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BRANES IN THE EUCLIDEAN AdS$_3$

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In this work we propose an exact microscopic description of maximally symmetric branes in a Euclidean AdS_3 background. As shown by Bachas and Petropoulos, the most important such branes are localized along a Euclidean $AdS_2 \subset AdS_3$. We provide explicit formulas for the coupling of closed strings to such branes (boundary states) and for the spectral density of open strings. The latter is computed in two different ways first in terms of the open string reflection amplitude and then also from the boundary states by world-sheet duality. This gives rise to an important Cardy type consistency check. All the results are compared in detail with the geometrical picture. We also discuss a second class of branes with spherical symmetry and finally comment on some implications for D-branes in a 2D back hole geometry.

1. INTRODUCTION

String theories on Anti-deSitter (AdS) spaces have received enormous attention over the last years because of their conjectured duality with gauge theories on the boundary of the AdS space (see [1] and references therein). Unfortunately, strings moving in AdS_p are rather difficult to study and therefore most of the tests and uses of the duality have been restricted to a super-gravity limit in which the AdS space is only weakly curved. For $p = 3$ the situation is much better because the string equations of motion for AdS_3 can be solved with a non-vanishing NSNS 3-form field strength so that there is no need for non-zero RR background fields. This allows to study the AdS/CFT correspondence in a truly stringy regime.

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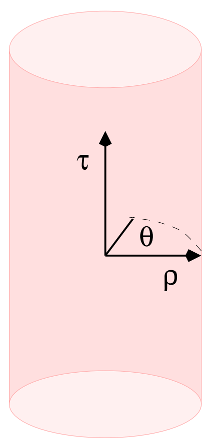
In this paper we shall work with the Euclidean counterpart H_3^+ of AdS_3. Let us be a bit more specific and describe the model for H_3^+ we will be using. To this end, we identify AdS_3 with the group manifolds of $SL(2,\mathbb{R})$. In fact,

$$\begin{pmatrix} X_0 + X_1 & X_2 + X_3 \\ X_2 - X_3 & X_0 - X_1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \tag{1.1}$$

implies that $X_0^2 - X_1^2 - X_2^2 + X_3^2 = 1$ which is the defining equation of $AdS_3 \subset \mathbb{R}^4$. One can imagine this space as an infinite solid cylinder which is parametrized by the global coordinates (ρ, θ, τ) such that (see Figure)



$$X_0 + iX_3 = e^{i\tau} \cosh\rho \quad , \tag{1.2}$$

$$X_1 + iX_2 = e^{i\theta} \sinh\rho \quad . \tag{1.3}$$

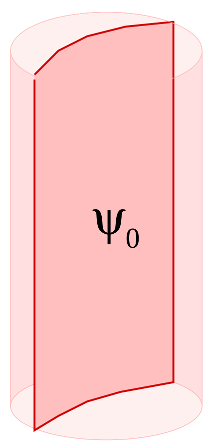
Upon rotation to a Euclidean time $\tau \rightarrow i\tau$, the coordinate X_3 gets replaced by iX_3. When we make this substitution in the matrices (1.1) above then we end up the the space H_3 of hermitian 2×2 matrices h with $\det h = 1$. It consists of two components and the component of the identity matrix is given by

$$H_3^+ = \{ h \in \mathrm{SL}(2,\mathbb{C}) \,|\, h^\dagger = h \,,\, \mathrm{tr} h > 0 \} \,.$$

This is the space on which we want to study string theory. We have used the AdS/CFT correspondence as our main motivation. Let us note, however, that there are various other good reasons to be interested in H_3^+. Part of them are related to the fact that one can descend from H_3^+ to the coset $H_3^+/\mathbb{R}_\tau$ describing a 2D Euclidean black hole [2]. The relevant action of $\mathbb{R}$ on H_3^+ is given by constant shifts in the Euclidean time τ. The black hole geometry appears as part of many interesting string backgrounds. One example is the near horizon geometry of non-extremal NS5-branes [3, 4]. Furthermore, it can emerge as a factor when Calabi-Yau spaces develop an isolated singularity [5].

Recently, there has been considerable progress towards the construction of perturbative closed string theory on AdS_3, see [6, 7, 8] and references therein. These works show that partition function and scattering amplitudes of string theory on AdS_3 can be constructed with the help of the H_3^+ gauged WZNW model. The procedure of constructing amplitudes for string theory on AdS_3 from correlation functions associated to a Euclidean target may be seen as some analog of the usual Wick-rotation. It is therefore crucial for the success of such a procedure to have sufficient control over the H_3^+ model. The first important step was the calculation of the partition function [9] which allows to determine the spectrum. Crossing symmetric correlation functions on the sphere were constructed in [10, 11] from the three-point functions of the model. The latter were first obtained in [12, 13, 10].

In the present paper we want to study D-branes on backgrounds containing AdS_3. For the Lorentzian models some possible brane geometries were first analyzed by [14] using the relation between AdS_3 and the group $\mathrm{SL}(2,\mathbb{R})$ along with results from [15] which show that branes on group manifolds can wrap conjugacy classes. It was later shown by Bachas and Petropoulos [16] that the most interesting branes on AdS_3 are associated with twined conjugacy classes in the sense of [17]. These can be localized along $AdS_2 \subset AdS_3$ (see Figure) and they are parametrized by a single real parameter Ψ_0. In addition one can have branes localized along H_2, dS_2, the light cone, as well as point-like branes. Not all of these geometric possibilities correspond to physical brane configurations, though: The branes localized along dS_2, for example, were found to have a supercritical electric field on their world-volume [16].



In view of the above-mentioned possibility to construct perturbative closed string theory on AdS_3 via the H_3^+ WZNW model it is natural to expect that at least the physical branes in AdS_3 can also be described by means of the corresponding Euclidean model.

Our main focus in this paper will be on the Euclidean counterparts of the AdS_2 branes. We shall also find analogues for the point-like branes as well as branes along the two connected components $H_2^\pm$ of H_2. In addition, it is possible to localize branes along 2-spheres, though the exact solution will tell us later that these spheres have an imaginary radius. Let us note, however, that the branes on $H_2^\pm$ are related to the Euclidean AdS_2 branes by a symmetry transformation on H_3^+ so that is suffices to study the latter.

In this paper we will analyse branes using microscopic techniques of boundary conformal field theory (BCFT). As we mentioned already, we shall concentrate on the AdS_2 branes since their analysis is much more difficult than for the point-like and spherical branes. This is related to the fact that the former are non-compact and extend to the boundary of AdS_3. Hence, in the exact solution we will find a continuous open string spectrum with a rather non-trivial spectral density. For completeness, we shall also spell out all the relevant formulas that are needed to treat the point-like and spherical branes.

Let us now explain our main results in more detail. In string theory, D-branes can be characterized by their couplings to closed string states. In the case of maximally symmetric branes on H_3^+, the relevant states are associated with bulk fields $\Phi^j(u|z)$ where $j \in -\frac{1}{2} + \mathbb{R}^+$ and $u \in \mathbb{C}$. These fields live on the upper half plane $\Im z \geq 0$. Couplings of closed string modes to the brane are encoded in the one-point functions of the bulk fields.

$$\langle \Phi^j(u|z) \rangle_{BC} \;=\; \frac{A^j(u)^{BC}}{|z-\bar{z}|^{2\Delta_j}} \;\; . \tag{1.4}$$

Here, Δ_j are the conformal dimensions of the fields Φ^j and the label BC refers to the choice of the boundary condition. The form of the 1-point functions is fixed by conformal invariance up to some constants $A^j(u)^{BC}$. The latter contain the same information as the boundary state.

Another interesting quantity in boundary conformal field theory is the partition function. It encodes information on spectrum of open strings that are living on the brane. For maximally symmetric branes the partition function can be expanded in terms of characters χ^j of the chiral algebra, i.e. very schematically one has

$$Z_{BC}(q) \;=\; \int_{\mathbb{S}_{BC}} dj\; \rho^{BC}(j)\; \chi_j(q) \;\; . \tag{1.5}$$

Here, the integration extends over a set that might depend on the branes and it might be either continuous or discrete. In the latter case, the integral gets replaced by a sum. When $\mathbb{S}_{BC}$ is continuous, the partition function involves a non-trivial spectral density function ρ_{BC} which describes the density of open string modes with ‘momentum’ j. Following Cardy, the partition function may be computed from the boundary state of the brane by world-sheet duality.

But there is another way of obtaining ρ_{BC}. It involves one more interesting quantity to study in case of non-compact branes: the so-called *reflection amplitude* of open strings. Open strings states can be created by boundary operators $\Psi^j_{BC}(u,x)$. Here u is a real variable which carries an action of the space-time symmetry that is left unbroken by the brane and x is the usual coordinate for the boundary of the world-sheet. One can then study the amplitude that describes the scattering of an open string that is sent in with momentum j_1 from the boundary of AdS_3 into an outgoing open string with momentum j_2. This defines the reflection amplitude,

$$\langle \Psi^{j_1}(u_1|x_1)\, \Psi^{j_2}(u_2|x_2) \rangle_{BC} \;\sim\; \delta(j_1-j_2)\; R(j)^{BC}\, \frac{1}{|x_1-x_2|^{\Delta_j}} \;\; . \tag{1.6}$$

Here, $j_i \in -1/2 + i\mathbb{R}^+$ and we omitted some (j_i, u_i) dependent factor that is determined by the unbroken symmetry. It is one of the fundamental observations in scattering theory that one can often recover the spectral density ρ_{BC} from such a reflection amplitude. For the reader’s convenience we have included a review of this relation in Appendix B. Comparison between the two ways of obtaining $\rho_{BC}(j)$ is an important consistency check.

The aim of this work is to determine the one-point functions (1.4), the open string reflection amplitude (1.6) and the open string spectral density (1.5) for all maximally symmetric branes of the model. As we have explained, these branes split into two classes. The first one consists of branes which break the $\mathrm{SL}(2,\mathbb{C})$ symmetry of the background to a subgroup $\mathrm{SL}(2,\mathbb{R})$. All branes from this class are related by a symmetry transformation to a Euclidean AdS_2-brane. Their solution is given by eqs. (3.35,4.13,5.11) below. Analogous results are also provided for a second class of D-branes in H_3^+ which possess an $\mathrm{SU}(2)$ symmetry (see eqs. (3.41,5.24)). They behave as if they were localized along a discrete set of 2-spheres with an *imaginary* radius.

In our exposition we shall begin with a discussion of the semi-classical limit of the model (Section 2) where the stringy corrections are turned off. This allows us to introduce all the relevant objects in a rather familiar and simple setup. It is also reassuring to see later that the results we obtain by very different methods in the full string theory do indeed possess the expected semi-classical behavior. The one-point functions are then constructed in Section 3 by solving certain factorization constraints. Similar techniques are also employed in Section 4 in order to find the reflection amplitude for open strings. The consistency between these data is then discussed in Section 5 where we show that they are related by world-sheet duality.

Several recent publications have addressed the problem of constructing branes in AdS_3 [18, 19, 20, 21, 22, 23]. However, it seems to us that even the most basic of the relevant data, namely the one-point function, is not available so far.[4] The discussion in [18, 19] focuses mainly on a series of boundary conformal field theories that includes the point-like brane and the ones that we called ‘spherical’ above. As we shall show below, however, the dependence of the one-point functions on the coordinate u of the CFT on the boundary of AdS_3 has not been stated correctly in those papers. The authors of [18] did observe that their one-point functions produced some puzzling singularities at the boundary of AdS_3. The correct formulas turn out to be regular, as one would have expected. The AdS_2 branes in AdS_3 were even less well understood. Semi-classical expressions for the one-point functions have been proposed in [20] and it was also suggested that these formulas might hold true in the string regime. Our analysis shows that this is not the case. We will comment more on the discrepancies with the existing literature as we proceed.

2. STRINGS ON H_3^+ – THE SEMI-CLASSICAL LIMIT

2.1. Bulk geometry and the closed string action

Geometry of H_3^+. As we have explained and motivated in the introduction, we are interested in studying string theory on the space H_3^+ of Hermitian 2×2 matrices h with determinant

[4] When this paper was nearly completed, we were informed by H. Ooguri that P. Lee, H. Ooguri and J. Park have also found the exact expression for the one-point function of AdS_2-branes. We thank H. Ooguri for kindly sending us a draft of their paper. It has some overlap with the discussion in Sections 2 and 3.

$\det h = 1$ and positive trace. It is convenient to parametrize this space through coordinates $(\phi, \gamma, \bar{\gamma})$ such that

$$h \;=\; \begin{pmatrix} e^{\phi} & e^{\phi}\bar{\gamma} \\ e^{\phi}\gamma & e^{\phi}\gamma\bar{\gamma} + e^{-\phi} \end{pmatrix} \;\; . \tag{2.1}$$

Here, ϕ runs through the real numbers and γ is a complex coordinate with conjugate $\bar{\gamma}$. We can visualize the geometric content of these coordinates most easily by expressing them in terms of the more familiar global coordinates (ρ, τ, θ) that we also used in the introduction,

$$\gamma \;=\; e^{\tau + i\theta} \, \tanh \rho \quad \text{ and } \quad e^{\phi} \;=\; e^{-\tau} \cosh \rho \;\; .$$

At fixed $\gamma, \bar{\gamma}$, the boundary of H_3^+ is reached in the limit of infinite ϕ. The boundary is now represented as the complex plane with coordinates $\gamma, \bar{\gamma}$, which are related to the coordinates (τ, θ) via the usual conformal mapping from the cylinder to the complex plane, $\gamma \;=\; e^{\tau + i\theta}$.

Let us note in passing that H_3^+ admits an action of the group $\mathrm{SL}(2, \mathbb{C})$ which is defined as follows

$$h \;\longrightarrow\; g \, h \, g^{\dagger} \quad \text{ for } \quad g \in SL(2, \mathbb{C}) \;\; . \tag{2.2}$$

Since the stabilizer of this action is isomorphic to the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$ we can identify H_3^+ with the coset $H_3^+ = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$.

The space H_3^+ comes equipped with the following metric and H-field,

$$ds^2 \;\;=\;\; d\phi^2 + e^{2\phi} \, d\gamma d\bar{\gamma} \;\;\; , \tag{2.3}$$

$$H \;\;=\;\; 2 \, e^{2\phi} \, d\phi \wedge d\bar{\gamma} \wedge d\gamma \;\; . \tag{2.4}$$

We shall introduce 2-form potentials B for the 3-form H later on.

The string action. To write down the action functional, we need to choose some 2-form potential B' for H. Note that the space H_3^+ is topologically trivial which implies that such a potential always exists and, moreover, that the resulting action for closed strings does not depend on the particular choice we make. For the moment, we shall work with

$$B' \;=\; e^{2\phi} \, d\gamma \wedge d\bar{\gamma} \;\; .$$

Putting all this information together, we arrive at the following action functional for closed strings moving on H_3^+,

$$S(\phi, \gamma, \bar{\gamma}) \;=\; \frac{k}{\pi} \int \, dz \, d\bar{z} \;\left(\partial\phi\bar{\partial}\phi + e^{2\phi} \, \partial\gamma\bar{\partial}\bar{\gamma} \right) \;\; . \tag{2.5}$$

One should note that this model has some obvious defect, namely it has an imaginary B-field that causes the theory to be non-unitary. The problem is quite easy to understand. Recall that the string equations of motion relate the curvature R of the background to the square of the H-field (provided that the dilaton is constant). Now it is also well known that strings on a 3-sphere have a perfectly unitary description. When we pass to H_3^+, the curvature changes its sign and we have to multiply the 3-form H with $\sqrt{-1}$ to be consistent with the string equations of motion. This factor $\sqrt{-1}$ is then certainly passed on to the potential B'. Such problems disappear when we descend to the black hole geometry $H_3^+/\mathbb{R}$ since the latter has vanishing H-field for purely dimensional reasons.

The currents. We want to conclude this subsection with a few remarks on the chiral currents of the model. Let us introduce the following matrices

$$T_+ \;=\; \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\,, \quad T_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \quad T_0 \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.6}$$

These are matrix representatives of the Lie algebra $\mathrm{SL}(2,\mathbb{R})$, i.e. they obey the relations $[T_0, T_\pm] = \pm T_\pm$ and $[T_-, T_+] = 2T_0$. For the chiral currents we use

$$J(z) \;:=\; k\, h^{-1}\bar\partial h \qquad \bar J(\bar z) \;=\; -k\,\partial h\, h^{-1}\;.$$

When we expand them according to $J(z) = T_+J^+ + T_-J^- + 2T_0J^0$, we obtain expressions for the components

$$J^-(z) \;\;:=\;\; k\, e^{2\phi}\,\bar\partial\gamma \tag{2.7}$$

$$J^0(z) \;\;:=\;\; k\,\left(\bar\partial\phi - e^{2\phi}\,\bar\gamma\,\bar\partial\gamma\right) \tag{2.8}$$

$$J^+(z) \;\;:=\;\; k\,\left(\bar\gamma^2\, e^{2\phi}\,\bar\partial\gamma - \bar\partial\bar\gamma - 2\,\bar\gamma\,\bar\partial\phi\right)\;\;. \tag{2.9}$$

The components of the anti-holomorphic currents are constructed in an analogous way. Both sets of currents are related by complex conjugation $(J^\pm)^* = (\bar J)^\mp$ and $(J^0)^* = -\bar J^0$.

2.2. **Brane geometry and the boundary conditions**

General results. In this section we want to present the possible geometries for branes in H_3^+ which preserve half of the $\mathrm{SL}(2,\mathbb{C})$ symmetry (2.2). Let us recall that $\mathrm{SL}(2,\mathbb{C})$ contains two important 3-parameter subgroups, namely the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. We shall analyse equations of the form

$$\mathrm{tr}(Ch) \;=\; c\;\;, \tag{2.10}$$

where C is a 2×2-matrix and $h \in H_3^+$. It turns out that there are two important cases to distinguish. If the matrix C is of the form [5]

$$C \;=\; U^\dagger U \quad \text{where} \quad U \in \mathrm{SL}(2,\mathbb{C})\;\;.$$

[5] One could admit matrices C of a slightly more general form and with $U \in$ GL(2,$\mathbb{C}$) but this extra freedom can be absorbed in a rescaling of the constant c.

then the equation (2.10) preserves a subgroup of $\mathrm{SL}(2,\mathbb{C})$ that is conjugate to SU(2). More precisely, for every $g \in$ SU(2) the element $U^{-1}gU \in \mathrm{SL}(2,\mathbb{C})$ is a symmetry of the equation. A second possibility is to impose eq. (2.10) with a matrix C of the following form:

$$C \;=\; U^{\dagger}\,\omega\,U \quad \text{where} \quad \omega := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U \in \mathrm{SL}(2,\mathbb{C}) \;\; .$$

In this case, the equation preserves a subgroup of $\mathrm{SL}(2,\mathbb{C})$ that is conjugate to $\mathrm{SL}(2,\mathbb{R})$, i.e. it is left invariant by the action with $U^{-1}gU \in \mathrm{SL}(2,\mathbb{C})$ for all $g \in \mathrm{SL}(2,\mathbb{C})$ that satisfy $\omega(g^{\dagger}) = g^{-1}$, where

$$\omega(g) := \omega\, g\, \omega^{-1} \;\; .$$

It is easy to see that this condition implies that the matrices g must be of the form $g = \left(\begin{smallmatrix} \alpha & i\beta \\ i\gamma & \delta \end{smallmatrix}\right)$ and therefore generate a subgroup of $\mathrm{SL}(2,\mathbb{C})$ that is conjugate to $\mathrm{SL}(2,\mathbb{R})$.

For each of these two cases it suffices to consider the the special choice $U = 1$. In fact, the submanifold defined by an equation with nontrivial U is obtained from the one corresponding to $U = 1$ through the symmetry transformation $U \in$ SL(2,$\mathbb{C}$). This means that it suffices to consider just two different choices of C. These will be described in more detail now.

The AdS_2 branes. The first case corresponds to surfaces which are characterized by the equations

$$\mathrm{tr}(\,\omega\, h\,) = \; c \, .$$

Solutions of these equations form Euclidean AdS_2-planes ending at $\theta = \pm\pi/2$ on the boundary. In terms of the coordinates introduced above one gets the equations

$$e^{\phi}\,(\gamma + \bar{\gamma}) \;=\; c \quad \text{or} \quad 2\,\sinh(\rho)\cos(\theta) \;=\; c \;\; .$$

It is convenient to introduce a new set of coordinates (ψ, ν, χ) on H_3^+ in which these branes are coordinate planes $\psi = r$. We can achieve this by setting

$$h \;=\; c(\nu,\chi)\cdot h_{\psi}\cdot c^{\dagger}(\nu,\chi) \;\; , \tag{2.11}$$

where

$$h_{\psi} \;\equiv\; \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}, \;\; \text{and} \;\; c(\nu,\chi) \;\equiv\; \begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ i\nu e^{\frac{\chi}{2}} & e^{-\frac{\chi}{2}} \end{pmatrix}.$$

Definition (2.11) is equivalent to:

$$h \;\equiv\; \begin{pmatrix} e^{\chi}\cosh\psi & \sinh\psi + i\nu e^{\chi}\cosh\psi \\ \sinh\psi - i\nu e^{\chi}\cosh\psi & (e^{-\chi} + \nu^2 e^{\chi})\cosh\psi \end{pmatrix} \;\; . \tag{2.12}$$

In these coordinates the metric ds^2 and the B-field have the from

$$ds^2 \;=\; d\psi^2 + \cosh^2\psi\,(\,e^{2\chi}\,d\nu^2 + d\chi^2\,) \tag{2.13}$$

$$B \;=\; 2i\,\left(\frac{1}{2}\sinh 2\psi \;+\psi\right)\,e^{\chi}\,d\nu\wedge d\chi\;\;. \tag{2.14}$$

The AdS_2 branes are described by the equation $\psi = r$. Note that in the coordinates (ψ,χ,ν), the boundary of an AdS_2-brane is at $\chi=\infty$. It is parametrized by $\nu=\pm e^{\tau}$. The $SL(2,\mathbb{C})$-invariant measure has the form

$$dh \;=\; 2d\nu\,d\chi e^{\chi}\,d\psi\cosh^2\psi\;\;.$$

Given the expressions for the metric and the B-field, it is straightforward to write down the open string action for the fields ψ,ν,χ. Vanishing of the boundary terms in the variation of this action is equivalent to the following simple boundary conditions for currents

$$J^{\pm}(z) \;=\; \bar{J}^{\pm}(\bar{z}) \quad,\quad J^{0}(z) \;=\; \bar{J}^{0}(\bar{z})\;\;. \tag{2.15}$$

holding all along the boundary $z=\bar{z}$. Note that these gluing conditions are consistent with the *-operation and they imply that the boundary current obeys $(J^{\pm})^* = J^{\mp}$ and $(J^0)^* = -J^0$.

It seems worth noting that the $\mathrm{SL}(2,\mathbb{C})$-translates of the Euclidean AdS_3-branes include branes which correspond to $H_2^{\pm}$ in the Minkowskian picture: These branes are characterized by the equation

$$\mathrm{tr}\begin{pmatrix} -\frac{i}{4} & 0 \\ 0 & \frac{i}{4}\end{pmatrix}\,h \;=\; c'\;\;.$$

As in the previous cases we display this equation in our coordinates:

$$e^{\phi}(\gamma\bar{\gamma}-1)+e^{-\phi} \;=\; c \quad\text{or}\quad 2\cosh\rho\sinh\tau \;=\; c\;\;.$$

It is now easy to see that these solutions are extended along H_2^- for $c<0$ and along H_2^+ for $c>0$. For $c=0$ one gets a disc at $\tau=0$. Although they look quite differently from the AdS_2 branes, they are related to the latter by an SL(2,ℂ) transformation U of the form

$$U \;=\; \frac{1}{\sqrt{2}}\,\begin{pmatrix} e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ -e^{-i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}}\end{pmatrix}\;\;.$$

Spherical branes. To get an idea about the subsets that preserve an $\mathrm{SU}(2)$-symmetry let us study equations of the form

$$\mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\,h \;=\; c\;\;.$$

When rewritten in terms of the coordinates $(\phi,\gamma,\bar{\gamma})$ or the global coordinates (ρ,τ,θ) these equations read

$$e^{\phi}(\gamma\bar{\gamma}+1)+e^{-\phi} \;=\; c \quad\text{or}\quad 2\cosh\rho\cosh\tau \;=\; c\;\;.$$

Solutions exist for $c \geq 2$ and they are point-like when $c = 2$ and spherical otherwise. None of them extends to the boundary because near to the boundary the equation would become $\gamma\bar{\gamma} + 1 = 0$.

It is convenient to introduce a new set of coordinates (Λ, ϕ, μ) on H_3^+ in which these branes are coordinate planes $\Lambda = \Lambda_0$. We can achieve this by setting

$$h \;=\; c(\mu,\varphi) \cdot h_\Lambda \cdot c^\dagger(\mu,\varphi) \tag{2.16}$$

where

$$h_\Lambda \;\equiv\; \begin{pmatrix} \cosh\Lambda & \sinh\Lambda \\ \sinh\Lambda & \cosh\Lambda \end{pmatrix}, \quad \text{and} \quad c(\mu,\varphi) \;\equiv\; \begin{pmatrix} e^{i\frac{\mu}{2}}\cos\frac{\varphi}{2} & e^{-i\frac{\mu}{2}}\sin\frac{\varphi}{2} \\ -e^{i\frac{\mu}{2}}\sin\frac{\varphi}{2} & e^{-i\frac{\mu}{2}}\cos\frac{\varphi}{2} \end{pmatrix}.$$

Definition (2.16) is equivalent to:

$$h \;\equiv\; \cosh\Lambda\, \mathbf{1}_2 + \sinh\Lambda \begin{pmatrix} \cos\mu\sin\phi & \cos\mu\cos\phi + i\sin\mu \\ \cos\mu\cos\phi - i\sin\mu & -\cos\mu\sin\phi \end{pmatrix}. \tag{2.17}$$

In the new coordinates the metric ds^2 and the B-field have the from

$$ds^2 \;=\; d\Lambda^2 + \sinh^2\Lambda\,(\,\cos^2\mu\, d\varphi^2 + d\mu^2\,) \tag{2.18}$$

$$B' \;=\; 2i\left(\frac{1}{2}\sinh 2\Lambda \;-\Lambda\right)\,\cos\mu\, d\varphi\wedge d\mu\;\;. \tag{2.19}$$

The spherical branes are described by the equation $\Lambda = \Lambda_0 \geq 0$. Note that in the coordinates (Λ, φ, μ), the boundary of AdS_3 is at $\Lambda = \infty$. The $SL(2,\mathbb{C})$-invariant measure is given by $dh = 2d\varphi\, d\mu\cos\mu\, d\Lambda\sinh^2\Lambda$. A straightforward computation shows that the currents must satisfy

$$J^\pm \;=\; \bar{J}^\mp \quad,\quad J^0 \;=\; -\bar{J}^0\;\;. \tag{2.20}$$

along the boundary $z = \bar{z}$ in order for the boundary terms in the variation of the action to vanish. Once more this is consistent with the $*$-structure but this time the induced action on the boundary currents is $(J^\pm)^* = J^\mp$ and $(J^0)^* = J^0$, i.e. we have an su(2) current algebra on the boundary of the world-sheet.

2.3. **Semi-classical limit of closed string couplings**

Our aim in this subsection is to study the semi-classical limit of the closed string couplings to the brane. Following [24], we will read them off by expanding the δ-functions that describe localization on the branes in a basis of eigen-functions for the Laplace operator on H_3^+. The latter are in one-to-one correspondence with the primary fields of the bulk theory. We will explain this in more detail after a short review of the harmonic analysis on H_3^+ [25, 26].

Harmonic analysis on H_3^+. Any wave function on H_3^+ can be expanded in terms of eigen-functions of the Laplace operator on H_3^+. We recall that there exists an action of $\mathrm{SL}(2,\mathbb{C})$ on H_3^+ which commutes with the Laplace operator. This implies that each eigen-space must carry some representation of $\mathrm{SL}(2,\mathbb{C})$. It is not difficult to show that the possible eigenvalues are given $j(j+1), j = -\frac{1}{2} + iP$, where P is a non-negative real number and that the associated eigen-spaces carry the irreducible representation D_j from the principal continuous series. Explicitly, the eigen-functions are given by the following formula

$$\begin{aligned} \Phi^j(u|\phi,\gamma,\bar{\gamma}) &= -\frac{2j+1}{\pi}\,(v_u\,h\,v_u^\dagger)^{2j} \\ &= -\frac{2j+1}{\pi}\left(|u-\gamma|^2e^{\phi}+e^{-\phi}\right)^{2j} \end{aligned} \tag{2.21}$$

Here, u is a complex coordinate and $v_u = (-u, 1)$. In the second line we inserted the parametrization (2.1) of H_3^+. The transformation law of the functions (2.21) under the action of $\mathrm{SL}(2,\mathbb{C})$ is easily worked out,

$$\Phi^j(u|ghg^\dagger) \;=\; |\beta u+\delta|^{4j}\;\Phi^j(g\cdot u|h) \quad \text{where} \quad g\cdot u \;=\; \frac{\alpha u+\gamma}{\beta u+\delta}\;, \tag{2.22}$$

and $\alpha, \beta, \gamma, \delta$ are the four matrix elements of $g \in \mathrm{SL}(2,\mathbb{C})$. For later use we shall also spell out the asymptotics of the eigen-functions near the boundary of H_3^+,

$$\Phi^j(u|\phi,\gamma,\bar{\gamma}) \quad \overset{\phi\to\infty}{\sim} \quad -I^j(u|\gamma)\,e^{2j\phi} +\; \delta(\gamma-u)\,e^{-2(j+1)\phi} \tag{2.23}$$

$$\text{where} \qquad I^j(u|\gamma) \;:=\; \frac{2j+1}{\pi}|\gamma-u|^{4j} \tag{2.24}$$

is the integral kernel of the unitary intertwiner that implements the equivalence between the representations D_j and D_{-j-1} of $\mathrm{SL}(2,\mathbb{C})$ [25]. The functions $\Phi^j(u|h)$ can be considered as the wave-function of some particle that was created with 'radial momentum' j at the boundary point with coordinates $u, \bar{u}$ [27]. They are in one-to-one correspondence with the ground states of the bulk conformal field theory on H_3^+ and form a basis in the space of square integrable functions on H_3^+.

The AdS_2-branes. We now want to determine the semi-classical one-point function $\langle\Phi^j\rangle_r$ which is supposed to describe the amplitude for absorption/emission of closed string modes with asymptotic radial momentum j by the brane. These amplitudes can be regarded as Fourier-transforms of the amplitudes $\langle\Phi^{h'}\rangle_r$ for the absorption/emission of point-like localized closed string modes $\Phi^{h'}$ with wave-functions $\Phi^{h'}(h) = \delta(h-h')$. The latter must of course vanish away from the surface $\psi = r$. Moreover, homogeneity of the brane world-volume (equivalent to its $\mathrm{SL}(2,\mathbb{R})$-symmetry) imply that the amplitude can only depend on the transverse coordinate ψ. Hence, we conclude that $\langle\Phi^{h'}\rangle_r \propto \delta(\psi - r)$ up to a constant.

Altogether this means that the one-point function $\langle\Phi^j\rangle_r$ can be read off from the Fourier-expansion of $\delta(\psi - r)$ w.r.t. the basis formed by Φ^j. This expansion takes the form

$$\begin{aligned}\delta(\psi - r) \;&=\; \kappa_1 \int_{\mathbb{S}} dj \int du^2\ (\Phi^j(u|\psi,\chi,\nu))^*\ \Big(d_0^j(u)\cosh r(2j+1) \\ &\qquad\qquad - d_1^j(u)\sinh r(2j+1)\Big)\end{aligned} \tag{2.25}$$

$$\text{where} \qquad d_\epsilon^j(u) \;=\; |u+\bar{u}|^{2j}\,\mathrm{sgn}^\epsilon(u+\bar{u}) \ ,$$

and κ_1 is defined as $\kappa_1 = (2/\pi i)\cosh r$. To prove this statement we make use of the following auxiliary formula

$$\cosh\psi \int d^2u\ (\Phi^j(u|\psi,\chi,\nu))^*\ d_\epsilon^j(u) \;=\; \begin{cases} \cosh\psi(2j+1) & \text{for } \epsilon = 0 \\ \sinh\psi(2j+1) & \text{for } \epsilon = 1 \end{cases} \tag{2.26}$$

which is derived in Appendix A. The main idea is to show first that the integral is constant along the surfaces of constant ψ, i.e. the orbits of the $\mathrm{SL}(2,\mathbb{R})$ action on H_3^+. At this point one makes use of the transformation law (2.22) of the functions Φ^j together with the fact that the functions d_ϵ^j depend only on the sum $u+\bar{u}$. Then one exploits that Φ^j are eigen-functions of the Laplace operator and derives a second order differential equation for the ψ-dependence of the integrals. The latter has two independent coefficients which can finally be determined by studying the integral near the boundary of H_3^+, i.e. in the limit $\phi \to \infty$ where Φ^j is known to behave according to formula (2.23).

Once the auxiliary formula is established, it is straightforward to obtain eq. (2.25). In fact, one has

$$\begin{aligned}&\int_{\mathbb{S}} dj\ (\cosh\psi(2j+1)\cosh r(2j+1) - \sinh\psi(2j+1)\sinh r(2j+1)) \\ &\qquad = \; \frac{i}{2}\int_{-\infty}^{\infty} dP\ e^{2i(\psi-r)P} = \frac{\pi i}{2}\delta(\psi - r) \ .\end{aligned}$$

The reason we have gone through these technical steps here was to show how the δ-functions arises from the two terms involving d_0 and d_1. One single term alone would give an answer that is either symmetric or anti-symmetric under the reflection $\psi \to -\psi$. Only if the two terms work together, we can obtain a $\delta-$ function that is localized at a single point r on the real line.

The spherical branes. A similar analysis can be performed for the 2-spheres in H_3^+. The formula for the decomposition of the δ function of a 2-sphere characterized by $\Lambda = \Lambda_0$ is given by

$$\delta(\Lambda - \Lambda_0) \;=\; \kappa_2 \int_{\mathbb{S}} dj \int du^2\ (\Phi^j(u|\psi,\chi,\nu))^*\ \sinh\Lambda_0(2j+1)\ (u\bar{u}+1)^{2j} \tag{2.27}$$

where $\kappa_2 = (4i/\pi)\sinh\lambda_0$. Note that $\mathrm{sgn}(u\bar{u}+1) = 1$. Correspondingly, there appears only one term in the expansion of the spherically symmetric δ-function in contrast to what we found for the AdS_2-branes above. The proof of this formula follows the same ideas as described in the previous paragraph, but of course one now has to use the SU(2) action on H_3^+. The counterpart of the auxiliary formula (2.26) turns out to be

$$\sinh\Lambda \int d^2u\ (\Phi^j(u|\Lambda,\varphi,\mu))^*(u\bar{u}+1)^{2j} \ = \ \sinh\Lambda(2j+1) \ \ .$$

In the end one must recall that the coordinate Λ is restricted to non-negative values so that

$$\int_{\mathbb{S}} dj\ \ \sinh\Lambda(2j+1)\sinh\Lambda_0(2j+1) \ = \ \frac{\pi}{4i}\delta(\Lambda-\Lambda_0) \ \ .$$

It is the restriction $\Lambda \geq 0$ that really allows us to decompose the spherically symmetric δ-functions in the way we have described with only a single u-dependent term appearing at each momentum j.

In conclusion, the expansions of the δ-functions on the Euclidean AdS_2- and the S^2-branes lead us to expect that the semiclassical limits of the one-point functions are given by

$$\langle\Phi^j(u|z)\rangle_r^{AdS_2} \quad \overset{k\to\infty}{\sim} \quad |u+\bar{u}|^{2j}\exp(-\,\mathrm{sgn}(u+\bar{u})r(2j+1)) \tag{2.28}$$

$$\langle\Phi^j(u|z)\rangle_{\Lambda_0}^{S^2} \quad \overset{k\to\infty}{\sim} \quad (u\bar{u}+1)^{2j}\sinh\Lambda_0(2j+1) \tag{2.29}$$

Let us anticipate that we shall indeed find an expression for $\langle\Phi^j(u|z)\rangle_r^{AdS_2}$ with the expected semi-classical behavior. For the spherical branes, however, the 1-point functions will have the form (2.29) with an imaginary parameter Λ_0 corresponding to an 'imaginary radius' of the 2-spheres.

2.4. **Semi-classical limit of open string spectra**

Finally, we would like to understand the semi-classical (point-particle) limit of the open string theory on the brane. The wave functions of open strings can be expanded in eigen-functions of the Laplace operator on the brane. In order to get an idea of the spectrum of open strings on our branes we have to understand the spectrum of the Laplace operator. We will discuss this for the two different cases separately.

The AdS_2-brane. With the data and notations provided in Subsection 2.2. we can easily write down the Laplace operator on the brane AdS_2^r,

$$Q_r \ \sim \ \partial_\chi^2 + \partial_\chi + e^{-2\chi}\partial_\nu^2 \ \ . \tag{2.30}$$

Once more, it is easy to write down an explicit formula for these eigen-functions

$$\begin{aligned}\Xi^j(u|r;\nu,\chi) \quad &= \quad (v'_u\, h\, {v'}_u^{\dagger})^j|_{\psi=r} \\ &= \quad \cosh^j r\ \left((u-\nu)^2e^\chi + e^{-\chi}\right)^j\end{aligned} \tag{2.31}$$

Here, u is a real coordinate and $v'_u = (iu, 1)$. In the second line we inserted the parametrization (2.1) of H_3^+.

Recall that the AdS_2 branes admit an action of $\mathrm{SL}(2,\mathbb{R})$ which commutes with Q_r so that the eigen-functions of the Laplace operator form representations for $\mathrm{SL}(2,\mathbb{R})$. It turns out that eigen-functions for a given eigenvalue $j(j+1), j = -\frac{1}{2} + iP$, carry an irreducible representation $\mathcal{P}_j$ from the principal continuous series. The transformation law of the functions (2.31) under the action of $\mathrm{SL}(2,\mathbb{R})$ is easily worked out,

$$\Xi^j(u|ghg^\dagger) \;=\; |\beta u + \delta|^{2j}\, \Xi^j_{g\cdot u}(h) \quad \text{where} \quad g \cdot u \;=\; \frac{\alpha u + \gamma}{\beta u + \delta} \;, \tag{2.32}$$

and $g = \left(\begin{smallmatrix} \delta & -i\beta \\ i\gamma & \delta \end{smallmatrix}\right)$ is a $\mathrm{SL}(2,\mathbb{C})$-matrix conjugate to an element of the $\mathrm{SL}(2,\mathbb{R})$ subgroup that preserves the Euclidean AdS_2. The asymptotics of these solutions are given by

$$\begin{aligned} c^{-1}(j)\; \Xi^j(u|r;\nu,\phi) \quad &\overset{\phi\to\infty}{\sim} \quad J^j(u|\nu)\, e^{j\phi} + \\ &+ \cosh^{2j+1} r \frac{c(-j-1)}{c(j)}\, \delta(\nu - u)\, e^{-(j+1)\phi}, \end{aligned} \tag{2.33}$$

$$\text{where} \qquad J^j(u|\nu) \;=\; |u-\nu|^{2j}\, c^{-1}(j) \;\;, \quad c(j) \;:=\; \sqrt{\pi}\, \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+1)} \;\; . \tag{2.34}$$

$J^j(u|\nu)$ is the integral kernel of the unitary transformation that implements the isomorphism between the two representations $\mathcal{P}_j$ and $\mathcal{P}_{-j-1}$ of $\mathrm{SL}(2,\mathbb{R})$ [25]. The normalizing factor $c(j)$ is the so-called Harish-Chandra c-function which plays a central role in the harmonic analysis of non-compact groups.

The set of functions $\{\Xi^j; j \in -1/2 + i\mathbb{R}^+\}$ forms a basis for the space of wave-functions of a particle on the Euclidean AdS_2. We shall see that they are in one-to-one correspondence with the ground states of the boundary conformal field theory. An important piece of information is the prefactor of the second term in (2.33): The first term describes a plane wave that is injected with some momentum parametrized by j at the boundary of the AdS_2-brane. Accordingly, the second term gives the outgoing signal, leading us to interpret the non-trivial coefficient in front of the second term as a semi-classical reflection amplitude,

$$R_{\rm c}(P) \;\equiv\; R_{\rm c}(r;P) \;=\; -(\cosh r)^{2iP} \frac{\Gamma(1-iP)}{\Gamma(1+iP)} \frac{\Gamma\big(\frac{1}{2}+iP\big)}{\Gamma\big(\frac{1}{2}-iP\big)}. \tag{2.35}$$

We will determine the rather nontrivial stringy corrections to this formula in Section 4.

Let us note that some important motivation to be interested in such reflection amplitudes derives from their relation with relative spectral densities. This relation is reviewed in Appendix B. It allows to predict how the density $\rho(P)$ of states in a quantum mechanical system changes when the scattering potential is varied. In our case, we shall fix one AdS_2-brane with parameter r_* and use it as a reference to compare with the spectral densities of the other branes. The precise relation is

$$\rho_{\rm rel}(P|r,r_*) \;=\; \frac{1}{2\pi i}\,\frac{\partial}{\partial P}\,\log\frac{R_{\rm c}(r;P)}{R_{\rm c}(r_*;P)} \;=\; \frac{1}{\pi}\,\log\frac{\cosh r}{\cosh r_*}\;.$$

Informally one may think of $\rho_{\rm rel}(P|r,r_*)$ as $\rho^{\rm c}_0(P) - \rho^{\rm c}_*(P)$. In the semiclassical limit, this quantity is is completely unrelated to the closed string couplings we described in the previous subsection. But this changes when we turn to the stringy analogue. In fact, in string theory the couplings of closed strings to the brane allow to compute the open string spectral density by using world sheet duality ("Cardy computation"). As we shall see below, the stringy couplings to the brane do indeed provide a formula for the spectral density that reduces to the semi-classical expression when the stringy corrections are turned off.

Spherical branes. For the spherical branes our discussion of the semi-classical limit of open string theory can be rather short as this is a lot simpler than for the AdS_2 branes. In addition, the following remarks can at best serve as some kind of guiding ideas since it will turn out that the conformal field theory does not seem to allow one to construct branes that are localized along a finite 2-sphere.

A priori, we would expect the following picture to emerge. As is well known, the space of functions on a 2-sphere is spanned by spherical harmonics $\Psi^j_m(\phi,\mu), j = 0,1,\dots; |m| < j$. They are eigen-functions of the standard Laplace operator on S^2 with eigen-value $j(j+1)$ and they transform according to the $2j+1$-dimensional representation of SU(2). Now let us take into account that the spherical branes come equipped with a non-vanishing B-field. By standard arguments, this implies that the space of wave functions must be finite dimensional with a dimension that grows as we increase the parameter Λ_0 of the 2-sphere. Since the number of states is an integer, we conclude that Λ_0 must be quantized too, i.e. boundary theories will only exist for a discrete set of Λ_0. All these expectations are essentially copied from the findings for spherical branes in S^3 [28, 29] and they do give rise to a rather accurate picture of the open string sector for the SU(2)-symmetric branes on H_3^+. But let us stress one more that the open string couplings will not quite fit into this geometric framework.

3. THE CLOSED STRING SECTOR

We shall now look for quantum corrections to the expressions for the semi-classical closed string couplings that we constructed in the previous section. In other words, our aim here is to obtain the exact 1-point functions $\langle\Phi^j(u|z)\rangle^{\rm BC}$ that describe maximally symmetric branes on H_3^+.[6] These 1-point functions are strongly constrained by the gluing conditions (2.15) or (2.20) for chiral currents. The latter are certainly required for conformal invariance of the boundary conformal field theory, as usual. Throughout most of this

[6]Note that these one-point functions contain the same information as the boundary state $|BC\rangle$ of the corresponding BCFT.

section we shall concentrate on the AdS_2 branes which are characterized by

$$J^{\pm} \;=\; \bar{J}^{\pm}, \qquad J^0 \;=\; \bar{J}^0 \quad . \tag{3.1}$$

A short subsection on the case of spherical branes appears at the end of the section. Obviously, the gluing condition (3.1) is not sufficient for the construction of a consistent boundary conformal field theory. In addition, one has to satisfy consistency conditions that arise from the factorization properties of correlation functions. The most important condition arises from the factorization of 2-point functions of bulk operators in the presence of a boundary [30, 31]. Together, the gluing condition and the factorization constraints can be expected to determine the 1-point functions completely. Let us emphasize that in this approach there is really no need for any geometric intuition of the type we have gained in the previous section.

The analysis of gluing and factorization constraints will lead us to a rather plausible candidate for the quantum corrections to the semi-classical boundary state. There is a caveat, though, which comes from the fact that we can evaluate only one special factorization constraint that arises from considering 2-point functions in which one of the two bulk fields corresponds to a degenerate current algebra representation. It turns out that this condition is not sufficient to fully determine the form of the 1-point function. Additional requirements have to be imposed in order to narrow down the remaining freedom. We shall later use a non-rational analogue of the Cardy condition for that purpose.

3.1. **Primary bulk fields**

Some basic facts. Let us collect some basic properties of the primary bulk fields that will be used in the present paper (see [12, 10] for more details). We are interested in bulk fields $\Phi^j(u|z)$, $\Im z \geq 0$, which obey the following operator product expansion (OPE) with respect to the currents,

$$J^a(z)\,\Phi^j(u|w) \;=\; \frac{1}{z-w}\,\mathcal{D}^a_{j,u}\Phi^j(u|w) \;\;,\;\; \bar{J}^a(\bar{z})\,\Phi^j(u|w) \;=\; \frac{1}{\bar{z}-\bar{w}}\bar{\mathcal{D}}^a_{j,u}\Phi^j(u|w)\;, \tag{3.2}$$

where the differential operators $\mathcal{D}^a_{j,u}$ are defined by

$$\mathcal{D}^+_{j,u} \;=\; -u^2\partial_u + 2ju \qquad \mathcal{D}^0_{j,u} \;=\; -u\partial_u + j \qquad \mathcal{D}^-_{j,u} \;=\; -\partial_u \tag{3.3}$$

and the same expressions with $\bar{u}$ instead of u are used to define $\bar{\mathcal{D}}^a_{j,\bar{u}}$. The fields $\Phi^j(u|z)$ are primary also w.r.t. the Sugawara Virasoro algebra with conformal dimensions

$$\Delta_j \;=\; -\frac{1}{k-2}j(j+1) \;=\; -b^2 j(j+1)\;\;. \tag{3.4}$$

In this expression and throughout most of our text, we parametrize k through $b^2 \equiv (k-2)^{-1}$.

Semi-classically one may think of the fields $\Phi^j(u|z)$ as being related to the functions $\Phi^j(u|h)$ that were defined in eq. (2.21) by identifying h with the field $h(z)$ that appears in the action of the H_3^+ WZNW model,

$$\Phi^j(u|z) \;=\; \Phi^j(u|h(z)) \;\; .$$

In terms of the coordinates $(\phi,\gamma,\bar\gamma)$ this amounts to

$$\Phi^j(u|z) \;=\; \frac{2j+1}{\pi}\Big((\gamma(z)-u)(\bar\gamma(z)-\bar u)e^{\phi(z)}+e^{-\phi(z)}\Big)^{2j} \;\; . \tag{3.5}$$

Normalization. A useful way to fix the normalization of these primary fields is to specify their asymptotic behavior near the boundary of H_3^+ [10]. The fields $\Phi^j(u,z)$ are normalized such that

$$\Phi^j(u|z) \;\sim\; :e^{2(-j-1)\phi(z)}:\delta^2(\gamma(z)-u)+B(j):e^{2j\phi(z)}:|\gamma(z)-u|^{4j} \;\; . \tag{3.6}$$

This should be compared with the asymptotic behavior (2.23) of the functions $\Phi^j(u|h)$. The only difference is that the the coefficient function $B(j)$ which appears for the fields $\Phi^j(u|z)$ is now given by[7]

$$B(j) \;=\; -\nu_b^{2j+1}\frac{2j+1}{\pi}\frac{\Gamma\big(1+b^2(2j+1)\big)}{\Gamma\big(1-b^2(2j+1)\big)}, \quad \nu_b \;=\; \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)} \;\; . \tag{3.7}$$

This expression reduces to the corresponding coefficient in rel. (2.23) in the limit $b\to 0$. Hence, the asymptotic behavior (3.6) is a deformation of rel. (2.23) which includes stringy effects at finite curvature of the background.

Although the normalization fixed by rel. (3.6) is the most natural one from the point of view of string theory on H_3^+ (cf. [10]), we find another set of fields more convenient from the mathematical point of view. The new set is introduced by

$$\Theta^j(u|z) \;\equiv\; B^{-1}(j)\,\Phi^j(u|z) \;\; .$$

This of course amounts to setting the prefactor of the second term in rel. (3.6) equal to one,

$$\Theta^j(u|z) \;\sim\; :e^{2j\phi(z)}:|\gamma(z)-u|^{4j}+B^{-1}(j):e^{2(-j-1)\phi(z)}:\delta^2(\gamma(z)-u) \;\; . \tag{3.8}$$

The two-point function of the fields $\Theta^j(u|z)$ on the complex plane is then given by an expression of the following form

$$\begin{aligned}&\big\langle\Theta^{-j_2-1}(u_2|z_2)\Theta^{j_1}(u_1|z_1)\big\rangle\,|z_2-z_1|^{4\Delta_{j_1}} = \\ &= \frac{\pi^3}{4P_1^2}\delta(P_2-P_1)\,\delta^{(2)}(u_2-u_1)+\frac{\pi\delta(P_2+P_1)}{B\big(-\frac12-iP_1\big)}|u_2-u_1|^{4j_1} \;\; ,\end{aligned} \tag{3.9}$$

[7] The following expression differs by a factor of π in the expression for ν_b from the one given in [10]. This means that the primary fields denoted $\Phi^j(x|z)$ in [10] differ by a factor π^j from our primary fields $\Phi^j(u|z)$.

where $j_i = -\frac{1}{2} + iP_i$, $P_i \in \mathbb{R}^+$ for $i = 1, 2$. In our analysis we will mainly work with the fields Θ^j, but all of our results will finally be rewritten in terms of the fields Φ^j above. The translation between the different normalizations is straightforward.

Reflection property. Except from a simple factor $(2j+1)/\pi$, one can identify the coefficient $B(j)$ with a reflection amplitude $R(j)$ for closed strings on H_3^+. Having fixed the normalization of operators by (3.8) one may re-express the reflection of closed strings as a linear relation between the operators $\Theta^j(u|w)$ and $\Theta^{-j-1}(u|w)$,

$$\Theta^j(u|z) \;=\; -R(-j-1)(\mathcal{I}_j\Theta^{-j-1})(u'|z) \;, \tag{3.10}$$

$$\text{where} \qquad R(j) \;=\; -\nu_b^{2j+1}\frac{\Gamma\big(1+b^2(2j+1)\big)}{\Gamma\big(1-b^2(2j+1)\big)} \;, \tag{3.11}$$

and $\mathcal{I}_j$ is the intertwining operator that establishes the equivalence of the $\mathrm{SL}(2,\mathbb{C})$-representations P_{-j-1} and P_j,

$$(\mathcal{I}_j\Theta^{-j-1})(u|z) \;=\; \frac{2j+1}{\pi}\int_{\mathbb{C}} d^2u' \; |u-u'|^{4j}\,\Theta^{-j-1}(u'|z) \;. \tag{3.12}$$

The operator $\mathcal{I}_j$ is normalized such that $\mathcal{I}_{-j-1}\circ\mathcal{I}_j = \mathrm{Id}$. This normalization ensures its unitarity for $j \in -\frac{1}{2} + i\mathbb{R}$.

3.2. **Constraints from the gluing condition**

To begin with, let us now analyse the constraints on the form of the one-point function that arise from the gluing condition (3.1). As usual, the z-dependence of the one-point functions can be determined from the behavior under conformal transformations of the world-sheet theory. This gives

$$\langle\Theta^j(u|z)\rangle_r \;=\; \frac{\mathcal{A}_u(j|r)}{|z-\bar{z}|^{2\Delta_j}} \;.$$

The dependence w.r.t. the variable u is likewise restricted by Ward identities associated with the currents $J^a, \bar{J}^a$. They can be derived using the operator product expansions of the chiral currents with the primary bulk fields. The resulting differential equations for $\mathcal{A}_u(j|r)$ are of the form

$$(\mathcal{D}^a_{j,u} - \mathcal{D}^a_{j,\bar{u}})\mathcal{A}_u(j|r) \;=\; 0 \;. \tag{3.13}$$

These equations are locally solved by $|u+\bar{u}|^{2j}$. However, $u+\bar{u}=0$ is a singular point so that there are two linearly independent distributional solutions of eq. (3.13). Hence, we have to consider the two solutions $|u+\bar{u}|^{2j}$ and $|u+\bar{u}|^{2j}\,\mathrm{sgn}(u+\bar{u})$. This means that the gluing conditions (3.13) restrict the one-point function to be of the form

$$\langle\Theta^j(u|z)\rangle_r \;=\; \frac{|u+\bar{u}|^{2j}A_\sigma(j|r)}{|z-\bar{z}|^{2\Delta_j}} \;, \tag{3.14}$$

where $A_\sigma(j|r)$ still depends on the variable u through the function $\sigma \equiv \mathrm{sgn}(u+\bar{u})$.

Let us furthermore note that additional restrictions arise from the reflection property (3.10) of the bulk primary fields. In the Appendix A.3 we prove the identity

$$\begin{aligned}\frac{2j+1}{\pi}\int_{\mathbb{C}} d^2u\ |u+\bar{u}|^{-2j-2}\,\mathrm{sgn}^{\epsilon}(u+\bar{u})\ |u-\gamma|^{4j} = \\ = \ -(-)^{\epsilon}|\gamma+\bar{\gamma}|^{2j}\,\mathrm{sgn}^{\epsilon}(\gamma+\bar{\gamma})\ \ .\end{aligned} \tag{3.15}$$

It implies a nice reflection property for $A_\sigma(j|r)$. The latter is most easily expressed in terms of the coefficients $A^\epsilon(j|r)$ which appear in the expansion

$$A_\sigma(j|r)\ \equiv\ A^0(j|r)+\sigma A^1(j|r)$$

with respect to σ (note that $\sigma^2 = 1$). With the help of the property (3.10), the coefficients A^i are easily shown to satisfy

$$A^\epsilon(j|r)\ =\ (-)^\epsilon\, R(-j-1)\, A^\epsilon(-j-1|r)\ \ . \tag{3.16}$$

The formula (3.14) along with the reflection property (3.16) encode all the information one can extract without considering further constraints.

3.3. **Constraints from two-point functions with degenerate fields**

Introductory remarks. As we have mentioned before, the simplest factorization constraints are obtained from the two-point functions of the theory. This correlation function can be factorized in two different ways: If one imagines the two bulk fields close to each other it is most natural to use the bulk operator product expansion to get a factorization in the closed string channel, leading to a representation of the two-point function as sum over one-point functions. The configuration where the two fields are far from each other is projectively equivalent to the situation where the fields are close to the boundary. In the latter case it is more natural to factorize in the open string channel by writing the bulk fields as sum over fields localized on the boundary. This yields an expression which is bilinear in the corresponding bulk-boundary operator product coefficients.

In rational conformal field theories one can exploit the equivalence between these two ways of factorizing the two-point function by concentrating on the contribution of the identity boundary field in the open string channel. One thereby gets a powerful quadratic equation for the one-point functions. In non-compact models, however, it is usually not possible to mimic this strategy since the identity may not appear in the open string channel at all.

Fortunately, there exists a way out. In fact, the fields Θ^j we have considered so far are not the only ones in the theory. They are the fields that are in one-to-one correspondence with the normalizable states of the model. By analytic continuation in j, however, we obtain additional fields which are still perfectly well defined even though they do not correspond to any normalizable state. For certain discrete values of j, the new fields are associated with degenerate representations of the current algebra. This implies that the operator product of these degenerate fields with any other field of the theory contains only finitely many blocks and that the factorization in the open string channel includes a contribution from the identity boundary field.

The fact that analytic continuation in j allows to recover the degenerate fields is not a priori obvious, though. The power of the results obtained by *assuming* that this is the case, cf. e.g. [12, 31], illustrates that one should consider it as a rather profound property of the theory. Therefore we would like to emphasize that this assumption can now be rigorously justified with the help of the results in [10, 11].

Here we shall only consider the simplest of the degenerate fields, $\Theta^{1/2}$, and study the following two-point functions

$$\mathcal{G}^j_r\left(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\right) \;\equiv\; \langle \Theta^{\frac{1}{2}}(u_2|z_2)\Theta^j(u_1|z_1)\rangle_r \, . \tag{3.17}$$

The special feature of the degenerate field $\Theta^{\frac{1}{2}}$ is that it satisfies the following differential equations

$$\partial_u^2 \Theta^{\frac{1}{2}}(u|z) \;=\; 0 \quad , \quad \partial_{\bar{u}}^2 \Theta^{\frac{1}{2}}(u|z) \;=\; 0 \; . \tag{3.18}$$

They become obvious when we identify $\Theta^{1/2}$ with the fundamental matrix-valued $h(z)$ through the familiar relation

$$\Theta^{\frac{1}{2}}(u|z) \;=\; (-u, 1)\cdot h(z) \cdot \begin{pmatrix} -\bar{u} \\ 1 \end{pmatrix} \quad . \tag{3.19}$$

Indeed, the expression is linear in both u and $\bar{u}$ so that the second derivatives ∂_u^2 and $\partial_{\bar{u}}^2$ vanish.

Differential equations. The form of the two-point function (3.17) is strongly constrained by various differential equations which we now want to discuss.

To begin with, there are six differential equations that arise from the symmetry of the theory under the *two actions of* $\mathrm{SL}(2,\mathbb{R})$ on the world-sheet coordinates z_i and the parameters u_i, respectively. The first is generated by the Virasoro modes $L_n, |n| \leq 1$, while the second is associated with the zero modes of the currents. The resulting equations imply that the two-point function can *locally* be represented as

$$\mathcal{G}^j_r\left(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\right) \;=\; \frac{|z_1 - \bar{z}_1|^{2(\Delta - \Delta_j)}}{|z_1 - \bar{z}_2|^{4\Delta}} |u_1 + \bar{u}_1|^{2j-1} |u_1 + \bar{u}_2|^2 G^j_r(u|z) \;\; . \tag{3.20}$$

Here we have abbreviated $\Delta \equiv \Delta_{\frac{1}{2}}$ and introduced the cross-ratios u and z as

$$z \;=\; \left|\frac{z_2 - z_1}{z_2 - \bar{z}_1}\right|^2 \;, \qquad u \;=\; \left|\frac{u_2 - u_1}{u_2 + \bar{u}_1}\right|^2 \;. \tag{3.21}$$

Next we are going to exploit the *null vector decoupling* equations (3.18). They were motivated above and express the decoupling of the null vector in the Verma module of spin $\frac{1}{2}$ of the $\mathfrak{sl}_2$ algebra. From these equations we easily conclude that G^j can be expanded in the form

$$G^j_r(u|z) \;=\; G^j_{r,0}(z) + u G^j_{r,1}(z) \quad .$$

Finally, we have to take the Knizhnik-Zamolodchikov equations into account which can be derived with the help of the Sugawara construction. With our choice of the gluing condition (3.1) they read

$$\begin{aligned} tz(z-1)\partial_z G^j_r(u|z) \;&=\; \big[\, u(u-1)(u-z)\partial_u^2 - (2ju(z-1)+(u+z)(u-1))\partial_u \\ &\qquad + u - \tfrac{z}{2} + j(z-1)\,\big]\, G^j_r(u|z) \;\; . \end{aligned} \tag{3.22}$$

The null vector decoupling equations furthermore imply that (3.22) reduces to a 2×2 matrix equation, and that it therefore has a two-dimensional space of solutions. Two canonical bases $\mathcal{F}^{\rm s}_\epsilon$ and $\mathcal{F}^{\rm t}_\epsilon$, $\epsilon = \pm$, for the space of solutions to (3.22) are introduced in Appendix C.

It is important to note that in our particular case the Knizhnik-Zamolodchikov equations are nonsingular for $z \in (0,1)$ and $u \in (0,\infty)$.[8] As a consequence, the two-point function (3.17) can be specified uniquely through its asymptotic behavior for $z \to 0$, followed by $u \to 0$. These asymptotics are what we will determine next.

Asymptotics $z_2 \to z_1$. The decomposition of $\mathcal{G}^j_r$ into conformal blocks can be obtained with the help of the operator product expansion

$$\begin{aligned} \Theta^{\frac12}(u_2|z_2)\Theta^j(u_1|z_1) \;\underset{z_2\to z_1}{\sim}\; \sum_{\epsilon=\pm} |z_2-z_1|^{2(\Delta_{j+\frac{\epsilon}{2}}-\Delta_j-\Delta)} |u_1-u_2|^{1-\epsilon} \times \\ \times\, C_\epsilon(j)\big(\Theta^{j+\frac{\epsilon}{2}}(u_1|z_1) + \mathcal{O}(z_2-z_1) + \mathcal{O}(u_2-u_1)\big) \;\; . \end{aligned} \tag{3.23}$$

To be specific, let us spell out the explicit expressions for the operator product coefficients $C_\epsilon(j)$, $\epsilon = \pm$ that appear on the right hand side (see [12, 10])

$$C_+(j) \;\equiv\; 1 \;\; , \qquad C_-(j) \;=\; \frac{1}{\nu_b}\frac{\Gamma(-b^2(2j+1))\Gamma(1+2b^2j)}{\Gamma(1+b^2(2j+1))\Gamma(-2b^2j)} \;\; . \tag{3.24}$$

In this way, we have completely determined the expression for $\mathcal{G}^j_r$ in terms of the coefficients $A_\sigma(j|r)$,

$$\begin{aligned} \mathcal{G}^j_r\big(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\big) \;=\; |z_1-\bar z_1|^{2(\Delta-\Delta_j)}|z_1-\bar z_2|^{-4\Delta}|u_1+\bar u_1|^{2j-1}|u_1+\bar u_2|^2 \times \\ \times \sum_{s=\pm} C_\epsilon(j)\mathcal{F}^{\rm s}_\epsilon(u|z) A_{\sigma_1}(j+\tfrac{\epsilon}{2}|r) \;\; . \end{aligned} \tag{3.25}$$

Here, $\mathcal{F}^{\rm s}_\epsilon$ is one of the bases in the space of solutions of eqs. (3.22) that we have mentioned before (see Appendix C for details).

[8] This is not true for generic values of $j_1,...,j_4$ where one has a singularity at $u = z$. This plays an important role e.g. in [8].

Asymptotics for $\Im z_2 \to 0$. The field $\Theta^{\frac{1}{2}}(u|z)$ becomes singular for $\Im z \to 0$. It is possible to describe this singular behavior through an expansion into boundary fields. This expansion is restricted by null vector decoupling and Knizhnik-Zamolodchikov equations and therefore has the form

$$\begin{aligned}\Theta^{\frac{1}{2}}(u|z) \underset{\Im z\downarrow 0}{\sim} \; & A(\tfrac{1}{2},0|r)\;(\Im z)^{\frac{3}{2}b^2}\,(u+\bar{u})\,\mathrm{id} \\ & + A(\tfrac{1}{2},1|r)\;(\Im z)^{-\frac{1}{2}b^2}\big(\Psi_-(x)-\tfrac{1}{2}(u-\bar{u})\Psi_0(x)-u\bar{u}\Psi_+(x)\big)\;\;.\end{aligned} \tag{3.26}$$

The three fields $\Psi_m = \Psi^1_m$ are boundary fields which are associated with the degenerate spin $j=1$ representation of $\mathrm{SL}(2,\mathbb{R})$.

$A(\frac{1}{2},0|r)$ is a natural device to parametrize the boundary conditions in the quantum theory as one can see from the following short computation [9]

$$\mathrm{Tr}(\omega_0 h(z)) \;=\; [(\partial_u+\partial_{\bar{u}})\Theta^{\frac{1}{2}}(u|z)]_{u=0} \underset{\Im z\downarrow 0}{\sim} (2\Im z)^{\frac{3}{2}b^2}\,A(\tfrac{1}{2},0|r) \tag{3.27}$$

where we have inserted eq. (3.19) in the first step. Equation (3.27) can be regarded as a natural quantum counterpart of the equation $\mathrm{Tr}(\omega_0 h) = 2\sinh r$ that defines the boundary condition in the classical theory. This motivates us to focus on the term proportional to the identity in (3.26). Formally one may project out the second term in (3.26) by considering

$$\begin{aligned}\mathcal{P}\mathcal{G}^j_r\big(\begin{smallmatrix}u_2 & u_1\\ z_2 & z_1\end{smallmatrix}\big) \;&\equiv\; \tfrac{1}{2}(u_2+\bar{u}_2)\big[(\partial_{u_2}+\partial_{\bar{u}_2})\mathcal{G}^j_r\big(\begin{smallmatrix}u_2 & u_1\\ z_2 & z_1\end{smallmatrix}\big)\big]_{u_2=0}\;\;, \\ &\equiv\; \tfrac{1}{2}(u_2+\bar{u}_2)\;\big\langle \mathrm{Tr}(\omega_0 h(z_2))\Theta^j(u_1|z_1)\big\rangle_r\;\;.\end{aligned}$$

The bulk-boundary expansion (3.26) then implies that the leading asymptotics of $\mathcal{P}\mathcal{G}^j_r$ for $\Im z_2 \to 0$ is given by

$$\mathcal{P}\mathcal{G}^j_r\big(\begin{smallmatrix}u_2 & u_1\\ z_2 & z_1\end{smallmatrix}\big) \underset{\Im z_2\to 0}{\sim} |z_2-\bar{z}_2|^{\frac{3}{2}b^2}(u_2+\bar{u}_2)\;A(\tfrac{1}{2},0|r)\;\langle\Theta^j(u_1|z_1)\rangle_r. \tag{3.28}$$

REMARK 1. — In rational conformal field theories one can conclude from (3.26) that $A(\frac{1}{2},0|r)$ is proportional to the one-point function of the operator $\Theta^{\frac{1}{2}}(u|z)$ and this results in a stronger version of the factorization constraint. In non-rational theories, however, one can not expect to find a *simple* relation between $A(\frac{1}{2},0|r)$ and $\langle\Theta^{\frac{1}{2}}(u|z)\rangle_r$, as was first observed for Liouville theory by Fateev et al. [31], cf. also our introductory remarks in this Subsection.

[9] It should be possible to calculate $A(\frac{1}{2},0|r)$ by means of a free field calculation similar to what was done in the case of Liouville theory [31].

Comparison of the asymptotics for $\Im z_2 \to 0$. Finally, we can combine all the information we have collected and derive the factorization constraint we were looking for. We achieve this by comparing eq. (3.28) with the expression (3.25) for the two-point function. The limit $\Im z_2 \to 0$ implies $z \to 1$ so that the asymptotic behavior of the conformal blocks is given in terms of the corresponding fusion coefficients $F_{st}(j)$ which can be found in Appendix C,

$$\mathcal{PG}^j_r\left(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\right) \underset{\Im z_2 \to 0}{\sim} |z_1 - \bar{z}_1|^{2(\Delta-\Delta_j)}|z_1 - \bar{z}_2|^{-4\Delta}|u_1 + \bar{u}_1|^{2j-1}|u_1 + \bar{u}_2|^2 \times \\ \times \left(\frac{4\Im z_2 \Im z_1}{|z_2 - \bar{z}_1|^2}\right)^{-2\Delta} \frac{4\Re u_1 \Re u_2}{|u_1 + \bar{u}_2|^2} \sum_{\epsilon=\pm} F_{\epsilon-}\; C_\epsilon(j)\; A_{\sigma_1}(j + \tfrac{\epsilon}{2}|r)\ . \tag{3.29}$$

In order to simplify this result we introduce some new objects $E_\sigma(j|r)$ which are related to $A_\sigma(j|r)$ by

$$A_\sigma(j|r) \; = \; \nu_b^{-j-\frac{1}{2}}\, \Gamma(-b^2(2j+1))\, E_\sigma(j|r)\ . \tag{3.30}$$

After inserting the explicit expressions for F_{s-} and $C_s(j)$ we can now rewrite (3.29) in the form

$$\mathcal{PG}^j_r\left(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\right) \underset{\Im z_2 \to 0}{\sim} \\ \underset{\Im z_2 \to 0}{\sim} |z_1 - \bar{z}_1|^{-2\Delta_j}|z_2 - \bar{z}_2|^{-2\Delta}|u_1 + \bar{u}_1|^{2j}(u_2 + \bar{u}_2)\,\mathrm{sgn}(u_1 + \bar{u}_1) \times \\ \times \nu_b^{-j-1}\Gamma(-b^2(2j+1))\frac{\Gamma(-2b^2)}{\Gamma(-b^2)}\Big[E_\sigma\big(j + \tfrac{1}{2}|r\big) - E_\sigma\big(j - \tfrac{1}{2}|r\big)\Big]\ . \tag{3.31}$$

We conclude that (3.28) will hold provided that $E_\sigma(j|r)$ satisfies the following functional equation,

$$E_\sigma\big(j + \tfrac{1}{2}|r\big) - E_\sigma\big(j - \tfrac{1}{2}|r\big) \; = \; \sigma\,\sqrt{\nu_b}\,\frac{\Gamma(-b^2)}{\Gamma(-2b^2)} A(\tfrac{1}{2}, 0|r) E_\sigma(j|r)\ . \tag{3.32}$$

This is the factorization constraint for the one-point function that we have been looking for. Its solution $E_\sigma(j|r)$ determines the coefficients $A_\sigma(j|r)$ of the one-point functions (3.14) through eq. (3.30).

3.4. **Solutions of the factorization constraint**

We shall propose the following expression as the relevant solution of the factorization constraint

$$A_\sigma(j|r) \; = \; A_b\, \nu_b^{-j-\frac{1}{2}}\, \Gamma(-b^2(2j+1))\, e^{-r(2j+1)\sigma} \tag{3.33}$$

with some constant A_b that is arbitrary for the moment. The parameter r is related to $A(\frac{1}{2}, 0|r)$ by

$$A(\tfrac{1}{2}, 0|r) \; = \; -\frac{1}{\sqrt{\nu_b}}\frac{\Gamma(-2b^2)}{\Gamma(-b^2)}\; 2\sinh r\ . \tag{3.34}$$

This expression can easily be recognized as the most natural quantum "deformation" of the corresponding classical expression that is compatible with the factorization constraint and the reflection property. If we insert this expression back into eq. (3.14) and take care of difference between the normalizations of Θ^j and Φ^j, we finally obtain

$$\langle \Phi^j(u|z)\rangle_r \;=\; |u+\bar{u}|^{2j}\, \nu_b^{j+\frac{1}{2}}\, \Gamma(1+b^2(2j+1))\, e^{-r(2j+1)\sigma}\, \frac{A_b/\pi b^2}{|z-\bar{z}|^{2\Delta_j}} \;\; . \tag{3.35}$$

Up to some overall normalization, this agrees with our expectation (2.28) in the limit where b is sent to zero.

We should note, however, that (3.33) is *not* the most general solution to (3.32). By using [33, Appendix C] one may show that the most general solution of (3.32) can be written in the following form

$$E_\sigma(j|r) \;=\; \big(e^0(j|r) + e^1(j|r)\sigma\big) e^{-r(2j+1)\sigma} \;\; , \tag{3.36}$$

where the coefficients e^ϵ must be periodic in j with period $1/2$, i.e. they have to satisfy $e^\epsilon(j+1/2|r) = e^\epsilon(j|r)$. Since we also want the one-point function to obey the reflection property (3.16), we need to impose the condition $e^\epsilon(-j-1|r) = (-)^\epsilon e^\epsilon(j|r)$. In particular, the latter excludes solutions with $e^1(j|r)$ being constant in j.

The freedom that is left by the factorization constraint is therefore considerable. Under rather mild assumptions [32], however, it is possible to show that the periodicity requirement on $e^\epsilon(j|r)$ forbids a nontrivial *phase*. One should therefore supplement the factorization constraint with conditions that restrict the *absolute value* of $e^\epsilon(j|r)$. Such a condition will be provided by the analogue of the Cardy condition that we shall discussed in section 5.

REMARK 2. — When appealing to the result of [32], the most important assumption one has to make is analyticity of $A_\sigma(j|r)$ in a strip of width $1/2$ around the axis $j+\frac{1}{2} \in i\mathbb{R}$. The necessity of such an assumption can be seen by considering a *three* point function in which exactly one of the fields is $\Phi^{\frac{1}{2}}$. By associativity of the OPE one can get two different representations as sum over one-point functions, where the contours over which the variable j is integrated will differ by shifts of $1/2$. In order to relate these two representations one will have to shift contours, which motivates our assumption concerning the analyticity of $A_\sigma(j|r)$.

3.5. **The case of the spherical branes**

Let us briefly discuss the modifications of the previous analysis that are necessary to treat the spherical branes. The different gluing conditions (2.20) force us to modify the ansatz (3.14) for the one-point function to

$$\langle \Theta^j(u|z)\rangle_s \;=\; \frac{(1+u\bar{u})^{2j} A'(j|s)}{|z-\bar{z}|^{2\Delta_j}} \;\; . \tag{3.37}$$

Note that this time the function $1 + u\bar{u}$ has no singularities and hence it dependence on u is entirely fixed by the Ward identities for currents. In the analysis of the factorization constraint one finds instead of (3.25) the expression

$$\mathcal{G}_s^j\left(\begin{smallmatrix} u_2 & u_1 \\ z_2 & z_1 \end{smallmatrix}\right) = |z_1 - \bar{z}_1|^{2(\Delta-\Delta_j)}|z_1 - \bar{z}_2|^{-4\Delta}(1+|u_1|^2)^{2j-1}|1+u_1\bar{u}_2|^2 \times \\ \times \sum_{s=\pm} (-)^{\frac{1-s}{2}} C_s(j)\mathcal{F}_s(u|z)A'\big(r|j+\tfrac{s}{2}\big) \ , \tag{3.38}$$

where u and $1-u$ are now modified to

$$u = -\frac{|u_2-u_1|^2}{|1+u_2\bar{u}_1|^2} \ , \quad \text{and} \quad 1-u = \frac{(1+|u_1|^2)(1+|u_2|^2)}{|1+u_2\bar{u}_1|^2} \ ,$$

respectively. Contrary to (3.28), we shall now require that the two-point functions factorize into the product of two one-point functions when we take the arguments of the fields far apart, cf. remark 1 below eq. (3.28). If we introduce $E'(j|s)$ by

$$A'(j|s) = \nu_b^{-j}\,\Gamma(-b^2(2j+1))E'(j|s) \ ,$$

the factorization constraint turns into the following functional relation for $E'(j|s)$:

$$E'\big(j+\tfrac{1}{2}|s\big) + E'\big(j-\tfrac{1}{2}|s\big) = \Gamma(-b^2)E'(\tfrac{1}{2}|s)E'(j|s) \ . \tag{3.39}$$

The relevant solution turns out to be of the form

$$A_\sigma(j|s) = \frac{\Gamma(-b^2(2j+1))}{2\,\nu_b^j\,\Gamma(-b^2)}\,\frac{\sin s(2j+1)}{\sin s} \ . \tag{3.40}$$

The solution can be inserted back into the ansatz (3.37) and gives the following one-point functions for the primary bulk fields Φ^j,

$$\langle \Phi^j(u|z)\rangle_s = (1+u\bar{u})^{2j}\Gamma(1+b^2(2j+1))\,\frac{\sin s(2j+1)}{\sin s}\,\frac{-\nu_b^{j+1}}{2\pi\Gamma(1-b^2)}\,\frac{1}{|z-\bar{z}|^{2\Delta_j}} \ . \tag{3.41}$$

Upon taking the semi-classical limit $b \to 0$, this almost reduces to the expression we anticipated in eq. (2.29) except that the one has to identify Λ_0 with is. It appears that a consistent boundary conformal field theory only exists for a discrete set of real values for s, see also our remarks in subsection 5.6. This would imply that the parameter Λ_0 which controls the radius of the spherical branes is imaginary.

REMARK 3. — The ansatz for the one-point function that was studied in [18] differs from (3.37) by replacing the factor $(1+u\bar{u})^{2j}$ with $(u-\bar{u})^{2j}$, where $(u-\bar{u})^{2j}$ is defined to be $\exp(2\pi ij)|u-\bar{u}|^{2j}$ whenever $\Im u < 0$. It turns out to be impossible to satisfy the factorization constraint with this ansatz. Thereby we resolve a puzzle pointed out in [18]: A dependence like $(u-\bar{u})^{2j}$ would create a singularity on the boundary of H_3^+, which one would not expect to appear in the case of spherical or instantonic branes. In fact, such branes are localized in the interior of H_3^+. The problem is clearly avoided with the correct u dependence of the form $(1+u\bar{u})^{2j}$.

4. THE OPEN STRING SECTOR

The aim of this section will be to study the open string sector of the H_3^+ model, i.e. the boundary operators and some simple correlations functions thereof. Our main goal is to determine the stringy corrections to the semi-classical reflection amplitude we computed in Section 2. Since there is no such quantity for the spherical branes, we shall restrict all the discussion of this section to the boundary conditions that preserve an $SL(2,\mathbb{R})$ subgroup of the $SL(2,\mathbb{C})$ symmetry in the bulk, i.e. to the Euclidean AdS_2 brane.

4.1. **The spectrum of boundary fields**

One can rewrite the Lagrangian (2.5) for closed strings on H_3^+ in a first order formalism (see e.g. [12]). In such a formulation, the interaction terms turn out to vanish near the boundary of H_3^+ so that the theory becomes a free field theory in the asymptotic region. This remains true in the corresponding boundary problem. In fact, the currents that we use to write down the boundary conditions approach the currents of the asymptotic free field theory and hence the boundary conditions become usual Dirichlet/Neumann boundary conditions when ϕ tends to infinity. The boundary conditions therefore do not introduce any interaction in the asymptotic model.

Following previous experience from Liouville theory [35] and the H_3^+ model with periodic boundary conditions [12, 10] one therefore expects that states and the corresponding bulk and boundary fields in the H_3^+ model on the upper half plane can be characterized by their asymptotic behavior at the boundary. We have already used this fact for the bulk fields.

Near the boundary we may describe the theory in terms of free fields ϕ, γ, $\bar{\gamma}$ and their canonical conjugate momenta. For gluing conditions for the AdS_2 branes imply usual Neumann boundary conditions for ϕ and $\nu = -\Im\gamma$, i.e. only these two fields possess non-vanishing zero modes ϕ_0 and ν_0, respectively. For the third field we impose Dirichlet boundary conditions that specify the 'transverse position' r of the brane. Canonical quantization of the free field theory then leads to a space of states of the form

$$\mathcal{H}^{\text{free}} \;=\; L^2(\mathbb{R}\times\mathbb{R}; e^{\phi_0}d\phi_0 d\nu)\otimes\mathcal{F}, \tag{4.1}$$

where $\mathcal{F}$ is a Fock-space that realizes the action of the non-zero modes of the three fields (cf. [12]). As indicated in eq. (4.1) we assume the zero modes ϕ_0 and ν to be realized as multiplication operators. Note that this space does not depend on the Dirichlet parameter r.

We want to rewrite the space $\mathcal{H}^{\text{free}}$ as a sum of sectors $\mathcal{R}_j$ for the current algebra $\widehat{\mathfrak{sl}}_2$. The latter is generated by the components $J^a_n, a = 0, \pm,$ of the three currents in the boundary problem. To begin with, we consider the subspace

$$\mathcal{H}^{\text{free}}_0 \;\equiv\; L^2(\mathbb{R}\times\mathbb{R}; d\phi_0 e^{\phi_0} d\nu)\otimes\Omega\ \ ,$$

where Ω is the Fock vacuum of $\mathcal{F}$ which is annihilated by the modes J^a_n with $n > 0$ of the current algebra. It is not difficult to see that the realization of the zero modes J^a_0 on

$\mathcal{H}_0^{\text{free}}$ can be decomposed into standard principal series representations $\mathcal{P}_j$ of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra. In fact, the zero mode part in the Hamiltonian of the free theory is simply given by $H_0 \sim \hat{P}^2 - i\hat{P}$ where $\hat{P} = i\partial_{\phi_0}$ is the momentum conjugate to ϕ_0.[10] Hence, the Fourier transformation defined by the basis of plane waves $\exp(j\phi_0)$ with $j \in -\frac{1}{2} + i\mathbb{R}$ provides a decomposition of $\mathcal{H}_0^{\text{free}}$ into a direct sum of representations $\mathcal{P}_j$ from the principal continuous series of $\mathrm{SL}(2,\mathbb{R})$. Each representation $\mathcal{P}_j$ of the zero mode algebra canonically extends to a so-called prolongation module $\mathcal{R}_j$ which is generated by acting with the creation operators J_n^a with $n < 0$. In conclusion we have argued that

$$\mathcal{H}^{\text{free}} \; = \; \int_{\mathbb{R}} dP \;\, \mathcal{R}_{-\frac{1}{2}+iP} \quad . \tag{4.2}$$

Note that the integral extends over the full real line, i.e. $P \in \mathbb{R}$ runs through all the allowed eigenvalues of the momentum operator $\hat{P}$.

So far we have been talking about the spectrum of the free theory which we obtained by dropping the interactions in the H_3^+ model. The spectrum of the original theory can be embedded into $\mathcal{H}^{\text{free}}$ because each state $|\psi\rangle$ of the interacting model behaves like a state for the free theory when we approach the boundary of H_3^+. In other words, its asymptotic behavior assigns a unique element of $\mathcal{H}^{\text{free}}$ to each state $|\psi\rangle$. In particular, primary states $|j;u\rangle_{rr'}$ are represented asymptotically by wave-functions

$$\psi^j_{rr'}(u|\gamma) \;\equiv e^{-(j+1)\phi_0}\delta(\gamma - u) \; + \; S(j|r,r')c^{-1}(j)e^{j\phi_0}|\gamma - u|^{2j} \quad , \tag{4.3}$$

where the normalizing factor $c(j)$ was defined in (2.34) and we are thinking about a theory in which the open strings can end on two possibly different AdS_2 branes associated with the two labels r and r'. Of course we do not expect the coefficient $S(j|r,r')$ of the outgoing plane wave to be arbitrary: As there is only one asymptotic region in H_3^+, the outgoing signal should be uniquely determined by the incoming signal. Therefore, the quantity $S(j|r,r')$ implicitly describes the (stringy) geometry in the interior of H_3^+. In particular, it is the first place where the dependence on the boundary parameters r and r' enters. The relation between incoming and outgoing signals implies that the state space of the boundary H_3^+ contains only "half" of the representations that we found in $\mathcal{H}^{\text{free}}$, i.e.

$$\mathcal{H}^{\text{int}}_{rr'} \; = \; \int_{\mathbb{S}}^{\oplus} dj \;\, \mathcal{R}^j \; , \quad \mathbb{S} = -\tfrac{1}{2} + i\mathbb{R}_0^+ \quad . \tag{4.4}$$

In other words, the integration over the momentum P is now restricted to a half line.

Normalization. With each of the normalizable states $|j;u\rangle_{r_2r_1}$ that we discussed above there comes a unique boundary field $\Psi^j_{r_2r_1}(u|x)$. The normalization of the state and the associated field is fixed by the asymptotic behavior (4.3). As in the case of the bulk primary fields, cf.

[10] H_0 coincides with the quadratic Casimir (2.30) for large values of the radial coordinate χ.

Subsection 3.1, we shall find it more convenient to work with a different set of boundary fields Ξ^j which are related to the fields Ψ^j by

$$\Xi^j_{r_2 r_1}(u|x) \;\equiv\; S^{-1}(j|r_2,r_1)\, c(j)\, \Psi^j_{r_2 r_1}(u|x) \;\; .$$

In comparison to (4.3), the second term in an asymptotic expansion of Ξ^j has a trivial coordinate independent pre-factor in front of the second term. On the complex plane, the two-point function of the fields Ξ is given by

$$\big\langle\, \Xi^{-j_2-1}_{r_1 r_2}(u_2|x_2)\, \Xi^{j_1}_{r_2 r_1}(u_1|x_1)\,\big\rangle \;=\; |x_2-x_1|^{-2\Delta_{j_1}}\; 2\pi\, \frac{\pi\delta(P_2-P_1)}{P_1 \tanh \pi P_1}\, \delta(u-u') \;\; , \quad (4.5)$$

with $j_i = -1/2 + iP_i$ and $P_i \in \mathbb{R}^+$ as usual. The factor on the right hand side of (4.5) is simply $|c(j)|^2$ and it may be identified as the inverse of the Plancherel measure for the representations $\mathcal{P}_j$.

As in [10] one can argue that the new choice of normalization fixes the leading short-distance behavior in the operator product expansions to be

$$\Xi^{j_2}_{r_2 r_2}(u_2|x_2)\; \Xi^{j_1}_{r_2 r_1}(u_1|x_1) \;\underset{x_2\to x_1}{\sim}\; |x_2-x_1|^{-2b^2(j_1+1)(j_2+1)}\; \Xi^{j_2+j_1}_{r_2,r_1}(u_1|x_1) \;\; . \qquad (4.6)$$

Here we assume that $\Re(j_1+j_2+\frac{1}{2})>0$ and there is a similar expansion when we multiply the boundary fields with equal boundary labels r from the right.

Reflection property. There is now also an open string version for the reflection of signals in the interior of the Euclidean AdS_2 similar to the reflection of closed string modes in the interior of of H_3^+. In order to state this more precisely, we extend the definition of the states $|j;u\rangle$ from the usual range of j to $j \in -\frac{1}{2} + i\mathbb{R}$. The states $|-j-1;u\rangle$ are certainly related to $|j;u\rangle$ by some linear transformation involving $S(j|r_2,r_1)$. For the fields Ξ^j such a relation implies

$$\Xi^j_{r_2,r_1}(u|x) \;=\; R(j|r_2,r_1)(\mathcal{J}^j\Xi^{-j-1}_{r_2,r_1})(u|x) \quad \text{with} \quad R(j|r_2,r_1) \;\equiv\; S^{-1}(j|r_2,r_1) \quad (4.7)$$

and $\mathcal{J}^j$ is the unitary $SL(2,\mathbb{R})$-intertwining operator that describes the equivalence of the representations $\mathcal{P}_j$ and $\mathcal{P}_{-j-1}$.

4.2. **Calculation of the reflection amplitude: Short description**

Sice the details of the analysis that we use to determine the reflection amplitude are somewhat tedious, we shall now present a short summary of the methods we employ and of the the main results we shall find. Readers who are not interested in the technical details may then skip Subsections 4.3-4.6 in which we provide a more thorough discussion.

In order to calculate $R(j|r_2,r_1)$ we shall make again use of a degenerate field much in the same way as in the previous section. But this time it is the degenerate boundary field $\Xi^1_r(u|x)$ corresponding to the finite dimensional representation with spin $j=1$ of the

$\mathfrak{sl}(2,\mathbb{R})$ zero mode algebra. It can be constructed from the degenerate bulk field $\Theta^{1/2}$ by

$$\Xi^1_r(u|\Re z) \;\equiv\; \lim_{\Im z\downarrow 0}(\Im z)^{\frac{1}{2}b^2}\Theta^{\frac{1}{2}}(u|z) \;\;, \tag{4.8}$$

where the limit is assumed to be taken in front of a boundary segment with boundary condition labeled by r. The properties of the degenerate field Ξ^1 constrain its operator product expansion with a generic boundary field to be of the form

$$\begin{aligned}\Xi^1_{r_2}(u_2|x_2)\,\Xi^j_{r_2,r_1}(u_1|x_1) \;\underset{x_2\to x_1}{\sim}\\ \sum_{s=-,0,+} e_s(j|r_2,r_1)(u_2-u_1)^{1-s}|x_2-x_1|^{\Delta_{j+s}-\Delta_j-\Delta_s}\,\Xi^{j+s}_{r_2,r_1}(u_1|x_1).\end{aligned} \tag{4.9}$$

A functional equation for $R(j|r_2,r_1)$ is then obtained by requiring consistency of the reflection relation (4.7) with the operator product expansion (4.9). This relation reads

$$\frac{R\big(j+\frac{1}{2}|r_2,r_1\big)}{R\big(j-\frac{1}{2}|r_2,r_1\big)} \;=\; \frac{2j}{2j+1}\,e_-\big(-j-\tfrac{1}{2}|r_2,r_1\big) \;\;. \tag{4.10}$$

Given the coefficient e_-, it can be shown under mild assumptions [32] that there is at most one unitary solution to (4.10). The central task is therefore to construct e_-. This is done by evaluating part of the consistency conditions that were first formulated by Cardy and Lewellen [30]. The dependence on the boundary conditions enters the analysis through the particular bulk-boundary coefficient $A(\frac{1}{2},0|r)$ which determines the contribution of the identity field in the expansion of $\Theta^{1/2}(u|z)$ near the boundary,

$$(2\Im z)^{-\frac{3}{2}b^2}\mathrm{tr}(\omega_0 h) \;\underset{\Im z\downarrow 0}{\sim}\; A(\tfrac{1}{2},0|r) \;\;.$$

We shall find that e_- is uniquely fixed by the consistency conditions. Explicitly, it is given by

$$\begin{aligned} e_-(j|r_2,r_1) \;=\; -\frac{\lambda_b}{\pi^3}\frac{\Gamma(1+b^2(2j-1))\Gamma(-b^2(2j+1))}{\sin\pi b^2 2j}\;\times\\ \times\prod_{s=\pm}\cos\big(\pi b^2 j+s\tfrac{i}{2}(r_2+r_1)\big)\sin\big(\pi b^2 j+s\tfrac{i}{2}(r_2-r_1)\big)\;\;.\end{aligned} \tag{4.11}$$

The constant λ_b shall be spelled out later (see eq. (4.49) below). The parameters r_i we use to label the boundary conditions are related to the coefficients $A(\frac{1}{2},0|r)$ via

$$\sqrt{\lambda_b}\sinh r \;=\; -\pi\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\;A(\tfrac{1}{2},0|r) \;\;. \tag{4.12}$$

Eq. (4.12) describes the quantum corrections to the classical relation $A(\frac{1}{2},0|r)=2\sinh r$. So far we have not made any assumptions about the range of the parameters r_i that appear in the expression (4.11) for e_-, although the semi-classical correspondence certainly suggests that the r_i should be real. It is quite interesting to note, however, that the same conclusion can be reached without any input from the expected geometry. In fact, requiring the reflection amplitude to be unitarity restricts the range for the r_i. More precisely, it follows from unitarity of R that the l.h.s. of (4.10) is an absolute square for $j \in \mathbb{S}$. Therefore, the functional equation (4.10) can only have unitary solutions if the object on the r.h.s. of (4.10) is positive for all $j \in \mathbb{S}$. This can be seen to require not only $r_i \in \mathbb{R}$ but furthermore $r_1 = r_2$. We shall comment on the second requirement later.

The unique solution for $R(j|r) \equiv R(j|r,r)$ is then finally given by the expression

$$R(j|r) \;=\; \left(\frac{\lambda_b}{4\pi^2}\right)^{iP} \frac{\Gamma_k^2(b^{-2} - iP + \frac{1}{2})}{\Gamma_k^2(b^{-2} + iP + \frac{1}{2})} \frac{\Gamma_k(b^{-2} + 2iP)}{\Gamma_k(b^{-2} - 2iP)} \frac{S_k(2R + P)}{S_k(2R - P)} \;, \tag{4.13}$$

where $R \equiv r/2\pi b^2$, $j = -\frac{1}{2} + iP$, and the special function $S_k(x)$ and Γ_k are defined through,

$$\log S_k(x) \;=\; i \int_0^\infty \frac{dt}{t} \left(\frac{\sin 2tb^2 x}{2 \sinh b^2 t \sinh t} - \frac{x}{t} \right) \;, \tag{4.14}$$

$$\Gamma_k(x) \;=\; b^{b^2 x(x - b^{-2})} (2\pi)^{\frac{x}{2}} \Gamma_2^{-1}(x|1, b^{-2}) \;. \tag{4.15}$$

Here, $\Gamma_2(x|\omega_1, \omega_2)$ denotes Barnes Double Gamma function [34].

REMARK 4. — Another outcome of the analysis seems to be worth noting: The operator product coefficients e_s, $s = -, 0, +$ turn out to be related to fusion matrices of particular conformal blocks, similar to what was found in rational CFT (see [36, 37, 28, 38]) and in Liouville theory [33]. We find that there exists a change of normalization of the boundary operators such that the re-normalized boundary fields have operator products coefficients

$$E_s(j|r_2, r_1) \;\equiv\; F_{j(r_2)\; j+s}\left[\begin{smallmatrix} 1 & j \\ j(r_2) & j(r_1) \end{smallmatrix}\right] \;. \tag{4.16}$$

instead of e_s. The spins $j = j(r)$ that one has to insert into the fusing matrix on the right hand side depend on the parameter r which labels the boundary conditions,

$$j(r) \;=\; -\frac{1}{2} - \frac{1}{4b^2} + i\frac{r}{2\pi b^2} \;. \tag{4.17}$$

We find it remarkable that the range of the representation labels j one uses in this eq. (4.16) for a real boundary parameter r is not identical to the spectrum $\mathbb{S}$. This represents a marked difference to previous cases where relations similar to (4.16) were found.

4.3. **The basic bulk field**

To obtain the results we have sketched above, we consider a setup on the upper half plane in which the boundary condition r_1 is imposed along the negative real line, while we impose a possibly different boundary conditions r_2 on the other side of the boundary. As in the previous section, we shall exploit the special properties of the primary bulk field

$\Theta^{\frac{1}{2}}(x|z)$ but now in this more general situation where the boundary condition jumps at $x=0$. We find it useful to decompose this field into chiral vertex operators,

$$\begin{aligned}\Theta^{\frac{1}{2}}(x|z) \; &= V_+^{\frac{1}{2}}(x|z)V_+^{\frac{1}{2}}(-\bar{x}|\bar{z})\, a(\hat{\jmath}|r) + V_+^{\frac{1}{2}}(x|z)V_-^{\frac{1}{2}}(-\bar{x}|\bar{z})\, b(\hat{\jmath}|r) \\ &+ V_-^{\frac{1}{2}}(x|z)V_+^{\frac{1}{2}}(-\bar{x}|\bar{z})\, c(\hat{\jmath}|r) + V_-^{\frac{1}{2}}(x|z)V_-^{\frac{1}{2}}(-\bar{x}|\bar{z})\, d(\hat{\jmath}|r) \;\; .\end{aligned} \tag{4.18}$$

The operator $\hat{\jmath}$ is defined by $\hat{\jmath}\mathcal{P}_j = j\mathcal{P}_j$. Equation (4.18) can be read as an expansion into a complete set of solutions to the Knizhnik-Zamolodchikov equations. It also takes the null vector decoupling equations (3.18) and the gluing conditions (3.1) between left and right currents into account. The coefficients a, b, c, d that are introduced through (4.18) will now be determined by some of the consistency conditions that were formulated in [30]. To this end we study the behavior of the degenerate field with $j = 1/2$ near the boundary $\Im z = 0$.

As a preparation let us note that the chiral vertex operators $V_s^{\frac{1}{2}}(x|z)$, $s = +,-$ satisfy the following operator product expansion,

$$V_r^{\frac{1}{2}}(x'|z')\, V_s^{\frac{1}{2}}(x|z) \;=\; (z'-z)^{\Delta_1 - 2\Delta} V^1_{r+s}\big(\tfrac{x+x'}{2}|\tfrac{z+z'}{2}\big)\;, \tag{4.19}$$

in the case where $r+s = \pm 1$, whereas for $r+s = 0$ one finds an expansion of the form

$$\begin{aligned}V_{-s}^{\frac{1}{2}}(u_2|z_2)\, V_s^{\frac{1}{2}}(u_1|z_1) \quad &= \quad f_{s,-}\; (z_2 - z_1)^{-2\Delta}\; (u_2 - u_1) \\ + \, f_{s,+}\; (z_2 - z_1)^{\Delta_1 - 2\Delta}\; &\big(V^1_{0,-}(z_1) - \tfrac{1}{2}(u_2+u_1)V^1_{0,0}(z_1) + u_2 u_1 V^1_{0,+}(z_1)\big) \;\; .\end{aligned}$$

Here, the f_{st}, $s, t \in \{+,-\}$ are the special elements of the fusing matrix given by (see Appendix C.2 for explicit expressions)

$$f_{st}(j) \;\equiv\; F_{j+\frac{s}{2},\frac{1}{2}+\frac{t}{2}}\big[\begin{smallmatrix}\frac{1}{2} & \frac{1}{2}\\ j & j\end{smallmatrix}\big] \;\; . \tag{4.20}$$

By means of these operator product expansions one indeed finds a singular behavior of the expected form

$$\begin{aligned}\Theta^{\frac{1}{2}}(u|z) \underset{\Im z \downarrow 0}{\sim} \; & A(\tfrac{1}{2},0|r_\nu)\; (\Im z)^{\frac{3}{2}b^2}\; (u+\bar{u})\, \mathrm{id} \\ & + A(\tfrac{1}{2},1|r_\nu)\; (\Im z)^{-\frac{1}{2}b^2}\big(\Xi_-(x) - \tfrac{1}{2}(u-\bar{u})\,\Xi_0(x) - u\bar{u}\,\Xi_+(x)\big)\end{aligned} \tag{4.21}$$

where $\nu = 1$ for $\Re z < 0$ and $\nu = 2$ otherwise. Let us note that we certainly are want the H_3^+ model to be local. This implies that the coefficients $A(\frac{1}{2}, i|r)$, $i = 0,1$ depend only on the boundary condition r assigned to the segment of the boundary that our bulk field approaches.

Keeping this in mind, let us now determine $A(\frac{1}{2},0|r_\nu)$ from our general ansatz (4.18) by sending the degenerate bulk field to the boundary once along the positive real axis and then again somewhere along the negative half line. This will provide expressions for $A(\frac{1}{2},0|r_\nu)$

in terms of the b, c which we can solve for the latter. If we let the bulk field approach the real line with $\Re z > 0$ we obtain the first formula to be given by

$$A(\tfrac{1}{2},0|r_2) \;=\; e^{\frac{3}{4}\pi i b^2}\big(f_{--}\,b\,+f_{+-}\,c\big) \;. \tag{4.22}$$

In order to treat the case where $\Re z < 0$, it is useful to note that the degenerate chiral vertex operators with $j = 1/2$ satisfy the braid relation

$$V_r^{\frac{1}{2}}(x|e^{\pi i}z) \;=\; e^{\pi i(\Delta_{j+\frac{r}{2}}-\Delta_j-\Delta)}\,V_r^{\frac{1}{2}}(x|z) \;. \tag{4.23}$$

With the help of this relation it is then obvious that the operator product coefficient $A(\frac{1}{2},0|r_1)$ must be of the form

$$A(\tfrac{1}{2},0|r_1) \;=\; e^{\frac{3}{4}\pi i b^2}\big(e^{-\pi i b^2(2j+2)}f_{--}\,b\,+\,e^{+\pi i b^2 2j}f_{+-}\,c\big) \;. \tag{4.24}$$

The two expressions for the coefficients of the bulk boundary expansion can now be solved for b, c,

$$\begin{aligned} b(j|r_1,r_2) \;&=\; +\,e^{\frac{1}{4}\pi i b^2}\,f_{--}^{-1}\,\frac{A_1 e^{\pi i b^2 2j}-A_2}{2i\sin(\pi i b^2(2j+1))} \\ c(j|r_1,r_2) \;&=\; -\,e^{\frac{1}{4}\pi i b^2}\,f_{+-}^{-1}\,\frac{A_1 e^{-\pi i b^2(2j+2)}-A_2}{2i\sin(\pi i b^2(2j+1))} \;. \end{aligned} \tag{4.25}$$

Here, we have use the shorthand notation $A_\nu \equiv A(\frac{1}{2},0|r_\nu)$ and explicit formulas for f_{rs} are spelled out in Appendix C.2.

4.4. **The basic boundary field**

Up to now we have focused on the contribution proportional to the identity in the bulk-to-boundary OPE (4.21). But there exists one non-trivial boundary field in eq. (4.21) which is extracted from the expansion of the degenerate bulk field by

$$\Xi^1_{r_\nu}(u|\Re z) \;\equiv\; \lim_{\Im z\downarrow 0}(\Im z)^{\frac{1}{2}b^2}\Theta^{\frac{1}{2}}(u|z) \;, \tag{4.26}$$

where the value of $\nu \in \{1,2\}$ depends on whether z approaches the left or right real half-axes. The formula defines a boundary operator $\Xi^1_r(u|x)$ that transforms in the degenerate $j = 1$ representation of $\mathrm{SL}(2,\mathbb{R})$ and it possesses an expansion into chiral vertex operators $V^1_s(u|x)$, $s = -,0,+$ that takes the following form,

$$\Xi^1_r(u|x) \;=\; \sum_{s=-1}^{+1}\; e_s(\hat{\jmath}|r_1,r_2)\,V^1_s(u|x) \;. \tag{4.27}$$

It is rather easy to express the coefficient $e_0(j|r_1,r_2)$ in terms of the coefficients b, c which we determined in eqs. (4.25) of the previous subsection,

$$\begin{aligned} e_0(j|r_1,r_2) &= e^{-\frac{\pi i}{4}b^2}\big(b(j|r_1,r_2)f_{-+} + c(j|r_1,r_2)f_{++}\big) \\ &= \frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\frac{\Gamma(1+2b^2j)\Gamma(-b^2(2j+2))}{2i\sin(\pi b^2(2j+1))}\times \\ &\quad \times\Big(A_2\big(\sin\pi b^2(2j+2)+\sin\pi b^2(2j)\big) - A_1\sin\pi b^2(4j+2)\Big) \ . \end{aligned} \tag{4.28}$$

We have also inserted formulas for the elements of the fusing matrix from the Appendix C.2. Let us furthermore note that the normalization (4.6) implies $e_+ \equiv 1$. It therefore remains to determine e_-. We now want to show how e_- can be computed from e_0, e_1 and elements of the fusing matrix, even though the procedure will be rather complicated at first.

CLAIM 1. — *The conditions of associativity of the operator product expansion yield an equation that expresses* $e_-(j|r_1,r_2)$ *algebraically in terms of* $e_s(j|r_1,r_2)$, $s = +, 0$ *and certain fusion coefficients.*

To prove the claim, let us study the constraints from associativity of the boundary operator product expansion. It suffices to analyze the following special four-point functions in which only two fields are non-degenerate,

$$\begin{aligned} \langle\, \Xi^{-j-1}_{r_1,r_2}(u_4|\infty)\,\Xi^1_{r_2}(u_3|x)\,\Xi^1_{r_2}(u_2|x')\,\Xi^j_{r_2,r_1}(u_1|0)\,\rangle \ &= \\ &= \ \langle j,u_4|\,\Xi^1(u_3|x)\,\Xi^1(u_2|x')\,|j,u_1\rangle \ . \end{aligned} \tag{4.29}$$

Since the boundary field $\Xi^1_r(u|x)$ is degenerate with $j=1$, it satisfies the null vector decoupling equation $\partial^3_u\,\Xi^1_r(u|x) = 0$. This restricts its operator products with other boundary fields to have the form

$$\begin{aligned} \Xi^1_{r_2}(u_2|x_2)\,\Xi^j_{r_2,r_1}(u_1|x_1) \quad &\underset{x_2\to x_1}{\sim} \\ \sum_{s=-1}^{+1} e_s(j|r_1,r_2)(u_2-u_1)^{1-s}&|x_2-x_1|^{\Delta_{j+s}-\Delta_j-\Delta_s}\,\Xi^{j+s}_{r_2,r_1}(u_1|x_1) \ . \end{aligned} \tag{4.30}$$

In the case $j=1$ one finds the degenerate boundary fields with $j=0,1,2$ on the right hand side eq. (4.30) in which Ξ^0 is the identity field. If the product of two fields with $j=1$ is inserted into the correlation function (4.29) each of the three contributions to the operator product yields one equation. With $s=-1$ we obtain

$$e_+(j-1)e_-(j)F^1_{--} + (e_0(j))^2F^1_{0-} + e_-(j+1)e_+(j)F^1_{+-} \ = \ e_-(1) \ , \tag{4.31}$$

where we have used the following abbreviations

$$e_s(j) \ \equiv \ e_s(j|r_1,r_2) \quad , \quad F^1_{st} \ \equiv \ F^1_{st}(j) \ \equiv \ F_{j+s\ 1+t}\big[\begin{smallmatrix}1 & 1\\ j & j\end{smallmatrix}\big] \ . \tag{4.32}$$

Explicit expressions for the fusion coefficients $F^1_{st}(j)$ can be found in the Appendix C.2. From the case $s = 0$ we infer

$$e_+(j-1)e_-(j)F^1_{-0} + (e_0(j))^2 F^1_{00} + e_-(j+1)e_+(j)F^1_{+0} \; = \; e_0(1)e_0(j) \;\; . \tag{4.33}$$

By imposing $e_+(j) \equiv 1$ and combining (4.31) and (4.33) we get an algebraic equation involving $e_-(j)$ and $e_-(1)$. When this is specialized to $j = 1$ it yields an equation that expresses $e_-(1)$ algebraically in terms of $e_0(1)$ and the elements $F^1_{st}|_{j=1}$ of the fusing matrix. The expression for $e_-(1)$ can now be inserted into the equation for $e_-(j)$ and this then leads to a formula which determines $e_-(j)$ in terms of $e_0(j)$ and the fusion coefficients $F^1_{st}(j)$. This proves our Claim 1.

4.5. **Relation with fusion coefficients**

At the end of the last subsection we sketched a procedure that allows to determine the last coefficient $e_-(j)$ we are missing for our construction of the degenerate boundary field (4.27). The recipe looks rather complicated but it turns out that there is a way of solving the conditions on $e_-(j)$ directly which is also conceptually very interesting. It is based on the observation [36] that the conditions (4.31)(4.33) are satisfied if we insert the following quantity for $e_s(j)$,

$$E_s(j|\rho_2,\rho_1) \;\equiv\; F_{\rho_2,j+s}\big[\begin{smallmatrix} 1 & j \\ \rho_2 & \rho_1 \end{smallmatrix}\big] \;\; . \tag{4.34}$$

Validity of eqs. (4.31)(4.33) with $e_s(j)$ replaced by $E_s(j)$ is assured by the pentagon identity for the fusion coefficients [39]. Let us stress that the equations hold true for any choice of the parameters ρ_2, ρ_1. This freedom is related to the choice of boundary conditions r_i but the precise relation between ρ_i and r_i will turn out to be non-trivial, unlike in the rational case. While $E_s(j)$ solve eqs. (4.31)(4.33), it is not normalized in the same way as $e_s(j)$, i.e. in particular $E_+(j) \neq 1$. To achieve proper normalization it is useful to observe that there is a whole family of solutions to eqs. (4.31) (4.33) which is generated by setting

$$E^{\mathrm{N}}_s(j|\rho_2,\rho_1) \;\equiv\; E_s(j|\rho_2,\rho_1)\frac{\mathrm{N}(j|\rho_2,\rho_1)\mathrm{N}(1|\rho_2,\rho_2)}{\mathrm{N}(j+s|\rho_2,\rho_1)} \;\; . \tag{4.35}$$

As we shall show now, one can find functions $\mathrm{N}(j,\rho_2,\rho_1)$ and $\rho_i = \rho_i(r_i)$ such that E^N_s is normalized in the same way as e_s. We can then use this E^N_s to construct the degenerate boundary field Ξ^1, i.e. we can set

$$\begin{aligned} e_-(j|r_2,r_1) \;\; &\equiv \;\; E^{\mathrm{N}}_0\big(j|\rho_2(r_2),\rho_1(r_1)\big) \; = \; F_{\rho_2,j-1}\big[\begin{smallmatrix} 1 & j \\ \rho_2 & \rho_1 \end{smallmatrix}\big]\frac{\mathrm{N}(j|\rho_2,\rho_1)\mathrm{N}(1|\rho_2,\rho_2)}{\mathrm{N}(j-1|\rho_2,\rho_1)} \\ &= \;\; F_{\rho_2,j-1}\big[\begin{smallmatrix} 1 & j \\ \rho_2 & \rho_1 \end{smallmatrix}\big]F_{\rho_2,j}\big[\begin{smallmatrix} 1 & j-1 \\ \rho_2 & \rho_1 \end{smallmatrix}\big]\mathrm{N}^2(1|\rho_2,\rho_2) \;\; . \end{aligned} \tag{4.36}$$

A precise formula for the proper choice of $\mathrm{N}(j,\rho_2,\rho_1)$ is spelled out in the process of proving the following claim.

CLAIM 2. — *There exists a function* $\mathrm{N}(j|\rho_2,\rho_1)$ *together with a choice for the parameters* $\rho_2=\rho_2(r_2)$, $\rho_1=\rho_1(r_1)$ *such that*

$$E_0^{\mathrm{N}}\big(j|\rho_2(r_2),\rho_1(r_1)\big) \;=e_0\big(j|r_2,r_1\big) \tag{4.37}$$

$$E_+^{\mathrm{N}}(j|\rho_2(r_2),\rho_1(r_1)) \;=1\;\;. \tag{4.38}$$

More precisely, these properties are ensured if we construct $\mathrm{N}(j|\rho_2,\rho_1)$ *by eq. (4.40) and choose* ρ_i *such that eqs. (4.41) are satisfied (see below).*

In order to prove our claim, we start by solving equation (4.38) which is equivalent to the functional equation

$$1\;\equiv\;F_{\rho_2,j+1}\big[\begin{smallmatrix}1 & j\\ \rho_2 & \rho_1\end{smallmatrix}\big]\,\frac{\mathrm{N}(j|\rho_2,\rho_1)\mathrm{N}(1|\rho_2,\rho_2)}{\mathrm{N}(j+1|\rho_2,\rho_1)}\;\;. \tag{4.39}$$

It is not difficult to see that the functional equation (4.39) can be solved in terms of a special function $\Gamma_k(x)$ satisfying the functional equation $\Gamma_k(x+1)=\Gamma(b^2x)\Gamma_k(x)$. With this functional equation and the explicit expression for the relevant fusion coefficient given in the Appendix C one can easily verify that the expression

$$\begin{aligned}\mathrm{N}(j|\rho_2,\rho_1)\;=\;&\lambda_b^{\frac{j}{2}}\frac{\Gamma_k(2+2\rho_2+b^{-2})\Gamma_k(-2\rho_2)}{\Gamma_k(j+\rho_2+\rho_1+2+b^{-2})\Gamma_k(j-\rho_2-\rho_1)}\times\\ &\times\frac{\Gamma_k(1+b^{-2})\Gamma_k(2j+1+b^{-2})}{\Gamma_k(j+\rho_2-\rho_1+1+b^{-2})\Gamma_k(j+\rho_1-\rho_2+1+b^{-2})}\end{aligned} \tag{4.40}$$

does the job for some positive constant λ_b. Now we can compute E_0^N using this choice of the function $\mathrm{N}(j|\rho_2,\rho_1)$,

$$\begin{aligned}E_0(j|\rho_2,\rho_1)\;=\;F_{\rho_2,j}\big[\begin{smallmatrix}1 & j\\ \rho_2 & \rho_1\end{smallmatrix}\big]\mathrm{N}(1|\rho_2,\rho_2)\;=\;\frac{\sqrt{\lambda_b}}{2\pi}\frac{\Gamma(1+2b^2j)\Gamma(-b^2(2j+2))}{\sin(\pi b^2(2j+1))}\times\\ \Big(\cos\pi b^2(2\rho_1+1)\big(\sin\pi b^2(2j+2)+\sin\pi b^2(2j)\big)-\cos\pi b^2(2\rho_2+1)\sin\pi b^2(4j+2)\Big)\;.\end{aligned}$$

Comparing the expression for E_0 to the one for e_0 given in eq. (4.28) it becomes obvious that the normalization (4.37) can be satisfied by a suitable choice of $\rho_2=\rho_2(r_2)$, $\rho_1=\rho_1(r_1)$,

$$\frac{\sqrt{\lambda_b}}{2\pi}\cos\pi b^2(2\rho_i+1)\;=\;\frac{1}{2i}\frac{\Gamma(-b^2)}{\Gamma(-2b^2)}\;A(\tfrac{1}{2},0|r_i)\;\;. \tag{4.41}$$

This proves the claim. Let us note briefly that in this way we have obtained an explicit formula for the missing coefficient e_-,

$$\begin{aligned}e_-(j|r_2,r_1)\;=\;&\frac{\lambda_b}{\pi^3}\frac{\Gamma(1+b^2(2j-1))\Gamma(-b^2(2j+1))}{\sin\pi b^2 2j}\times\\ &\times\prod_{s=\pm}\sin\pi b^2(j+s(\rho_2+\rho_1+1))\sin\pi b^2(j+s(\rho_2-\rho_1))\;\;.\end{aligned} \tag{4.42}$$

Along with the normalization $e_+ = 1$ and our formula (4.28) this completely determines the boundary field Ξ^1.

REMARK 5. — It is clear that eq. (4.41) determines the boundary parameters ρ_i only up to $\rho_i \to \rho_i + b^{-2}$. We will find restrictions that eliminate that ambiguity in the next subsection.

4.6. **Reflection amplitude**

With an explicit formula for basic boundary field Ξ^1, we can now turn to the construction of the reflection amplitude $R(j|r_2, r_1)$. A functional equation for this quantity is obtained by requiring consistency of the reflection relation (4.7) with the OPE (4.9). We can either use (4.7) first followed by (4.9) or do it the other way around and then compare the leading asymptotics for $x_2 \to x_1$ followed by $u_2 \to u_1$. This gives

$$\frac{R\big(j+\frac{1}{2}|\rho_2,\rho_1\big)}{R\big(j-\frac{1}{2}|\rho_2,\rho_1\big)} \; = \; \frac{2j}{2j+1}\, e_-\big(-j-\tfrac{1}{2}|\rho_2,\rho_1\big) \ . \tag{4.43}$$

By definition, the reflection amplitude $R(j|r_2, r_1)$ must also satisfy the simple condition

$$R(j|r_2,r_1)R(-j-1|r_2,r_1) \; = \; 1 \ .$$

If we finally take into account that $R(j|r_2, r_1)$ should be unitary, i.e. $|R(j|r_2, r_1)| = 1$ for all $j \in \mathbb{S}$, then we arrive at important restrictions on the allowed values of ρ_2, ρ_1. In fact, the unitarity of $R(j|r_2, r_1)$ allows to rewrite eq. (4.43) as

$$|R\big(j+\tfrac{1}{2}|\rho_2,\rho_1\big)|^2 \; = \; \frac{2j}{2j+1}\, e_-\big(-j-\tfrac{1}{2}|\rho_2,\rho_1\big) \ , \tag{4.44}$$

which implies that the right hand side of eq. (4.44) has to be positive for all $j \in \mathbb{S}$. To analyze the resulting restrictions, we introduce $j = -\frac{1}{2} + iP$, $q_\pm = -i(\rho_1 + \frac{1}{2} \pm \rho_2 + \frac{1}{2})$, and then re-write eq. (4.44) in the form

$$\begin{aligned} |R\big(j+\tfrac{1}{2}|\rho_2,\rho_1\big)|^2 \; = \; & \frac{\lambda_b}{\pi^4}|\Gamma(1-b^2+2ib^2P)\Gamma(2ib^2P)|^2 \times \\ & \times (\cosh 2\pi b^2 P - \cosh 2\pi b^2 q_+)(\cosh 2\pi b^2 P - \cosh 2\pi b^2 q_-) \ . \end{aligned} \tag{4.45}$$

The right hand side of this relation can only be positive if either

i) the $q_\pm$ are of the form $q_\pm = q'_\pm + ik_\pm/2b^2$, $q_\pm \in \mathbb{R}$, $k_\pm \in 1 + 2\mathbb{Z}$, or

ii) q_s vanishes for one $s \in \{+,-\}$, $q_{-s} = q'_{-s} + ik_{-s}/2b^2$, $q'_{-s} \in \mathbb{R}$ and $k_{-s} \in 1 + 2\mathbb{Z}$.

Let us note that the first of these cases does not seem to be of physical interest. One would have parameters k_i $i = 1, 2$ defined by $k_\pm = k_1 \pm k_2$ labeling boundary conditions to the left and the right of $x = 0$. However, $k_\pm \in 1 + 2\mathbb{Z}$ implies that $k_1 \neq k_2$. As we will be interested in the possibility to put the same boundary condition along the whole real line, we are forced to discard possibility i).

Hence, we are left with possibility ii). Since the two sub-cases $q_\pm = 0$ are very similar, we shall focus on the case $q_- = 0$. This means that $\rho_i = -1/2 - 1/4b^2 + iR$ and the functional equation (4.43) then takes the form

$$\frac{R\big(j+\frac{1}{2}|\rho,\rho\big)}{R\big(j-\frac{1}{2}|\rho,\rho\big)} = \frac{\lambda_b}{(2\pi)^2}\left|\frac{\Gamma(1-b^2+2ib^2P)\Gamma(1+2ib^2P)}{\Gamma^2(1+ib^2P)}\right|^2 \times \cosh b^2(2R+P)\cosh b^2(2R-P) > 0 \ . \tag{4.46}$$

A unitary solution to this equation can be constructed in terms of the special functions Γ_k and S_k which were defined in eqs. (4.15,4.14) above. These functions possess a number of nice properties which are relevant for us,

$$\Gamma_k(x+1) = \Gamma(b^2x)\Gamma_k(x) \ , \quad \frac{S_k\big(x-\frac{i}{2}\big)}{S_k\big(x+\frac{i}{2}\big)} = 2\cosh\pi b^2 x \ , \quad |S_k(x)|^2 = 1 \ . \tag{4.47}$$

They are used to verify that the following expression represents a unitary solution of (4.43)

$$R(j|\rho,\rho) = \left(\frac{\lambda_b}{4\pi^2}\right)^{iP}\frac{\Gamma_k^2(b^{-2}-iP+\frac{1}{2})}{\Gamma_k^2(b^{-2}+iP+\frac{1}{2})}\frac{\Gamma_k(b^{-2}+2iP)}{\Gamma_k(b^{-2}-2iP)}\frac{S_k(2R+P)}{S_k(2R-P)} \ . \tag{4.48}$$

This concludes the construction of the unitary reflection amplitude $R(j|\rho,\rho)$ for the special choice $\rho = \rho_1 = \rho_2$. There is no physical unitary solution for $\rho_1 \neq \rho_2$.

REMARK 6. — We have obtained two expressions for $A(\frac{1}{2},0|r)$, eq. (3.34) and eq. (4.41). Comparing these two expressions finally allows us to determine λ_b,

$$\lambda_b = (2\pi)^2\frac{\Gamma(1+b^2)}{\Gamma(1-b^2)} \ . \tag{4.49}$$

5. ANALOGS OF THE CARDY CONDITION

At this point we have found the full string corrections for the semi-classical closed string couplings and the reflection amplitude. The latter determines the relative open string spectral density as a logarithmic derivative of the ratio of two reflection amplitudes (see Appendix B). In the point particle limit of our theory, there is no relation between the bulk couplings to the brane and the spectral density. But the situation is entirely different within string theory: here, the closed string couplings determine the annulus amplitude which is related to the open string partition function by world-sheet duality. The partition function involves the open string spectral density. For rational boundary conformal field theory, the analysis of world-sheet duality is a crucial ingredient in obtaining exact solutions [40]. Similarly, world-sheet duality gives rise to an important consistency condition also in the case of non-rational models [41]. Our aim here is to explore this “Cardy condition”.

5.1. **Boundary state**

We shall now introduce boundary states for our AdS_2-branes. To this end, we re-interpret the one-point function of Θ^j as a linear form ${}_{\rm B}\langle r|$ on the space of closed string states. More precisely, we define

$$ {}_{\rm B}\langle r|j;u\rangle \;:=\; {}_{\rm B}\langle\Theta^j(u|\tfrac{i}{2})\rangle_r \;\; . \tag{5.1} $$

We have placed a subscript B on the boundary state to distinguish it from the usual closed string states. For the following it will be useful to perform a Fourier-transformation over the variable u, i.e. to evaluate the boundary state in a new basis of closed string states $|j;n,p\rangle$,

$$ {}_{\rm B}\langle r|j;n,p\rangle \;=\; \int_{\mathbb{C}} d^2u\, e^{-in\arg(u)}|u|^{-2j-2-ip}{}_{\rm B}\langle r|j;u\rangle \;\; . \tag{5.2} $$

The integral can be performed (see subsection A.3 in Appendix A) and yields ${}_{\rm B}\langle r|j;n,p\rangle = 2\pi\delta(p)A(j,n|r)$, where

$$ \begin{aligned} A(j,n|r) \;=\; & 2\pi\, A_b\, \Gamma(-b^2(2j+1))\; d^j_n \;\times \\ & \times\big(\pi^0_n \cosh r(2j+1) - \pi^1_n \sinh r(2j+1)\big), \end{aligned} \tag{5.3} $$

where d^j_n and π^ϵ_n, $\epsilon = 0,1$ are defined by

$$ d^j_n \;=\; \frac{\Gamma(2j+1)}{\Gamma(1+j+\frac{n}{2})\Gamma(1+j-\frac{n}{2})}, \quad \pi^\epsilon_n \;=\; \begin{cases} 1-\epsilon \;\text{ if }\; n \;\text{ even} \\ \quad\;\; \epsilon \;\text{ if }\; n \;\text{ odd.} \end{cases} \tag{5.4} $$

Note that in the new basis, n runs through integers and p is a real number.

5.2. **The annulus amplitude**

Following Cardy [40], we would like to consider the quantity ${}_{\rm B}\langle r|\tilde{q}^{\frac{1}{2}\mathsf{H}^{\rm cyl}}|r\rangle_{\rm B}$, where $\tilde{q} = \exp(-2\pi i/\tau)$ and $\mathsf{H}^{\rm cyl}$ is the Hamiltonian for the theory with periodic boundary conditions, i.e. $\mathsf{H}^{\rm cyl} = \mathsf{L}_0 + \bar{\mathsf{L}}_0 - c/12$. In our case, this expression is ill-defined due to various divergencies. The strategy will be to first cut off all potentially divergent summations by introducing a regularized boundary state. After identifying similar divergencies in the partition function on the strip in the next subsection, we shall discuss the interpretation of these divergencies and the relations between the respective cut-offs in Subsection 5.4. This naturally leads to the identification of physically meaningful quantities that one can compare between the open and closed string channels.

The "regularized boundary state" we want to work with automatically removes the divergent contributions coming from the summations over $j = -\frac{1}{2} + iP$, n and p. It is defined by

$$ {}_{\rm B,reg}\langle r|j;n,p\rangle \quad := \quad \Theta(\delta - P)\,\Theta_\lambda(n)\,2\pi\delta_T(p)\;A(j,n|r) \;\; , \tag{5.5} $$

$$ \text{where }\; \Theta_\lambda(n) \quad = \quad \begin{cases} 1 & \text{for }\; n \in \{-\lambda+1, -\lambda+2, \dots, \lambda\} \\ 0 & \text{otherwise} \end{cases} \quad , $$

Θ is the usual step function, i.e. $\Theta(x) = 1$ for $x > 0$ and it vanishes for negative arguments, and $2\pi\delta_T(p) = \frac{2}{p}\sin Tp$. Having introduced the cut-offs δ, λ, T, we may now study the regularized annulus amplitude

$$\begin{aligned} & {}_{\rm B,reg}\langle r|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r\rangle_{\rm B,reg} \; \equiv \\ & \equiv \; -\int_{\mathbb{S}} \frac{dj}{\pi^3}\,(2j+1)^2\;\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\;\int_{\mathbb{R}}\frac{dp}{2\pi}\;\chi^j(\tilde{q})\;{}_{\rm B,reg}\langle r|j;n,p\rangle\langle j;n,p|r\rangle_{\rm B,reg}\;\;. \end{aligned} \tag{5.6}$$

It involves the characters $\chi^j(q) = q^{b^2P^2}\eta^{-3}(\tau)$ with $q = \exp(2\pi i\tau)$. The Dedekind eta-function $\eta(\tau)$ is familiar from flat space theories and it appears here because there are no singular vectors in the modules of the current algebra when $j = -1/2 + iP$. After inserting (5.5) and performing some straightforward calculations, one can bring the regularized annulus amplitude into the following form,

$${}_{\rm B,reg}\langle r|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r\rangle_{\rm B,reg} \; = \; \int_\delta^\infty dP\,\chi^j(\tilde{q})\,N_{\rm ann}(P|r)\;\;. \tag{5.7}$$

We have changed to an integration over the variable P using the standard relation $j = -\frac{1}{2} + iP$. The density $N_{\rm ann}(P|r)$ is given by

$$N_{\rm ann}(P|r) \;\equiv\; 2\lambda\;4\pi T\;\frac{|A_b|^2}{\pi^3 b^2}\frac{\cos^2 2rP\cosh^2\pi P + \sin^2 2rP\sinh^2\pi P}{\sinh 2\pi P\sinh 2\pi b^2 P}\;\;. \tag{5.8}$$

The previous two equations provide an explicit formula for the regularized annulus amplitude that comes with the brane r.

5.3. **The open string partition function**

We now want to discuss the second important quantity that enters Cardy's consistency condition, namely the partition function on the strip with the boundary condition r imposed along both boundaries. The naive definition of the partition function on the strip would be to take the trace of $\exp(2\pi i\tau\mathsf{H}_r^{\rm strip})$ over the space $\mathcal{H}^{\rm strip}$. This trace turn out to be divergent. There are two sources of divergencies: One is the unboundedness of the spectrum for the $SL(2,\mathbb{R})$-generators in the principal series representations. One may e.g. consider a basis for the principal series representation $\mathcal{P}_j$ that consists of vectors $|j;m\rangle$ which diagonalize the generator of the compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. The variable m will then take on any integer value. We will regularize this divergence by considering traces that are performed over subspaces $\mathcal{H}_\kappa^{\rm strip} \subset \mathcal{H}^{\rm strip}$ where $\kappa \in \mathbb{Z}^{>0}$. They are generated by acting with current algebra generators on the states $|j;m\rangle$, $m \in \{-\kappa+1, -\kappa+2, \dots, \kappa\}$.

More interesting is the divergence of the open string partition function which comes from the infinite volume of the radial coordinate on H_3^+. We shall imagine having restricted the coordinate ϕ in H_3^+ to be smaller than some cut-off L so that H_3^+ gets effectively replaced by a compact space. Standard arguments suggest (see Appendix B) that the leading behavior of the spectral density $N_{\rm str}(P|r)$ for $L\to\infty$ is of the form

$$N_{\rm str}(P|r) \underset{L\to\infty}{\sim} \frac{L}{\pi}+\frac{1}{2\pi i}\frac{\partial}{\partial P}\log R\big(-\tfrac{1}{2}+iP|r\big). \tag{5.9}$$

There are two ways of dealing with such a divergence: One possibility is to simply divide by the "volume" L of the radial coordinate. We are thus lead to an object $Z_\kappa(q)$ which does not depend at all on the boundary condition r. But there is a second more interesting possibility which exploits that the divergence for $L\to\infty$ is universal. This suggests to focus attention on the sub-leading term in eq. (5.9) by subtracting the spectral density $N_{\rm str}(P|r_*)$ for a fixed reference boundary condition r_*. The corresponding *relative partition function*

$$Z_{\rm rel}(q|r;r_*) \;\equiv\; Tr_{\mathcal{H}_\kappa^{\rm strip}}\big(q^{\mathsf{H}_r^{\rm strip}}-q^{\mathsf{H}_{r_*}^{\rm strip}}\big) \tag{5.10}$$

is indeed well-defined and given by the expression

$$Z_{\rm rel}(q|r;r_*) \;\equiv\; \int\limits_0^\infty dP\;\frac{1}{2\pi i}\frac{\partial}{\partial P}\log\frac{R\big(-\frac{1}{2}+iP|r\big)}{R\big(-\frac{1}{2}+iP|r_*\big)}\;\chi_\kappa^j(q)\;\;. \tag{5.11}$$

We have used the notation $\chi_\kappa^j(q)$ for the following regularized characters,

$$\chi_\kappa^j(q) \;=\; 2\kappa\; q^{b^2P^2}\eta^{-3}(\tau)\;\;. \tag{5.12}$$

The previous two formulas provide an explicit expression for the regularized partition function. We claim that it is related with the annulus amplitude by world-sheet duality. In order to make this more precise, we need to compare the various cut-offs we have introduced so far.

5.4. **Comparison of cut-offs**

First, we observe that the cut-off T does not have a direct counterpart in our discussion of the partition function on the strip. T was introduced to regularize the factor $\delta(p)$ in the boundary state. The variable p is related via Fourier transformation to the time-variable t on the cylinder that describes the boundary of AdS_3. Hence, we can interpret $2T$ as the length of a time-interval to which one has restricted the path-integral of the model. This interpretation suggests to consider the transition amplitude per unit of time which is obtained by simply dropping the factor $\delta_T(p)$ in (5.5). Let us note that an analogous prescription is implicit in the identification of the partition function on the strip with a trace over the space of states.

Furthermore, the constants λ and κ can both be considered as ultraviolet cut-offs for the periodic angular variable θ of the asymptotic cylinder. In fact, the variable n that is cut off by λ is related to θ by Fourier transformation. To find the precise relation between λ and κ, let us note that the annulus amplitude can be naturally interpreted in terms of the radial evolution that is generated by $\mathsf{H}^{\rm cyl}$ in an annulus bounded by two concentric circles in the complex plane. One must take the conformal transformation of the currents that is induced by the change of coordinates from the strip to the complex plane into account. For definiteness, let us assume that the annulus in the complex plane is bounded by two concentric circles around the origin with inner and outer radii R_1 and R_2, respectively. The parameter τ is related to R_1 and R_2 through

$$\log \frac{R_2}{R_1} \;=\; \frac{\pi i}{\tau} \;\; .$$

λ is cutting off the spectrum of the zero mode J^3_0 for the current $J^3(z)$. Here z is the usual coordinate on the complex plane. In order to relate that current to the one used in a Hamiltonian formulation of the theory on the strip, we perform the canonical mapping from the annulus to the strip,

$$z \;=\; R_1 e^{aw}, \quad a \equiv \frac{i}{\pi} \log \frac{R_2}{R_1} \;=\; \frac{i}{\tau} \;\; .$$

The current $J^3(z)$ on the annulus is then related to its counterpart $J^3(w)$ on the strip via $zJ^3(z) = aJ^3(w)$. This implies that the cut-offs λ and κ should obey [11]

$$-i\tau\lambda \;=\; \kappa \;\; . \tag{5.13}$$

We finally need to relate the infra-red cut-off δ to the "volume" cut-off L. Standard arguments from Fourier analysis lead to the relation

$$L \;=\; (2\pi\delta)^{-1} \;\; . \tag{5.14}$$

The two relations (5.13,5.14) must be taken into account when we compare the regularized annulus amplitude with the regularized partition function.

5.5. **Analogs of the Cardy condition**

We are now ready to formulate and verify analogues of the Cardy condition: One possibility is to simply divide the expression (5.7) by $2\lambda\, 2T\, \delta^{-1}$, and compare the resulting quantity to $Z_\kappa(q)/2\kappa$, as defined in subsection 5.3. Using the modular transformation of the characters,

$$\chi_P\big(-\tfrac{1}{\tau}\big) \;=\; 2\sqrt{2} b \frac{i}{\tau} \int_0^\infty dP' \;\cos(4\pi b^2 PP')\chi_{P'}(\tau) \;\; , \tag{5.15}$$

and the identifications (5.13,5.14) between the different cut-offs, it becomes trivial to verify that the two expressions agree provided that A_b satisfies

$$2\sqrt{2}|A_b|^2 \;=\; \pi^2 b^3 \;\; . \tag{5.16}$$

[11] The fact that λ and κ were originally defined as integers, whereas a is not integer in general will not matter when taking λ, κ to infinity.

More interesting is the comparison of the sub-leading terms in the divergence for $L \to \infty$. We shall say that the *relative* Cardy condition is fulfilled if

$$0 \;=\; \lim_{\kappa\to\infty}\Bigg(\frac{1}{2\kappa} Z_{\rm rel}(q|r;r_*) \;- \\ -\lim_{\delta\to 0}\;\frac{1}{2\lambda\,2T}\Big({}_{\rm B,reg}\langle r|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r\rangle_{\rm B,reg} - {}_{\rm B,reg}\langle r_*|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r_*\rangle_{\rm B,reg}\Big)_{\lambda=a\kappa}\Bigg)\;. \tag{5.17}$$

In order to verify that our expression for the boundary state indeed satisfies this condition, let us start by considering the second term in (5.17). It follows from (5.7) and the simple identity $\sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a$ that

$$\lim_{\delta\to 0}\frac{1}{4\lambda T}\Big({}_{\rm B,reg}\langle r|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r\rangle_{\rm B,reg} - {}_{\rm B,reg}\langle r_*|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|r_*\rangle_{\rm B,reg}\Big) \;= \\ =\; 2\pi\;\frac{|A_b|^2}{\pi^3 b^2}\;\int_0^\infty dP\,\chi^j(\tilde{q})\frac{\cos^2(2rP)-\cos^2(2r_*P)}{\sinh(2\pi P)\sinh(2\pi b^2 P)}\;. \tag{5.18}$$

Inserting the modular transformation law for the characters, equation (5.15), leads to an expression of the desired form $\frac{i}{\tau}\int_0^\infty dP'\;N(P'|r;r_*)\;\chi_{P'}(\tau)$, with $N(P'|r;r_*)$ given by

$$N(P|r;r_*) \;= \\ =\frac{\sqrt{2}|A_b|^2}{\pi^3 b^3}\frac{\partial}{\partial P}\int_0^\infty \frac{dt}{t}\;\frac{\sin 2tb^2\big(P+\frac{r}{\pi b^2}\big)+\sin 2tb^2\big(P-\frac{r}{\pi b^2}\big)-(r\leftrightarrow r_*)}{2\sinh t\sinh b^2 t}\;. \tag{5.19}$$

With the help of the special function $S_k(x)$ that we defined in eq. (4.14) one finally rewrites the spectral density in the following form

$$N(P|r;r_*) \;=\; \frac{\sqrt{2}|A_b|^2}{\pi^3 b^3}\,\frac{1}{i}\frac{\partial}{\partial P}\;\log\frac{S_k(P+2R)S_k(P-2R)}{S_k(P+2R_0)S_k(P-2R_0)}\;, \tag{5.20}$$

where $R \equiv \frac{r}{2\pi b^2}$, $R_0 \equiv \frac{r_*}{2\pi b^2}$. This expression should be compared to (5.11). We find agreement provided that A_b satisfies (5.16).

5.6. **The spherical branes**

Towards the end of this section let us briefly discuss the verification of Cardy's condition for the spherical branes. We will not find any divergence in this case, reflecting the fact that the spherical or instantonic branes are compact and do not to extend to the boundary of H_3^+.

We start by introducing the boundary state ${}_{\rm B}\langle s|$ of the spherical branes through the standard prescription ${}_{\rm B}\langle s|j;u\rangle \;\equiv\; {}_{\rm B}\langle\Theta^j(u|\tfrac{i}{2})\rangle_s$. Since there is no divergence, we can now study the usual expression for the annulus amplitude,

$$ {}_{\rm B}\langle s'|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|s\rangle_{\rm B} \;\equiv \\ \equiv\; -\int_{\mathbb{S}} \frac{dj}{\pi^3}\,(2j+1)^2\;\int_{\mathbb{C}} d^2u\;\chi^j(\tilde{q})\;{}_{\rm B,reg}\langle s'|j;u\rangle\langle j;u|s\rangle_{\rm B,reg}\;\;. \tag{5.21} $$

The integral over u is trivial to carry out and it leads us to an expression

$$ {}_{\rm B}\langle s'|\tilde{q}^{\frac{1}{2}\mathsf{H}_{\rm P}}|s\rangle_{\rm B} \;\propto\; \frac{2}{\sin s'\sin s}\int_0^\infty dP\,P\,\frac{\sinh 2s'P\sinh 2sP}{\sinh 2\pi b^2 P}\,\chi_P(\tilde{q})\;\;. \tag{5.22} $$

If one now restricts the values of s, s' to be elements of the following discrete set

$$ \mathbb{S}_{\rm deg} \equiv \left\{\pi b^2(2J+1);\, J=0,\tfrac{1}{2},1,\ldots\right\}\;\;, $$

then one can proceed with the verification of the Cardy condition along the lines of [18, Section 4], see also [42] for a very similar previous discussion for the case of Liouville theory. The crucial ingredient of their discussion is the identity

$$ \frac{\sinh 2\pi n b^2 P\sinh 2\pi m b^2 P}{\sinh 2\pi b^2 P} \;=\; \sum_{l=0}^{\min(n,m)-1}\;\sinh 2\pi b^2(n+m-2l-1)P\;\;. \tag{5.23} $$

This allows them to rewrite the r.h.s. of eq. (5.22) as a sum of terms which can be directly identified as characters for the theory on the strip by using a close relative of the modular transformation formula (5.15). After a short computation one obtains the open string partition function

$$ Z(q|s,s') \;=\; \sum_{J=|R_1'-R_2'|}^{R_1'+R_2'-1}\;\chi^J(q)\;\;. \tag{5.24} $$

Here $\chi^J(q)$ are characters for the sectors which are generated by the current algebra from ground states with a $2J+1$-fold degeneracy. This is somewhat similar to the spectrum of maximally symmetric D-branes on a 3-sphere with infinite radius [28].

REMARK 7. — For labels of the boundary parameters outside of $\mathbb{S}_{\rm deg}$, it does not seem possible to satisfy the Cardy condition. This is supported by the evaluation of the factorization constraint for a second degenerate field in [18, 19]. We have also argued for the discreteness of the parameter s in our semi-classical analysis of the open string spectrum.

6. **CONCLUSIONS AND OPEN PROBLEMS**

In this work we have proposed an exact solution for all the maximally symmetric branes in the Euclidean AdS_3. They fall into one of two classes depending on whether they preserve a $\mathrm{SL}(2,\mathbb{R})$ or a $\mathrm{SU}(2)$ subgroup of $\mathrm{SL}(2,\mathbb{C})$. Branes within each class are related by symmetries of the background. The most interesting representative of the former class are branes localized along a Euclidean $AdS_2 \subset AdS_3$. The boundary states for these branes are given by formula (5.3). Branes preserving an SU(2) symmetry are point-like or spherical (with imaginary radius) and their exact solution is encoded in the one-point function (3.41).

Our strategy in solving these theories was to look at factorization constraints on the one-point functions in the boundary conformal field theory. The latter arise from considering two-point functions of bulk fields in the model. We have evaluated one such constraint explicitly that uses the degenerate field $\Phi^{1/2}$. It would certainly be interesting to check our solution against factorization constraints involving other degenerate fields like the one with label $j = 1/2b^2$ that was used in [12]. This would further restrict the freedom left by the first factorization constraint (see discussion in Subsection 3.4).

The analysis we have carried out also provides important information on the open string sector of the model. In particular, we have derived the open string spectral density in two different ways. For the AdS_2 branes the answer is given in eq. (5.11) whereas the partition function for the spherical branes has been spelled out in eq. (5.24). The latter is discrete and agrees with the findings of [18]. Let us note, however, that there is more information in the open string sector than just the spectrum. It would certainly be interesting to compute scattering amplitudes of open string states. The latter require to derive and solve the factorization constraints on the three-point functions of the boundary fields, similar to what was done in [33] for the case of Liouville theory. So far we have only considered a boundary field that corresponds to a degenerate representation of the current algebra. The results of Section 4 imply that the three point functions involving this degenerate boundary field can be expressed in terms of special fusion coefficients (see 4.16). The only difference is that there appears an interesting shift (4.17) in the identification of boundary conditions and sectors of the model. This observation supports our expectation that a calculation of the three point function of generic boundary fields should be possible along similar lines as in [33].

The boundary two-point functions are non-trivial in the case of non-compact branes and it allow to read off the reflection amplitude for open strings. We have computed the latter for open strings ending on the same brane and our expression is manifestly unitary. If one considers open strings which have one end on a AdS_2 brane r and the other on $r' \neq r$, however, a unitary reflection amplitude does not exist. This might be related to the fact that all the AdS_2-branes from the family parametrized by r end along the same lines on the boundary of the Euclidean AdS_3. The geometry suggests that unitarity can be retained by considering *all* the open string modes that that exist in the brane configuration consisting of two branes with parameters r and r' (and similarly for any larger number of branes).

As we have remarked in the introduction, one of the most interesting applications of our results concerns the coset $H_3^+/\mathbb{R}_\tau$ which describes a 2D Euclidean black hole geometry. In this construction, $\mathbb{R}_\tau$ acts by constant shifts of the Euclidean time. Since our AdS_2 branes are left unchanged by translations along the time direction, they descend trivially to the black hole geometry. In particular, their boundary state is simply given by omitting the factor $\delta(p)$ from the boundary states of the AdS_2 brane, i.e.

$$\begin{aligned} {}_{\mathrm{B}}\langle r|j;n\rangle^{BH} \; = \; & (2\pi)^2\, C_b\, \Gamma(-b^2(2j+1))\, d^j_n \, \times \\ & \times \big(\pi^0_n \cosh r(2j+1) - \pi^1_n \sinh r(2j+1)\big) \; . \end{aligned} \tag{6.1}$$

The definition of the constants d^j_n and π^0_n can be found after eq. (5.3) and the constant C_b can be determined by a Cardy type computation as above. The expression (6.1) describes the coupling of closed string modes that do not wind around the semi-infinite cigar as $\phi \to \infty$.

Geometrically, the boundary state (6.1) corresponds to a one-dimensional brane that is stretched in between two antipodal points on the asymptotic circle at the boundary of the black hole geometry and extends into 2-dimensional space. It crosses the tip of the semi-infinite cigar for $r = 0$. In Section 2 we have remarked that the bulk and boundary theories for H_3^+ do not possess a positive definite state space because of the imaginary B-field. This problem disappears in the coset model for purely dimensional reasons both for the closed and the open string sector. It would also be interesting to study the holographic dual of the branes (6.1) in a black hole geometry using the matrix model description that was proposed in [43]. Following [44, 45], these would show up as point-like defects on the asymptotic circle. We hope to return to all these issues in the near future.

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Appendix A. ANALYSIS ON THE EUCLIDEAN AdS_2 AND AdS_3

A.1. Some integrals for Subsection 2.3

Our aim here is to compute the integrals (2.26). To this end let us introduce the following distributions D^j_ϵ on $\mathcal{S}(H_3^+)$,

$$D^j_\epsilon(f) \;\equiv\; \int_{\mathbb{C}} d^2u\; |u+\bar{u}|^{2j}\,\mathrm{sgn}^\epsilon(u+\bar{u}) F^j_u[f] \;\; . \tag{A.1}$$

Here, F^j_u denotes the generalized Fourier transformation on H_3^+ that is defined by

$$F^j_u[f] \;=\; \int_{H_3^+} dh\; \big(\Phi^j(u|h)\big)^* f(h). \tag{A.2}$$

PROPOSITION A.1. — *The distributions D^j_ϵ can be represented in the following simple form*

$$D^j_\epsilon(f) \;=\; \int_{H_3^+} dh\; D^j_\epsilon(h) f(h) \qquad \textit{where} \tag{A.3}$$

$$D^j_0(h) \;=\; \frac{\cosh\psi(2j+1)}{\cosh\psi} \qquad\qquad D^j_1(h) \;=\; \frac{\sinh\psi(2j+1)}{\cosh\psi}. \tag{A.4}$$

Here, h is an element of H_3^+ which is parametrized by (ψ,χ,ν) that we introduced in equation (2.11).

Proof. — Let us begin by proving the following simple lemma about the transformation properties of D^j_ϵ under the action of $\mathrm{SL}(2,\mathbb{R})$.

LEMMA 1. — *The functionals D^j_ϵ are invariant under the subgroup of matrices $g \in$ $\mathrm{SL}(2,\mathbb{C})$ that satisfy $\omega(g^\dagger) = g^{-1}$,*

$$D^j_\epsilon(T_g f) \;=\; D^j_\epsilon(f) \;\; . \tag{A.5}$$

Here, $\mathrm{SL}(2,\mathbb{C})$ *acts on functions $f \in L^2(H_3^+)$ according to* $(T_g f)(h) \;=\; f\big(g^{-1}h(g^{-1})^\dagger\big)$.

Proof. — We begin by noting that the generalized Fourier transformation $F^j_u(f)$ satisfies the following intertwining property,

$$F^j_u[T_g f] \;=\; |\beta u+\delta|^{-4j-4} F^j_{g\cdot u}[f] \;\;\text{where}\;\; g = \big(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\big) \;\;\text{and}\;\; g\cdot u \equiv \frac{\alpha u+\gamma}{\beta u+\delta}.$$

This follows easily from $\mathrm{SL}(2,\mathbb{C})$-invariance of the measure dh on H_3^+ and the identity (2.22). One may then substitute the u-integration by an integration over $u' \equiv g\cdot u$. It remains to observe that

$$u+\bar{u} \;=\; 2\Re\left(\frac{-\delta\bar{\beta}u'\bar{u}' + u'(\delta\bar{\alpha}+\beta\bar{\gamma}) - \gamma\bar{\alpha}}{|-\beta u'+\alpha|^2}\right) \;=\; \frac{2(u'+\bar{u}')}{|-\beta u'+\alpha|^2} \;, \tag{A.6}$$

where the second equality holds true for all g which satisfy $\omega(g^\dagger) = g^{-1}$. This completes the proof of the lemma. □

We can use the invariance of the functionals D_ϵ^j to express them as averages over the Euclidean AdS_2 branes $\mathcal{C}_\psi$,

$$D_\epsilon^j(f) \;=\; E_\epsilon^j(\mathcal{A}f)\,, \quad (\mathcal{A}f)(\psi) \equiv \int_{\mathcal{C}} dc\; (T_c f)(h_\psi) \;\;, \tag{A.7}$$

where h_ψ was defined in eq. (2.11, the measure dc is given by $dc = dtd\chi\cosh\chi$, and E_ϵ^j is a distribution on functions $f(\psi)$ of a single real variable. The Casimir Q acts diagonally on the functions E_ϵ^j, i.e. $E_\epsilon^j(Qf) = j(j+1)E_\epsilon^j(f)$. For functions which are constant in χ, ν, the Casimir takes the simple form

$$Q_\psi \;=\; \frac{1}{4}\partial_\psi^2 + \frac{1}{2}\frac{\sinh\psi}{\cosh\psi}\partial_\psi \;=\; \frac{1}{4}\frac{1}{\cosh\psi}\big(\partial_\psi^2 - 1\big)\cosh\psi. \tag{A.8}$$

By inserting this into the eigenvalue equation for the functions E_ϵ^j we can now easily conclude that

$$E_\epsilon^j(f) \;=\; \int_{\mathbb{R}} d\psi \cosh^{-1}\psi\; \big(K_+ e^{(2j+1)\psi} + K_- e^{-(2j+1)\psi}\big) f \;\;. \tag{A.9}$$

In order to fix the two unknown coefficients $K_\pm$ one may consider $D_\epsilon^j(f)$ for functions f that are supported near the boundary of H_3^+. More precisely, let us consider the set of functions f_r that vanish if $|\psi - r| > \delta$ for sufficiently small $\delta > 0$. We will be interested in the asymptotic behavior of $D_\epsilon^j(f_r)$ for large values of $|r|$. From the asymptotic behavior (2.23) of the functions Φ_u^j we obtain

$$F_u^j[f] \underset{|r|\to\infty}{\sim} \int\limits_{H_3^+} dh\; f(h)\left(e^{2j\phi}\delta(\gamma - u) + \frac{2j+1}{\pi} e^{-2(j+1)\phi}|\gamma - u|^{-4j-4}\right)$$

up to terms of $\mathcal{O}(e^{-r})$. Inserting this expression into (A.1) of $D_\epsilon^j(f)$ and using (3.15) yields

$$D_\epsilon^j(f) \underset{|r|\to\infty}{\sim} \int\limits_{H_3^+} dh\; f(h)\,\mathrm{sgn}^\epsilon(\gamma+\bar\gamma)\,\Big(\big(e^\phi|\gamma+\bar\gamma|\big)^{2j} + (-)^\epsilon\big(e^\phi|\gamma+\bar\gamma|\big)^{-2j-2}\Big)\;\;.$$

We finally note that $(\gamma + \bar\gamma)\exp\phi \,=\, 2\sinh\psi \,\sim\, \exp|\psi|$ in order to rewrite the previous property as

$$D_\epsilon^j(f) \underset{|r|\to\infty}{\sim} \int_{H_3^+} dh\; f(h) e^{-|\psi|}\Big(e^{(2j+1)\psi} + (-)^\epsilon e^{-(2j+1)\psi}\Big)\;\;. \tag{A.10}$$

We can finally conclude from here that $K_+ = 1/2$ and $K_-\;(-1)^\epsilon/2$. This completes the proof of Proposition A.1. □

Let us finally state without proof the corresponding result for the spherical branes. To this end we introduce

$$\tilde D^j(f) \;\equiv\; \int\limits_{\mathbb{C}} d^2u\; |u\bar u + 1|^{2j} F_u^j[f]\;\;. \tag{A.11}$$

Using the same ideas as in the proof of the previous proposition one can establish the following result.

PROPOSITION A.2. — *The distributions* $\tilde{D}^j$ *can be represented in the following simple form*

$$\tilde{D}^j(f) \;=\; \int_{H_3^+} dh\, \frac{\sinh \Lambda(2j+1)}{\sinh \Lambda} f(h) \;\; . \tag{A.12}$$

Here, h *is an element of* H_3^+ *which is parametrized by* (ψ,χ,ν) *that we introduced in equation (2.16).*

A.2. **Harmonic analysis on the** AdS_2 **branes**

As we discussed in some detail above, the space of wave functions on the Euclidean AdS_2-brane carries an action of $\mathrm{SL}(2,\mathbb{R})$. Let us denote the surface on which the brane is localized by $\mathcal{C}_\psi$ and recall that it comes equipped with a measure $dc = 2e^\chi d\chi d\nu$. The action of $\mathrm{SL}(2,\mathbb{R})$ on the Hilbert space $L^2(\mathcal{C}_\psi, dc)$ is easily shown to be unitary. Our claim is that $L^2(\mathcal{C}_\psi, dc)$ decomposes into irreducible representations $\mathcal{P}_j, j = -\frac{1}{2} + ip$, from the principal series of $\mathrm{SL}(2,\mathbb{R})$.

THEOREM 1. — *There exists an isomorphism between the following representations of* $\mathrm{SL}(2,\mathbb{R})$

$$L^2(\mathcal{C}_\psi, dc) \;\simeq\; \int_{\mathbb{S}}^{\oplus} d\mu_{\mathrm{P}}(j)\; \mathcal{P}_j \;\; , \qquad d\mu_{\mathrm{P}}(j) \;=\; \frac{2j+1}{2\pi} \coth \pi j \; dj \;\; . \tag{A.13}$$

This isomorphism can be realized explicitly by a generalized Fourier transform. It implies that any function $f \in L^2(\mathcal{C}_\psi, dc)$ *may be decomposed in the form*

$$f(c) \;\;=\;\; \int_{\mathbb{S}} d\mu_{\mathrm{P}}(j) \int_{\mathbb{R}} du\; \Xi^j(u|\psi;c)\; \mathcal{F}^j_u[f] \tag{A.14}$$

$$\textit{where} \qquad \Xi^j(u|\psi;c) \;=\; \Big(v'_u\; h\; {v'_u}^{\dagger} \Big)^j \tag{A.15}$$

with $v'_u = (iu, 1)$ *and the functions* $\mathcal{F}^j_u[f]$ *are the generalized Fourier coefficients of* f*, i.e.*

$$\mathcal{F}^j_u[f] \;=\; \int_{\mathcal{C}_\psi} dc\; \big(\Xi^j(u|\psi;c)\big)^* \; f(c) \;\; . \tag{A.16}$$

The Fourier transformation $\mathcal{F}$ *diagonalizes the action of* $\mathrm{SL}(2,\mathbb{R})$ *on* $\mathcal{C}_\psi$ *in the sense that*

$$\begin{aligned} \mathcal{F}^j[T_g f] \;=\; P^j_g \mathcal{F}^j[f] \quad \textit{where} \quad & T_g f(c) \;=\; f(g^{-1} c (g^{-1})^\dagger) \;\; , \\ \textit{and} \quad & P^j_g h(u) \;=\; |\beta u + \delta|^{2j}\; h(g\cdot u) \end{aligned} \tag{A.17}$$

for all $g \in \mathrm{SL}(2,\mathbb{C})$ *of the form* $g = \left(\begin{smallmatrix} \delta & -i\beta \\ i\gamma & \delta \end{smallmatrix}\right)$*,* $\alpha,\beta,\gamma,\delta \in \mathbb{R}$*. Moreover, the Fourier transform satisfies a reflection property which is related to the equivalence* $\mathcal{P}_j \simeq \mathcal{P}_{-j-1}$*,*

$$\mathcal{F}_u^{-j-1}[f] \; = \; -(\cosh\psi)^{2j+1} \int_{\mathbb{R}} du' \; J^j(u,u') \; \mathcal{F}_u^j[f] \;\; , \tag{A.18}$$

where J^j *is the integral kernel that we have defined in eq. (2.33) above.*

Sketch of proof. To begin with let us note that the action of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ on the surface $\mathcal{C}_\psi$ is represented by the differential operators

$$\begin{aligned} \mathcal{D}^+ \; &= \; -\,\partial_\nu \\ \mathcal{D}^- \; &= \; -\,(\nu^2 - e^{-2\chi})\partial_\nu + 2\nu\partial_\chi \end{aligned} \qquad \mathcal{D}^0 \; = \; -\nu\partial_\nu + \partial_\chi \;\; . \tag{A.19}$$

Here we use the coordinates from eq. (2.16). The expression the Laplace operator on $\mathcal{C}_\psi$ is

$$Q \; = \; \partial_\chi^2 + \partial_\chi + e^{-2\chi}\partial_\nu^2 \;\; . \tag{A.20}$$

It is easy to determine the common spectral decomposition of Q and $\mathcal{D}^+$. The latter is diagonalized by the Fourier transformation w.r.t. ν and on the eigen-spaces of $\mathcal{D}^+$ the operator Q gets represented by $Q_k \equiv \partial_\chi^2 + \partial_\chi - k^2 \exp(-2\chi)$ where ik is the eigen-value of $\mathcal{D}^+$. By a simple re-definition of the eigen-functions one can see that the spectral problem for Q_k in $L^2(\mathbb{R}, d\chi e^\chi)$ is equivalent to the spectral problem for the Hamilton-operator $H_k = -\partial_\chi^2 + k^2 \exp(-2\chi)$, for which the solution is well-known [46]. One thereby finds that the set of functions

$$\Psi_k^j(\chi,\nu) \; := \; \frac{2\left(\frac{|k|}{2}\right)^{-j-\frac{1}{2}}}{\Gamma\left(-j-\frac{1}{2}\right)} e^{-ik\nu} e^{-\frac{1}{2}\chi} K_{j+\frac{1}{2}}(|k|e^{-\chi}), \quad j \in \mathbb{S}, \;\; k \in \mathbb{R} \tag{A.21}$$

forms a basis for $L^2(\mathbb{R}^2, d\nu d\chi e^\chi)$ with normalization given by

$$\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\chi \, e^\chi \; \Psi_{-k}^{-\frac{1}{2}-ip}(\chi,\nu) \, \Psi_{k'}^{-\frac{1}{2}+ip'}(\chi,\nu) \; = \; (2\pi)^2 \delta(p-p')\delta(k-k'). \tag{A.22}$$

Let us compare this with the Fourier transform of $\Xi^j(u|\psi;c)$

$$\begin{aligned} \hat{\Xi}^j(k|\psi;x) \;\; &= \;\; \int_{\mathbb{R}} du \; e^{-iku} \; \Xi^j(u|\psi;c) \\ &= \;\; \cosh^j\psi \; \frac{\left(\frac{|k|}{4}\right)^{-j-\frac{1}{2}}}{\Gamma(-j-\frac{1}{2})} \; e^{-ik\nu} e^{-\frac{1}{2}\chi} K_{j+\frac{1}{2}}(|k|e^{-\chi}) \;\; . \end{aligned}$$

The second line follows easily from one of the standard integral representation for the Bessel function K_ρ. Hence we conclude from the completeness and orthogonality of the $\Psi_k^j(\chi,\nu)$ that eq. (A.14) defines a generalized Fourier transformation on $L^2(\mathcal{C}_\psi, dc)$.

The intertwining property (A.17) is easily verified by a direct calculation. To finally verify the reflection property (A.18) one may note that $K_\rho = K_{-\rho}$ implies a simple reflection property for the $\Xi^j(k|\psi;c)$. This is translated into (A.17) by means of the following formula [47] for the Fourier transformation of the distribution $|x|^{2j}$,

$$\int_{\mathbb{R}} dx\; e^{ikx}|x|^{2j} \;=\; \sqrt{\pi}\,\Big|\frac{2}{k}\Big|^{2j+1}\frac{\Gamma\big(j+\frac{1}{2}\big)}{\Gamma(-j)} \quad . \tag{A.23}$$

Thereby we did prove all the assertions of our theorem describing the harmonic analysis of the Euclidean AdS_2 branes.

A.3. The distributions d^j_ϵ

Our aim here is to prove a reflection property for the tempered distributions d^j_ϵ that are defined by

$$d^j_\epsilon(f) \;\equiv\; \int_{\mathbb{C}} d^2u\; |u+\bar{u}|^{2j}\,\mathrm{sgn}^\epsilon(u+\bar{u})\, f(u). \tag{A.24}$$

PROPOSITION A.3. — *The tempered distributions d^j_ϵ satisfy the following reflection property*

$$d^j_\epsilon(f) \;=\; -(-)^\epsilon\, d^j_\epsilon(\mathcal{I}_j f) \quad , \tag{A.25}$$

where $\mathcal{I}_j$ is the intertwining operator that establishes the equivalence of the $SL(2,\mathbb{C})$-representations P_{-j-1} and P_j (see eq. (3.12))

REMARK 8. — Proposition A.3 can be re-interpreted as the following identity between the corresponding distributions,

$$\begin{aligned}\frac{2j+1}{\pi}\int_{\mathbb{C}} d^2u\; |u+\bar{u}|^{2j}\,\mathrm{sgn}^\epsilon(u+\bar{u})\; |u-\gamma|^{-4j-4} &= \\ &= \;(-)^\epsilon|\gamma+\bar{\gamma}|^{-2j-2}\,\mathrm{sgn}^\epsilon(\gamma+\bar{\gamma})\,.\end{aligned} \tag{A.26}$$

Proof. — It will be convenient to use a kind of Fourier-transformed version of the definition (A.24). The function $f(u)$ may be represented as

$$f(u) \;=\; (2\pi)^{-2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dp\; e^{in\arg(u)}|u|^{-2j-2+ip}\,F^j_{np}(f) \quad , \tag{A.27}$$

where the Fourier-transform $F^j_{np}(f)$ of f is defined by

$$F^j_{np}(f) \;=\; \int_{\mathbb{C}} d^2u\; e^{-in\arg(u)}|u|^{2j-ip}\, f(u) \quad . \tag{A.28}$$

LEMMA 2. — *The distribution $d^j_\epsilon(f)$ can be represented in terms of $F^j_{np}(f)$ as follows*

$$d^j_\epsilon(f) \;=\; \sum_{n\in\mathbb{Z}} d^{j,\epsilon}_n\, F^j_{n0}(f) \quad , \tag{A.29}$$

where the coefficients $d_n^{j,\epsilon}$ *are given by the expression* $d_n^{j,\epsilon} = d_n^j \pi_n^\epsilon$ *with*

$$d_n^j \;=\; \frac{\Gamma(2j+1)}{\Gamma(1+j+\frac{n}{2})\Gamma(1+j-\frac{n}{2})}, \quad \pi_n^\epsilon \;=\; \begin{cases} 1-\epsilon \;\text{ if }\; n \;\text{ even} \\ \quad\;\; \epsilon \;\text{ if }\; n \;\text{ odd}\;. \end{cases} \tag{A.30}$$

Proof. — Inserting eq. (A.27) into eq. (A.24) immediately leads to a representation of the form (A.29) with $d_n^{j,\epsilon}$ given through

$$d_n^{j,\epsilon}\delta(p) \;=\; (2\pi)^{-2}\int_{\mathbb{C}} d^2u\; e^{in\arg(u)}|u|^{-2j-2+ip}|u+\bar{u}|^{2j}\,\mathrm{sgn}^\epsilon(u+\bar{u})\;. \tag{A.31}$$

It is straightforward to reduce the resulting integral for $d_n^{j,\epsilon}$ to the form

$$d_n^{j,\epsilon} \;=\; e^{-in\frac{\pi}{2}}\int_0^\pi d\varphi\;(e^{in\varphi}+(-)^\epsilon e^{-in\varphi})(2\sin\varphi)^{2j}\;. \tag{A.32}$$

By studying the behavior of the integrand under $\varphi\to\pi-\varphi$ one may verify that $d_n^{j,\epsilon}=\pi_n^\epsilon d_n^j$. The integral for d_n^j can be found e.g. on p. 427 of [25]. □

LEMMA 3. — *The Fourier-transform* $F_{np}^j(f)$ *satisfies the following reflection property* $F_{np}^j(f) = -r_{np}^j F_{np}^{-j-1}(\mathcal{I}_j f)$, *where*

$$r_{np}^j \equiv \frac{\Gamma(-2j-1)}{\Gamma(2j+1)}\frac{\Gamma\big(1+j-\frac{1}{2}(n+ip)\big)\Gamma\big(1+j-\frac{1}{2}(n-ip)\big)}{\Gamma\big(-j-\frac{1}{2}(n-ip)\big)\Gamma\big(-j-\frac{1}{2}(n+ip)\big)}\;. \tag{A.33}$$

Proof. — The claim follows easily with the help of

$$\begin{aligned}\int_{\mathbb{C}} d^2x'\;|x-x'|^{-4j-4}x'^{j-m}\bar{x}'^{j-\bar{m}} \;=& \\ =\pi\,\frac{\Gamma(1+j-m)\Gamma(1+j+\bar{m})}{\Gamma(-j-m)\Gamma(-j+\bar{m})}\frac{\Gamma(-2j-1)}{\Gamma(2j+2)}\,x^{-j-1-m}\bar{x}^{-j-1-\bar{m}}\;.&\end{aligned} \tag{A.34}$$

This can be obtained by a minor generalization from an integral calculated in [48]. □

To complete the proof, one may note that the functional relation for the Gamma function implies $r_{n0}^j d_n^j/d_n^{-j-1} = (-)^n$. From this we conclude

$$r_{n0}^j\, d_n^{j,\epsilon} \;=\; (-)^\epsilon d_n^{-j-1,\epsilon}\,. \tag{A.35}$$

Inserting the result of Lemma 3 into eq. (A.29) yields

$$d_\epsilon^j(f) \;=\; \sum_{n\in\mathbb{Z}} d_n^{j,\epsilon}(-r_{n0}^j)F_{n0}^{-j-1}(\mathcal{I}_j f)\,. \tag{A.36}$$

If we finally take eq. (A.35) into account and use eq. (A.29) again we can establish the Proposition A.3. □

Appendix B. RELATIVE PARTITION FUNCTIONS

In this appendix we are going to review the relation between reflection amplitudes and spectral densities in a quantum mechanical setting. This is certainly well-known to many people, but may not be familiar to all potential readers. For a mathematical rigorous treatment the reader may consult e.g. [49].

Assume we are given a quantum mechanical system with a Hamiltonian $H = p^2 + V(q)$, where the potential $V(q)$ rapidly approaches zero for $q \to -\infty$, but diverges for $q \to \infty$. The vanishing of the potential for $q \to -\infty$ implies that eigen-functions of the Hamiltonian can be specified by their asymptotic behavior in this region,

$$\Xi_E(q) \sim A_p\, e^{ipq} + B_p\, e^{-ipq} , \qquad p = \sqrt{E} . \tag{B.1}$$

In general, the eigenvalue equation $H\psi = E\psi$ has two linearly independent solutions. But due to the divergence of the potential for $q \to \infty$, a generic eigen-function will have a similar divergence and there can be at most one solution which is well-behaved for $q \to \infty$. From this solution we may read off the ratio $R(p) = B_p/A_p$. This is the quantum mechanical characteristics of a totally reflecting potential: An incoming plane wave e^{ipq} is reflected[12] into an outgoing wave e^{-ipq} times the *reflection amplitude* $R(p)$. The reflection amplitude is a functional of the potential.

Having introduced the reflection amplitude $R(p)$ we want to analyse how it is related with the partition function of the system. The definition of partition functions gets subtle in the case of systems with continuous spectrum. One might hope that partition functions could be represented in the following form

$$\mathrm{Tr}\big(\mathcal{O}(H)e^{-\beta H}\big) = \int dE\; \rho(E)\; \mathcal{O}(E)e^{-\beta E} , \tag{B.2}$$

where $\rho(E)$ is some spectral density. Intuitively one would consider $\rho(E)$ to represent the "density of eigenvalues".

The most naive version of this idea does not quite work: It is instructive to consider the system obtained by putting a perfectly reflecting wall at $q = -L$, with large positive L, and taking the limit $L \to \infty$. For any finite value of L one then has a system with discrete spectrum, but when L increases, the number $N(E)$ of eigenvalues corresponding to energies $E' < E$ will likewise increase. The average density of eigenvalues in an interval $[E-\delta, E]$ is

$$\rho_\delta(E) = \frac{N(E) - N(E-\delta)}{\delta} .$$

[12] Of course the qualification "incoming" resp. "outgoing" requires consideration of the problem of asymptotic time-evolution of wave-packets that in the asymptotic past approximate an incoming plane wave. The relation between time-asymptotics of the scattering problem and space-asymptotics of the eigen-functions follows by applying stationary phase methods to the eigenfunction expansion of the time-dependent wave-function.

Now one needs to consider the behavior of such quantities for $L \to \infty$. Quantization of the energy eigenvalues for finite L is a consequence of the boundary condition

$$\psi_E(q)\big|_{q=L} = 0 \ . \tag{B.3}$$

If L is large enough one may approximate ψ_E by its asymptotic behavior

$$\psi_E(q) \ \sim \ e^{ipq} + R(p)e^{-ipq} \ .$$

This is the basic source for the relation between reflection amplitude and spectral density: The quantization condition (B.3) turns into an equation that determines the possible eigenvalues from the reflection amplitude

$$R(p) \ = \ -e^{-2ipL} \quad \text{for any eigenvalue } E = p^2. \tag{B.4}$$

By introducing the function $\Delta_L(p) = p - \frac{i}{2L}\ln(R(p))$ one may express the positions p_n of eigenvalues in terms of the inverse function Δ_L^{-1} as

$$p_n = \Delta_L^{-1}\left(\frac{2n+1}{2L}\pi\right) \ .$$

One will have to consider the eigenvalues near a fixed value $E_n = E(p_n)$. When taking $L \to \infty$ one obviously needs to consider values of the eigen-value label n of the same order as L. The spacing $\delta p \equiv p_{n+1} - p_n$ of two momenta can then be estimated as

$$\begin{aligned} \delta p \equiv p_{n+1} - p_n \ &= \ \Delta_L^{-1}\left(\frac{2n+3}{2L}\pi\right) - \Delta_L^{-1}\left(\frac{2n+1}{2L}\pi\right) \\ &\sim \ \frac{\pi}{L}\frac{\partial}{\partial y}\Delta_L^{-1}(y)\big|_{y=\frac{2n+1}{2L}} \\ &= \ \frac{\pi}{L}\frac{1}{\Delta_L'(p_n)} \ . \end{aligned} \tag{B.5}$$

The average density ρ_δ of eigenvalues is therefore approximately

$$\rho_\delta(p) \ \sim \ \frac{L}{\pi}\frac{\partial}{\partial p}\Delta_L(p) \ = \ \frac{L}{\pi} + \frac{1}{2\pi i}\frac{\partial}{\partial p}\ln R(p) \ . \tag{B.6}$$

This quantity diverges for $L \to \infty$. It follows that traces like (B.2) do not make sense in a theory with continuous spectrum. However, one may note that this divergence is universal, i.e. to a large extend independent of the interaction $V(q)$. Interesting objects to study are therefore the *relative* partition functions, which compare the spectrum in the potential V to the spectrum of a fixed reference Hamiltonian $H_* = p^2 + V_*$

$$\text{Tr}_{\text{rel}}\left(\mathcal{O}(H)\right)_{H_*} \equiv \text{Tr}\big(\mathcal{O}(H) - \mathcal{O}(H_*)\big) \ . \tag{B.7}$$

We assume that $V_*(q)$ belongs to the same class of potentials as $V(q)$, i.e. it also rapidly approaches zero for $q \to -\infty$ and diverges for $q \to \infty$. The spectra of H and H_* will therefore have the same continuous part. If we denote the reflection amplitude for V_* by $R_*(p)$, we immediately get the following *relative trace formula* from eq. (B.6)

$$\mathrm{Tr}_{\mathrm{rel}}\big(\mathcal{O}(H)\big)_{H_*} \;=\; \int d\mu(E)\;\rho_{\mathrm{rel}}\,(E)\;\mathcal{O}(E)\;\;, \tag{B.8}$$

where the relative eigenvalue density $\rho_{\mathrm{rel}}\,(E)$ is given by the expression

$$\rho_{\mathrm{rel}}\,(E) \;=\; \frac{1}{2\pi i}\frac{\partial}{\partial p}\ln\frac{R(p)}{R_*(p)}\bigg|_{p=\sqrt{E}}\;\;. \tag{B.9}$$

Appendix C. **CONFORMAL BLOCKS**

For the convenience of the reader, we want to gather some basic definitions and results concerning chiral vertex operators, conformal blocks and the associated fusion matrices. These results are mostly well-known, but it may be helpful to list the required formulae in a uniform notation. The only slightly unusual point comes from the definition (C.3) of the conformal blocks by means of the invariant bilinear form (C.2) on the representations $\mathcal{P}_j$, which accounts for some absolute value signs in the formulae below.

C.1. **Chiral vertex operators and conformal blocks**

We only need to consider one rather special class of chiral vertex operators. Consider operators $\mathsf{V}^j_m(u|z) : \mathcal{P}_j \to \mathcal{P}_{j+m}$ with $j = \frac{1}{2}, 1, \ldots$ and $m = -j, -j+1, \ldots, j$ that are uniquely defined by the properties

$$\begin{aligned}
&\text{(i)}\;\; J^a_n\mathsf{V}^j_m(u|z) - \mathsf{V}^j_m(u|z)J^a_n = z^n\mathcal{D}^a_{j,u}\mathsf{V}^j_m(u|z),\\
&\text{(ii)}\;\; \mathsf{V}^{j_2}_{m_2}(u_2|z)|j_1;u_1\rangle = z^{\Delta_{j_1+m_2}-\Delta_{j_2}-\Delta_{j_1}}(u_2-u_1)^{j_2-m_2}\times\\
&\qquad\qquad\times\big(|j_1+m_2;u_1\rangle + \mathcal{O}(z) + \mathcal{O}(u_2-u_1)\big)\;\;.
\end{aligned} \tag{C.1}$$

In this particular case the dependence of $\mathsf{V}^j_m(u|z)$ on its variable u happens to be polynomial, which means that $\mathsf{V}^j_m(u|z)$ satisfies an equation of the form $\partial_u^{2j+1}\mathsf{V}^j_m(u|z) \equiv 0$.

Conformal blocks can be defined with the help of the invariant bilinear form on $\mathcal{P}_j$ which can be described using the following object

$$B\big(\,|j,u_1\rangle\,,|j,u_2\rangle\,\big) \;=\; |u_2-u_1|^{2j}\;\;. \tag{C.2}$$

One can then introduce a class of conformal blocks ("s-channel") by

$$\begin{aligned}
&\mathcal{F}^{(s)}_{j_1+m_1}\big[\begin{smallmatrix} j_3 & j_2\\ j_4 & j_1\end{smallmatrix}\big](u_4,\ldots,u_1|z_3,z_2) \;\equiv\\
&\qquad\equiv\; B\big(\,|j_4,u_4\rangle\,,\,\mathsf{V}^{j_3}_{m_2}(u_3|z_3)\mathsf{V}^{j_2}_{m_1}(u_2|z_2)|j_1;u_1\rangle\,\big)
\end{aligned} \tag{C.3}$$

where $j_4 = j_1 + m_2 + m_1$. As usual, one finds that $\mathcal{F}^{(s)}$ can be expressed in terms of a function of the cross-ratios u, z,

$$\begin{aligned}\mathcal{F}^{(s)}_{j_1+m_1}\big[\begin{smallmatrix} j_3 & j_2 \\ j_4 & j_1 \end{smallmatrix}\big](u_4,\dots,u_1|z_3,z_2) \;&=\; |u_4-u_1|^{j_4+j_1-j_2-j_3}(u_4-u_3)^{j_4+j_3-j_2-j_1}\times \\ &(u_4-u_2)^{2j_2}\,(u_3-u_1)^{j_1+j_2+j_3-j_4}F^{(s)}_{j_1+m_1}\big[\begin{smallmatrix} j_3 & j_2 \\ j_4 & j_1 \end{smallmatrix}\big](u|z) \qquad \text{(C.4)} \\ \text{where} \qquad u \;&=\; \frac{(u_4-u_3)(u_2-u_1)}{(u_4-u_2)(u_3-u_1)} \quad , \quad z \;=\; \frac{z_2}{z_3}\end{aligned}$$

The Knizhnik-Zamolodchikov (KZ-) equations follow in the usual manner from the Sugawara-construction. The resulting equation for the "reduced" conformal blocks $F(u|z)$ takes the form [50]

$$tz(z-1)\partial_z F \;=\; \mathcal{D}^{(2)}_u F \;\;,$$

where the differential operator $\mathcal{D}^{(2)}_u$ is given by the expression

$$\begin{aligned}\mathcal{D}^{(2)}_u \;\;=\;\; & u(u-1)(u-z)\partial_u^2 + 2j_2\kappa(u-z) + 2j_1j_2(z-1) + 2j_2j_3z \\ & -\big((\kappa-1)(u^2-2zu+z) + 2j_1u(z-1) + 2j_2u(u-1) + 2j_3z(u-1)\big)\,\partial_u\end{aligned}$$

We have used the abbreviation $\kappa = j_1 + j_2 + j_3 - j_4$. In the presently considered case one has *polynomial* dependence on u, so that the KZ-equations have a finite dimensional space of solutions. Two canonical bases for the space of solutions ("s- and t-channel conformal blocks") can be defined by the asymptotics

$$\begin{aligned}&\mathcal{F}^{\rm s}_{j_{21}}(u|z) \underset{z\to 0}{\sim} z^{\Delta_{j_{21}}-\Delta_{j_2}-\Delta_{j_1}}x^{j_1+j_2-j_{21}}(1+\mathcal{O}(x)+\mathcal{O}(z)), \\ &\mathcal{F}^{\rm t}_{j_{32}}(u|z) \underset{z\to 1}{\sim} (1-z)^{\Delta_{j_{32}}-\Delta_{j_3}-\Delta_{j_2}}(1-x)^{j_2+j_3-j_{32}}(1+\mathcal{O}(1-x)+\mathcal{O}(1-z)),\end{aligned} \qquad \text{(C.5)}$$

where it is understood that the limits are first taken in the z-variable.

For our purposes it suffices to write down the solutions in the special case $j_1 = \frac{1}{2}$. To this end let us introduce the notation

$$\begin{aligned}&u = -\,b^2(j_1+j_3+j_4+\tfrac{3}{2})-1, \\ &v = -\,b^2(j_1+j_3-j_4+\tfrac{1}{2}),\end{aligned} \qquad w = -b^2(2j_1+1)\;\;. \qquad \text{(C.6)}$$

A set of normalized solutions for the s- and t-channel is then given by

$$\begin{aligned}
\mathcal{F}^{\rm s}_{+} &= z^{b^2 j_1}(1-z)^{b^2 j_3}\Big(F(u+1,v,w;z) - x\frac{v}{w}F(u+1,v+1,w+1;z)\Big)\\
\mathcal{F}^{\rm s}_{-} &= z^{-b^2(j_1+1)}(1-z)^{b^2 j_3}\Big(xF(u-w+1,v-w+1,1-w;z) - \\
&\qquad - z\frac{u-w+1}{1-w}F(u-w+2,v-w+1,2-w;z)\Big),\\
\mathcal{F}^{\rm t}_{+} &= (1-z)^{b^2 j_3} z^{b^2 j_1}\Big(F(u+1,v,u+v-w+1;1-z) + \\
&\qquad + (1-x)\frac{v}{u+v-w+1}F(u+1,v+1,u+v-w+2;1-z)\Big)\\
\mathcal{F}^{\rm t}_{-} &= (1-z)^{-b^2(j_3+1)} z^{b^2 j_1}\Big((1-x)F(w-u,w-v,w-u-v;1-z) - \\
&\qquad - (1-z)\frac{w-v}{w-u-v}F(w-u,w-v+1,w-u-v+1;1-z)\Big)\,.
\end{aligned}$$

C.2. **Fusion matrices**

The fusion matrices used in the main text can all be obtained from the following basic example

$$F_{st}(j|\rho_2,\rho_1) \;\equiv\; F_{\rho_2+\frac{s}{2}\; j+\frac{t}{2}}\Big[\begin{smallmatrix} \frac{1}{2} & j \\ \rho_2 & \rho_1 \end{smallmatrix}\Big] \;\equiv\; F_{\rho_2+\frac{s}{2}\; j+\frac{t}{2}}\Big[\begin{smallmatrix} j & \frac{1}{2} \\ \rho_1 & \rho_2 \end{smallmatrix}\Big]\,, \tag{C.7}$$

where s,t take the values ± 1. The matrix elements F_{st} have the following expressions

$$\begin{aligned}
F_{++} &= \frac{\Gamma(-b^2(2\rho_2+1))\Gamma(1+b^2(2j+1))}{\Gamma\big(1+b^2(j+\rho_1-\rho_2+\frac{1}{2})\big)\Gamma\big(b^2(j-\rho_1-\rho_2-\frac{1}{2})\big)}\\
F_{+-} &= \frac{\Gamma(-b^2(2\rho_2+1))\Gamma(-b^2(2j+1))}{\Gamma\big(-b^2(j+\rho_1+\rho_2+\frac{3}{2})\big)\Gamma\big(-b^2(j-\rho_1+\rho_2+\frac{1}{2})\big)}\\
F_{-+} &= \frac{\Gamma(1+b^2(2\rho_2+1))\Gamma(1+b^2(2j+1))}{\Gamma\big(1+b^2(j+\rho_1+\rho_2+\frac{3}{2})\big)\Gamma\big(1+b^2(j-\rho_1+\rho_2+\frac{1}{2})\big)}\\
F_{--} &= -\frac{\Gamma(1+b^2(2\rho_2+1))\Gamma(-b^2(2j+1))}{\Gamma\big(-b^2(j+\rho_1-\rho_2+\frac{1}{2})\big)\Gamma\big(1-b^2(j-\rho_1-\rho_2-\frac{1}{2})\big)}\,.
\end{aligned} \tag{C.8}$$

In Subsection 4.3 we need the following special case of these formulae

$$f_{st}(j) \;\equiv\; F_{j+\frac{s}{2}\; \frac{1}{2}+\frac{t}{2}}\Big[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{smallmatrix}\Big]\;. \tag{C.9}$$

The previously given expressions simplify to

$$\begin{aligned}
f_{++} &= \frac{\Gamma(1+2b^2)}{\Gamma(1+b^2)}\frac{\Gamma(-b^2(2j+1))}{\Gamma(-2b^2 j)} & f_{+-} &= \frac{\Gamma(-2b^2)}{\Gamma(-b^2)}\frac{\Gamma(-b^2(2j+1))}{\Gamma(-2b^2(j+1))}\\
f_{-+} &= \frac{\Gamma(1+2b^2)}{\Gamma(1+b^2)}\frac{\Gamma(1+b^2(2j+1))}{\Gamma(1+2b^2(j+1))} & f_{--} &= -\frac{\Gamma(-2b^2)}{\Gamma(-b^2)}\frac{\Gamma(1+b^2(2j+1))}{\Gamma(1+2b^2 j)}\;.
\end{aligned}$$

We finally need

$$F_s^1(j|\rho_2,\rho_1) \;\equiv\; F_{\rho_2\; j+s}\left[\begin{smallmatrix} 1 & j \\ \rho_2 & \rho_1 \end{smallmatrix}\right] \;. \tag{C.10}$$

These elements of the fusing matrix can be calculated in terms of F_{st} by means of the pentagon identity [39],

$$F_{\rho_2\; j+s}\left[\begin{smallmatrix} 1 & j \\ \rho_2 & \rho_1 \end{smallmatrix}\right] \;=\; \sum_{t=\pm} \frac{F_{t+}\big(\frac{1}{2}|j,j+s\big)}{F_{++}\big(\frac{1}{2}|\rho_2,\rho_2\big)}\; F_{-t}\big(j|\rho_2+\tfrac{1}{2},\rho_1\big)F_{+,s-t}\big(j+\tfrac{t}{2}|\rho_2,\rho_1\big) \;.$$

Explicit expressions for F_s^1 are given by (with $g(b)=\frac{\Gamma(1+b^2)}{\Gamma(1+2b^2)}$)

$$\begin{aligned}
F_+^1 \;&=\; \frac{g(b)\Gamma(1+b^2(2\rho_2+2))\Gamma(-2b^2\rho_2)\Gamma(1+b^2(2j+2))\Gamma(1+b^2(2j+1))}{\prod_s^{\pm}\Gamma\big(\frac{1}{2}+s(\frac{1}{2}+b^2(\rho_1+\rho_2+1))+b^2(j+1)\big)\Gamma\big(1+b^2(j+s(\rho_1-\rho_2)+1)\big)}, \\
F_0^1 \;&=\; \frac{\Gamma(1+b^2(2\rho_2+2))\Gamma(-2b^2\rho_2)\Gamma(1+2b^2j)\Gamma(-b^2(2j+2))}{2\pi\sin\pi b^2(2j+1)}\times \\
&\qquad \times\Big(\cos\pi b^2(2\rho_1+1)\big(\sin\pi b^2(2j+2)+\sin\pi b^2(2j)\big) \\
&\qquad\qquad -\cos\pi b^2(2\rho_2+1)\sin\pi b^2(4j+2)\Big)\;, \\
F_-^1 \;&=\; \frac{g(b)\Gamma(1+b^2(2\rho_2+2))\Gamma(-2b^2\rho_2)\Gamma(-2b^2j)\Gamma(-b^2(2j+1))}{\prod_{s=\pm}\Gamma\big(\frac{1}{2}-s(\frac{1}{2}+b^2(\rho_1+\rho_2+1))-b^2j\big)\Gamma\big(-b^2(j+s(\rho_1-\rho_2))\big)}\;.
\end{aligned}$$

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