Relativistic solution of Iordanskii problem in multi-constituent superfluid mechanics

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Abstract. Flow past a line vortex in a simple perfect fluid or superfluid gives rise to a transverse Magnus force that is given by the well known Joukowski lift formula. The problem of generalising this to multiconstituent superfluid models has been controversial since it was originally posed by the work of Iordanski in the context of the Landau 2-constituent model for ^{4}He at finite temperature. The present work deals not just with this particular case but with the generic category of perfect multiconstituent models including the kind proposed for a mixture of ^{4}He and ^{3}He by Andreev and Bashkin. It is shown here (using a relativistic approach) that each constituent will provide a contribution proportional to the product of the corresponding momentum circulation integral with the associated asymptotic current density.

1 Introduction

For a simple perfect fluid with asympotically uniform density $\overline{\rho}$ say, the Magnus effect of a uniform background flow with relative velocity v^i say in the rest frame of a vortex in the direction of a 3 dimensional unit vector ℓ^i results in a force per unit length given by the well known (non-relativistic) Joukowsky formula as

$$\mathcal{F}_i = \kappa\bar{\rho}\,\varepsilon_{ijk}\ell^j\overline{v}^k\,, \tag{1}$$

where κ is the relevant velocity circulation integral.

The question raised by Iordanskii [1] of how this formula should be generalisd to the case of Landau's 2 constituent model for superfluid ^{4}He at finite temperature has been a subject of controversy: the most widely accepted [2, 3] prescription is that of Sonin [4, 5], but various alternatives have been proposed by other authors [6, 7, 8]. The present work clarifies the issue by demonstrating the existence of an elegant generalisation of the

Joukowsky formula (1) for an extensive class of perfect multiconstituent fluid models, including, as well as the Landau model, the Andreev Bashkin model [9] for a mixture of superfluid ^{4}He with ("normal") ^{3}He. (However our analysis does not cover the more complicated subject [2] of superfluid ^{3}He).

For technical convenience (not to mention the consideration that it is more accurate in contexts such as that of neutron star matter) the work is carried out using a (special) relativistic formulation .

2 Perfect multiconstituent fluid dynamics

As well as including the original Landau model in the (not so simple) Galilean limit [10], two independent lines of early development of relativistic (and thus technically simpler) perfect multiconstituent fluid theory – using currents [11, 12] and momenta [13] respectively as independent variables in (suitably constrained) variational formulations – were subsequently shown to be entirely equivalent[14]: the independent momentum covectors $\mu^{\rm X}_\nu$ of the latter aproach are identifiable just as the dynamical conjugates of the independent conserved current vectors $n_{\rm X}^{\,\nu}$ on which the former approach was based. In the latter approach, which is the most convenient for our present purpose, the fundamental equation of state characterising a particular perfect multiconstituent fluid model is given by specifying the dependence of the relevant generalised pressure function Ψ on the independent momentum covectors $\mu^{\rm X}_\nu$, and the associated current vectors $n_{\rm X}^{\,\nu}$ are then obtained as the corresponding partial derivatives in the infinitesimal variation formula

$$\delta\Psi = -\sum_{\rm X} n_{\rm X}^{\,\nu}\delta\mu^{\rm X}_\nu\,. \tag{2}$$

For such a model, the complete set of equations of motion consists just of a set of particle and vorticity conservation laws of the form

$$\nabla_\nu n_{\rm X}^{\,\nu} = 0\,, \qquad n_{\rm X}^{\,\nu} w_{\nu\sigma} = 0\,. \tag{3}$$

where, for any particular constituent with label X the corresponding generalised vorticity 2-form is defined as the exterior derivative

$$w^{\rm X}_{\nu\sigma} = \nabla_\nu \pi^{\rm X}_\sigma - \nabla_\sigma \pi^{\rm X}_\nu\,, \qquad \pi^{\rm X}_\nu = \mu^{\rm X}_\nu + e^{\rm X} A_\nu \tag{4}$$

of the generalised momentum covector $\pi^{\rm X}_\nu$, whose specification[15] allows for the possibility of coupling to an electromagnetic field

$$F_{\nu\sigma} = \nabla_\nu A_\sigma - \nabla_\sigma A_\nu\,. \qquad \nabla_\sigma F^{\sigma\nu} = 4\pi J^\nu\,, \tag{5}$$

with current and electromagnetic stress tensor given by

$$J^\nu = \sum_{\rm X} e^{\rm X} n_{\rm X}^{\,\nu}\,, \qquad T_{\rm F}{}^{\sigma}{}_{\nu} = \frac{1}{4\pi}(F^{\mu\nu}F_{\mu\sigma} - \frac{1}{4}F^{\mu\rho}F_{\mu\rho}g^\sigma_\nu)\,, \tag{6}$$

where $e^{\rm X}$ is the electric charge, if any, per particle of the X th species.

In such a model a (flat space) conservation law of the usual form

$$\nabla_\nu T^\nu{}_\sigma = 0\,, \tag{7}$$

will be satisfied by the relevant total stress energy tensor, which takes the form

$$T^\sigma{}_\nu = \sum_{\rm X} n^\sigma_{\rm X}\mu^{\rm X}_\nu + \Psi g^\sigma_\nu + T_{\rm F}{}^\sigma{}_\nu = \sum_{\rm X} n^\sigma_{\rm X}\pi^{\rm X}_\nu + \Psi g^\sigma_\nu - J^\sigma A_\nu + T_{\rm F}{}^\sigma{}_\nu\,. \tag{8}$$

The transport law (3) for the vorticities is such that they will remain zero if they are zero initially. Thus we shall have

$$w^{\rm X}_{\nu\sigma} = 0\,, \tag{9}$$

not just for cases of superfluidity or superconductivity (i.e. cases for which the momentum covector is the gradient of a condensate phase scalar) but even for "normal" constituents in configurations of the kind to be considered here, in which a perturbing vortex moves through an *asymptotically uniform* medium characterised by vanishing of the asymptotic background value (indicated here by an overhead bar) not just of the current (as is necessary for uniformity) but also of the electromagnetic field, and (in an appropriate gauge) of its vector potential,

$$\overline{J}^\nu = 0 \qquad \overline{F}_{\nu\sigma} = 0\,, \qquad \overline{A}_\sigma = 0\,. \tag{10}$$

This must necessarily be the case (the Meissner effect), with the implication that the uniform background value of the stress energy density tensor will be given simply by

$$\overline{T}^\sigma{}_\nu = \sum_{\rm X} \overline{n}^\sigma_{\rm X}\overline{\pi}^{\rm X}_\nu + \overline{\Psi} g^\sigma_\nu\,, \tag{11}$$

whenever even just a single one of the uniform background constituents is superconducting (since $e^{\rm X} F_{\nu\sigma} = \nabla_\nu \pi^{\rm X}_\sigma - \nabla_\sigma \pi^{\rm X}_\nu - w^{\rm X}_{\nu\sigma}$).

3 Specification of lift force on vortex

The subject of this investigation is an asymptotically uniform vortex configuration that is stationary with respect to a rest frame characterised by a uniform timelike unit symmetry generating vector field k^μ say, and that is aligned in the direction of a uniform orthogonal spacelike unit symmetry generating vector field ℓ^μ in a flat background spacetime using Minkowski coordinates. Provided the corresponding conditions of stationarity and longitudinal symmetry apply to $F_{\nu\sigma}$ not just outside but even within the vortex core region, they will be applicable to the potential A_σ in a suitable gauge, and hence also, not just to the gauge independent covectors $\mu^{\rm X}_\nu$ but to the corresponding generalised momenta $\pi^{\rm X}_\nu$ as well. In view of the vanishing (9) of the vorticity vectors (4), the condition that the momentum covectors should be invariant with respect to the action of the the uniform symmetry generating vector fields k^ν and ℓ^ν can be seen to imply the uniformity of corresponding sets of generalised Bernouilli constants,

$$k^\nu \pi^{\rm X}_\nu = k^\nu \overline{\pi}^{\rm X}_\nu\,, \qquad \ell^\nu \pi^{\rm X}_\nu = \ell^\nu \overline{\pi}^{\rm X}_\nu\,. \tag{12}$$

The force per unit length, $\mathcal{F}_\nu$ acting on such a stationary longitudinally invariant vortex can be evaluated as the integral round a circuit s say surrounding the vortex in an orthogonal 2-plane in the form

$$\mathcal{F}_\nu = \oint f_\nu ds\,, \tag{13}$$

where ds is the proper distance element given by $ds^2 = g_{\nu\sigma} dx^\nu dx^\sigma$ and f_ν is the local force density that is given by

$$f_\nu = \nu_\sigma T^\sigma_{\ \nu}\,, \tag{14}$$

in terms of the unit normal covector ν_σ which will be given in terms of the antisymmetric background measure tensor $\varepsilon_{\lambda\mu\nu\sigma}$ by

$$\nu_\sigma ds = {}^\star\varepsilon_{\sigma\nu} dx^\nu\,, \qquad {}^\star\varepsilon_{\sigma\nu} = \ell^\rho \varepsilon_{\rho\sigma\nu}\,, \qquad \varepsilon_{\rho\sigma\nu} = k^\mu \varepsilon_{\mu\rho\sigma\nu}\,. \tag{15}$$

4 Generalised Joukowski theorem

It is to be observed that, as a consequence of the conservation law (7), it makes no difference what circuit is employed for evaluating $\mathcal{F}_\nu$. We are thus allowed to choose a circuit sufficiently far out for reliability of our smoothed fluid description (whose physical validity might be questionable near the core) to be ensured, and also for the deviation from the uniform background value $\overline{T}^\sigma_{\ \nu}$ to be evaluated as a linear perturbation:

$$T^\sigma_{\ \nu} - \overline{T}^\sigma_{\ \nu} = \delta T^\sigma_{\ \nu} + O\{\delta^2\}\,. \tag{16}$$

Since the force integral for the unperturbed uniform background must evidently vanish, $\overline{\mathcal{F}}_\nu = 0$ by symmetry, the corresponding value in the presence of the vortex will be given by

$$\mathcal{F}_\nu = \delta\mathcal{F}_\nu + O\{\delta^2\}\,. \tag{17}$$

Using (2) and (11) it can immediately be seen that the required first order variation will be given by

$$\delta T^\sigma_{\ \nu} = \sum_{\rm x} (\overline{\pi}^{\rm x}_\nu \delta n_{\rm x}^{\ \sigma} + \overline{n}_{\rm x}^{\ \sigma} \delta\pi^{\rm x}_\nu - g^\sigma_\nu \overline{n}_{\rm x}^{\ \rho} \delta\pi^{\rm x}_\rho)\,. \tag{18}$$

Using the decomposition of the 4-dimensional spacetime metric in the form $g^\nu_\sigma = \eta^\nu_\sigma + \perp^\nu_\sigma$ as the sum of the (rank 2) operators of projection respectively parallel to and orthogonal to the vortex given by $\eta^\nu_\sigma = -k^\nu k_\sigma + \ell^\nu \ell_\sigma$, and $\perp^\nu_\sigma = {}^\star\varepsilon^{\mu\nu}\,{}^\star\varepsilon_{\mu\sigma}$ and using the possibility of taking the Bernouilli constants (produced by parallel projection) outside the integration, (18) provides a result expressible simply as

$$\delta\mathcal{F}_\nu = \sum_{\rm x} (\overline{n}_{\rm x}^{\ \sigma}\,{}^\star\varepsilon_{\sigma\nu} \delta\mathcal{C}^{\rm x} + \overline{\pi}^{\rm x}_\nu \delta D_{\rm x})\,, \tag{19}$$

where for each species x the corresponding momentum circulation integral $\mathcal{C}^{\rm x}$ and current outflux integral $D_{\rm x}$ are defined by

$$\mathcal{C}^{\rm x} = \oint \pi^{\rm x}_\nu dx^\nu\,, \qquad D_{\rm x} = \oint n_{\rm x}^{\ \sigma} \nu_\sigma ds\,. \tag{20}$$

The irrotationality condition (9) ensures that $\mathcal{C}^{\rm X}$ is independent of the choice of circuit, and the current conservation law (3) ensures that the same will apply to $D_{\rm X}$, which furthermore will simply vanish, $D_{\rm X} = 0$, provided there is no current creation in the vortex core. Thus by the fact that the uniform background value of the circulation integrals must also vanish, $\overline{\mathcal{C}}^{\rm X} = 0$, and by taking the limit in which the circuit is taken to a very large distance outside, one obtains an exact net force formula of the simple form

$$\mathcal{F}_\nu = {}^\star\varepsilon_{\sigma\nu} \sum_{\rm X} \mathcal{C}^{\rm X} \overline{n}_{\rm X}^{\,\sigma} \,. \tag{21}$$

This result is the required (relativistic, multiconstituent) generalisation of Joukowsky's well known formula (1) for the single constituent case. What it means is that each constituent contributes an amount proportional to, but orthogonal to, its asymptotic current vector, with a coefficient given by the corresponding momentum circulation integral.

5 Application to the Landau model

The particular example that motivated this work is that of superfluid ^{4}He at finite temperature, as described by the Landau model in terms of just two constituents with conserved 3-dimensional current densities $n_{\alpha}^{\,i} = n_{\alpha} v_{\alpha}^{\,i}$ and $n_{\beta}^{\,i} = n_{\beta} v_{\beta}^{\,i}$ of which the first represents Helium atoms, i.e. "dressed" alpha particles, characterised by a "rest mass" m_{α}, and the second represents units of entropy, characterised by a vanishing rest mass $m_{\beta} = 0$. As in the less specialised case of the 2 constituent (zero temperature) limit of the Andreev Bashkin model for which the second constituent is ^{3}He with non vanishing rest mass, $m_{\beta} \simeq 3m_{\alpha}/4$, the Newtonian limit description can be formulated in terms of a total mass density and 3 dimensional mass current

$$\rho = \rho_{\alpha} + \rho_{\beta} \,, \qquad \rho^i = \rho_{\alpha} v_{\alpha}^{\,i} + \rho_{\beta} v_{\beta}^{\,i} \,, \tag{22}$$

where $\rho_{\alpha} = m_{\alpha} n_{\alpha}$ and $\rho_{\beta} = m_{\beta} n_{\beta}$, so that the latter vanishes in the particular case of the Landau model. The total mass current is identifiable with the total momentum density $\rho_i = n_{\alpha} \mu_i^{\alpha} + n_{\beta} \mu_i^{\beta}$, in which, due to the effect of "entrainment" (which is describable in terms of "effective masses" that are different from the bare masses) the vanishing of the second contribution to the mass current does not imply absence of the second momentum contribution proportional to μ_i^{β}.

The pseudo-velocity $v_{\rm S}^{\,i}$ that is commonly referred to as the "superfluid velocity" is defineable by $v_{{\rm S}i} = m_{\alpha}^{-1} \mu_i^{\alpha}$. In the Landau case (unlike the generic Andreev Bashkin case) it is not possible to define an analogous pseudo velocity for the other consituent, because of the vanishing of m_{β}, and the quantity commonly denoted as $v_{\rm N}^{\,i}$ and known as the "normal" velocity is simply identifiable with the velocity of the entropy current, i.e. $v_{\rm N}^{\,i} = v_{\beta}^{\,i}$. In a mass and momentum decomposition of the commonly used (effectively "mongrel") form

$$\rho = \rho_{\rm S} + \rho_{\rm N} \,, \qquad \rho^i = \rho_{\rm S} v_{\rm S}^{\,i} + \rho_{\rm N} v_{\rm N}^{\,i} \,, \tag{23}$$

the coefficients $\rho_{\rm S}$ and $\rho_{\rm N}$ must not be confused with ρ_{α} and ρ_{β} (of which the latter is zero in the Landau case characterised by by $\rho = \rho_{\alpha}$). In the Landau case, as well as in the generic Andreev Bashkin case, there are two independently conserved momentum circulation integrals,

$$\mathcal{C}^\alpha = \oint \mu^\alpha_i \, dx^i \,, \qquad \mathcal{C}^\beta = \oint \mu^\beta_i \, dx^i \,, \tag{24}$$

of which, by the superfluidity property, the former is quantised, $\mathcal{C}^\alpha = h$. Since $m_\alpha \neq 0$ we can write

$$\mathcal{C}^\alpha = m_\alpha \kappa_{\rm S} \,, \qquad \mathcal{C}^\beta = m_\beta \kappa_{\rm S} + \oint \frac{\rho_{\rm N}}{n_\beta}(v_{{\rm N}i} - v_{{\rm S}i}) dx^i \,, \qquad \kappa_{\rm S} = \oint v_{{\rm S}i} \, dx^i \,, \tag{25}$$

in terms of the pseudo-velocity circulation integral $\kappa_{\rm S}$, which has no "normal" analogue in the Landau case, because of the vanishing of m^β. Thus (in the rest frame of the vortex) using the notation ${}^\star\varepsilon_{ij} = \varepsilon_{ijk}\ell^k$, one finally obtains a non-relativistic force formula of the form

$$\mathcal{F}_i = {}^\star\varepsilon_{ij}(\kappa_{\rm S}\overline{\rho}_\alpha \overline{v}^j_\alpha + \mathcal{C}^\beta \overline{n}_\beta \overline{v}^j_\beta) = \kappa_{\rm S}\overline{\rho}_{\rm S} \, {}^\star\varepsilon_{ij}\overline{v}^j_{\rm S} + \mathcal{F}_{{\rm I}i} \,, \tag{26}$$

where in this last version the first term is what is commonly referred to as the "superfluid Magnus force" contribution, while the remaining "Iordanskii" correction term is found to be given by

$$\mathcal{F}_{{\rm I}i} = (\mathcal{C}^\beta \overline{n}_\beta + \kappa_{\rm S}(\overline{\rho}_{\rm N} - \overline{\rho}_\beta)) \, {}^\star\varepsilon_{ij}\overline{v}^j_{\rm N} \,. \tag{27}$$

The third term in this expression is needed for the generic Andreev Bashkin case, but drops out for the special Landau case characterised by $\rho_\beta = 0$.

If, as well as setting ρ_β to zero, one adopts the plausible supposition that the "normal" circulation will vanish, $\mathcal{C}^\beta = 0$, then the first term also drops out so that our formula will reduce to a form that is in exact agreement with the result that was derived by Sonin[4, 5] and confirmed, on the basis of a more rigorous microscopic analysis of phonon dynamics, by Stone[3].

This widely accepted conclusion has however been vigorously contested by Thouless and coworkers [7, 8] who have used a more sophisticated – though not obviously more reliable – kind of microscopic analysis to argue that the Iordanskii force term $\mathcal{F}_{{\rm I}i}$ vanishes, leaving just the purely "superfluid" term (namely ${}^\star\varepsilon_{ij} \, \kappa_{\rm S}\overline{\rho}_{\rm S} \, \overline{v}^j_{\rm S}$) in (26). As *prima facie* evidence in favour of this dissident conclusion, it is to be observed that in the limit when there is no relative flow at all (i.e. $\overline{v}^i_{\rm S} = \overline{v}^i_{\rm N} = 0$) then – as a requirement for compatibility with strict stationarity – the long term effect of the small "normal" viscosity contribution that was neglected in the preceeding analysis will impose a rigidity condition to the effect that $v^i_{\rm N} = 0$ throughout. This imperative entails small deviations from strict irrotationality of the normal constituent except in the incompressible case for which the ratio $\rho_{\rm N}/n_\beta$ is exactly uniform, and it ensures in any case by (25) that the normal momentum circulation round a circuit at large distance will be given by the formula $\mathcal{C}^\beta = (m_\beta - \overline{\rho}_{\rm N}/\overline{n}_\beta)\kappa_{\rm S}$ whose substitution in (27) does indeed give, $\mathcal{F}_{{\rm I}i} = 0$. However this simple counter argument is inconclusive because – as shown every time an ordinary light aircraft takes off – the stationary circulation value due to the long term effect of slight deviations from strictly inviscid behaviour will change as a function of the relative flow velocity.

To sum up, the present work shows how the Iordanskii force is given simply as a function of the (in the inviscid limit conserved) “normal” momentum circulation integral $\mathcal{C}^\beta$, but the issue of the appropriate value for this parameter in a realistic steady flow confguration is beyond the scope of a perfectly conducting fluid treatment such as is provided here. Experience with the analogous aerofoil problem in the context of aircraft engineering suggests that the final resolution of this issue may involve subtleties that have have eluded even the most sophisticated analysis available so far.

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References

[1] S.V. Iordanskii, *Sov. Phys. J.E.T.P.* **22**, 160 (1966).

[2] N.B. Kopnin, G.E. Volovik, U. Parts, *Europhys. Lett.* **32**, 651-656 (1995)

[3] M. Stone, “Iordanskii force and gravitational Aharonov-Bohm effect on a moving vortex”. [cond-mat/9909313]

[4] E.B. Sonin, *Sov. Phys. J.E.T.P.* **42**, 469 (1976).

[5] E.B. Sonin, *Phys. Rev.* **B55**, 485 (1997). [cond-mat/9606099]

[6] C.F. Barenghi, R.J. Donelly, W.F. Vinen, *J. Low. Temp. Phys.* **52**, 189 (1983).

[7] D.J. Thouless, Ping Ao, Quian Niu *Phys. Rev. Letters* **76**, 3758-3760 (1996).

[8] C. Wexler, D.J. Thouless, *Phys. Rev. B58*, 8897 (1998). [cond-mat/9804118]

[9] A.F. Andreev, E.P. Bashkin, *Sov. Phys. J.E.T.P.,* **42**, 164-646 (1975).

[10] B. Carter, I.M. Khalatnikov, *Rev. Math. Phys.* **6**, 277-305 (1994).

[11] B. Carter, in Journées Relativistes 1979, ed. I. Moret-Bailly & C. Latremolière, 166-182 (Faculté des Sciences, Angers, 1979).

[12] B. Carter, in *A random walk in Relativity and Cosmology* ed. N. Dadhich, J. Krishna Rao, J.V. Narlikar, C.V. Visveshwara, 48-62 (Wiley Eastern, Bombay, 1985).

[13] V.V. Lebedev, I.M. Khalatnikov, *Sov. Phys. J.E.T.P.* **56**, 923-930 (1982).

[14] B. Carter, I.M. Khalatnikov, *Phys. Rev.* **D45**, 4536-4544 (1992).

[15] B. Carter, D. Langlois, “Relativistic models for superconducting superfluid mixtures”, *Nuclear Phys.* **B531**, 478-504 (1998). [gr-qc/9806024].