Global Prescribed Mean Curvature foliations in cosmological spacetimes with matter
Part II

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Abstract

This second part is devoted to the investigation of global properties of Prescribed Mean Curvature (PMC) foliations in cosmological spacetimes with local $U(1) \times U(1)$ symmetry and matter described by the Vlasov equation. It turns out, that these spacetimes admit a global foliation by PMC surfaces, as well, but the techniques to achieve this goal are more complex than in the cases considered in part I.

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1 Introduction

For the main motivation see the introduction in part I, [H3]. Only some remarks, special to the present situation remain.
The structure of this second part is as close as possible to the structure of part one. Although the preliminary section 2 of part I has been omitted, the formulas here refer to it as well.

The spacetimes considered here are cosmological spacetimes with two commuting local Killing vector fields. This symmetry will be referred to as (local) $U(1) \times U(1)$ symmetry. In comparison with part I, there are now only two (local) Killing fields instead of three, generalizing the plane symmetric case of [H3]. The absence of a third Killing field requires a more detailed description of the geometry and a deeper analysis to control the momenta of the Vlasov particles. Thus the estimates in this work rely on the simple structure of the Vlasov equation. Despite this there seems to exist no crucial obstructions for other types of 'well behaved' matter.

2 Spacetimes with local $U(1) \times U(1)$ symmetry

2.1 The geometry of spacetimes with local $U(1) \times U(1)$ symmetry

Following the analysis in [R], we consider now the globally hyperbolic spacetime (M, g) with topology $\mathbb{R} \times \Sigma$, where Σ denotes a bundle with base S^1 and compact orientable fibre F. As usual it is assumed, that the submanifolds $\{t\} \times \Sigma$ are Cauchy hypersurfaces in M. The coordinates of Σ are denoted by $(x, y^2, y^3) = (x, y^A)$, $A = 2, 3$, where (y^A) denote coordinates on F. As usual, Greek indices range in the interval $0, \ldots, 3$, lower case Latin indices from the middle of the alphabet take values in $1, \ldots, 3$, while those from the beginning of the alphabet take the values $0, 1$ and upper case ones are confined to the values $2, 3$.
The covering map $\mathbb{R} \longrightarrow S^1$ defines a pullback of the bundle Σ with base $\mathbb{R}$, hence we get a trivial bundle. If $\hat{F}$ denotes the universal covering space of F, we get a natural covering $\hat{\Sigma} = \mathbb{R} \times \hat{F}$ of Σ with canonical projection p. Now we can associate a spacetime $(\hat{M}, \hat{g})$, where $\hat{M} = \mathbb{R} \times \hat{\Sigma}$ and $\hat{g}$ is the pullback of g under the projection $\mathrm{id} \times p$.

The fibres $\hat{F}$ in the trivial bundle $\hat{M}$ are assumed to be the orbits of a two dimensional translation group G of isometries of $\hat{g}$. Hence $\hat{F}$ is the Euclidean space form E_2 and $F = E_2/\Gamma$, for a discrete subgroup Γ of G. The compactness and orientability of F implies then $F = T^2$, so Γ can be represented by a two-parameter lattice, Σ turns out to be a torus bundle over the circle and the induced action of G on (M, g) is given by the quotient action $G/\Gamma = U(1) \times U(1)$ with orbits F. As in [H3] we will call the orbits surfaces of symmetry, hypersurfaces in M diffeomorphic to $\{t\} \times \Sigma$, which consist of a union of surfaces of symmetry will be called symmetric surfaces and we call (M, g) a spacetime with local $U(1) \times U(1)$ symmetry.
The induced action of $U(1) \times U(1)$ on (M, g) is local, if the bundle is non-trivial. In that case we have to deal with non-trivial transformations for the transition $x \longmapsto x + 2\pi$ in S^1 by lattice preserving translations and automorphisms $\mathrm{GL}(2, \mathbb{Z})$ of G. To represent the metric $\overline{g}$ of the orbits in M at first locally for $x \in [0, 2\pi[$, define the area radius by $r := \sqrt{(4\pi)^{-1} \operatorname{Vol}(F)}$, and write the metric as

$$\overline{g} = r^2 \tilde{g} \quad , \tag{1}$$

with a metric $\tilde{g}$ of unit determinant. Due to symmetry both quantities, r and $\tilde{g}$ do not depend on the points of F, and one easily verifies, that the curvature of $\overline{g}$ vanishes, as required. For $\tilde{g}$ there are two remaining degrees of freedom, V and W, and we can use them to parametrize the metric:

$$\tilde{g} = \begin{pmatrix} e^W \cosh V & \sinh V \\ \sinh V & e^{-W} \cosh V \end{pmatrix} \quad . \tag{2}$$

If the bundle is trivial, then this representation of the metric is global, but if the bundle is not trivial, then the translation of 2π in S^1 induces a transformation of $\overline{g}$ by an element Z of $\mathrm{GL}(2, \mathbb{Z})$,

$$\overline{g}(x + 2\pi) = Z^T \overline{g}(x) Z \quad , \tag{3}$$

which fixes the geometry of the spacetime.

Given a globally acting symmetry we can specialize to some well known geometries: If $V = W = 0$ then we get the plane symmetric case $\epsilon = 0$ of [H3]. Setting only $V = 0$

we get a symmetry called polarized, corresponding to the reflection symmetries $y^2 \longmapsto -y^2$ or $y^3 \longmapsto -y^3$ respectively. If the composition of these reflections is an isometry (regardless whether the individual reflections act isometrically), we call this symmetry to be of *Gowdy–type*, since the Gowdy spacetimes are defined by this symmetry and the additional requirement that the spacetime is vacuum. Thus, in general $V \neq 0$ in a spacetime with Gowdy-type symmetry, but if V vanishes also we call the Gowdy-type symmetry polarized.

We construct the coordinate system $\{x^\mu\}$ mentioned in the beginning of this section locally by the following procedure. Consider first an arbitrary symmetric Cauchy surface S in M. Then we extend the coordinates (y^A) of F to a Gaussian neighbourhood of F in S. Later on we will do some rescaling along S^1, such that we choose the coordinates, such that the metric h of S takes the general form $h = A^2\, dx + \overline{g}$. Now we embed this structure into the spacetime. In general the time coordinate t defines lapse N and shift $\nu = \nu^i\, \partial_i$, thus the spacetime metric g is not in block diagonal form. The components g_{0A} which prevent g being block diagonal can be represented by $\overline{\nu}_A := \overline{g}_{AB}\, \nu^B$.

2.1.1 The 2+2–geometry

Now we want to investigate the geometry of $(F, \overline{g})$ in (M, g) more closely. As before let S denote a symmetric Cauchy surface in M, foliated by its symmetric surfaces. The unit normal of F in $S \subset M$ will be denoted by m and n is the unit normal of S in M, as usual. Then there are some canonical geometrical objects induced:

(a) the two second fundamental forms

$$\lambda(v,w) = g({}^4\nabla_v w, m) \qquad \text{(4a)}$$
$$\kappa(v,w) = g({}^4\nabla_v w, n) \quad , \qquad \text{(4b)}$$

for $v, w \in TF$ with arbitrary extensions to vector fields, in order to make the expressions well defined. In the sequel it turns out to be convenient, to deal with the trace free parts $\tilde\lambda$ and $\tilde\kappa$ instead, defined by $\lambda = \tilde\lambda + \frac{1}{2}\operatorname{tr}\lambda\, \overline{g}$ and $\kappa = \tilde\kappa + \frac{1}{2}\operatorname{tr}\kappa\, \overline{g}$ respectively. In the given coordinates we have explicit formulas:

$$\begin{aligned}
\lambda_{AB} &= -\tfrac{1}{2} m(\overline{g}_{AB}) = -\tfrac{1}{2} A^{-1} (\overline{g}_{AB})' \\
\operatorname{tr}\lambda &= -\tfrac{1}{2}\overline{g}^{AB}\, m(\overline{g}_{AB}) = -\tfrac{1}{2}\frac{m(\det\overline{g})}{\det\overline{g}} = -\tfrac{2}{r} m(r) = -2A^{-2}A' \\
\kappa_{AB} &= -\tfrac{1}{2} n(\overline{g}_{AB}) = -\tfrac{1}{2} N^{-1} (\overline{g}_{AB})^{\cdot} - N^{-1} A \nu^1 \lambda_{AB} \\
\operatorname{tr}\kappa &= -\tfrac{2}{r} n(r) \quad ,
\end{aligned}$$

reflecting the definition of the area radius r as a volume measure and the second fundamental forms as its rate of change. In the above formulas $\overline{g}$ is considered as intrinsic to F (and we maintain this from now on) and $\overline{g}^{AB} = r^{-2}\tilde g^{AB}$ denotes the inverse of $\overline{g}_{AB}$. Differentiating $\tilde g$ directly one gets from its relation to $\overline{g}$ and the formulas above the following representations of the trace free parts of the second fundamental forms:

$$\begin{aligned}
\tilde\lambda_{AB} &= -\tfrac{1}{2} r^2\, m(\tilde g_{AB}) & \qquad \tilde\lambda^{AB} &= \tfrac{1}{2} r^{-2}\, m(\tilde g^{AB}) \\
\tilde\kappa_{AB} &= -\tfrac{1}{2} r^2\, n(\tilde g_{AB}) & \qquad \tilde\kappa^{AB} &= \tfrac{1}{2} r^{-2}\, n(\tilde g^{AB})
\end{aligned}$$

(b) the connection in the normal bundle $T^\perp F$. This can be represented by a single one form η, defined as

$$
(5)\qquad \begin{aligned} \eta(v) &= g({}^4\nabla_v n, m) = -g({}^4\nabla_v m, n) \\ &= -\frac{1}{2}\, g([n,m], v) \end{aligned} \qquad v \in TF
$$

From this formula one can see, that $\eta = 0$ is equivalent to $[n,m] \in T^\perp F$, thus $[T^\perp F, T^\perp F] \subset T^\perp F$ and the theorem of Frobenius tells us, that $T^\perp F$ is an integrable distribution of two planes in TM. If $T^\perp F$ is integrable, then we can decompose the spacetime into a direct sum $(M,g) = (F^\perp, \underline{g}) + (F, \overline{g})$, with $g = \underline{g} \oplus \overline{g}$. Sufficient for the existence of an integral manifold $F^\perp$ of the distribution $T^\perp F$ is the Gowdy–type symmetry $y^A \longmapsto -y^A$, $A = 2,3$, since then it follows $\eta = 0$ immediately, because $v \longmapsto \eta(v)$ is an antisymmetric map.

Whether or not $T^\perp F$ is integrable, we can orthogonally split the tensor bundle over M, following Kundu ([K]). To perform this task we start with the two dimensional Riemannian manifold $(F, \overline{g})$. With our choice of spacetime coordinates we have Killing fields $Y_A = \partial_A$. Their spacetime components define projection operators

$$
\begin{aligned} p_A{}^\mu &:= Y_A{}^\mu\,, & p_A &= \partial_A \\ p^A{}_\mu &:= \overline{g}^{AB} g_{\mu\nu} p_B{}^\nu\,, & p^A &= \overline{\nu}^A dt + dy^A \quad , \end{aligned}
$$

where $\overline{\nu}^A := \overline{g}^{AB}\,\overline{\nu}_B = \nu^A$, and the metric components g_{1B} are zero by our definition of the coordinates.
The projection operators into $T^\perp F$ can now defined as follows: Let the unit normal vectors n and m serve as projection operators $q_a{}^\mu$ and define their duals by the relations $q_a{}^\mu q^b{}_\mu = \delta_a^b$ and $p_A{}^\mu q^b{}_\mu = 0$. The result is

$$
\begin{aligned} q_0 &= n = N^{-1}(\partial_t - \nu^i \partial_i)\,, & q_1 &= m = A^{-1}\partial_1 \\ q^0 &= N\,dt\,, & q^1 &= A\nu^1 dt + A\,dx \end{aligned}
$$

Now we are ready to define the transversal metric $\underline{g}$ in $T^\perp F$ by

$$
(6)\qquad \underline{g}_{ab} = g_{\mu\nu} q_a{}^\mu q_b{}^\nu \quad ,
$$

thus in the given frame $\underline{g}$ and its inverse are represented by the two dimensional Minkowski metric and from now on we consider $\underline{g}$ as an intrinsic object in the tensor bundle over $T^\perp F$. In summary we have constructed a complete set of projection operators, projecting orthogonally tensors over (TM, g) into $(TF, \overline{g})$ and $(T^\perp F, \underline{g})$, characterized by the relations

$$
\begin{aligned} & p_A{}^\mu p^A{}_\nu + q_a{}^\mu q^a{}_\nu = \delta^\mu_\nu \quad \text{(completeness)} \quad , \\ & p_A{}^\mu p^B{}_\mu = \delta_A^B \qquad\qquad p_A{}^\mu q^b{}_\mu = 0 \\ & q_a{}^\mu p^B{}_\mu = 0 \qquad\qquad q_a{}^\mu q^b{}_\mu = \delta_a^B \quad \text{(orthogonality)} \quad , \end{aligned}
$$

with $p_A = \partial_A$ and $q_0 \perp q_1$ (these two relations fix the component representations shown above).

Now we define the convention already used in some expressions above, that wherever confusion might arise, we attach a bar to quantities, which will be considered as intrinsic to the associated bundle, and a tensor index furnished with such a bar denotes projected components, which also can be considered as intrinsic. For example $\underline{T}_{ab}$ denotes a tensor in the bundle $(T^{\perp}F, \underline{g})$, but $T_{\underline{ab}} = q_a{}^{\mu} q_b{}^{\nu} T_{\mu\nu}$, too, although T might be a spacetime tensor. The philosophy lying beyond this notation is, that quantities with a bar attached to them or to their indices can be manipulated by the associated metric, while indices without a bar always denote component indices corresponding to the bundle the tensor is intrinsic to. If we apply this notational convention to the metric itself we get the definitions for $\underline{g}$ and $\overline{g}$ back, for example we get the identities $g_{\underline{ab}} = q_a{}^{\mu} q_b{}^{\nu} g_{\mu\nu} = \underline{g}_{ab}$ and $g_{\overline{A}\overline{B}} = p_A{}^{\mu} p_B{}^{\nu} g_{\mu\nu} = \overline{g}_{AB}$ In particular we have the relations

$$\underline{\nabla}_a = q_a{}^{\mu}\, {}^4\nabla_{\mu}$$
$$\overline{\nabla}_A = p_A{}^{\mu}\, {}^4\nabla_{\mu} \quad ,$$

defining the Levi-Civita connection in the projected bundles, and the algebraic identity

$$T^{\mu}_{\mu} = T^{\underline{a}}_{\underline{a}} + T^{\overline{A}}_{\overline{A}} \quad .$$

For later use we need the projected components of the Ricci tensor. First note, that the normal connection η on $(F, \overline{g})$ reads in our new notation

$$\overline{\eta}^A = -\tfrac{1}{2}\, [q_0, q_1]^{\overline{A}} = -\tfrac{1}{2}\, p^A{}_{\mu} [q_0, q_1]^{\mu} = \tfrac{1}{2}\, dp^A(q_0, q_1) \quad ,$$

or equivalently $\epsilon_{ab}\overline{\eta}^A = 1/2\, dp^A(q_a, q_b)$, making use of the relation $0 = \nabla(p^A{}_{\mu} q_a{}^{\mu})$, which gives $\nabla q_a{}^{\mu} = -q_a{}^{\nu} p_B{}^{\mu} \nabla p^B{}_{\nu}$. The ϵ_{ab} here has its standard meaning as the totally anti-symmetric symbol (independent of the frame used, so it is not necessary here to attach a bar to it). On $T^{\perp}F$ we find $dp_A = d(\overline{g}_{AB}\, p^B) = \overline{g}_{AB}\, dp^B$, so that index manipulations with $\overline{g}$ can be applied, as desired.
After some calculations we arrive at formulas for the projected components of the Ricci tensor:

(7a) $${}^4R_{\underline{ab}} = \underline{R}_{ab} - 2r^{-1}\underline{\nabla}_a \underline{\nabla}_b\, r + 2r^{-2}(\underline{\nabla}_a r)(\underline{\nabla}_b r) + \tfrac{1}{4}(\underline{\nabla}_a\, \overline{g}_{AB})(\underline{\nabla}_b\, \overline{g}^{AB}) + 2\underline{g}_{ab}|\eta|^2$$

(7b) $${}^4R_{\overline{A}\underline{b}} = r^{-2}\epsilon^c{}_b \underline{\nabla}_c (r^2 \overline{\eta}_A)$$

(7c) $${}^4R_{\overline{A}\overline{B}} = -\tfrac{1}{2}\overline{g}_{AD}\, \underline{\nabla}^c(\overline{g}^{CD} \underline{\nabla}_c\, \overline{g}_{BC}) - r^{-1}(\underline{\nabla}^c r)(\underline{\nabla}_c\, \overline{g}_{AB}) - 2\overline{\eta}_A\, \overline{\eta}_B$$

Contraction of the last equation yields a formula for the Laplacian of r (Note, that the Laplacian is really a wave operator in the present situation):

(8) $$\underline{\Delta} r = \tfrac{1}{2}{}^4R^{\overline{A}}_{\overline{A}} + r^{-1}(\underline{\nabla}_c r)(\underline{\nabla}^c r) + r|\overline{\eta}|^2$$

Finally we consider other frames of reference and their relation to the frames already in use in the projected spaces. At first we concentrate on $(TF, \overline{g})$, which is canonically endowed with the coordinate vector fields ∂_A. Alternatively we can introduce the orthonormal frame $\{\overline{e}_A\}$, defined by $\overline{e}_A := r^{-1}\tilde{e}_A$, with

$$\tilde{e}_2 = \quad e^{-W/2} \cosh V/2\, \partial_2 \quad - \quad e^{W/2} \sinh V/2\, \partial_3$$
$$\tilde{e}_3 = -e^{-W/2} \sinh V/2\, \partial_2 \quad + \quad e^{W/2} \cosh V/2\, \partial_3 \quad .$$

The dual frame $\{\overline{\sigma}^A\}$ is defined by $\overline{\sigma}^A := r\tilde{\sigma}^A$, with $(\tilde{\sigma}^A)_{\mu}(\tilde{e}_B)^{\mu} = \delta^A_B$, thus

$$\tilde{\sigma}^2 = e^{W/2} \cosh V/2\, dy^2 \quad + \quad e^{-W/2} \sinh V/2\, dy^3$$
$$\tilde{\sigma}^3 = e^{W/2} \sinh V/2\, dy^2 \quad + \quad e^{-W/2} \cosh V/2\, dy^3 \quad .$$

Finally we introduce the two dimensional quotient manifold $B := M/(U(1)\times U(1))$. Then we have isomorphisms $TB \simeq TM/TF \simeq T^{\perp}F$ and we can consider the metric $\underline{g}$ as acting on TB, so having constructed the two-dimensional (quotient) spacetime $(B, \underline{g})$. If we take (t, x) as coordinates of B, then the metric $\underline{g}$ has coordinate components $\underline{g}_{ab} = -(q^0)^2 + (q^1)^2 = (-N^2 + (A\nu^1)^2)\,dt^2 + 2A^2\nu^1\,dt\,dx + A^2\,dx^2$, giving an alternative way to describe tensor components in $(T^{\perp}F, \underline{g})$, even if this bundle is not integrable. The dual base $\{\underline{e}_a\}$ to $\{\underline{\sigma}^a = q^a\}$ in $(B, \underline{g})$ has the coordinate components $\underline{e}_0 = N^{-1}(\partial_t - \nu^1\,\partial_x)$, $\underline{e}_1 = A^{-1}\,\partial_x$. The component notation just introduced conflicts with the conventions described above, and in the following sections we are forced to make clear, which conventions we will follow.

2.1.2 The 3+1–geometry

To describe the 3–geometry of an arbitrary symmetric Cauchy surface S, we construct the coordinate system (x', y^A) explicitly. On the surface of symmetry $F \subset S$ we have already coordinates, such that the scaled metric $\tilde{g}$ takes the form (2). In a Gaussian neighbourhood of F the metric h of S then has the form $h = dx'^2 + \sqrt{|h|}\tilde{g}(x')$. Now we introduce a scale factor $a = 2\pi\left(\int_0^L |h(z)|^{-1/4}\,dz\right)^{-1}$, where L denotes the length of a (projected) geodesic along all of S^1 orthogonal to the orbits F. Define a new coordinate x by $x(x') = a\int_0^{x'} |h(z)|^{-1/4}\,dz$, then we get a convenient representation of h as

(9) $$h(x) = A(x)^2\left(dx^2 + a^2\tilde{g}(x)\right) \quad ,$$

with $A(x) = a^{-1}|h(x)|^{1/4}$ defined on S^1, and $\tilde{g}$ transforms under some element of $\mathrm{GL}(2,\mathbb{Z})$ like $\overline{g}$ in (3). By the way, from this explicit formula and the definition of $\overline{g}$ in (1) it follows easily that the relation $r = Aa$ holds.
The Laplacian of S acting on functions ψ then has the form

$$\begin{aligned}\Delta\psi &= -h^{11}\nabla_1\nabla_1\psi + h^{AB}\Gamma^1_{AB}\psi' + \bar{\Delta}\psi \\ &= -A^{-2}(\psi'' + A^{-1}A'\psi') + \bar{\Delta}\psi \quad ,\end{aligned}$$

with $\bar{\Delta}\psi = -h^{AB}\partial_A\partial_B\psi$ since $\bar{\Gamma}^C_{AB} = 0$.
Finally we cast the second fundamental form k of S into a convenient form. Set $K := k(m,m) = A^{-2}k_{11}$ and observe $k_{1B} = -{}^4\nabla_B n_1 = -A\eta_B$ we get

(10) $$k(x) = A^2K\,dx - 2A\eta_B\,dx\,dy^B + \kappa_{AB}\,dy^A\,dy^B \quad ,$$

where all quantities on the right-hand side depend on x. Taking the trace yields the relation

$$H - K = \mathrm{tr}\,\kappa \quad .$$

2.1.3 The 4–geometry

We will describe the 4–geometry (locally) in terms of a PMC foliation by symmetric Cauchy surfaces (Σ, h). It turns out, that there is not much left to do. Of course, we need the strong energy condition in M, and we can choose Σ without loss of generality to have non-vanishing second fundamental form. In view of theorem 2.2 of part I this is enough to guarantee the existence of a local in time PMC foliation of a neighbourhood of Σ in M. The remaining question is, if the leaves of the foliation turn out to be symmetric. But the arguments given in the corresponding place in [H3] do apply to the present situation, since the Laplacian of Σ splits into the Laplacian of F and a part depending only on x, which coincides with the one in the plane symmetric case and we end up with

2.1. Proposition.

Let (M,g) be a globally hyperbolic, spatially compact spacetime with local $U(1)\times U(1)$ symmetry, obeying the strong energy condition.

Then there exists a local in time PMC foliation $\{S_t\}$ in M, covering a neighbourhood $]t_1,t_2[\times\Sigma$ of $\Sigma = S_0$, in M. Moreover, if Σ is symmetric, then all the leaves of the foliation are symmetric, too, and there are coordinates $(x^\mu) = (t,x,y^2,y^3)$ adapted to the foliation, which cast the metric into the form

$$g = \begin{pmatrix} -N^2+(A\nu^1)^2+|\overline{\nu}|^2 & A^2\nu^1 & \overline{\nu}_2 & \overline{\nu}_3 \\ A^2\nu^1 & A^2 & 0 & 0 \\ \overline{\nu}_2 & 0 & \overline{g}_{22} & \overline{g}_{23} \\ \overline{\nu}_3 & 0 & \overline{g}_{32} & \overline{g}_{33} \end{pmatrix} \quad , \tag{11}$$

where N, A, ν^1, $\nu^A = \overline{\nu}^A$, $\overline{\nu}_A = \overline{g}_{AB}\,\overline{\nu}^B$, $\overline{g} = r^2\tilde{g}$ are functions on $]t_1,t_2[\times S^1$, with $r = Aa$ and $a = a(t)$ only. The quantities with an overbar transform under some representation of $\mathrm{GL}(2,\mathbb{Z})$ after each transition $x \longmapsto x+2\pi$ in S^1. The shift functions are fixed by the conditions $\nu^1(t,0) = \nu^1(t,2\pi) = 0$ and $\nu^A(t,0) = 0$.

The orthonormal frames $\{{}^4e_\mu\}$ and $\{{}^4\sigma^\mu\}$ associated with the $3+1$–split read explicitly

$$\begin{aligned}
{}^4e_0 &= n = N^{-1}(\partial_t - \nu^i\partial_i) \\
{}^4e_1 &= m = A^{-1}\,\partial_1 \\
{}^4e_2 &= e_2 = (Aa)^{-1}\left(+e^{-W/2}\cosh V/2\;\partial_2 \;-\; e^{W/2}\sinh V/2\;\partial_3\right) \\
{}^4e_3 &= e_3 = (Aa)^{-1}\left(-e^{-W/2}\sinh V/2\;\partial_2 \;+\; e^{W/2}\cosh V/2\;\partial_3\right) \\
{}^4\sigma^0 &= N\,dt \\
{}^4\sigma^1 &= A(\nu^1\,dt + dx) \\
{}^4\sigma^2 &= (Aa)\left(e^{W/2}\cosh V/2\,(\overline{\nu}^2\,dt + dy^2) \;+\; e^{-W/2}\sinh V/2\,(\overline{\nu}^3\,dt + dy^3)\right) \\
{}^4\sigma^3 &= (Aa)\left(e^{W/2}\sinh V/2\,(\overline{\nu}^2\,dt + dy^2) \;+\; e^{-W/2}\cosh V/2\,(\overline{\nu}^3\,dt + dy^3)\right)
\end{aligned}$$

Finally, the Ricci rotation coefficients will be calculated. Some expressions are not explicitly given, since they turn out to be too complicated. Instead, the dependence on the geometric quantities involved will be specified in square brackets:

$$\begin{aligned}
{}^4\gamma^0_{01} &= {}^4\gamma^1_{00} = A^{-1}N^{-1}N' \\
{}^4\gamma^0_{0B} &= {}^4\gamma^B_{00} = 0 \\
{}^4\gamma^0_{11} &= {}^4\gamma^1_{01} = -K \\
{}^4\gamma^0_{1B} &= {}^4\gamma^B_{10} = (Aa)^{-1}\overline{e}_B{}^C\eta_C \\
{}^4\gamma^0_{AB} &= {}^4\gamma^B_{A0} = -\overline{e}_A{}^C\overline{e}_B{}^D\,\kappa_{CD} \\
{}^4\gamma^1_{01} &= 0 \\
{}^4\gamma^1_{0B} &= {}^4\gamma^1_{0B}[N^{-1},\overline{\nu},\overline{\nu}',A,A^{-1},A',a,a^{-1},V,W,V',W',\dot{V},\dot{W}] = -{}^4\gamma^B_{01} \\
{}^4\gamma^C_{0B} &= {}^4\gamma^C_{0B}[N^{-1},\nu^1,A,A^{-1},A',a,a^{-1},V,W,V',W',\dot{V},\dot{W}] \\
{}^4\gamma^k_{ij} &= \gamma^k_{ij}[A^{-1},A',V,W,V',W'] \quad ,
\end{aligned}$$

where the dependence on the derivatives of V and W is linear.

2.2 The field equations

Again we represent the matter quantities by $\rho = T(n,n)$, $j = -T(n,m) = A^{-1} j_1$ and $S_{ij} = T({}^4e_i, {}^4e_j)$.

2.2.1 2+1–decomposition of the constraints

First consider the Hamiltonian constraint

$$R + H^2 - |k|^2 = 16\pi\rho \quad .$$

Decomposing R by the Gauss–Codazzi formula and k by (10) we get the following expression:

$$m(\operatorname{tr}\lambda) = 8\pi\rho - H \operatorname{tr}\kappa + |\eta|^2 + \tfrac{1}{2}\big(\tfrac{3}{2}(\operatorname{tr}\lambda)^2 + |\tilde{\lambda}|^2\big) + \tfrac{1}{2}\big(\tfrac{3}{2}(\operatorname{tr}\kappa)^2 + |\tilde{\kappa}|^2\big) \quad . \tag{12}$$

The first component of the momentum constraint

$$\nabla^i k_{i1} - \nabla_1 H = 8\pi j_1$$

can be decomposed into

$$m(\operatorname{tr}\kappa) = -8\pi j_1 - H \operatorname{tr}\lambda + \tfrac{3}{2} \operatorname{tr}\kappa \operatorname{tr}\lambda + \tilde{\lambda}^{AB}\tilde{\kappa}_{AB} \tag{13}$$

(where the symmetry forced $\operatorname{div}\eta = 0$). The other components of the momentum constraints are calculated analogously to (again some terms cancel out due to symmetry)

$$m(\eta_B) = -8\pi j_B + (\operatorname{tr}\lambda)\eta_B \quad . \tag{14}$$

As in [H3] a convenient form of the equation is achieved by introducing the null expansions

$$\vartheta_\pm := -(\operatorname{tr}\lambda \pm \operatorname{tr}\kappa) = \frac{2}{r} k_\pm(r) \tag{15}$$

(compare the formulas for λ and κ in 2.1.1), which yields an easy formula for the Hawking mass

$$\text{M} := m_H(F) = -\tfrac{1}{8} r (r\vartheta_+)(r\vartheta_-) = -\tfrac{1}{2} r ({}^4\nabla^\alpha r)({}^4\nabla_\alpha r) \quad , \tag{16}$$

by $\vartheta_+\vartheta_- = 4/r^2 ({}^4\nabla^\alpha r)({}^4\nabla_\alpha r)$.

Adding and subtracting the constraint equations now yields

$$m(\vartheta_\pm) = -8\pi(\rho \mp j_1) - |\eta|^2 - \tfrac{1}{2}|\tilde{\lambda} \pm \tilde{\kappa}|^2 \mp H\vartheta_\pm - \tfrac{3}{4}\vartheta_\pm^2 \tag{17}$$

and the transition to the variables $\omega_\pm := r\vartheta_\pm$ casts this equation into

$$\begin{aligned} m(\omega_\pm) &= m(r)\vartheta_\pm + r m(\vartheta_\pm) \\ &= -8\pi r(\rho \mp j_1) - r|\eta|^2 - \tfrac{1}{2} r |\tilde{\lambda} \pm \tilde{\kappa}|^2 \mp H\omega_\pm + \tfrac{1}{4\pi}\left(\omega_+\omega_- - 2\omega_\pm^2\right) \quad , \end{aligned} \tag{18}$$

where the identity $m(r)\vartheta_\pm = \frac{1}{4\pi}((\omega_+ + \omega_-)\omega_\pm)$ has been used.

2.2.2 2+2–decomposition of the field equations

With the 2+2–geometry in mind, where we denote tensor components with respect to the frame of projection operators $\{q_a, p_A\}$, we write the equation (7a) for ${}^4R_{\underline{ab}}$ in terms of $\tilde{g}$,

$$ {}^4R_{\underline{ab}} = \underline{R}_{ab} - 2r^{-1}\underline{\nabla}_a\underline{\nabla}_b\, r + \tfrac{1}{4}(\underline{\nabla}_a\, \tilde{g}_{AB})(\underline{\nabla}_b\, \tilde{g}^{AB}) + 2\underline{g}_{ab}|\eta|^2 \quad , $$

as a field equation in the quotient spacetime $(B, \underline{g})$ for the radius function r. We still have to eliminate the unknown $\underline{R}_{ab} = -\underline{K}\,\underline{g}_{ab}$, where $\underline{K} := \underline{R}_{0101}$ denotes the Gaussian curvature of $(B, \underline{g})$ in (M, g). Contracting the equation for $\underline{R}_{ab}$ and replacing $\underline{\Delta} r$ by (8) yields an equation for $\underline{K}$:

$$ \underline{K} = \tfrac{1}{2}(-{}^4R^{\underline{c}}_{\underline{c}} + {}^4R^{\underline{A}}_{\underline{A}}) + r^{-2}(\underline{\nabla}^c r)(\underline{\nabla}_{\underline{c}} r) + \tfrac{1}{8}(\underline{\nabla}^c \tilde{g}_{AB})(\underline{\nabla}_c \tilde{g}^{AB}) + 3|\eta|^2 $$

Inserting this into the equation for ${}^4R_{\underline{ab}}$ results in

$$ \begin{aligned} \underline{\nabla}_a\underline{\nabla}_b r = &- \tfrac{1}{2}r^{-1}(\underline{\nabla}^c r)(\underline{\nabla}_c r)\underline{g}_{ab} - \tfrac{1}{2}r\Big({}^4R_{\underline{ab}} - \tfrac{1}{2}({}^4R^{\underline{c}}_{\underline{c}} - {}^4R^{\underline{A}}_{\underline{A}})\underline{g}_{ab}\Big) \\ &+ \tfrac{1}{8}r\Big((\underline{\nabla}_a \tilde{g}_{AB})(\underline{\nabla}_b \tilde{g}^{AB}) - \tfrac{1}{2}(\underline{\nabla}^c \tilde{g}_{AB})(\underline{\nabla}_c \tilde{g}^{AB})\underline{g}_{ab}\Big) - \tfrac{1}{2}r|\eta|^2\underline{g}_{ab} \end{aligned} $$

Now we are going to interpret the first three terms appearing on the right-hand side of the field equation: Recalling the definition of the mass function M in (16) we can write the first term as $\frac{\mathrm{M}}{r^2}\underline{g}_{ab}$. The curvature expression in the second term can be rewritten using Einstein's equations in terms of the projected energy momentum tensor $\underline{T}$ in the form $8\pi(\underline{T}_{ab} - \operatorname{tr}\underline{T}\,\underline{g}_{ab})$.

To simplify the third term, there is more structure involved than already presented. The space of two-dimensional metrics $\mathrm{Bil}(2, \mathbb{R})$ is a three dimensional real vector space, whose elements we denote by $\tilde{\phi} = \tilde{\phi}^i\, E_i$, where the basis $\{E_i\}$ has been chosen as $E_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $E_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, $E_3 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. In this picture the determinant is a quadratic form on $\mathrm{Bil}(2, \mathbb{R})$, and the matrix representation with respect to the given basis turns out to be $-\eta_{ij}$, where $\eta_{ij} = \mathrm{diag}(-++)$ denotes the standard Minkowski metric in three dimensional Minkowski space. The restriction of this space to the hypersurface $\eta(\tilde{\phi}, \tilde{\phi}) = -1 \iff \det\tilde{\phi} = 1$ is the well known hyperbolic plane H^2, and we have established a correspondence between the two-dimensional unit-determinant metrics $\tilde{g}$ and the elements of the hyperbolic plane. Considering $\tilde{\phi}$ as a function on the quotient manifold B, introduced in 2.1.1 we have a map $\tilde{\phi} : (B, \underline{g}) \longrightarrow (H^2 \subset \mathbb{R}^{2,1}, \eta_{ij})$, $(t, x) \longmapsto \tilde{\phi}^i(\tilde{g}(t, x))E_i$. Now let us interpret $\tilde{\phi}$ for a moment as a wave map from B to H^2, then the energy momentum tensor ${}^{\phi}\underline{T}$ associated with $\tilde{\phi}$ reads

$$ (19) \qquad {}^{\phi}\underline{T}_{ab} = \underline{\nabla}_a\tilde{\phi}^i\underline{\nabla}_b\tilde{\phi}_i - \tfrac{1}{2}(\underline{\nabla}^c\tilde{\phi}^i\underline{\nabla}_c\tilde{\phi}_i)\underline{g}_{ab} $$

and with the correspondences $\tilde{\phi}_i = \eta_{ij}\tilde{\phi}^j = -\tilde{g}^{AB}$ and $\tilde{\phi}_i\tilde{\phi}^i = -\frac{1}{2}\,\tilde{g}_{AB}\,\tilde{g}^{AB}$, we see that the third term in the field equation for r can be rewritten as $-\frac{1}{4}r\,{}^{\phi}\underline{T}_{ab}$.

Collecting all these things together we get the field equation for r in a convenient form:

$$ (20) \qquad \underline{\nabla}_a\underline{\nabla}_b r = \frac{\mathrm{M}}{r^2}\underline{g}_{ab} - 4\pi r(\underline{T}_{ab} - \operatorname{tr}\underline{T}\,\underline{g}_{ab}) - \tfrac{1}{4}r\,{}^{\phi}\underline{T}_{ab} - \tfrac{1}{2}r|\eta|^2\underline{g}_{ab} \quad . $$

Differentiating the mass function we find for the mass flux equation the formula

$$ (21) \qquad \underline{\nabla}_a\mathrm{M} = \Big(4\pi r^2(\underline{T}_{ab} - \operatorname{tr}\underline{T}\,\underline{g}_{ab}) + \tfrac{1}{4}r\,{}^{\phi}\underline{T}_{ab}\Big)(\underline{\nabla}^b r) + \tfrac{1}{2}r^2|\eta|^2\underline{\nabla}_a r \quad . $$

The contribution of ${}^\phi\underline{T}$ to the right-hand sides of these equations is known in so far, that ${}^\phi\underline{T}$ obeys the dominant and strong energy conditions. Furthermore, we can again rewrite its components using the formulas for the trace free parts of λ and κ in 2.1.1 (remember, that $\underline{\nabla}_0$, $\underline{\nabla}_1$ act as n, respectively m, on $\tilde{g}$, in the formula for ${}^\phi\underline{T}_{ab}$). The result is

(22a) $\quad {}^\phi\rho = {}^\phi p = |\tilde{\lambda}|^2 + |\tilde{\kappa}|^2$

(22b) $\quad {}^\phi j_{\underline{1}} = -2\tilde{\lambda}_{AB}\tilde{\kappa}^{AB}$

Now we want to analyse the underlying wave map. To make things more explicit, consider the parameters V, W of $\tilde{g}$ as coordinates of H^2 in view of the correspondence $\tilde{\phi}^i E_i$. The pullback under this parametrization yields a representation $\hat{h}$ of the metric $(\eta_{ij})_{|H^2}$, with the explicit form $\hat{h}_{IJ} = dV^2 + (\cosh V)^2\, dW^2$, leading to $\hat{\Gamma}^3_{23} = \tanh V$ and $\hat{\Gamma}^2_{33} = -\sinh V \cosh V$ as non-vanishing Christoffel symbols. The pullback of the map $\tilde{\phi}$ under this parametrization induces a map $\phi : (B, \underline{g}) \longrightarrow (H^2, \hat{h})$, $\phi(t,x) = (V(t,x), W(t,x))$. The wave map ϕ is then defined to obey

(23) $$\underline{\nabla}^c\underline{\nabla}_c\phi^K + \hat{\Gamma}^K_{IJ}\underline{\nabla}^c\phi^I\underline{\nabla}_c\phi^J = 0 \quad .$$

Let us turn back to the field equations. Writing equation (7c) in the form

$$ {}^4R_{\overline{A}\overline{B}} + 2\overline{\eta}_A\overline{\eta}_B = -\tfrac{1}{2}r^{-2}\overline{g}_{AD}\underline{\nabla}^c(r^2\overline{g}^{CD}\underline{\nabla}_c\overline{g}_{BC}) = -\tfrac{1}{2}\tilde{g}_{AD}\underline{\nabla}^c(\tilde{g}^{CD}\underline{\nabla}_c r^2\tilde{g}_{BC})$$

we arrive after some calculation at

$$\begin{aligned}\underline{\nabla}^c(r^2\underline{\nabla}_c\tilde{g}_{AB}) =& r^2\tilde{g}^{CD}(\underline{\nabla}^c\tilde{g}_{AD})(\underline{\nabla}_c\tilde{g}_{BC}) \\ &- 16\pi\big(\overline{T}_{AB} - \tfrac{1}{2}\operatorname{tr}\overline{T}\overline{g}_{AB}\big) - 4\big(\overline{\eta}_A\overline{\eta}_B - \tfrac{1}{2}|\eta|^2\overline{g}_{AB}\big) \quad ,\end{aligned}$$

where $\overline{T}$ denotes the orthogonal projection of the energy momentum tensor T into $(F, \overline{g})$. More explicitly we get

(24a) $$\begin{aligned}\underline{\nabla}^c(r^2\underline{\nabla}_c V) =& r^2\sinh V\cosh V(\underline{\nabla}^c W)(\underline{\nabla}_c W) \\ &- 2r^2(\cosh V)^{-1}\Big((T_{23} - \tfrac{1}{2}\tilde{g}^{AB}T_{AB}\tilde{g}_{23}) - \tfrac{1}{2}(\eta_2\eta_3 - \tfrac{1}{2}\tilde{g}^{AB}\eta_A\eta_B\tilde{g}_{23})\Big)\end{aligned}$$

(24b) $$\begin{aligned}\underline{\nabla}^c(r^2\underline{\nabla}_c W) =& -r^2\tanh V(\underline{\nabla}^c W)(\underline{\nabla}_c V) \\ &- r^2(\cosh V)^{-1}\Big((e^{-W}T_{22} - e^W T_{33}) - \tfrac{1}{2}(e^{-W}(\eta_2)^2 - e^W(\eta_3)^2)\Big)\end{aligned}$$

or

(25a) $$\begin{aligned}\underline{\nabla}^c\underline{\nabla}_c V - \sinh V\cosh V(\underline{\nabla}^c W)(\underline{\nabla}_c W) =& -2/r(\underline{\nabla}^c r)(\underline{\nabla}_c V) \\ &- 2(\cosh V)^{-1}\Big((T_{23} - \tfrac{1}{2}\tilde{g}^{AB}T_{AB}\tilde{g}_{23}) - \tfrac{1}{2}(\eta_2\eta_3 - \tfrac{1}{2}\tilde{g}^{AB}\eta_A\eta_B\tilde{g}_{23})\Big)\end{aligned}$$

(25b) $$\begin{aligned}\underline{\nabla}^c\underline{\nabla}_c W + \tanh V(\underline{\nabla}^c W)(\underline{\nabla}_c V) =& -2/r(\underline{\nabla}^c r)(\underline{\nabla}_c W) \\ &- (\cosh V)^{-1}\Big((e^{-W}T_{22} - e^W T_{33}) - \tfrac{1}{2}(e^{-W}(\eta_2)^2 - e^W(\eta_3)^2)\Big)\end{aligned}$$

The left-hand sides of equations (25) coincide with the the left-hand side of (23) and we conclude, that V and W solve an inhomogeneous wave map.

2.2.3 3+1–decomposition of the field equations

Now we want to calculate the ADM equations and bring them into a form most similar to the equations in the plane symmetric case of part I. The formulas concerning the 3–geometry in 2.1.2 are of particular importance for the calculations here, as well as the definitions of λ, κ and η in 2.1.1. A straightforward calculation then yields the following set of equations.

The constraint equations

$$(A^{1/2})'' = \tfrac{1}{8}A^{5/2}\Big(H^2 - \tfrac{1}{2}(H-K)^2 - K^2 - |\tilde\kappa|^2 - |\tilde\lambda|^2 - 2|\eta|^2 - 16\pi\rho\Big) \tag{26}$$

$$K' = -3A^{-1}A'K + A^{-1}A'H + H' - A\tilde\kappa_{AB}\tilde\lambda^{AB} + 8\pi Aj \tag{27}$$

$$\eta'_B = -2A^{-1}A'\eta_B - 8\pi A j_B \quad . \tag{28}$$

The equations fixing the foliation

$$N'' = -A^{-1}A'N' + A^2N\Big(\tfrac{1}{2}(H-K)^2 + K^2 + |\tilde\kappa|^2 + 2|\eta|^2 + 4\pi(\rho + \operatorname{tr} S)\Big) - A^2 \tag{29}$$

$$\dot H = 1 + \nu^1 H' \tag{30}$$

$$(\partial_t - \nu^1\partial_x)H' = \nu^{1\prime}H' \quad . \tag{31}$$

The evolution equations

$$\dot A = -NAK + A\nu^{1\prime} + A'\nu^1 \tag{32}$$

$$\dot a = -\tfrac{1}{2}Na(H-3K) - a\nu^{1\prime} \tag{33}$$

$$\begin{aligned}\dot K =& \nu^1K' - A^{-2}(N'' - A^{-1}A'N')\\ &+ N\Big(-4A^{-5/2}(A^{1/2})'' + (A^{-2}A')^2 + HK\\ &\qquad - |\tilde\lambda|^2 + 2|\eta|^2 - 8\pi(A^{-2}S_{11} + \tfrac{1}{2}(\rho - \operatorname{tr} S))\Big)\end{aligned} \tag{34}$$

$$\begin{aligned}\dot\eta_B =& \big(\tfrac{1}{2}N(H+K) - 2A^{-1}A'\nu^1\big)\eta_B - 2N\tilde\kappa_{BC}\eta^C\\ &+ \tfrac{1}{2}A^{-2}A'(H-K)\nu_B - \big(\tilde\kappa_B^D\tilde\lambda_{DC} + \tfrac{1}{2}(H-K)\tilde\lambda_{BC} - A^{-2}A'\tilde\kappa_{BC}\big)\nu^C\\ &- 8\pi A\nu^1 j_B\end{aligned} \quad . \tag{35}$$

As in [H3] integration of (33) over the circle yields the analogous equation for the first component of the shift vector,

$$\nu^{1\prime} = -\tfrac{1}{2}N(H-3K) + \tfrac{1}{2}\int_{S^1} N(H-3K) \tag{36}$$

and the definition of the second fundamental form provides some additional equations, namely

$$\nu^{B\prime} = -2NA\eta^B \tag{37}$$

and the coordinate representations of λ and κ in 2.1.1. The remaining equations for V and W are still missing. To this end we supplement the system (26)–(37) by the field equations (24) or (25), and end up with the full system of equations.

As in the plane symmetric case, we can formulate an existence and uniqueness theorem for solutions of the equations (26)–(37), (25). First we define a *symmetric initial data set*

for a spacetime with local $U(1)\times U(1)$ symmetry by the smooth collection (Σ, h, k), where Σ denotes a (possibly non-trivial) torus bundle over the circle and the fundamental forms h and k are represented in a suitable coordinate system as (9) and (10), respectively. If there are matter fields, then we assume the matter data and equations to be smooth and symmetric, leading to a well posed Cauchy problem coupled to the reduced field equations in harmonic coordinates.
On the universal cover $\hat{\Sigma}$ of Σ the induced data then is invariant under the action of G, hence G acts isometrically on the whole Cauchy development, which induces a local $U(1)\times U(1)$ symmetry on the Cauchy development of (Σ, h, k). Assuming $\lambda := |k|^2 + 4\pi(\rho + \operatorname{tr} S) > 0$ somewhere on Σ, we get for $t_0 \in \mathbb{R}$ a unique symmetric local in time PMC foliation, defined on some time interval containing t_0, with $\Sigma = S_{t_0}$, and coordinates described in proposition 2.1 and we just have proved:

2.2. Proposition.

Let (Σ, h, k) be a symmetric initial data set for a spacetime with local $U(1)\times U(1)$ symmetry, with matter obeying the strong energy condition and $\lambda > 0$ somewhere on Σ. Further, let t_0 denote an arbitrary real number.
Then there exists a $\delta > 0$ and a PMC foliated surface symmetric spacetime $(\bar{M}, \bar{g})$ diffeomorphic to $]t_0 - \delta, t_0 + \delta[\times\Sigma$ with an embedding $\iota : \Sigma \longrightarrow \bar{M}$, satisfying $\iota(\Sigma) = S_{t_0}$ and $\iota_* h$, $\iota_* k$ are the first and second fundamental form of S_{t_0} in $(\bar{M}, \bar{g})$. $(\bar{M}, \bar{g})$ obeys the strong energy condition and $\bar{g}$ can be written in the form described in proposition 2.1. This construction is unique up to the choice of t_0 and δ.

2.2.4 The expanding model

Now we can proceed in close analogy to the corresponding analysis in part I. The formula (16) for the Hawking mass shows, that $\operatorname{grad} r$ is timelike as long as $\mathrm{M} > 0$. Indeed, proposition 3.1 in [R] proves this, provided the spacetime is not flat and the dominant energy condition holds.

Therefore, under these conditions $\operatorname{grad} r$ is timelike and ϑ_+, ϑ_- have fixed and opposite signs. Without loss of generality we choose the time orientation, such that $\operatorname{grad} r$ is past pointing and dr is future pointing (by the induced time orientation of the cotangent bundle). Then $\vartheta_+ > 0$ and $\vartheta_- < 0$ which classifies the spacetime as expanding in the sense described in part I.
Again we expect the singularity in the distant past from our symmetric initial data Cauchy surface Σ and any symmetric Cauchy surface S in M is not maximal with mean curvature not everywhere positive on S.

2.3 A priori estimates for the field equations

Assume the dominant and strong energy condition to be fulfilled in (M, g). Let Σ be a symmetric Cauchy surface in M with strictly negative mean curvature H and denote by $\{S_t\}$, $t \in]t_1, t_2[$ the local in time PMC foliation with $\Sigma = S_0$ and the time orientation chosen in correspondence with 2.2.4, such that H decreases with decreasing PMC time.
We consider the past $D^-(\Sigma)$ of Σ. In $D^-(\Sigma)$ the mean curvature is bounded from above by some $\bar{H} < 0$, and $H = \bar{H}$ only on Σ. Thus $|H|$ is bounded from below and as long as H remains finite, we find the following estimates.

Consider first the constraint equation in the form (18). The dominant energy condition gives the inequality

$$m(\omega_\pm) \le \mp H\omega_\pm + \tfrac{1}{4\pi}\left(\omega_+\omega_- - 2\omega_\pm^2\right) \quad ,$$

from which we infer the basic estimate

$$(38) \qquad |\vartheta_\pm| \le 4|H| \qquad \Longrightarrow \qquad |A^{-2}A'| \le C\,, \quad |K| \le C \quad .$$

Now we can perform nearly the same estimates, as we have done in the corresponding place in [H3]. The additional terms appearing here in the more general equations (26)–(27), (29)–(33) and (36) do not cause any problems. There is only one additional estimate, which will become important in our further analysis, resulting from the integration of equation (26) along S^1, bounding not only $\int_{S^1}\rho$, but also $\int_{S^1}(|\tilde\lambda|^2 + |\tilde\kappa|^2 + |\eta|^2) = \int_{S^1}({}^\phi\rho + |\eta|^2)$, and we can state the analogous

2.3. Proposition.

Let (M,g) be a globally hyperbolic, spatially compact spacetime with local $U(1)\times U(1)$ symmetry, obeying the dominant and strong energy condition. Assume the existence of a symmetric Cauchy surface Σ with strictly negative mean curvature. In particular we get from proposition 2.1 a PMC time coordinate t, ranging in $]t_1,t_2[$ with $\Sigma = \{t=0\}$ and H decreases with decreasing t.
Then we have uniformly on $]t_1,0]$

$$|A|, |A^{-1}|, |A'|, |a|, |a^{-1}|, |H|, |H'|, |K|, |N|, |N^{-1}|, |N'|, |\nu^1|, |\nu^{1\prime}| \le C \quad .$$

Until now the control over the coefficients of the fundamental forms h and k is not complete. We still need bounds for the metric coefficients V, W, the components η and κ of k and the remaining components ν^B of the shift vector. We will get bounds for most of these quantities here in this section, using the bound for $\int_{S^1}(\rho + {}^\phi\rho + |\eta|^2)$.

2.4. Proposition.

Under the hypotheses of proposition 2.3 we get uniform bounds for

$$|V|, |W|, |\eta_B|, |\nu^B|, |\nu^{B\prime}|$$

on $]t_1,0]$.

Proof.
The bounds for η_B, ν^B and $\nu^{B\prime}$ are simple consequences of the bounds for V and W, since having bounded them, we can conclude as follows: The bounds for V and W allow coordinate components to be bounded in terms of components according to the orthonormal frame vectors $\overline{e}_2, \overline{e}_3$ in $(F, \overline{g})$ defined in 2.1.1. We adopt the convention, that a hat above indices denotes components with respect to an orthonormal frame. Having this in mind we see, that $\int_{S^1}\rho$ bounds $\int_{S^1} j_B \le C\int_{S^1} j_{\hat B}$ by the dominant energy condition, so integration of the constraint (28) yields a bound for the difference $A^2\eta_B|_{x_1}^{x_2}$, which is independent of t. Using $\eta_B \le C\eta_{\hat B} \le C(1+|\eta|^2)$ we can bound the integral $\int_{S^1}|A^2\eta_B|$ by $C\int_{S^1}(1+|\eta|^2)$, which together with the estimated difference yields the desired estimate for η_B. Now we get immediately a bound for $\nu^{B\prime}$ by inspection of equation (37), and the condition $\nu^B(t,0) = 0$ bounds ν^B.

Let us now investigate the field equations (24) for V and W, considered as equations on $(B, \underline{g})$ endowed with the coordinates (t, x), as described in the end of 2.1.1. Then we calculate explicitly in these coordinates

$$\begin{aligned}|\tilde{\lambda}|^2 + |\tilde{\kappa}|^2 &= \tfrac{1}{4}(m(\tilde{g}_{AB})m(\tilde{g}^{AB}) + n(\tilde{g}_{AB})n(\tilde{g}^{AB})) \\ &= \tfrac{1}{2}A^{-2}\left(V'^2 + (\cosh V)^2 W'^2\right) \\ &+ \tfrac{1}{2}N^{-2}\left((\dot{V} - \nu^1 V')^2 + (\cosh V)^2(\dot{W} - \nu^1 W')^2\right) \quad ,\end{aligned}$$

and we see, that the first term on the right-hand side of equations (24) can be bounded in terms of this expression.

For the second term on the right-hand side of (24) we exploit the special structure appearing there. The first aim is, to express the tensor components with respect to an orthonormal frame. Looking at the definitions of the coframe $\{\overline{\sigma}^B\}$ defined in 2.1.1 we recognize the relations

$$\begin{aligned}e^{W/2}dy^2 &= r^{-1}\left(+\cosh V/2\,\overline{\sigma}^2 - \sinh V/2\,\overline{\sigma}^3\right) \\ e^{-W/2}dy^3 &= r^{-1}\left(-\sinh V/2\,\overline{\sigma}^2 + \cosh V/2\,\overline{\sigma}^3\right) \quad .\end{aligned}$$

If we mark tensor components expressed in this basis by a tilde, we see that the terms in brackets in both equations (24) are easily rewritten with respect to this basis, by simply putting a tilde above each tensor component and deleting the factors $e^{\pm W/2}$. But we have schematically $\eta = \eta_B dy^B = \tilde{\eta}_B\, e^{\pm W/2}\, dy^B = f_B(\cosh V/2)\tilde{\eta}_B\, \overline{\sigma}^B$, with $f_B(\cosh V/2)\tilde{\eta}_B = \eta_{\hat{B}}$ is the component according to the orthonormal frame. A similar relation holds for T_{AB}. Since we have $\eta_{\hat{B}} \leq 1 + |\eta|^2$ and $T_{\hat{A}\hat{B}} \leq \rho$ we conclude, that the terms in brackets in (24) are bounded by $C\cosh V(1 + \rho + |\eta|^2)$, which leads together with our result about the first term on the right-hand side of (24) to

$$|\underline{\nabla}^c(r^2\underline{\nabla}_c\phi)| \leq C(1 + \rho + |\tilde{\lambda}|^2 + |\tilde{\kappa}|^2 + |\eta|^2) \quad ,$$

and since the integration of the right-hand side along S^1 is already bounded we get

$$\int_{S^1} |\underline{\nabla}^c(r^2\underline{\nabla}_c\phi)| \leq C$$

uniformly for each $t \in]t_1, 0]$. Note, that we use the wave map $\phi = (V, W)$ defined in 2.2.2 in order to abbreviate some formulas only.

On the other hand, we can interpret the integrated quantity as a divergence, a fact we will take advantage of to get rid of the integral sign. To this end consider a point (t, x) in the quotient manifold B, with $t \in]t_1, 0[$. We need estimates uniformly in t for t approaching t_1. We define the (upside down) characteristic triangle T in B, by its counterclockwise oriented boundary $\partial T = \gamma_+ + \gamma_0 + \gamma_-$, where $\gamma_\pm$ denote the characteristic curves of (24), with $\dot{\gamma}_\pm = k_\pm = m \pm n$ connecting (t, x) with $(0, x_\pm)$ and γ_0 is the curve in the hypersurface $\{t = 0\}$ from $(0, x_+)$, to $(0, x_-)$ with $\dot{\gamma}_0 = -m$. Define $v \in T^\perp F$ by $v^a = r^2\underline{\nabla}^a\phi$, then the corresponding 1-form ω reads $\omega = \iota(v)\underline{\Omega}$, where $\underline{\Omega}$ denotes the volume form with respect to $\underline{g}$. Expressing tensor components with respect to the projected orthonormal frame $\{\underline{\sigma}_a = \underline{q}_a\}$ (see 2.1.1) we have $v^a = r^2\begin{pmatrix} -\underline{\nabla}_0\phi \\ \underline{\nabla}_1\phi \end{pmatrix}$, and $\omega = -r^2\left(m(\phi)\underline{\sigma}^0 + n(\phi)\underline{\sigma}^1\right)$. The pullback of ω along ∂T has the three parts $\gamma_\pm^*\omega = \mp r^2 k_\pm(\phi \circ \gamma_\pm(u))\, du$, $\gamma_0^*\omega = r^2 n(\phi(x)) A\, dx$.

Thus, setting $\tau := \sqrt{2}t$

$$\begin{aligned}\int_{\partial T}\omega &= \int_\tau^0 -r^2k_+(\phi(\gamma_+(u)))\,du + \int_{x_+}^{x_-} n(\phi(x))A\,dx + \int_0^\tau +r^2k_-(\phi(\gamma_-(u)))\,du\\ &= -\int_\tau^0 r^2k_+(\phi(\gamma_+(u)))\,du - \int_\tau^0 r^2k_-(\phi(\gamma_-(u)))\,du + \int_{x_+}^{x_-} n(\phi(x))A\,dx\\ &= -r^2\phi\Big|_{\gamma_+(\tau)}^{\gamma_+(0)} - r^2\phi\Big|_{\gamma_-(\tau)}^{\gamma_-(0)} + \int_\tau^0 k_+(r^2)\phi + \int_\tau^0 k_-(r^2)\phi + \int_{x_+}^{x_-} n(\phi(x))A\,dx\end{aligned}$$

where integration by parts has been carried out in the last step. Note, that the last term is bounded, since that integration takes place on the line $t=0$ in B, where ϕ is smooth. In addition $k_\pm(r^2)$ is bounded everywhere by proposition 2.3. Further, the first terms can be evaluated to $-r^2\phi\big|_{\gamma_+(\tau)}^{\gamma_+(0)} - r^2\phi\big|_{\gamma_-(\tau)}^{\gamma_-(0)} = 2(r^2\phi)(t,x) - (r^2\phi)(0,x_+) - (r^2\phi)(0,x_-)$, where the last two terms live on the line $t=0$, and therefore are bounded.
Stokes' theorem now applies to the present situation, $\int_{\partial T}\omega = \int_T dw = \int_T(\operatorname{div} v)\,\underline{\Omega}$. Putting all this together we get an estimate

$$\|r^2\phi(t)\|_{L^\infty} \le C\left(1 + \int_t^0 \|r^2\phi(s)\|_{L^\infty}\,ds + \int_t^0\int_{S^1}|\underline{\nabla}^c(r^2\underline{\nabla}_c\phi)|\,ds\right)$$

where the last term is already known to be bounded uniformly in t by our previous analysis. Hence we can apply Gronwall's inequality and the bounds for r, r^{-1} obtained in proposition 2.3 complete our argument, arriving finally at $|V|,|W| \le C$ uniformly on $]t_1, 0[$. ■

Unfortunately, there is still a lack of control over the components κ of k (respectively its traceless part, since $(\operatorname{tr}\kappa)\,\underline{g} = (H-K)\underline{g} \le C$). It turns out, that the necessary bounds for the first derivatives of ϕ depend on a bound for the matter quantities and vice versa. The analysis of this dependence will be performed in the next section, where we have to take a specific matter model into account.

2.4 Higher order estimates

2.4.1 First order estimates

We start with the derivation of first order estimates for the metric coefficients $\phi = (V,W)$. To keep things simple, we consider collisionless matter, as described by the Einstein Vlasov system.

The particle density f on the mass shell is governed by the geodesic spray X with $X = v^\alpha {}^4e_\alpha - {}^4\gamma^\kappa_{\mu\nu}v^\mu v^\nu \frac{\partial}{\partial v^\kappa}$, and the killing vector fields ∂_A define two conserved quantities $g(X,\partial_A)$, which bound v^A. To bound v^1 we consider the characteristics of the Vlasov equation (compare the explicit formulas in [H3]):

$$\begin{aligned}\frac{dv^1}{ds} &= -\left(e_1(N)v^0 + N(-k_{rs}{e_1}^r{e_j}^s + {}^4\gamma^1_{0j})v^j + N\gamma^1_{jk}v^jv^k/v^0\right)\\ &\le C\left(1 + Q_1(s) + v^1(s)\right) \quad ,\end{aligned}$$

where $Q_1 := \|\underline{D}\phi\|_{L^\infty}$, $\underline{D} := (\partial_t, \partial_x)$ and we used the fact, that in view of the boundedness of v^A, the vanishing of ${}^4\gamma^1_{01}$ and the special form of the non-vanishing rotation coefficients no term involving the product $Q_1 v^1$ occurs. Thus we find for the quantity $\bar{P}_f(s) = \{\sup|v| \,|\, v \in \operatorname{supp} f\}$ (which measures the matter quantities, compare [H3]) the Gronwall-like estimate

$$1 + \bar{P}_f(t) \leq C \left(1 + \bar{P}_f(0) + \int_t^0 (\bar{P}_f(s) + Q_1(s))\, ds \right)$$

This inequality shows, that we indeed have to estimate the matter quantities together with the second fundamental form κ. The next step consists in finding a complementary inequality for Q_1, which yields in combination with the inequality just obtained a true Gronwall estimate for $1 + \bar{P}_f + Q_1$.

We need again the field equations for V and W, but now it is more convenient to analyse the equations in the 'wave map form' (25). First consider the left-hand side of the field equation. We want to express this derivative operator in terms of the characteristic vector fields $k_\pm = m \pm n$ tangent to the characteristic curves $\gamma_\pm$ introduced in the previous section (in the proof of proposition 2.4). In the quotient manifold $(B, \underline{g})$ one calculates the non-vanishing Ricci rotation coefficients to

$$\begin{array}{l} \underline{\gamma}^0_{01} = \underline{\gamma}^1_{00} = (AN)^{-1} N' \\ \underline{\gamma}^0_{11} = \underline{\gamma}^1_{01} = (AN)^{-1} (\dot{A} - (A\nu^1)') \end{array} \quad ,$$

thus $\underline{\nabla}_{\underline{e}_a} \underline{e}_b$ is bounded. The characteristics $k_\pm$ expressed in the coordinates of B are simple linear combinations of the $\underline{e}_a$, hence $\underline{\nabla}_{k_\pm} k_\pm$ is also bounded in B. We will use this fact to estimate commutators such as $[m, n]$ or $[k_+, k_-]$. Transforming from the orthonormal frame $\{\underline{e}_a\}$ in $(B, \underline{g})$ to the frame $\{k_\pm\}$, we can write the left-hand side of (25) as

$$\underline{\nabla}^c \underline{\nabla}_c \phi^K + \hat{\Gamma}^K_{IJ} \underline{\nabla}^c \phi^I \underline{\nabla}_c \phi^J = \underline{\nabla}_{k_+} \left(k_-(\phi^K) \right) + \hat{\Gamma}^K_{IJ}\, k_+(\phi^I) k_-(\phi^J) + [k_+, k_-](\phi^K)$$

and there is an analogous equation for k_+ and k_- interchanged.

Let us turn now to the right-hand side of (25). The terms not known to be bounded are of the form $k_\pm(\phi)$ and T_{AB}. Fortunately we have control over the matter term, since the bounds on V, W and v^A justify the inequality $T_{AB} \leq C\, \bar{P}_f$ and we obtain

$$\underline{\nabla}_{k_+} \left(k_-(\phi^K) \right) + \hat{\Gamma}^K_{IJ}\, k_+(\phi^I) k_-(\phi^J) \leq C \Big(1 + k_\pm(\phi) + \bar{P}_f \Big) \quad .$$

The left-hand side is still non-linear in the first derivatives of ϕ, so this inequality is not in the appropriate form to apply some kind of Gronwall estimate. But we can overcome this difficulty by adapting an observation of Gu ([Gu], see also [R] for explanation), who considered wave maps defined on two dimensional Minkowski space. We already performed the first step, consisting of a transformation to characteristic coordinates. Now we define the vector field $\hat{k}_\pm$ over the map ϕ,

$$\hat{k}_\pm := \phi_*(k_\pm) = k_\pm(\phi) = k_\pm(\phi^K) \frac{\partial}{\partial \phi^K} = k_\pm(V) \frac{\partial}{\partial V} + k_\pm(W) \frac{\partial}{\partial W} \quad ,$$

thus bounding the length of $\hat{k}_\pm$ accomplishes the estimate of Q_1. The vector field $\hat{k}_\pm$ over ϕ is a section in the bundle $\phi^*(TH^2)$ over B and the covariant derivative operator $\phi^* \hat{\nabla}$ over ϕ acts like $(\phi^* \hat{\nabla})_v \hat{X} = v(\hat{w}^K) \frac{\partial}{\partial \phi^K} + \hat{w}^K \hat{\nabla}_{\phi_* v} \frac{\partial}{\partial \phi^K}$ for every vector $v \in TB$ and every $\hat{X} = \hat{w}^K \frac{\partial}{d\phi^K}$ with $w^K : B \to H^2$.

Evaluating this expression with $v = k_+$ and $\hat{X} = \hat{k}_-$ gives

$$(\phi^* \hat{\nabla})_{k_+} \hat{k}_- = k_+(k_-(\phi^K)) \frac{\partial}{\partial \phi^K} + \hat{\Gamma}^K_{IJ}\, k_+(\phi^I) k_-(\phi^J) \frac{\partial}{\partial \phi^K}$$

(and analogously for $k_\pm$ interchanged). The components in this equation are similar to the left-hand side of the inequality already obtained for the field equations. The remaining term involves $k_\pm(\phi)$ times the connection coefficients in B with respect to the frame $\{k_\pm\}$, which we know to be bounded. Thus we get for the field equations the inequality

$$|(\phi^*\hat{\nabla})_{k_+}\hat{k}_-| \leq C\Big(1 + k_\pm(\phi) + \bar{P}_f\Big) \quad ,$$

expressing an estimate about the growth of $\hat{k}_\pm$ during the transport along the characteristic curve $\gamma_\mp$. This inequality is in the appropriate form for a Gronwall-like estimate: Taking the maximum for each fixed t allows us, to combine the inequalities for $|\hat{k}_+|$ and $|\hat{k}_-|$, replaced collectively in terms of Q_1. Hence

$$Q_1(t) \leq C\left(1 + Q_1(0) + \int_t^0 (\bar{P}_f(s) + Q_1(s))\, ds\right) \quad ,$$

where $\gamma_\pm$ has been reparametrized in terms of t. Combining this with the inequality for $\bar{P}_f$ we arrive at

$$1 + \bar{P}_f(t) + Q_1(t) \leq C\left(1 + \bar{P}_f(0) + Q_1(0) + \int_t^0 (1 + \bar{P}_f(s) + Q_1(s))\, ds\right) \quad .$$

Performing a Gronwall argument we have proven

2.5. Proposition.

Let (M, g, f) be a globally hyperbolic, spatially compact solution of the Einstein-Vlasov system with local $U(1)\times U(1)$ symmetry, which possesses a symmetric Cauchy surface Σ with strictly negative mean curvature. The PMC time coordinate t ranges in $]t_1, t_2[$ with $\Sigma = \{t = 0\}$ and H decreases with decreasing t.
Then we get uniform bounds for

$$|\dot{V}|, |\dot{W}|, |V'|, |W'|, \rho$$

on $]t_1, 0]$.

2.4.2 Second order estimates

We still have to do one further step before it will be possible to apply some iteration scheme. We need to establish bounds for the second derivatives of $\phi = (V, W)$ together with bounds for the first derivatives of the matter variables. Again it turns out, that it is not possible to get separate estimates for these quantities.

First we differentiate the field equation (25) and obtain an equation of the form

$$k_\pm\left(k_\mp(\partial_x\phi^K)\right) + 2\hat{\Gamma}^K_{IJ}\, k_\pm(\phi^I)k_\mp(\partial_x\phi^J) = [...]\partial_x T_{AB} + \text{lower order terms} \quad ,$$

where [...] is an abbreviation for some term involving already bounded quantities, in particular we see by a quick look at the 3+1–field equations, whose right-hand sides are now bounded, that for the Ricci rotation coefficients $\partial_x\underline{\gamma} \leq C$ holds. Note, that the differentiation kills the nonlinearity in the equation, but we have to deal with first order derivatives of the matter variables instead. These quantities cause serious trouble, because if we differentiate the Vlasov equation, terms involving the second derivatives of ϕ times first derivatives of f come up, thus there is no direct Gronwall argument possible. To attack

this difficulty we have to combine the equations, using an idea of [GS].
First we integrate the equation along the characteristics $\gamma_\pm$ to get an integral inequality. In order to apply a Gronwall argument all what remains to do is to bound the term $\int_{\gamma_\pm} \partial_x T_{AB}$. Then we express T_{AB} in terms of its components with respect to the orthonormal frame, producing merely some already bounded quantities and the frame components of the energy-momentum tensor for Vlasov-type matter look like $\int_{\gamma_\pm} \left(\int v_A v_B / v^0 \, \partial_x f \, dv\right)$. Finally we represent ∂_x by a linear combination of $k_\pm$ and $\underline{X} := \partial_t + (NA^{-1}v^1/v^0 - \nu^1)\partial_x$, which is the part of the characteristic vector field for the Vlasov equation, lying in B. The transformation to this basis is obviously bounded and we can proceed as follows. The part involving $k_\pm$ permits a direct application of the integration by parts rule, which contributes something bounded by proposition 2.5. The remaining part can be treated in a similar fashion, after inserting the Vlasov equation into $\underline{X}(f)$. This yields terms involving only first order derivations of ϕ (arising from the ${}^4\gamma$'s) times $\frac{\partial f}{\partial v}$, and again we can perfom an integration by parts with respect to the velocity integral. The result consists in terms already bounded and we are through. Applying now Gronwall's inequality we arrive at a bound for $\|\partial_x \underline{D}\phi\|_{L^\infty}$ and inserting this result into the field equation (25) we get $Q_2 := \|\underline{D}^2\phi\|_{L^\infty}$ bounded.
An immediate consequence is, that we also have established a bound for the first derivatives of f. Differentiating the Vlasov equation with respect to x or v yields an equation for $\partial_x f$ or $\partial_v f$ respectively, with bounded characteristics and an inhomogeneous term, consisting of $({}^4e_\mu)^\alpha$, ${}^4\gamma^\kappa_{\mu\nu}$ and their derivatives with respect to x. These terms are either bounded by what has been said above or by inspection of the field equations (26)–(37). Having bounded the spatial and velocity derivatives of f, the structure of the Vlasov equation bounds immediately $\partial_t f$ and we have proven

2.6. Proposition.
Under the hypotheses of proposition 2.5 we get uniform bounds for

$$|\ddot{V}|, |\dot{V}'|, |V''|, |\ddot{W}|, |\dot{W}'|, |W''| \quad \textit{and} \quad |\dot{\rho}|, |\rho'|$$

on $]t_1, 0]$.

2.4.3 The iteration scheme

Following the analysis of part I we need some additional framework.

2.7. Definition.

$$\mathcal{F} := \big(A, a, V, W, N, \nu^1, \nu^B, H, K, \eta_B\big)$$

$$\Phi := (\rho, {}^\phi\rho)$$

It turns out, that what we have done in the previous subsections are the first steps towards the matter regularity property of part I. Here we will define this property adapted to the present situation with the notational conventions introduced as follows: α denotes a multi index for derivatives in $(B, \underline{g})$ and $\underline{D} := (\partial_t, \partial_x)$. Then the matter regularity property reads

$$(39) \qquad |\underline{D}^\alpha \mathcal{F}| \leq C \qquad \Longrightarrow \qquad |\underline{D}^\alpha \Phi| \leq C$$

and we state

2.8. Proposition.

Under the hypotheses of proposition 2.5 (39) *holds for all α uniformly on* $]t_1, 0]$.

Proof.
We prove the statement by induction with respect to $|\alpha|$. The induction hypotheses for $\alpha = 0$ is contained in the statements of the propositions 2.5 and 2.6. To proceed further let us assume $|\underline{D}^\alpha \mathcal{F}| \leq C$ for some α with $|\alpha| =: p \geq 1$. Then we can assume $|\underline{D}^{p-1}\Phi| \leq C$ by induction. We have to show $|\underline{D}^p \Phi| \leq C$, or extending our previous notation, $P_p := \|\underline{D}^p f\|_{L^\infty} \leq C$ and $Q_{p+1} := \|\underline{D}^{p+1}\phi\|_{L^\infty} \leq C$.

Following the analysis of the previous subsection we set $q := p - 1$ and differentiate the field equations (25) with $\partial_x \underline{D}^q$, which yields schematically

$$
\begin{aligned}
k_\pm\big(k_\mp(\partial_x \underline{D}^q \phi)\big) =& [\partial_x \underline{D}^q \underline{\gamma}, \partial_x \underline{D}^p r, \partial_x \underline{D}^q \phi]\big(k_\mp \partial_x \underline{D}^q \phi\big) \\
&+ [\underline{D}^q \phi]\, \partial_x \underline{D}^q T_{AB} + [\partial_x \underline{D}^q \phi, \partial_x \underline{D}^q \eta, \underline{D}^q T_{AB}] \quad ,
\end{aligned}
$$

where quantities in square brackets abbreviates some expressions formed by them, whose detailed structure is not important for our analysis here. We want to perform the same kind of argument as we have done in the proof of the second order estimates. In order to do this we must bound the quantities in the square brackets:

- The terms in the second and third square bracket are bounded by the induction hypotheses.
- For the first square bracket we can proceed as follows. $\partial_x \underline{D}^q \phi$ is already bounded. For $\partial_x \underline{D}^p r$ we have to estimate $\partial_x \underline{D}^p A = \partial_x^2 \underline{D}^q A$. This can be done by applying $\underline{D}^q$ to (26) in a strictly analogous manner as in the first item in the proof of lemma 3.6 in [H3], applied to equation (26). The definition of $\underline{\gamma}$ in 2.4.1 shows, that the terms in $\partial_x \underline{D}^q \underline{\gamma}$ not already bounded are $\partial_x^2 \underline{D}^q \nu^1$, $\partial_x^2 \underline{D}^q \overline{N}$ and $\partial_x \partial_t \underline{D}^q A$. Applying $\underline{D}^p$ to (36) gives a bound for the first quantity. Using $\underline{D}^q$ on (29), then the argument in the fourth step in the proof of lemma 3.6 in [H3], applied to (29) bounds the second quantity. Now all quantities appearing on the right-hand side of ($\partial_x \underline{D}^q$ applied to) (32) are bounded, and this bounds $\partial_t \partial_x \underline{D}^q A$.

Turning now to the Vlasov equation and applying $\partial_x \underline{D}^q$ yields an inhomogeneous equation for $\partial_x \underline{D}^q f$ with the same characteristics. Thus we can apply the same trick as in the previous subsection, substituting ∂_x by $k_\pm$ and $\underline{X}$. Then again we are concerned with integration by parts, which yields only bounded terms, and the inhomogeneous term of the differentiated Vlasov equation, which is also bounded by induction hypotheses. All together we can apply Gronwall's inequality to find $\|\partial_x \underline{D}^p \phi\|_{L^\infty} \leq C$, and immediately $Q_{p+1} \leq C$ by inserting the spatial bounds into the field equation. The Vlasov equation for $\partial_x \underline{D}^q f$ now bounds $\partial_x \underline{D}^q f$ and analogously for $\partial_v \underline{D}^q f$, which automatically bounds $\partial_t \underline{D}^q f$ by inserting the spatial and velocity bounds into the differentiated Vlasov equation. Thus we have also bounded P_p, which completes the proof of the proposition. ■

To extend the local in time PMC foliation, our aim is to find uniform C^∞ bounds of all geometric and matter quantities, when the PMC time t approaches t_1. Property (39) shows, that it is enough to bound the geometric quantities $\underline{D}^\alpha \mathcal{F}$ for all α as long as t or respectively the mean curvature remains finite . Therefore all we need is the analogue of the lemmas 3.6 and 3.7 in [H3], starting with the propositions 2.3, 2.4, 2.5 and 2.6.
The first part is straightforward, given an arbitrary multi index α and $|\underline{D}^\alpha \mathcal{F}| \leq C$ we

find $|\partial_x \underline{D}^\alpha \mathcal{F}| \leq C$ by inspection of the relevant (differentiated) field equations (26)–(37) (compare the arguments given in the proof of lemma 3.6 in [H3] where necessary), with the terms on the right-hand sides bounded by property (39). Moreover, proposition 2.8 provides not only bounds for the spatial derivatives, but also for the time derivatives of ϕ by the definition of ${}^\phi\rho$. This allows us to proceed as follows: First we bound, starting from $\underline{D}^\alpha \mathcal{F}$ all spatial derivatives $\partial_x^k \underline{D}^\alpha \mathcal{F}$, $k = 1, 2, \ldots$ Then we bound, starting successively from $\partial_x^k \underline{D}^\alpha \mathcal{F}$, $k = 1, 2, \ldots$ the quantities $\underline{D}^\beta \mathcal{F}$ for all multi indices β with $|\beta| = |\alpha| + k$, redistributing the spatial derivatives into time derivatives. This procedure will successively bound all derivatives of $\mathcal{F}$, taking advantage of the fact, that in each step all lower order derivatives with the same order of time derivatives have been bounded as well as at least one more spatial order derivative in the step with one order less in time derivatives. In a more compact formulation, we have to show the boundedness of $\partial_t \underline{D}^\alpha \mathcal{F}$, having already bounds for $\partial_x \underline{D}^\alpha \mathcal{F}$ and $\underline{D}^\alpha \mathcal{F}$.

We accomplish this step for each member of $\mathcal{F}$, by again considering some of the (differentiated by $\underline{D}^\alpha$) 3+1–field equations. We see immediately from (32) and (33), that we have bounds for $\partial_t \underline{D}^\alpha A$, $\partial_t \underline{D}^\alpha a$. The bounds for $\partial_t \underline{D}^\alpha$ applied on H, K and η are straightforward, too. Of course, $\partial_t \underline{D}^\alpha \phi$ is already bounded by property (39). Moreover we can strengthen the regularity. Since we have already a bound for $\partial_x \underline{D}^\alpha \mathcal{F}$ proposition 2.8 provides us with bounds for $\partial_x \underline{D}^\alpha$ applied to ρ and $\underline{D}\phi$. Applying $\partial_x \underline{D}^\alpha$ to the field equation (25) we first get a bound for $\partial_x \underline{D}^\alpha \underline{D}\phi$ and then for $\partial_t^2 \underline{D}^\alpha \phi$ by inserting the first result into the field equation. This in turn used together with the bounds for the differentiated matter variable in the differentiated Vlasov equation, bounds $\partial_t \underline{D}^\alpha \rho$.
Now we turn to the analysis of the differentiated lapse equation. First we see, that the bound for $\partial_t \underline{D}^\alpha \nu$ follows from the bound of $\partial_t \underline{D}^\alpha N$. For the latter one, we follow the argument given in the corresponding place in the proof of lemma 3.7 in [H3], where all that is needed, has already been bounded by the arguments just given here and we are done.

Therefore we end up with

2.9. Theorem.

Let (M, g, f) be a globally hyperbolic, spatially compact solution of the Einstein-Vlasov system with local $U(1)\times U(1)$ symmetry, which possesses a symmetric Cauchy surface Σ with strictly negative mean curvature $H \leq \bar{H} < 0$ and $H = \bar{H}$ somewhere on Σ.
Then all of the past of Σ admits a PMC foliation $\{S_t\}$, where t takes all values in the interval $]-\infty, 0]$ and H takes all values in $]-\infty, \bar{H}]$.

2.4.4 Improving the result

Here we will try to do the same construction as in the corresponding place in [H3], to get rid of the restriction concerning the fixed sign of the mean curvature H on the Cauchy surfaces. So we assume M to be non-flat and denote by Σ an arbitrary symmetric Cauchy surface in M. In order to follow the steps performed in the plane symmetric case in part I, remember first from 2.2.4, that the spacetime is expanding. Note further, that the Einstein-Vlasov system fulfills the dominant and strong energy condition as well as the non-negative pressures condition.

Then we find for the past domain of dependence $D^-(\Sigma)$

- Since the spacetime is expanding, dr is future pointing, thus r is bounded in $D^-(\Sigma)$. By assumption, M^{-1} is bounded on the compact surface Σ. Unfortunately, unlike in the plane symmetric case we cannot conclude from the non-negative pressures condition and the fact that dr is future pointing, that $d\text{M}$ is past pointing, since in equation (21) for the mass flux the term involving η contributes with the wrong sign (luckily, the contribution of the energy momentum tensor ${}^{\phi}\underline{T}$ (19) for the wave map ϕ does not cause any trouble, due to the energy conditions automatically fulfilled by ${}^{\phi}\underline{T}$).
  To overcome this difficulty we impose the Gowdy–type symmetry condition, thus η vanishes identically in the spacetime (and by the gauge condition $\nu^B(t,0) = 0$ this is also true for $\bar{\nu}$ in view of (37)). Then we conclude, that M^{-1} is bounded in $D^-(\Sigma)$ as desired.

  Now we want to apply theorem 2.1 in [B] to establish a bound for the length of all timelike curves in $D^-(\Sigma)$. For this we have to adapt the proof a little bit. Inspection of the proof shows, that the argument relies on the inequality $\ddot{r} \leq -\frac{\text{M}}{r^2}$ satisfied by the area radius along timelike geodesics, where the dot denotes differentiation along the curve. But looking at the field equation (20) for the area radius easily establishes this relation (since η vanishes) and we are through.

- The basic estimate (38) together with (16) shows, that r^{-1} and M are bounded.

- Lemma 2 in [BR] applies and we get bounds for the volume and its inverse for any Cauchy surface in $D^-(\Sigma)$.

The volume $V(t)$ for the leaf S_t of the PMC foliation is given by $V(t) = (4\pi)^2 a^{-1} \int_{S^1} r^3$ and a closer look at the estimates given in the corresponding place in [H3] shows, that the arguments apply literally. This relies on the facts, that on the one hand the additional equations (25) are unaffected by the reparametrization of the foliation, thus the estimates for ϕ and its derivatives done in the previous subsections hold. On the other hand the construction in part I is based mainly on the structures introduced by the matter regularity property and the lemmas 3.6 and 3.7, a program we adapted successfully to the more general situation here.

So we finally arrive at

2.10. Theorem.

Let (M, g, f) be a globally hyperbolic, spatially compact solution of the Einstein-Vlasov system with Gowdy–type local $U(1) \times U(1)$ symmetry and Σ be a symmetric Cauchy surface.
If (M, g) is non-flat then we can foliate all of the past of Σ by PMC hypersurfaces, where the time function takes on all values in the interval $]-\infty, 0]$ and the mean curvature of the leaves tends uniformly to $-\infty$ for $t \to -\infty$.

Using the PMC leaves as barrier surfaces we get the

2.11. Corollary.

In the situation of theorem 2.10 $D^-(\Sigma)$ possesses a CMC Cauchy surface for each value of the mean curvature in $]-\infty, \min_\Sigma H[$.

3 Conclusion and outlook

We have seen in this second part, that the program initiated in part I has been successfully extended to the more general case of local $U(1) \times U(1)$ symmetric spacetimes in close analogy to the plane symmetric spacetimes. Therefore all what has been mentioned in the corresponding place in [H3] applies.

A corresponding analysis for spacetimes with local $U(1) \times U(1)$ symmetry has been independently performed in the work [An] of Andréasson. His construction leads to stronger results (as the work [Re] of Rein in the surface symmetric case of part I), but the time functions used are defined in terms of the symmetry. Thus it is not clear, how to generalize them. In view of Andréasson's work, the present analysis can be seen as a suggestion pointing in a slightly different direction.

In comparison with the spacetimes considered in part one the control of the momenta of the Vlasov particles turned out to be more complicated, in particular the coupling to gravitational waves required second order estimates before the iteration procedure could be performed. In this process we took advantage from the wave-map structure of the dynamical part of the geometry driven by the simple form of the Vlasov equation.
Nevertheless one can hope, that the approach may be generalized to other matter models and geometries, since the obstructions seem to be of technical nature only.

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