system  $(M, g, f)$ . Note, that we assume the non-negative pressures condition not only for the spherically symmetric case. Indeed we will see, that this energy condition is necessary for the arguments given in this section, thus we cannot apply them to matter involving electromagnetic fields.

As pointed out in 3.2.1 there is an important difference between the possible types of surface symmetry and it turns out to be a good idea, to analyse the spherically symmetric case separately.

### 3.5.1 Spherically symmetric spacetimes

There are some results already obtained elsewhere, so we can exclude some special cases from our analysis. Namely for the Einstein-Vlasov system (as well as for the massless scalar field) it has been proven in [R4] and [BR], that given an arbitrary symmetric constant mean curvature Cauchy surface, there exist a global CMC foliation with the mean curvature taking all real values. So we may assume, that the mean curvature  $H$  on  $\Sigma$  is not constant, and we immediately get from theorem 2.2 the existence of a local in time PMC foliation  $]t_1, t_2[ \times \Sigma$  of a neighbourhood of  $\Sigma$  in  $M$ .

There are some a priori estimates:

- It is shown in [B], that  $r$  and  $M^{-1}$  are bounded. Theorem 2.1 in [B] then shows, that all timelike curves have finite length.
- Inserting the bound for  $r$  on the right-hand side of the estimate (34), we get from (31) bounds for  $r^{-1}$  and  $M$  on any finite time interval.

- Section III in [BR] shows, that in light of the fact, that all timelike curves have finite lengths, the volumes of two arbitrary Cauchy surfaces  $\Sigma_1$  and  $\Sigma_2$  are related by

$$\text{Vol}(\Sigma_2) \leq \text{Vol}(\Sigma_1) \left( 1 + C \sup_{\Sigma_1} |H| \right)^3 .$$

This implies, that the volumes of all Cauchy surfaces  $S$  in  $M$  are bounded above by the volume of  $\Sigma$  and, since  $H$  is bounded on each finite time interval, interchanging the roles of  $\Sigma$  and  $S$  shows, that the volumes are bounded from below, too, on each finite time interval.

If we denote the volume form of a PMC leaf  $S_t$  by  $\Omega$  and the volume of  $S_t$  by  $V(t)$  we have  $\Omega = \sqrt{|h|}$  and  $V(t) = \int \Omega dx d\vartheta d\varphi = 4\pi a^{-1} \int_{S^1} r^3$ .

With the bounds of  $V, V^{-1}, r, r^{-1}$  we get now  $a, a^{-1} \leq C$ , and again using the bounds for the radius function we get  $A, A^{-1} \leq C$ , thus the first fundamental form of the leaves is bounded from above and below on each finite time interval.

Moreover, as shown in section 3.3, the estimate (34) bounds  $|A^{-2}A'|$  and  $|K|$ , so that the second fundamental form is also bounded (from above) on each finite time interval as well as  $|A'| = |A^2| |A^{-2}A'|$ .

Consider now the lapse equation (21), written in the form

$$(AN')' = A^3 N \left( \frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right) - A^3 ,$$

where the term in brackets is non-negative. Therefore, setting this term to zero, we get the estimate  $(AN')' \geq -C$ , hence, for arbitrary  $p, q \in S_t$ :  $(AN')(p) - (AN')(q) \geq -C$ . Now choosing  $q$  to be a critical point of  $N$ , yields  $N'(p) \geq -C$ , and similarly choosing  $p$  as a critical point of  $N$  gives  $N'(q) \leq C$ . From this we see, that  $|N'|$  is bounded, since  $p$ , respectively  $q$  is arbitrary on  $S_t$ .

The difference to the situation considered in section 3.3 is, that we are no longer confined to a mean curvature, having everywhere fixed sign. The difficulty with the estimates done there is, that they provide no information about the behaviour of the lapse function, when  $H$  becomes zero. The estimates done here so far, do not rely on this fact and the uniform bound of  $N'$  shows, that either lapse remains finite or diverges uniformly to infinity.

In order to prove global existence of a symmetric PMC foliation assume, that  $]t_1, t_2[$  is the maximal time interval of existence. Without loss of generality let us consider only a possible extension towards the past, where  $H$  decreases with decreasing  $t$ . Thus, we are looking for regular symmetric initial data for  $t = t_1$  in the sense of definition 3.2.

Then there are two cases: First,  $H(t)$  is everywhere positive or everywhere negative near  $t_1$ , the arguments in section 3.3 apply (the fixed sign of  $H$  is enough to perform the estimates, whether  $H$  is positive or negative), extending the foliation and we get a contradiction to the maximality of the time interval, hence  $t_1 = -\infty$ .

Second,  $\lim_{t \rightarrow t_1} H(t)$  possesses no unique sign near  $t_1$ . Then we have to prove proposition 3.4 and the two subsequent lemmas 3.6 and 3.7. The discussion at the beginning of this section has shown, that some of these quantities are already bounded. The crucial step is, to find a bound for lapse and its derivatives.

The idea is to reparametrize the foliation as has been done for a CMC foliation in [BR]. The effect of the reparametrization on lapse and shift was outlined in section 2.1. Let us introduce a function  $\tau$  by

$$\tau := t + \int_{t_1}^t N(u, x_0) du ,$$

where  $\int_{t_1}^t$  means  $\lim_{s \rightarrow t_1} \int_s^t$ . It is well defined, since  $\int N(\gamma(u)) du$  along the integral curves of the normals of the leaves measures the length of  $\gamma$ , which we already know to be finite, and  $|N'| \leq C$  ensures the integrability of  $N(t, x_0) \leq N(\gamma(t)) + C$ . This construction works in either case, whether  $N$  is bounded or diverges uniformly to infinity. The function  $t \mapsto \tau(t)$  is monotone since  $\frac{d\tau}{dt} = 1 + N(t, x_0)$  and turns out to be an orientation preserving diffeomorphism  $]t_1, t_2[ \rightarrow ]\tau_1 = t_1, \tau_2[$ , thus  $\tau$  can be used as a new time function for the foliation, which squeezes the time differences of neighbouring leaves compared with the normal vector by adding to  $t$  the normal component of the length of the piece of the  $x_0 = \text{const}$  line connecting  $(t_1, x_0)$  with  $(t, x_0)$ . This produces a new lapse function and a new shift vector:

$$\begin{aligned}\tilde{N} &= \left(\frac{d\tau}{dt}\right)^{-1} N = \frac{N}{1 + N(t, x_0)} \\ \tilde{\nu} &= \left(\frac{d\tau}{dt}\right)^{-1} \nu = \frac{\nu}{1 + N(t, x_0)} \quad .\end{aligned}$$

Conversely,  $t$  stretches time differences of adjacent leaves, such that we get the inverse transformation by  $t = \tau - \int_{\tau_1}^{\tau} \tilde{N}(u, x_0) du$ , which subtracts from  $\tau$  the former added 'length' of the  $x_0 = \text{const}$  line now expressed in terms of the new time coordinate. By the inverse transformation, we see that  $\frac{dt}{d\tau} = 1 - \tilde{N}(\tau, x_0)$ , and therefore get the relation  $1 - \tilde{N}(\tau, x_0) = (1 + N(t, x_0))^{-1}$  in points with spatial coordinate  $x_0$ .

The benefit of this reparametrization is, that due to  $|N'| \leq C$ ,  $\tilde{N}$  and  $|\tilde{N}'|$  remain bounded, and we could try to analyse the field equations according to the reparametrized foliation. Inspection of these equations shows, that we get the field equations in the new coordinates from equations (19)-(27) by replacing everywhere  $\partial_t$  by  $\partial_\tau$ ,  $N$  by  $\tilde{N}$ ,  $\nu$  by  $\tilde{\nu}$  (since reformulating of the constraint and evolution equations preserves their form) while the lapse equation (21) and the PMC condition (22) depend on the parametrization, and hence have to be modified: The subtraction of  $A^2$  on the right-hand side stems from the term  $A^2 N n(H)$ , where the PMC condition sets  $N n(H)$  equal to one. Replacing  $N$  by  $\tilde{N}$  yields  $(N n(H))(1 + N(t, x_0))^{-1} = (1 + N(t, x_0))^{-1} = 1 - \tilde{N}(\tau, x_0)$ . The PMC condition (22), which holds in the old parametrization, can be formulated in the new coordinates by expressing  $\nu$  in terms of  $\tilde{\nu}$ , which produces an additional summand  $1 - \tilde{N}(\tau, x_0)$  on the right-hand side, besides the replacement of  $\nu$  by  $\tilde{\nu}$ .

Now we start to analyse the new field equations, trying to retain the same line of thought of section 3.3: The analysis already done here bounds  $A, A^{-1}, |A'|, a, a^{-1}, |H|, |K|$  as well as, of course,  $\tilde{N}$  and  $|\tilde{N}'|$ . The arguments carried out in section 3.3 bound now  $|\tilde{\nu}|, |\tilde{\nu}'|$  and  $|H'|$ . Finally we need a bound for  $\tilde{N}^{-1}$ . Since it is not clear, in the case where  $N$  diverges near  $t_1$ , whether  $N$  grows monotonically or not, we divide the time interval  $]t_1, t_2[$  into two subsets. First consider all points, where  $N(t, x_0) \geq M$ , with some suitable chosen real number  $M$ , which will be specified later. Then we get the estimate

$$\left|1 - \tilde{N}(\tau, x)\right| = \left|1 - \frac{N(t, x)}{1 + N(t, x_0)}\right| = \left|\frac{1 + N(t, x_0) - N(t, x)}{1 + N(t, x_0)}\right| \leq \frac{1 + C}{1 + M} \quad ,$$

where  $C$  is an upper bound for  $|N(t, x_0) - N(t, x)|$ , whose existence is guaranteed by the bounds for  $|N'|$  and the volume of the leaves of the foliation. Choose now  $M = 2C$  and we get  $|1 - \tilde{N}| \leq \frac{1+C}{1+2C} < 1$ , thus  $\tilde{N}$  is bounded away from zero for all points, where  $N(t, x_0) \geq M$ . Consider now the points, where  $N(t, x_0) \leq M$ . Then we have

$$\left|1 - \tilde{N}(\tau, x_0)\right| = \left|1 - \frac{N(t, x_0)}{1 + N(t, x_0)}\right| = \left|\frac{1 + N(t, x_0) - N(t, x_0)}{1 + N(t, x_0)}\right| \geq \frac{1}{1 + M} \quad .$$

Now we can apply the arguments in [R5] to the modified lapse equation, as has been done in section 3.3 (whereupon the factor  $1 - \tilde{N}(\tau, x_0)$  causes no trouble, since it is bounded from above and below) and we get a bound for  $\tilde{N}^{-1}$  for all points, where  $N(t, x_0) \leq M$ , and we are done.

Therefore we have estimated all quantities appearing in proposition 3.4. lemma 3.6 is also true for the new field equations without the need to modify the proof, while the proof of lemma 3.7 has to be modified, since the argument bounding  $|\partial_t^{m+1} \partial_x^n \tilde{N}|$  does not carry over. The lapse equation and the equation for its time derivatives are

$$\begin{aligned} (A\tilde{N}')' &= A^3 \tilde{N} \left( \frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right) - A^3 \left( 1 - \tilde{N}(\tau, x_0) \right) \\ (A\partial_t^m \tilde{N}')' &= A^3 \partial_t^m \tilde{N} \left( \frac{1}{2}(H - K)^2 + K^2 + 4\pi(\rho + \text{tr } S) \right) \\ &\quad - A^3 \left( 1 - \partial_t^m \tilde{N}(\tau, x_0) \right) + B \quad , \end{aligned}$$

where  $B$  denotes an already bounded quantity at the corresponding stage in the inductive argument in lemma 3.7. In this situation lemma 1 in [BR] applies literally, bounding the time derivatives of the lapse function, therefore completing the proof of lemma 3.7.

Putting all arguments together we are able to extend the foliation beyond  $t_1$ , contradicting the assumed maximality of the interval of existence, thus arriving at the following

### 3.11. Theorem.

*Let  $(M, g, f)$  be a surface symmetric solution of the Einstein-Vlasov system with spherical symmetry.*

*Then we can foliate the whole spacetime by a PMC foliation, where the time function takes on all real values and the mean curvature of the leaves tends uniformly to  $\pm\infty$  for  $t \rightarrow \pm\infty$ , thus producing crushing singularities.*

This theorem provides barrier surfaces (see [G]), establishing now the existence of CMC hypersurfaces for each value of the mean curvature, therefore proving the closed universe recollapse conjecture in this case:

### 3.12. Corollary.

*In the situation of theorem 3.11, the spacetime possesses a global CMC foliation and the mean curvature takes on all real values. In particular, the spacetime admits a maximal Cauchy surface.*

## 3.5.2 Spacetimes with plane and hyperbolic symmetry

We will see, that most of the arguments performed in the spherically symmetric case carry over to the past domain of dependence  $D^-(\Sigma)$  in the expanding models (compare 3.2.1 for a precise definition of this terminology). Due to this fact we perform the following analysis only on the half-open time interval  $]t_1, 0]$  and assume further,  $(M, g)$  to be non-flat in the plane symmetric case and the mass function (31) to be positive on  $\Sigma$  in the hyperbolic case.

Again we can exclude some cases, already investigated (although this does not matter, since the arguments here apply to these cases). It has been proven in [R4] and [BR], that given a symmetric Cauchy surface with negative constant mean curvature  $H_0$  (remember the restrictions on the mean curvature in the expanding models, compare 3.2.1), there exists a global CMC foliation with the mean curvature taking all values in the interval  $] -\infty, H_0]$ . So we can assume without loss of generality, that the mean curvature  $H$  on  $\Sigma$

is not constant and not everywhere positive. Again, this assumption ensures the existence of a local in time PMC foliation  $]t_1, t_2[ \times \Sigma$  of a neighbourhood of  $\Sigma$  in  $M$ .

Now we find similar a priori bounds

- In  $D^-(\Sigma)$   $r$  is bounded, since  $dr$  is future pointing timelike everywhere and  $\Sigma$  is compact.  $M^{-1}$  is bounded on  $\Sigma$  by assumption in the hyperbolic case and in the plane symmetric case this follows from lemma 2.4 in [R4]. By the mass flux equation (33) together with the non-negative pressures condition,  $M$  grows along past pointing timelike curves, thus  $M^{-1}$  is bounded in all of  $D^-(\Sigma)$ . Theorem 2.1 in [B] then shows, that all timelike curves in  $D^-(\Sigma)$  have finite length.
- Applying the arguments in the corresponding place of section 3.5.1 we get bounds for  $r^{-1}$  and  $M$  for any finite time interval of the form  $]t_1, 0]$ .
- Again, the corresponding argument performed in the spherically symmetric case holds, and we get upper and lower bounds for the volumes of arbitrary Cauchy surfaces in  $M$ .

Finally, the volume of the leaves  $S_t$  is given by  $V(t) = Ca^{-1} \int_{S^1} r^3$ , establishing bounds for the first fundamental form and its inverse (in  $D^-(\Sigma)$ ) as has been shown in 3.5.1, therefore all of the remaining arguments performed there apply and we get the

### 3.13. Theorem.

*Let  $(M, g, f)$  be a surface symmetric solution of the Einstein-Vlasov system with plane or hyperbolic symmetry and  $\Sigma$  a symmetric Cauchy surface in  $M$ .*

*If  $(M, g)$  is non-flat in the plane symmetric case and the mass function is positive on  $\Sigma$  in the hyperbolic case, then we can foliate all of the past of  $\Sigma$  by PMC hypersurfaces, where the time function takes on all values in the interval  $] -\infty, 0]$  and the mean curvature of the leaves tends uniformly to  $-\infty$  for  $t \rightarrow -\infty$ .*

Using again the PMC leaves as barrier surfaces we get the

### 3.14. Corollary.

*In the situation of theorem 3.13  $D^-(\Sigma)$  possesses a CMC Cauchy surface for each value of the mean curvature in  $] -\infty, \min_{\Sigma} H[$ .*

### 3.5.3 Comparing the results

To close this chapter, there are some remarks concerning the theorems 3.11 and 3.13 and the relation to the work done in the preceding sections.

At first, it is obvious, that theorem 3.11 generalizes theorem 3.8 in the spherically symmetric case. In the plane and hyperbolic case theorem 3.13 does not generalize the theorems 3.8, 3.9 and 3.10. The latter ones only establish a global PMC foliation unless the mean curvature of the leaves becomes zero somewhere (and again the leaves provide barrier surfaces for CMC Cauchy surfaces, establishing corollary 3.14 in the situation of those theorems). Theorem 3.13 is not restricted to this condition, but it needs the extra non-negative pressures condition, which excludes electromagnetic fields.

Apart from the non-negative pressures condition the assumption of positive mass on  $\Sigma$  is a non-trivial constraint in the hyperbolic case, while automatically fulfilled in the non-flat plane symmetric case by lemma 2.4 in [R4]. As shown above, the positivity of mass is needed to obtain a bound for the length of timelike curves in  $M$ , by applying the arguments in [B], thus necessary to the construction done here.

## 4 Conclusion and outlook

First I list the main results achieved in this work

- For the spacetimes considered so far the existence of a global PMC foliation has been shown, where several matter models have been taken into account.
- For the model, which is not expanding or contracting everywhere, the closed universe recollapse conjecture has been proved. In particular the foliation covers the whole Cauchy development of the initial hypersurface, with a crushing singularity both in the distant past and future. Moreover the spacetime possesses a maximal hypersurface.
- In the expanding models the foliation covers at least the whole past of the initial hypersurface towards a crushing singularity.

The choice of matter turned out to be important only in so far, as some energy conditions are satisfied and the matter fields do not become singular in a regular geometric background.

The necessary energy conditions are the dominant and strong energy conditions. Furthermore to obtain stronger results, which do not rely on a fixed sign condition for the mean curvature, the non-negative pressures condition is required, to ensure that the Hawking mass does not tend to zero, contributing enough attraction, that the lifetime of timelike curves become finite.

The rather strong results about locally spatially homogeneous spacetimes in [R3] likewise rely on this condition, which appears in the slightly relaxed form, that only the sum of the pressures has to be non-negative, due to the high degree of symmetry in those models. This relaxed condition even permits electromagnetic fields, a type of matter, which does not satisfy the non-negative pressures condition, thus not leading to the stronger results in the context of spacetimes with less symmetry.

The close analogy of the proofs for the global existence of CMC and PMC foliations indicates, that all results obtained for CMC foliations may also be proved for PMC foliations. If this conjecture turns out to be true (the second part of this paper will give another positive example for this conjecture), then we are no longer concerned with the topological restrictions imposed on the existence of CMC hypersurfaces, having a much more flexible tool at hand. Moreover, the global results on PMC foliations can be used to provide barrier surfaces, which guarantee the existence of a CMC foliation, where the mean curvature ranges between these barriers (compare the corollaries 3.12 and 3.14).

The results demonstrate, that in the cases considered in this work a satisfying answer has been given to the global existence problem for PMC foliations. Then the related question arises, whether the foliation covers the whole spacetime. The present work gives only a partial answer to this question. Denoting the initial Cauchy surface by  $\Sigma$ , we saw, that we have covered the whole past  $D^-(\Sigma)$  in the expanding spacetimes and the whole Cauchy development  $D(\Sigma)$  in the recollapsing models. These positive results have two limits

1. In the expanding models there remains an open question about the future of  $\Sigma$  in  $M$ . Either the mean curvature becomes zero somewhere or there is no control on the radius function towards the expanding direction. In each case the present techniques do not apply. In addition there are topological obstructions in the expanding direction, preventing the mean curvature to become positive, a foliation of the future of

$\Sigma$  must stop before this happens. But this does not imply, that the whole of  $D^+(\Sigma)$  can be covered by such a foliation, since the leaves of the foliation may become null or non-compact, where the general notion of singularity avoidance comes up.

Namely, compared with the strong results obtained in [R3] in the locally spatially homogeneous models, we require more information about the asymptotic behaviour of the foliation to obtain results on geodesic completeness.

2. In the contracting direction and in the recollapsing models things look different. In the contracting direction the theorems ensure the existence of crushing singularities, and all of the Cauchy development of  $\Sigma$  is covered by the foliation. This is a consequence of Hawking's famous incompleteness theorem for globally hyperbolic spacetimes, satisfying the timelike convergence condition, where the maximal time of existence is estimated by  $3/|H|$  and  $H \rightarrow -\infty$ . But there is no information about the boundary of  $D(\Sigma)$ , which might be either a curvature singularity or merely a Cauchy horizon.

The work in [R3] indicates roughly, what remains to do. Relating the particle current density to the energy density and investigating the asymptotic behaviour of the former may give rise to unbounded curvature and produce a curvature singularity.

For the spacetimes considered here, the work [Re] of Rein is important. He succeeded in finding satisfying answers to the above questions, by using a time function intimately tied to the symmetry. Unfortunately it is not clear how to generalize his approach to other spacetimes. Although the estimates done here also exploit the high symmetry of the models, the construction itself does not depend on it, motivating this work. A generalization to spacetimes with two commuting local Killing vector fields will be the content of the second part.

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