STRUCTURE CONSTANT OF BOUNDARY OPERATORS IN LIOUVILLE FIELD THEORY

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Liouville field theory is considered with conformal boundary. The analytical expression for the expectation value of the boundary three point function is explicitly given in terms of the fusion matrix determining the monodromy properties of the conformal blocks.

1 Liouville theory on the sphere

The celebrated two dimensional Liouville Field Theory has been attracting attention for the last 20 years due to its appearance in different contexts like two dimensional gravity, non critical string theory and D-branes physics. It is the most simple case of *non compact* conformal field theories, (*i.e.* CFT with a continuous spectrum of primary fields), and is a natural starting point for the developement of techniques for the exact solution of such conformal field theories.

Let us recall some results on Liouville theory, see [4] for more details and references: LFT on the sphere is semiclassically defined by the following action

$$\mathcal{A}_L = \int \left(\frac{1}{4\pi}(\partial_a \phi)^2 + \mu e^{2b\phi} \right) d^2x \tag{1}$$

with the following boundary condition on the Liouville field ϕ

$$\phi(z,\bar{z}) = -Q\log(z\bar{z}) + O(1) \qquad \text{at } |z| \to \infty. \tag{2}$$

The parameter b is the coupling constant, the scale parameter μ is the cosmological constant, and Q is the background charge

$$Q = b + 1/b.$$

It was first proposed in [1] that Liouville theory can be quantized as a conformal field theory with a space of states that decomposes as follows into irreducible unitary highest weight representations $\mathcal{V}_\alpha$ of the Virasoro algebra:

$$\mathcal{H} = \int_{\mathbb{S}} d\alpha\, \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha, \qquad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+ \tag{3}$$

The highest weight Δ_α of the representation V_α is parametrized as $\Delta_\alpha = \alpha(Q-\alpha)$. The action of the Virasoro algebra on $\mathcal{H}$ is generated by the modes of the energy momentum tensors:

$$\begin{aligned} T(z) &= -(\partial\phi)^2 + Q\partial^2\phi, \\ \bar{T}(\bar{z}) &= -(\bar{\partial}\phi)^2 + Q\bar{\partial}^2\phi. \end{aligned} \tag{4}$$

The central charge of the Virasoro algebra is then given in terms of b via

$$c_L = 1 + 6Q^2.$$

The local observables can be generated from the fields $V_\alpha(z,\bar{z})$ which semiclassically ($b \to 0$) correspond to exponential functions $e^{2\alpha\phi(z,\bar{z})}$ of the Liouville field. The fields $V_\alpha(z,\bar{z})$ transform as primary fields under conformal transformations with conformal weight Δ_α. Thanks to conformal symmetry the fields $V_\alpha(z,\bar{z})$ are fully characterized by the three point functions

$$C(\alpha_3,\alpha_2,\alpha_1) = \lim_{z_3\to\infty} |z_3|^{4\Delta_{\alpha_3}} \langle 0| V_{\alpha_3}(z_3,\bar{z}_3) V_{\alpha_2}(1,1) V_{\alpha_1}(0,0)|0\rangle$$

An explicit formula for the three point function was proposed in [2,3][a]

$$C(\alpha_3,\alpha_2,\alpha_1) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \frac{\Upsilon_0\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_3-\alpha_2)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)} \tag{5}$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, $\Upsilon_0 = \mathrm{res}_{x=0}\frac{d\Upsilon_b(x)}{dx}$.
These pieces of information indeed amount to a full characterization of Liouville theory on the sphere or cylinder: Multipoint correlation functions can be factorized into three point functions by summing over intermediate states. Let us consider the four point function $\langle 0|\prod_{i=1}^4 V_{\alpha_i}(z_i,\bar{z}_i)|0\rangle$. Such four point

[a] see Appendix for some definitions and properties of the special functions used in this article.

functions may be represented by summing over intermediate states from the spectrum (3) iff the variables $\alpha_4, \ldots, \alpha_1$ are restricted to the range[b]

$$\begin{aligned} &2|\mathrm{Re}(\alpha_1 + \alpha_2 - Q)| < Q,\ 2|\mathrm{Re}(\alpha_1 - \alpha_2)| < Q, \\ &2|\mathrm{Re}(\alpha_3 + \alpha_4 - Q)| < Q,\ 2|\mathrm{Re}(\alpha_3 - \alpha_4)| < Q. \end{aligned} \tag{6}$$

Inserting a complete set of intermediate states between $\langle 0|V_{\alpha_4}V_{\alpha_3}$ and $V_{\alpha_2}V_{\alpha_1}|0\rangle$ would lead to an expression of the following form:

$$\begin{aligned} &\langle V_{\alpha_4}(z_4,\bar{z}_4)V_{\alpha_3}(z_3,\bar{z}_3)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_1}(z_1,\bar{z}_1)\rangle = \\ &\int_0^\infty C(\alpha_4,\alpha_3,Q/2-iP)C(Q/2+iP,\alpha_2,\alpha_1)|\mathcal{F}^s(\Delta_{\alpha_i},\Delta,z_i)|^2 dP \end{aligned} \tag{7}$$

$\mathcal{F}^s(\Delta_{\alpha_i}, \Delta, z_i)$ is the s-channel conformal block which is completly determined by the conformal symmetry (although no closed formula is known for it in general).

$$\begin{aligned} \mathcal{F}^s(\Delta_{\alpha_i},\Delta,z_i) = & \\ & (z_4-z_2)^{-2\Delta_2}(z_4-z_1)^{\Delta_2+\Delta_3-\Delta_1-\Delta_4}(z_4-z_3)^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} \\ & (z_3-z_1)^{\Delta_4-\Delta_1-\Delta_2-\Delta_3} \times \mathcal{F}\left(\begin{matrix}\alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4\end{matrix} \mid P \mid \eta\right) \end{aligned}$$

where

$$\eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_4)(z_1-z_3)}$$

and $\Delta_{\alpha_i} = \alpha_i(Q-\alpha_i)$, $\Delta = \frac{Q^2}{4} + P^2$.
Locality of the fields V_α or associativity of the operator product expansion would lead to an alternative representation for $\langle 0|\prod_{i=1}^4 V_{\alpha_i}(z_i,\bar{z}_i)|0\rangle$ as sum over *t-channel* conformal blocks $\mathcal{F}^t$:

$$\begin{aligned} &\langle V_{\alpha_4}(z_4,\bar{z}_4)V_{\alpha_3}(z_3,\bar{z}_3)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_1}(z_1,\bar{z}_1)\rangle = \\ &\int_0^\infty C(\alpha_4,Q/2-iP,\alpha_1)C(Q/2+iP,\alpha_3,\alpha_2)|\mathcal{F}^t(\Delta_{\alpha_i},\Delta,z_i)|^2 dP \end{aligned} \tag{8}$$

[b] It turns out [3] that the four-point function defined in the range (6) permits a meromorphic continuation to generic values of $\alpha_4, \ldots, \alpha_1$.

For the equivalence of the two representations (7) and (8) it is crucial that there exist [4] invertible fusion transformations between s- and t-channel conformal blocks, defining the fusion coefficients:

$$\mathcal{F}^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}}, z_i) = \int_{\mathbb{S}} d\alpha_{32} F_{\alpha_{21}\alpha_{32}} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \mathcal{F}^t(\Delta_{\alpha_i}, \Delta_{\alpha_{32}}, z_i). \tag{9}$$

In [5], an explicit formula for this fusion matrix was proposed in terms of the Racah-Wigner coefficients for an appropriate continuous series of representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ with deformation parameter $q = e^{i\pi b^2}$. This formula was subsequently [4] confirmed by direct calculation. The resulting expression for the fusion coefficients is the following:

$$\begin{aligned}
&F_{\sigma_2\beta_3}\begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} = \\
&\frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(Q+\beta_2-\beta_1-\beta_3)}{\Gamma_b(2Q-\sigma_1-\beta_1-\sigma_2)\Gamma_b(\sigma_1+\sigma_2-\beta_1)\Gamma_b(Q-\beta_1-\sigma_2+\sigma_1)}\times \\
&\frac{\Gamma_b(\beta_3+\sigma_1+\sigma_3-Q)\Gamma_b(\sigma_1+\sigma_3-\beta_3)\Gamma_b(\sigma_3+\beta_3-\sigma_1)}{\Gamma_b(\beta_2+\sigma_2+\sigma_3-Q)\Gamma_b(\sigma_2+\sigma_3-\beta_2)\Gamma_b(\sigma_3+\beta_2-\sigma_2)}\times \\
&\frac{\Gamma_b(Q+\beta_3-\beta_2-\beta_1)\Gamma_b(Q-\beta_3-\sigma_1+\sigma_3)}{\Gamma_b(Q-\beta_1-\sigma_1+\sigma_2)\Gamma_b(Q-\beta_2-\sigma_2+\sigma_3)}\frac{\Gamma_b(2Q-2\sigma_2)\Gamma_b(2\sigma_2)}{\Gamma_b(Q-2\beta_3)\Gamma_b(2\beta_3-Q)} \\
&\times\frac{1}{i}\int_{-i\infty}^{i\infty} ds\, \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(Q+s)},
\end{aligned} \tag{10}$$

where:

$$\begin{aligned}
&U_1 = \sigma_2+\sigma_1-\beta_1, \qquad V_1 = Q+\sigma_2-\beta_3-\beta_1+\sigma_3, \\
&U_2 = Q+\sigma_2-\beta_1-\sigma_1, V_2 = \sigma_2+\beta_3+\sigma_3-\beta_1, \\
&U_3 = \sigma_2+\beta_2+\sigma_3-Q, V_3 = 2\sigma_2, \\
&U_4 = \sigma_2-\beta_2+\sigma_3.
\end{aligned}$$

An important identity satisfied by the fusion coefficients is the so-called pentagonal equation, which follows from a similar identity satisfied by the Racah-Wigner coefficient mentioned previously [6].

$$\begin{aligned}
\int_{\mathbb{S}} d\delta_1 F_{\beta_1\delta_1}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \beta_2 & \alpha_1 \end{bmatrix} F_{\beta_2\gamma_2}\begin{bmatrix} \alpha_4 & \delta_1 \\ \alpha_5 & \alpha_1 \end{bmatrix} F_{\delta_1\gamma_1}\begin{bmatrix} \alpha_4 & \alpha_3 \\ \gamma_2 & \alpha_2 \end{bmatrix} \\
= F_{\beta_2\gamma_1}\begin{bmatrix} \alpha_4 & \alpha_3 \\ \alpha_5 & \beta_1 \end{bmatrix} F_{\beta_1\gamma_2}\begin{bmatrix} \gamma_1 & \alpha_2 \\ \alpha_5 & \alpha_1 \end{bmatrix}
\end{aligned} \tag{11}$$

2 Liouville theory on domains with boundary

One may also be interested to define a version of Liouville theory on a simply connected domain Γ with nontrivial boundary $\partial\Gamma$. For definiteness, we will only consider the conformally equivalent cases where Γ is either the unit disk, the upper half plane or the infinite strip.

Semiclassically, one may define the theory by means of the action

$$A_{\text{bound}} = \int_{\Gamma} \left(\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) d^2x + \int_{\partial\Gamma} \left(\frac{Qk}{2\pi} + \mu_B e^{b\phi} \right) dx, \tag{12}$$

where k is the curvature of the boundary $\partial\Gamma$ and μ_B is the so-called boundary cosmological constant. For the description of exact results in the quantum theory it was found to be useful [7] to parametrize μ_B by means of a variable σ that is related to μ_B via

$$\cos\left(2\pi b(\sigma - \frac{Q}{2})\right) = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)}. \tag{13}$$

Requiring μ_B to be real one finds the two following regimes for the parameter σ:

1. if $\frac{\mu_B}{\sqrt{\mu}}\sqrt{\sin(\pi b^2)} > 1$, then σ is of the form $\sigma = Q/2 + iP$
2. if $\frac{\mu_B}{\sqrt{\mu}}\sqrt{\sin(\pi b^2)} < 1$, then σ is real.

Anticipating that all relevant objects will be found to possess meromorphic continuations w.r.t. the boundary parameters σ, we shall discuss only the first regime explicitly in the following.

The Hamiltonian interpretation of the theory [9] is simplest in the case where Γ is the infinite strip. The associated Hilbert space $\mathcal{H}_B$ was found in [9] to decompose as follows into irreducible representations of the Virasoro algebra:

$$\mathcal{H}^{\mathrm{B}} = \int_{\mathbb{S}}^{\oplus} d\beta \, \mathcal{V}_\beta. \tag{14}$$

The highest weight states generating the subrepresentation $\mathcal{V}_\beta$ in $\mathcal{H}^{\mathrm{B}}$ will be denoted $|\beta; \sigma_2, \sigma_1\rangle$, where σ_2 (resp. σ_1) are the parameters of the boundary conditions associated to the left (resp. right) boundaries of the strip. It was proposed in [7] [9] that the states $|\beta; \sigma_2, \sigma_1\rangle$ satisfy a reflection relation of the form

$$|\beta; \sigma_2, \sigma_1\rangle = S(\beta; \sigma_2, \sigma_1) |Q - \beta; \sigma_2, \sigma_1\rangle. \tag{15}$$

which expresses the totally reflecting nature of the Liouville potential in (12). The following formula was given in [7] for the reflection coefficient $S(\beta;\sigma_2,\sigma_1)$:

$$\begin{aligned} S(\beta,\sigma_2,\sigma_1) =& (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(Q-2\beta)}\times \\ &\times\frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}\frac{S_b(\sigma_2+\sigma_1-\beta)S_b(2Q-\beta-\sigma_1-\sigma_2)}{S_b(\beta+\sigma_2-\sigma_1)S_b(\beta+\sigma_1-\sigma_2)}. \end{aligned} \tag{16}$$

In addition to the fields $V_\alpha(z,\bar{z})$ localized in the interior of Γ, one one may now also consider operators ${}^{\sigma_2}B_\beta^{\sigma_1}(x)$ that are localized at the boundary $\partial\Gamma$. The insertion point x may separate segments of the boundary with different boundary conditions σ_2 and σ_1. The boundary fields ${}^{\sigma_2}B_\beta^{\sigma_1}(x)$ are required to be primary fields with conformal weight $\Delta_\beta = \beta(Q-\beta)$. They are therefore expected to create states $|\beta;\sigma_2,\sigma_1\rangle$ and $\langle\beta;\sigma_2,\sigma_1|$ via

$$\lim_{x\to 0} {}^{\sigma_2}B_\beta^{\sigma_1}(x)|0\rangle = |\beta;\sigma_2,\sigma_1\rangle, \quad \lim_{x\to\infty}\langle 0|{}^{\sigma_2}B_\beta^{\sigma_1}(x)|x|^{2\Delta_\beta} = \langle Q-\beta;\sigma_2,\sigma_1|. \tag{17}$$

To fully characterize LFT on the upper half plane, one needs to determine some additional structure functions beside the bulk three point function $C(\alpha_3,\alpha_2,\alpha_1)$

1. Bulk one point function: [7,8]

$$\langle V_\alpha(z,\bar{z})\rangle = \frac{U(\alpha|\mu_B)}{|z-\bar{z}|^{2\Delta_\alpha}}. \tag{18}$$

2. Boundary two point function: [7]

$$\left\langle {}^{\sigma_1}B_{\beta_1}^{\sigma_2}(x)\,{}^{\sigma_2}B_{\beta_2}^{\sigma_1}(0)\right\rangle = \frac{\delta(\beta_2+\beta_1-Q)+S(\beta_1,\sigma_2,\sigma_1)\delta(\beta_2-\beta_1)}{|x|^{2\Delta_{\beta_1}}}. \tag{19}$$

The appearance of the second term in (19) is a consequence of the reflection property (15).

3. bulk-boundary two point function [13][c]

$$\left\langle V_\alpha(z,\bar{z})\,{}^{\sigma}B_\beta^{\sigma}(x)\right\rangle = \frac{R(\alpha,\beta|\sigma)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}\,|z-x|^{2\Delta_\beta}} \tag{20}$$

[c]the bulk one point function is a special case of the bulk-boundary coefficient with $\beta = 0$

4. boundary three point function

$$\left\langle {}^{\sigma_1} B^{\sigma_3}_{\beta_3}(x_3)^{\sigma_3} B^{\sigma_2}_{\beta_2}(x_2)^{\sigma_2} B^{\sigma_1}_{\beta_1}(x_1) \right\rangle = \frac{C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{21}$$

Taking advantage of the reflection property (15), we shall consider instead of $C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}$ the related quantity

$$C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1} \equiv C^{(\sigma_3\sigma_2\sigma_1)}_{Q-\beta_3,\beta_2,\beta_1} \equiv S^{-1}(\beta_3;\sigma_1,\sigma_3)C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}. \tag{22}$$

Let us now turn to the determination of this last structure function.

2.1 Boundary three point function

Associativity condition

The basic consistency condition that the three-point function of boundary operators has to satisfy expresses the associativity of the product of boundary fields. Let us consider the 4 point function of boundary operators. Inserting a complete set of intermediate states between the first two and the last two fields leads to an expansion into conformal blocks of the following form: [d]

$$\left\langle {}^{\sigma_1} B^{\sigma_4}_{\beta_4}(x_4)^{\sigma_4} B^{\sigma_3}_{\beta_3}(x_3)^{\sigma_3} B^{\sigma_2}_{\beta_2}(x_2)^{\sigma_2} B^{\sigma_1}_{\beta_1}(x_1) \right\rangle = \\ = \int_{\mathbb{S}} d\beta_{21}\; C^{(\sigma_4\sigma_3\sigma_1)\beta_4}_{\beta_3,\beta_{21}} C^{(\sigma_3\sigma_2\sigma_1)\beta_{21}}_{\beta_2\beta_1} \mathcal{F}^s(\Delta_{\beta_i},\Delta_{\beta_{21}},x_i)$$

By using associativity of the operator product expansion one would get a second expansion (t-channel):

$$\left\langle {}^{\sigma_1} B^{\sigma_4}_{\beta_4}(x_4)^{\sigma_4} B^{\sigma_3}_{\beta_3}(x_3)^{\sigma_3} B^{\sigma_2}_{\beta_2}(x_2)^{\sigma_2} B^{\sigma_1}_{\beta_1}(x_1) \right\rangle = \\ = \int_{\mathbb{S}} d\beta_{32} C^{(\sigma_4\sigma_2\sigma_1)\beta_4}_{\beta_{32},\beta_1} C^{(\sigma_4\sigma_3\sigma_2)\beta_{32}}_{\beta_3\beta_2} \mathcal{F}^t(\Delta_{\beta_i},\Delta_{\beta_{32}},x_i)$$

[d] As in the case one restricts oneself to the case where $Re(\beta_i)$, $i = 1\ldots4$ are close enough to Q/2. In this case, β_{21} is of the form $Q/2 + iP$. Meromorphic continuation is understood otherwise.

Using the fusion transformations (9), the equivalence of the factorisation in the two channels can be rewritten:

$$\int_{\mathbb{S}} d\beta_{21}\; C^{(\sigma_4\sigma_3\sigma_1)\beta_4}_{\beta_3,\beta_{21}} C^{(\sigma_3\sigma_2\sigma_1)\beta_{21}}_{\beta_2\beta_1} F_{\beta_{21}\beta_{32}} \begin{bmatrix} \beta_3 & \beta_2 \\ \beta_4 & \beta_1 \end{bmatrix} = C^{(\sigma_4\sigma_2\sigma_1)\beta_4}_{\beta_{32},\beta_1} C^{(\sigma_4\sigma_3\sigma_2)\beta_{32}}_{\beta_3\beta_2} \tag{23}$$

By means of the pentagonal equation (11) it easy to verify that the following ansatz

$$C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1} = \frac{g(\beta_3,\sigma_3,\sigma_1)}{g(\beta_2,\sigma_3,\sigma_2)g(\beta_1,\sigma_2,\sigma_1)} F_{\sigma_2\beta_3} \begin{bmatrix} \beta_2 & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} \tag{24}$$

yields a solution to (23), as was noticed in [10]. The functions $g(\beta,\sigma_2,\sigma_1)$ appearing are unrestricted by (23), and do correspond to the normalization of the boundary operators. Let us remark however that we have already fixed the normalization by requiring the prefactor of the first delta-distribution on the right hand side of (19) to be unity.

Normalization of the boundary operators

The boundary three point function $C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}$ should be meromorphic w.r.t. the variables $\beta_3, \beta_2, \beta_1$. This assumption can be motivated in various ways: One may e.g. use arguments like those reviewed in section 3 of [4] concering the path integral for Liouville theory. These arguments exhibit the analytic properties of correlation functions as a reflection of the asymptotic behavior of the Liouville path integral measure in the region $\phi \to -\infty$ where the interaction terms vanish.

Such considerations lead in particular to the identification of the residues for the poles of $C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}$ with certain correlation functions in free field theory, which generalize the so-called screening-charge constructions of [11, 12]. The resulting prescription for the calculation of these residues was formulated in [7]. Most relevant for our purposes will be the observation that $C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}$ has a pole with residue 1 if $\beta_1 + \beta_2 + \beta_3 = Q$: The relevant correlation functions in free field theory do not contain any screening charges.

On the other hand it seems worth observing that the fusion coefficients themselves are meromorphic functions of all six variables they depend on. This means that the function $C^{(\sigma_3\sigma_2\sigma_1)}_{\beta_3\beta_2\beta_1}$ that is given by the expression (24) will be meromorphic iff the function $g(\beta;\sigma_2,\sigma_1)$ is meromophic w.r.t. β.

In the following, we shall consider the special boundary field ${}^{\sigma_1}B_{-b}{}^{\sigma_1}$, which corresponds to a degenerate representation of the Virasoro algebra. As pointed out in [7] it is in general not a trivial issue to decide when a boundary field that corresponds to a degenerate representation will satisfy the corresponding differential equations expressing null vector decoupling. Here, however, one may observe that one may create the boundary field ${}^{\sigma_1}B_{-b}{}^{\sigma_1}$ by sending the bulk field $V_{-b/2}$ to the boundary. It follows from the fact that $V_{-b/2}$ satisfies a second order differential equation that the asymptotic behavior when $V_{-b/2}$ approaches the boundary is described by an boundary field ${}^{\sigma_1}B_{-b}{}^{\sigma_1}$ that satisfies a third order differential equation. This last fact also implies that the operator product expansion of ${}^{\sigma_1}B_{-b}{}^{\sigma_1}$ with a generic boundary operator can only contain three types of contributions:

$$ {}^{\sigma_3}B^{\sigma_1}_{\beta_2}\,{}^{\sigma_1}B^{\sigma_1}_{-b} = c_+(\beta_2)^{\sigma_3}B^{\sigma_1}_{\beta_2-b} + c_0(\beta_2)^{\sigma_3}B^{\sigma_1}_{\beta_2} + c_-(\beta_2)^{\sigma_3}B^{\sigma_1}_{\beta_2+b}. \tag{25} $$

One may then consider the vacuum expectation values of the product of operators that is obtained by multiplying (25) with the boundary fields ${}^{\sigma_1}B^{\sigma_3}_{Q-\beta_2-sb}$, $s = -, 0, +$. Taking into account (19), one is lead to identify the structure functions $c_s(\beta)$ $(s = +, 0, -)$ with residues of the general three point function. As mentioned previously, the relevant residues can be represented as correlation function in free field theory. The structure function c_+ is nothing but a special case of the residue at $\beta_1 + \beta_2 + \beta_3 = Q$, which is 1.

This should be compared to what would follow from our ansatz (24). Let us note that the fusion coefficients indeed have a pole in the presently considered case. The corresponding residue is most easily calculated by recursion using the pentagon equation with, say, α_1 and α_2 equal to $-b/2$. The residue of the fusion matrix with one coefficient being $-b/2$ is a well-known 2×2 matrix, see appendix. We find

$$ \begin{aligned} &F_{\sigma_1,\beta_2-b}\begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix} = \\ &\frac{\Gamma(1+b^2)}{\Gamma(1+2b^2)}\frac{\Gamma(2b\sigma_1)\Gamma(2b(Q-\sigma_1))}{\Gamma(b(Q-\beta_2+\sigma_3-\sigma_1))\Gamma(b(Q-\beta_2+\sigma_1-\sigma_3))} \times \\ &\frac{\Gamma(b(Q-2\beta_2))\Gamma(b(Q-2\beta_2+b))}{\Gamma(b(\sigma_3+\sigma_1-\beta_2))\Gamma(b(2Q-\beta_2-\sigma_3-\sigma_1))}. \end{aligned} \tag{26} $$

Our ansatz (24) together with $c_+ \equiv 1$ therefore implies the following first order difference equation for g:

$$ 1 = \frac{g(\beta_2 - b, \sigma_3, \sigma_1)}{g(\beta_2, \sigma_3, \sigma_1)g(-b, \sigma_1, \sigma_1)} F_{\sigma_1,\beta_2-b}\begin{bmatrix} \beta_2 & -b \\ \sigma_3 & \sigma_1 \end{bmatrix}. \tag{27} $$

This functional equation is solved by the following expression:

$$g(\beta,\sigma_3,\sigma_1) = \frac{f(\sigma_3,\sigma_1)^{b^{-1}\beta/2}\Gamma_b(Q)\Gamma_b(Q-2\beta)\Gamma_b(2\sigma_1)\Gamma_b(2Q-2\sigma_3)}{\Gamma_b(2Q-\beta-\sigma_1-\sigma_3)\Gamma_b(\sigma_1+\sigma_3-\beta)\Gamma_b(Q-\beta+\sigma_1-\sigma_3)\Gamma_b(Q-\beta+\sigma_3-\sigma_1)}, \tag{28}$$

where $f(\sigma_3,\sigma_1)$ is an arbitrary function. Let us furthermore note that one may derive a second finite difference equation that is related to (27) by substituting $b \to b^{-1}$ if one considers ${}^{\sigma_1}B_{-b^{-1}}{}^{\sigma_1}$ instead of ${}^{\sigma_1}B_{-b}{}^{\sigma_1}$. Taken together, these two functional equations allow one to conclude that our solution (28) is unique at least for irrational values of b.

It is finally useful to note that we now have two possible ways to calculate the structure function $c_-(\beta;\sigma_2,\sigma_1)$: On the one hand one may use our ansatz (24), using (28) and the following residue of the fusion coefficients:

$$\begin{aligned} &F_{\sigma_1,\beta_2+b}\begin{bmatrix}\beta_2 & -b\\ \sigma_3 & \sigma_1\end{bmatrix} = \\ &\frac{\Gamma(1+b^2)}{\Gamma(1+2b^2)}\frac{\Gamma(2b\sigma_1)\Gamma(2b(Q-\sigma_1))}{\Gamma(b(\beta_2+\sigma_3-\sigma_1))\Gamma(b(\beta_2+\sigma_1-\sigma_3))}\times \\ &\frac{\Gamma(2b\beta_2-2bQ)\Gamma(2b\beta_2-1)}{\Gamma(b(\sigma_3+\sigma_1+\beta_2-Q))\Gamma(b(\beta_2-\sigma_3-\sigma_1+Q))} \end{aligned} \tag{29}$$

On the other hand, $c_-(\beta;\sigma_2,\sigma_1)$ is one of the cases where a representation in terms of free field correlation functions is available [7]:

$$\begin{aligned} c_-(\beta;\sigma_2,\sigma_1) &= -\frac{4\mu}{\pi}\frac{\Gamma(1+b^2)}{\Gamma(-b^2)} \\ &\times \Gamma(b(2\beta_2-Q))\Gamma(2b\beta_2-1)\Gamma(1-2b\beta_2)\Gamma(1-b(2\beta_2+b)) \\ &\times \sin\pi b(Q+\beta_2-\sigma_3-\sigma_1)\sin\pi b(\beta_2+\sigma_3+\sigma_1-Q) \\ &\times \sin\pi b(\beta_2+\sigma_3-\sigma_1)\sin\pi b(\beta_2+\sigma_1-\sigma_3). \end{aligned} \tag{30}$$

It is a nontrivial check of the consistency of our approach that the expressions which one obtains by following these two ways are indeed consistent and agree if the function $f(\sigma_3,\sigma_1)$ is chosen according to

$$f(\sigma_3,\sigma_1) = \pi\mu\gamma(b^2)b^{2-2b^2}. \tag{31}$$

By collecting the pieces, one finally arrives at the following expression for the three point function of boundary operators:

$$C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1} = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{1}{2b}(\beta_3-\beta_2-\beta_1)}$$
$$\frac{\Gamma_b(Q+\beta_2-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_1-\beta_2)}{\Gamma_b(Q-2\beta_1)\Gamma_b(Q)}$$
$$\times\frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(\beta_2+\beta_3-\beta_1)}{\Gamma_b(2\beta_3-Q)\Gamma_b(Q-2\beta_2)}$$
$$\times\frac{S_b(\beta_3+\sigma_1-\sigma_3)S_b(Q+\beta_3-\sigma_3-\sigma_1)}{S_b(\beta_2+\sigma_2-\sigma_3)S_b(Q+\beta_2-\sigma_3-\sigma_2)}$$
$$\times\frac{1}{i}\int_{-i\infty}^{i\infty} ds\ \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(Q+s)}$$

the coefficients U_i, V_i and $i = 1,\ldots,4$ read

$$\begin{array}{ll} U_1 = \sigma_1+\sigma_2-\beta_1 & V_1 = Q+\sigma_2-\sigma_3-\beta_1+\beta_3 \\ U_2 = Q-\sigma_1+\sigma_2-\beta_1 & V_2 = 2Q+\sigma_2-\sigma_3-\beta_1-\beta_3 \\ U_3 = \beta_2+\sigma_2-\sigma_3 & V_3 = 2\sigma_2 \\ U_4 = Q-\beta_2+\sigma_2-\sigma_3 & \end{array}$$

3 Remarks

The details of this section and some additional information can be found in [14]

1. It is possible to check explicitly that the boundary three point function is invariant w.r.t. cyclic permutations.

2. One recovers the expression for the boundary reflection amplitude from the boundary three point function the same way the bulk reflection amplitude was recovered from the bulk three point function in [3]. Using the fact that the fusion matrix depends on conformal weights only, and is thus invariant when $\beta_i \to Q-\beta_i$,

$$C^{(\sigma_3\sigma_2\sigma_1)Q-\beta_3}_{\beta_2\beta_1} = \frac{g(Q-\beta_3,\sigma_3,\sigma_1)}{g(\beta_3,\sigma_3,\sigma_1)}C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_2\beta_1} \qquad (32)$$

From the expression (28) for the function g, one indeed finds formula (16) for $S(\beta;\sigma_2,\sigma_1)$.

3. One may explicitly check that the two-point function (19) is recovered by taking e.g. the limit $\beta_1 \to 0$ if the three-point function:

$$\lim_{\beta_1 \to 0} \left\langle {}^{\sigma_1}B^{\sigma_3}_{Q-\beta_3}(\infty)^{\sigma_3}B^{\sigma_1}_{\beta_2}(1)^{\sigma_1}B^{\sigma_1}_{\beta_1}(0) \right\rangle = \delta(\beta_3 - \beta_2) + S(\beta_2; \sigma_3\,\sigma_1)\delta(\beta_3 + \beta_2 - Q) \tag{33}$$

Acknowledgments

B.P acknowledges support from TMR network with contract ERBFMRX CT960012 and from the GIF-Project Nr I-645-130-14/1999. J.T. is supported by DFG SFB 288 "Differentialgeometrie und Quantenphysik".

Appendix

Special functions

- Γ_b function
  The Double Gamma function introduced by Barnes [15] is defined by:

$$\log\Gamma_2(s|\omega_1,\omega_2) = \left(\frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0}$$

Definition: $\Gamma_b(x) \equiv \frac{\Gamma_2(x|b,b^{-1})}{\Gamma_2(Q/2|b,b^{-1})}$.
Functional relations:

$$\Gamma_b(x+b) = \frac{\sqrt{2\pi}b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x)$$

$$\Gamma_b(x+1/b) = \frac{\sqrt{2\pi}b^{-\frac{x}{b}+\frac{1}{2}}}{\Gamma(x/b)}\Gamma_b(x)$$

Γ_b is a meromorphic function of x, which poles are located at $x = -nb - mb^{-1}, n, m \in \mathbb{N}$
Integral representation convergent for $0 < Rex$

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q/2-x)^2}{2}e^{-t} - \frac{Q/2-x}{t}\right]$$

- S_b function
Definition: $S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$
Functional relations:

$$S_b(x+b) = 2\sin(\pi bx)S_b(x)$$
$$S_b(x+1/b) = 2\sin(\pi x/b)S_b(x)$$

$S_b(x)$ is a meromorphic function of x, which poles are located at $x = -nb - mb^{-1}, n, m \in \mathbb{N}$, and which zeros are located at $x = Q + nb + mb^{-1}, n, m \in \mathbb{N}$.
Integral representation convergent in the strip $0 < Rex < Q$

$$\log S_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{\sinh(\frac{Q}{2}-x)t}{2\sinh(\frac{bt}{2})\sinh(\frac{t}{2b})} - \frac{(Q-2x)}{t}\right]$$

- Υ_b function
Definition: $\Upsilon_b(x)^{-1} \equiv \Gamma_b(x)\Gamma_b(Q-x)$
Functional relations:

$$\Upsilon_b(x+b) = \frac{\Gamma(bx)}{\Gamma(1-bx)}b^{1-2bx}\Upsilon_b(x)$$
$$\Upsilon_b(x+1/b) = \frac{\Gamma(x/b)}{\Gamma(1-x/b)}b^{2x/b-1}\Upsilon_b(x)$$

$\Upsilon_b(x)$ is an entire function of x which zeros are located at $x = -nb - mb^{-1}$ and $x = Q + nb + mb^{-1}$, $n, m \in \mathbb{N}$.
Integral representation convergent in the strip $0 < Rex < Q$

$$\log\Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]$$

Residues of fusion coefficients for $\beta_1 = -b/2$

By definition

$$F_{s,s'} \equiv F_{\sigma_1 - \frac{sb}{2}, \beta_2 - \frac{s'b}{2}}\begin{bmatrix}\beta_2 & -b/2\\ \sigma_3 & \sigma_1\end{bmatrix}$$

where $s, s' = \pm$.

$$F_{++} = \frac{\Gamma(b(2\sigma_1 - b))\Gamma(b(b-2\beta_2)+1)}{\Gamma(b(\sigma_1 - \beta_2 - \sigma_3 + b/2)+1)\Gamma(b(\sigma_1 - \beta_2 + \sigma_3 - b/2))}$$

$$F_{+-} = \frac{\Gamma(b(2\sigma_1 - b))\Gamma(b(2\beta_2 - b) - 1)}{\Gamma(b(\sigma_1 + \beta_2 + \sigma_3 - 3b/2)-1)\Gamma(b(\sigma_1 + \beta_2 - \sigma_3 - b/2))}$$

$$F_{-+} = \frac{\Gamma(2 - b(2\sigma_1 - b))\Gamma(b(b-2\beta_2)+1)}{\Gamma(2 - b(\sigma_1 + \beta_2 + \sigma_3 - 3b/2))\Gamma(1 - b(\sigma_1 + \beta_2 - \sigma_3 - b/2))}$$

$$F_{--} = \frac{\Gamma(2 - b(2\sigma_1 - b))\Gamma(b(2\beta_2 - b) - 1)}{\Gamma(b(-\sigma_1 + \beta_2 + \sigma_3 - b/2))\Gamma(b(-\sigma_1 + \beta_2 - \sigma_3 + b/2)+1)}$$

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