

Coherent States for Canonical Quantum General Relativity and the Infinite Tensor Product Extension

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Abstract

We summarize a recently proposed concrete programme for investigating the (semi)classical limit of canonical, Lorentzian, continuum quantum general relativity in four spacetime dimensions. The analysis is based on a novel set of coherent states labelled by graphs. These fit neatly together with an Infinite Tensor Product (ITP) extension of the currently used Hilbert space. The ITP construction enables us to give rigorous meaning to the infinite volume (thermodynamic) limit of the theory which has been out of reach so far.

1 Introduction

Canonical Quantum General Relativity has matured over the last decade into a serious candidate theory of quantum gravity which is manifestly background independent. The most important developments include a rigorously defined mathematical framework, non-perturbative operator regularization methods, the prediction of a discrete or distributional (rather than smooth) quantum geometry at Planck scale, a microscopic explanation for the Bekenstein-Hawking black hole entropy and a fully diffeomorphism covariant state sum formulation. See [1] for pedagogical reviews, more or less covering these developments.

Recently, some effort [2] has also been devoted to the extraction of semi-classical physics from the non-perturbative quantum theory. By this one means that if the distances we probe are large compared to the Planck scale $\ell_p = \sqrt{\hbar\kappa}$ (κ is Newton's constant up to a numerical factor in units where $c = 1$) then the fluctuations of geometry can be neglected and we are close to a situation that one can treat with the methods of quantum field theory on curved spacetimes, which in turn also is valid only as long as the backreaction of geometry to quantum matter can be discarded. In particular, if the gravitational field is in a semi-classical state approximating the Minkowski metric then we should arrive at the quantum field theory describing the standard model plus corrections.

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What is a semi-classical gravitational state ? It is not entirely straight forward to answer this question as the manifest background independence of the quantization scheme forces us to adopt non-standard representations of the canonical commutation relations, in particular, the powerful Fock Hilbert space techniques are not available which would immediately allow us to construct the standard coherent states used in the literature. In what follows, the contents of a programme based on a new set of gravitational coherent states [3] is summarized which provides a proposal for a systematic construction of the classical limit.

The organization of the article is as follows :

In section two we describe in detail and in a non-technical way the main ideas of our proposal which is based on a new class of kinematical coherent states, that is, minimal uncertainty states for *both* connection and electric field operators labelled by a points in the classical, not necessarily constrained or reduced, phase space. Here we use the gauge theory canonically conjugate coordinates for this phase space described in [1]. As such, they are in sharp contrast to the weave states already constructed in the literature which are electric field operator eigenstates and thus completely outspread in the connection operator. For every classical observable we construct an approximate (effective) quantum operator which has the correct expectation values and fluctuations with respect to our coherent states. The term “effective” refers to the fact that these operators are, in general, different from those already constructed in the literature. This poses no problem because to say that certain operator algebras correspond to certain classical Poisson algebras is an empty statement without specifying with respect to which semi-classical states the correspondence is made. Our coherent states are simply not always adapted to the operators already constructed but we will point out how they can be modified in that respect.

We will see that three different scales emerge in the semi-classical analysis, the microscopic Planck scale, a mesoscopic scale defined by the edge length of an embedded graph (as measured by the three metric defined by a point in the classical phase space) and a macroscopic scale associated with the curvature of the four metric of that phase space point. When we ask that fluctuations be minimized we learn that the mesoscopic scale is a geometric mean of the other two scales. This result can be interpreted in the following way : Since the Planck length is proportional to a positive power of \hbar and if we understand the classical limit roughly as the limit $\hbar \rightarrow 0$ while the continuum limit corresponds to $\epsilon \rightarrow 0$, then we may say that *the continuum and classical limit merge into only one limit* ! More precisely, since \hbar is a small but finite number, also ϵ is a small but finite number and the actual continuum limit $\epsilon \rightarrow 0$ is unphysical, however, in an expansion in powers of ϵ the zeroth order term gives the correct classical continuum expression.

An important step needed is the combination of model coherent states labelled by points in a model phase space (the cotangent bundle over a compact gauge group) with the continuum phase space that we are actually interested in. We discuss advantages and shortcomings of our proposal at the end of section 2 and we supplement it by appendices A, B where we exhibit the derivation of an approximate version of the area operator and the appearance of optimal scales in fluctuation calculations.

In section three we sketch the construction of model coherent states in a hopefully pedagogical way, bearing, as much as possible on an analogy with the harmonic oscillator coherent states of which the present ones are a non-Abelian version. In particular, we try

to provide some intuition based on geometric quantization and heat kernel methods.

In section four we introduce a technical tool, the Infinite Tensor Product construction which is well known in statistical physics, in order to tackle the infinite volume limit of the theory which has been out of reach so far. We outline just the most basic mathematical definitions and discuss how these mathematical termini translate into physical concepts. This section is supplemented by appendix C which employs the mathematical notions used in a simple system, the infinite spin chain.

In the final section we summarize and list future applications of the concepts introduced, one of the most important being how to make contact with quantum field theory on curved spacetimes.

2 A Concrete Programme for the Construction of (Semi)classical Canonical Quantum General Relativity

To say that a state is semi-classical makes sense only if one specifies a set of elementary observables g_e (e belongs to some label set) which are to behave almost classically, not every possible operator on the Hilbert space can have a good classical limit in the sense of approximating a given function on the classical phase space (see the third reference in [2] for an illustration in the context of canonical quantum general relativity). The specification of a such a subset of the operator algebra is sometimes referred to as *coarse-graining*. One starts with a classical phase space \mathcal{M} which in general depends also on the three – manifold Σ underlying the canonical formulation, for example through boundary or fall-off conditions or through the differentiable structure (we refrain from displaying this dependence of \mathcal{M} on Σ in what follows). A state ψ_m labelled by a point m in this classical phase space (say a canonical pair consisting of a connection A_a^j and an electric field E_j^a) is said to be semi-classical for a set of elementary classical observables g_e if the expectation values of the corresponding self-adjoint operators assume their classical values at the point m and if their relative fluctuations are small, that is, $\langle \psi_m, \hat{g}_e \psi_m \rangle = g_e(m)$ and

$$(\Delta g_e)_{quant}(m) := \sqrt{\langle \psi_m, [\hat{g}_e]^2 \psi_m \rangle / [g_e(m)]^2 - 1} \ll 1 \quad (2.1)$$

Clearly, as remarked by Ashtekar, if $g_e(m) = 0$ then (2.1) blows up but this is just because the value zero is smaller than the fundamental absolute fluctuation of the operator which is bounded away from zero. Nobody would say that the harmonic oscillator coherent states break down at the origin of the phase space. If we replace g_e by the quantity $g_e + c$ (which classically captures the same information as g_e) for some constant c much larger than the absolute fluctuation then (2.1) is finite and small. Alternatively, we can work with dimensionless quantities g_e and consider only absolute fluctuations. In what follows we assume that one has taken care of these subtleties in either way and that (2.1) is well defined. Notice that the four – metric determined by the initial data set m introduces a first length scale into the analysis, namely its typical curvature radius $L(m)$ as measured by the three metric determined by m . As we will see in appendix B there are actually two such radii (an electric and magnetic one) but we will symbolize them by a single label $L(m)$ for the descriptive purposes of this section. If needed, we may want to restrict the phase space by allowing only those points m for which $L(m) \geq L_0$ is bounded from

below which could induce a restriction on the (Gauss, diffeomorphism and Hamiltonian) gauge choice involved in specifying m .

The states of the gravitational Hilbert space that we will consider are labelled by piecewise analytic graphs γ with an at most countably infinite number of edges. These graphs are embedded into the three dimensional spatial manifold Σ of given topology which underlies the canonical formulation classical general relativity. For a given state it is only at the locus of the corresponding embedded graph where the excitations of the gravitational field are probed. This raises immediately the question : What are these degrees of freedom, called graph degrees of freedom in what follows, that are probed ? As far as the configuration degrees of freedom are concerned, the answer is easy : they are the holonomies of the gravitational connection along the edges of the graph. However, for the momentum degrees of freedom this is less obvious. There is some freedom in choosing them and we will specify our choice below. Taken together, the observables associated with these degrees of freedom for given graph γ are good candidates for a set of elementary observables g_e with respect to which a state $\psi_{\gamma,m}$, now also labelled by the graph, is semi-classical and e will run through the set of edges of the graph.

Now the following problem appears : The above comments suggest to define semi-classical states over one single arbitrary but fixed graph γ . However, an arbitrary classical observable O generically cannot be written as a function of the elementary graph degrees of freedom g_e for the given graph γ and therefore it is not guaranteed that the corresponding continuum quantum operator will display semi-classical behaviour in the state $\psi_{\gamma,m}$. A way out is suggested by the observation that any O can be approximated by a function $O_\gamma = O_\gamma(\{g_e\})$ of the graph degrees of freedom in the sense that

$$(\Delta O)_{class}(m) := \left| \frac{O_\gamma(m)}{O(m)} - 1 \right| \ll 1 \quad (2.2)$$

for all $m \in \mathcal{M}$. This can be seen as follows : Notice that given γ, m automatically a second length scale ϵ is introduced, namely the typical length (measured by the three metric determined by m) of an edge of γ . Since L is the scale on which the four metric varies, the classical approximation will be good as long as $\epsilon \ll L$. If $L \geq L_0$ we just need to choose ϵ small enough to obtain a small value of (2.2) even uniformly in m .

This observation motivates to use the corresponding operators \hat{O}_γ as substitutes for the continuum operator \hat{O} . There are two worrisome issues about this proposal :

I.)

The operators \hat{O}_γ will, in general, not coincide with the cylindrical projections of the continuum operators already constructed in the literature, say the area operators [4]. Thus, even if one manages to produce diffeomorphism covariant, cylindrically consistently defined families of operators \hat{O}_γ corresponding to O one will get a different continuum operator this way.

II.)

The substitute function O_γ necessarily must make explicit use of a coordinate system. Thus, unless one is very careful in constructing it, it is conceivable that one breaks diffeomorphism covariance and background independence. This would be unsatisfactory because one of the strongest motivations and guidelines for this whole approach to quantum gravity is to keep background independence at every stage of the construction. To see how this might happen, consider the so-called ‘‘staircase problem’’ which we discovered in discussions with Ashtekar, Lewandowski and Pullin and which we discuss more explicitly

in appendix A : Suppose we would like to measure the area $O = A_S$ of a surface S in the state $\psi_{\gamma,m}$ where γ is topologically a cubic lattice. If the surface is built from the elementary surfaces S^e that come with the graph degrees of freedom as specified below then the cylindrical projection of the operator constructed in [4], which is manifestly diffeomorphism covariant, actually coincides with \hat{O}_γ and its expectation values agree with the classical values. However, if it lies transversally to them then its expectation value will be drastically off the classical value $A_S(m)$. The replacement $A_{S,\gamma}$ will have to take this error into account by using a coordinate system which enables us to tell how the elementary surfaces are embedded relative to S . (This coordinate system, however, does not depend on the point m). Now, while g_e transforms covariantly under diffeomorphisms, there could be an extra coordinate dependence which is guaranteed to disappear only in the limit $\epsilon \rightarrow 0$ and O_γ might not transform covariantly any longer. Thus, the operator \hat{O}_γ could be not only state dependent (it depends explicitly on γ but not on m) but could also be coordinate dependent. The former dependence can be removed if one can show that the \hat{O}_γ are cylindrical projections of one and the same operator \hat{O} which can always be achieved by introducing suitable projections, see below. However, it is possible that the extra coordinate dependence cannot be removed, at least not obviously, at any finite ϵ .

Of course, one could ignore this problem altogether and measure only operators that actually can be written in terms of the \hat{g}_e . In fact, as pointed out by Lewandowski, if one agrees that all physical information can be extracted, up to a certain accuracy set by a length scale, from electric and magnetic fluxes through surfaces of the size of that scale or bigger then, up to a negligible error, one can actually write these quantities purely in terms of the g_e due to the Gauss law and the Bianchi identity, see appendix B, in a manifestly coordinate independent way. While this is a working proposal, we want to be more ambitious and measure any operator quantum mechanically and not by first measuring elementary operators quantum mechanically and then assembling the values of these measurements by a classical formula into a value for the observable that we are interested in. Thus we have to deal with both problems I.), II.) in what follows.

We will show below that one can actually solve problem II.). Then one would still have problem I.), that is, for every classical observable O one has two candidate quantum observables \hat{O}', \hat{O} , the first one of the kind already constructed in the literature using background independent regularization techniques and the second one being constructed by the method outlined below. One may object that given a classical function O there must be a unique operator \hat{O} such that its classical limit is given by O and not several of them. However, in order to tell that a quantum theory given by a Hilbert space and a quantum commutator algebra has a given phase space and Poisson algebra as its classical limit one needs one more input : a selection of semi-classical states with respect to which the semi-classical limit is to be performed. It is perfectly possible to have two systems of operators and two systems of semi-classical states such that the classical limit of the first set of operators with respect to the first system of states agrees with the classical limit of the second set of operators with respect to the second system of states. There is no claim that this will continue to hold after exchanging the two systems of states. In fact, our current situation in quantum general relativity is the following : What we have are candidate operators \hat{O}' already constructed in the literature and we have candidate semi-classical states ψ constructed in [3]. The staircase problem alluded to above reveals that these states are not always semi-classical for the operators \hat{O}' . Thus there exist two

avenues to arrive at a satisfactory semi-classical analysis :

Either A) one keeps the operators \hat{O}' and modifies the states ψ to arrive at better behaved semi-classical states ψ' or B) one keeps the states ψ and modifies the operators \hat{O}' to arrive at better behaved operators \hat{O} . These are complimentary programmes aiming at the same goal whose methods can be fruitfully combined so that one meets somewhere in the middle. The advantage of avenue A) is that the \hat{O}' are explicitly known while the ψ' are not, for avenue B) the ψ are explicitly known while the \hat{O} have to be constructed graph by graph.

In this paper we will mainly focus on avenue B) but we will devote this short paragraph to describe avenue A) in some detail in order to reveal that both avenues can actually be combined.

Currently, avenue A) is being studied in the context of the *statistical geometry approach* [5]. The idea is to construct semiclassical states without making reference to a particular, arbitrary but fixed, graph γ or any additional extra structure at all. The state should be constructed just from one datum, the phase space point m , in such a way that a sufficiently coarse-grained and complete subset of the set of the operators \hat{O}' already constructed in the literature behaves semi-classically. Since these operators are manifestly diffeomorphism covariantly defined, problem II.) never arises. In order to remove effects similar to the staircase problem which are associated with a direction dependence on a given graph, one obviously has to sum or average over a huge number of graphs. One could consider either pure or mixed states but at this stage it is more natural to consider mixed states because then we do not need to choose the phases of the probability amplitudes involved but only their absolute values. Choosing this option, instead of looking at one pure coherent state $\psi_{\gamma,m}$ one constructs a mixed state by averaging the one-dimensional projections $\hat{P}_{\gamma,m}$ onto $\psi_{\gamma,m}$ over a subset Γ_m of the set of all allowed graphs (say piecewise analytic, compactly supported) which may depend on $m \in \mathcal{M}$ with a probability measure $d\mu_m(\gamma)$ on Γ_m . Here, one can be very general to begin with so that $\psi_{\gamma,m}$ is not necessarily one of our coherent states, see [5]. Thus, one tries to construct a density matrix (a trace class operator of unit trace)

$$\hat{\rho}_m := \int_{\Gamma_m} d\mu_m(\gamma) \hat{P}_{\gamma,m} \quad (2.3)$$

and now expectation values are given by $\langle \hat{O}' \rangle_m := \text{Tr}(\hat{\rho}_m \hat{O}')$ where \hat{O}' is an operator on the currently used Hilbert space and not necessarily a function of the \hat{g}_e . Notice that, if as in [5] the measure μ_m is absolutely continuous with respect to a Lebesgue measure, then as an operator on the continuum Hilbert space, $\hat{\rho}_m$ is (almost) the zero operator (and has actually zero trace) due to the diffeomorphism invariance of the inner product, however, the idea is to construct a new representation \mathcal{H}_m of the operator algebra via the GNS construction from this state. This should be compared with the strong equivalence class Hilbert spaces based on pure coherent states as cyclic vectors that we will discuss in section 4.

Now, if the set Γ_m is large enough then due to the averaging performed in (2.3) the above pointed out direction dependence (staircase problem) disappears. In fact, the explicit choice made in [5] for flat initial data m and compact Σ is such that $\hat{\rho}_m$ is *Euclidean invariant* (using a weave state [7] or one of our coherent states for $\psi_{\gamma,m}$). The interested reader is referred to this beautiful statistical geometry construction (based on the

Dirichlet-Voronoi method [6] for Euclidean spatial metrics) for more details. In general, there is certainly a lot of freedom in the choice of Γ_m, μ_m and it has to be understood how $\langle \hat{O}' \rangle_m$ depends on these choices.

However, whatever choice is made, our coherent state construction fits neatly in with the idea to construct density matrices as is obvious from the general formula (2.3). For instance, using the peakedness properties derived in the third reference of [3] it is easy to see that for flat initial data the coherent state is sharply peaked in the momentum representation *exactly at the value of the spin j that has been previously derived in the weave construction!* In other words, if we expand our coherent states in terms of spin-network states, then practically all the coefficients except for the one with spin label $j_e \ell_p^2 \approx \text{Ar}(S_e)$ are vanishing, where e is an edge of γ , S_e is a face of a polyhedral decomposition of Σ dual to γ (see below) which is intersected only by e and $\text{Ar}(S_e)$ its area measured by the flat three-metric. This dual polyhedral decomposition, an extra structure which is necessary to introduce in our coherent state construction below and which comes with some freedom to choose from, *is actually naturally fixed by the Dirichlet Voronoi construction!* This is a very surprising coincidence because the need for the polyhedral decomposition in the Dirichlet-Voronoi construction on the one hand and in the coherent state construction on the other hand are logically completely unrelated. Roughly then, if a macroscopic surface S is assembled from the S_e , the area expectation value can be practically taken with respect to a weave spin-network state at the above specified values of j_e and gives the correct result already reported earlier in the literature. If S is not assembled from the S_e it still might be possible, although no proof exists so far, that by using some of the freedom just mentioned in our coherent state construction (specifically in the the map $\Phi_{\gamma, \Sigma, X}$ derived below) we can simply use our coherent states in this averaging method in order to arrive at a satisfactory semi-classical state. These issues are subject to future research in collaboration with the authors of [5].

Let us now come to avenue B) for which at this point one will work with a single, sufficiently large, graph only rather than with an average. Let us stress that restricting to a single graph is only due to mathematical convenience and by no means crucial. In the future we might want to relax the condition to work with a single graph only and to use some kind of average over a *countably infinite* number of graphs in the fashion of (2.3). The measure μ_m would in this case become a discrete (counting) measure which would not result in a precisely Euclidean invariant density operator peaked on the Minkowski metric but this is maybe not bad because fundamentally there should not exist a precisely Euclidean invariant state in nature anyway if anything about the idea of a quantum spacetime foam at Planck scale is correct. On the other hand, using a countable number of graphs in the average would have the advantage that we obtain a non-vanishing trace class operator of unit trace on the continuum Hilbert space and do not need to pass to a new GNS Hilbert space.

The easiest single graphs to use that come to one's mind are regular lattices (say of cubic topology). This has the obvious disadvantage that such graphs have preferred directions and do not look isotropic at any scale larger than ϵ . One idea to remove this direction dependence motivated by [5] and suggested to us by Bombelli is to use for our γ not a graph that is embedded as and/or is topologically a regular lattice but rather a (generic) *random graph* of the Dirichlet-Voronoi type so that there are no preferred directions on scales larger than or equal to an *isotropy scale I* which in turn is much larger

larger than ϵ . As a result, in measuring macroscopic observables (those that depend on a large number of elementary ones) one does not detect the particulars of the graph any more. The isotropy scale I gets absorbed in the particulars of the classical error (2.2) as we will see in appendix B and as a net effect influences the size of the curvature scale L . Having said this, we will not explicitly display the dependence of L on I any more in this paper.

Our first task is to show that one can indeed solve problem II.), that is, the \hat{O}_γ have to be constructed in diffeomorphism covariant fashion. To do this, recall that in the canonical approach to quantum gravity one assumes that the four-dimensional manifold M has topology $\mathbb{R} \times \Sigma$ where Σ is an analytic three-dimensional manifold of given topology. The graphs are unions of edges which themselves are one-dimensional analytic submanifolds of Σ intersecting at most in their endpoints. Likewise, classical functions O_S on \mathcal{M} typically depend on analytic submanifolds S of Σ like curves, surfaces and regions. Also, as we will see below, in order to define the graph degrees of freedom we must actually also specify a polyhedral decomposition P_γ dual to γ and a set of paths Π_γ inside the corresponding faces. One can describe these structures mathematically by introducing a coordinate system X (to which we will refer as an *embedding* in what follows) and *model* graphs $\check{\gamma}$, faces $P_{\check{\gamma}}$, sets of paths $\Pi_{\check{\gamma}}$ and surfaces \check{S} in the model space $\check{\Sigma}$ of Σ (that is, \mathbb{R}^3) such that $\gamma = X(\check{\gamma})$, $P_\gamma = X(P_{\check{\gamma}})$, $\Pi_\gamma = X(\Pi_{\check{\gamma}})$, $S = X(\check{S})$. In what follows, model objects like these will carry a check sign as compared to their embedded counterparts. Given Σ , fix once and for all a coordinate system X_0 . Given a pair γ, S , choose once and for all a representant (γ_0, S_0) in their orbit $[(\gamma, \Sigma)] := \{(\varphi(\gamma), \varphi(S)); \varphi \in Diff(\Sigma)\}$ under diffeomorphisms and a diffeomorphism $\varphi_{S,\gamma}$ such that $\varphi_{S,\gamma}(\gamma_0) = \gamma$, $\varphi_{S,\gamma}(S_0) = S$. Furthermore, we require that γ_0 is the same graph for pairs (γ, S) , (γ', S') such that γ, γ' are diffeomorphic. Notice that $\gamma_0, S_0, \varphi_{S,\gamma}$ are far from unique and in order to choose them we make use of the axiom of choice.

Define the model graph and surface $\check{\gamma}, \check{S}$ respectively by $\gamma_0 = X_0(\check{\gamma})$, $S_0 = X_0(\check{S})$ and choose once and for all model faces and paths $P_{\check{\gamma}}, \Pi_{\check{\gamma}}$ for $\check{\gamma}$. Pick a prescription to construct a model function $O_{\check{S}, \check{\gamma}, P_{\check{\gamma}}, \Pi_{\check{\gamma}}}$, which depends only on the model graph degrees of freedom $g_{\check{e}}$. The model graph degrees of freedom are labelled explicitly by a model edge \check{e} (but no dual face and corresponding path system $S^{\check{e}}, \Pi_{\check{e}}$ respectively since we fixed them once and for all) and take values in a model phase space $\mathcal{M}_{\check{\gamma}}$ as we will see below. By definition, the numerical coefficients that the model function involves *do not explicitly depend either on X_0 or on m !* Rather they depend only on $\check{\gamma}, P_{\check{\gamma}}, \Pi_{\check{\gamma}}, \check{S}$ and possibly additional extra structures in $\check{\Sigma}$. Now, given X_0, Σ there is a natural map $\Phi_{\check{\gamma}, \Sigma, X_0}$ from \mathcal{M} into $\mathcal{M}_{\check{\gamma}}$ defined by $g_{\check{e}} := g_{e_0}(m) =: \Phi_{\check{\gamma}, \Sigma, X_0}(m)$ where g_{e_0} depends explicitly on the embedded structures $e_0 = X_0(\check{e})$, $S^{e_0} = X_0(S^{\check{e}})$, $\Pi_{e_0} = X_0(\Pi_{\check{e}})$ respectively. The point is now that if O_S was diffeomorphism covariantly and background independently defined then there is a natural way to construct this model function in such a way that $O_{\check{S}, \check{\gamma}, P_{\check{\gamma}}, \Pi_{\check{\gamma}}} \circ \Phi_{\check{\gamma}, \Sigma, X_0}$ is close to O_S pointwise in \mathcal{M} as we will explicitly demonstrate in appendices A, B.

We can then finish the construction of $O_{S,\gamma}$ through the definition

$$O_{S,\gamma}(m) := O_{\check{S}, \check{\gamma}, P_{\check{\gamma}}, \Pi_{\check{\gamma}}}(\{\varphi_{S,\gamma} \cdot g_{e_0}(m)\}_{e_0 \in E(\gamma_0)}) \quad (2.4)$$

where $(\varphi \cdot g_e)(m) := g_e(\varphi^{-1} \cdot m)$ is the natural action of $Diff(\Sigma)$ on the g_e induced by that on \mathcal{M} and $E(\gamma)$ denotes the set of edges of γ . Now it is easy to see that due to the diffeomorphism covariance of the functions g_e we actually have $\varphi_{S,\gamma} \cdot \Phi_{\check{\gamma}, \Sigma, X_0} = \Phi_{\check{\gamma}, \Sigma, X_{\varphi_{S,\gamma}}}$

where $X_\varphi := \varphi \circ X$. We can therefore write (2.4) more compactly as

$$O_{S,\gamma} := O_{\check{S},\check{\gamma},P_{\check{\gamma}},\Pi_{\check{\gamma}}} \circ \Phi_{\check{\gamma},\Sigma,X_{\varphi S,\gamma}} \quad (2.5)$$

which shows we have automatically a natural action of the diffeomorphism group on $O_{S,\gamma}$ given by

$$(\varphi \cdot O_{S,\gamma})(m) = O_{S,\gamma}(\varphi^{-1}(m)) \quad (2.6)$$

as expected from a function on the classical phase space. However, in general $\varphi \cdot O_{S,\gamma} \neq O_{\varphi(S),\varphi(\gamma)}$ in contrast to $\varphi \cdot O_S = O_{\varphi(S)}$ for the exact classical function. Since $\lim_{\epsilon \rightarrow 0} O_{S,\gamma}(m) = O_S(m)$ pointwise on \mathcal{M} by construction, this behaviour under diffeomorphisms is expected only in the continuum limit although it will be close to it for sufficiently small ϵ . This violation of a continuum property of the classical function is the price to pay for making the function graph dependent. There are two complementary points of view, reached in a discussion with Ashtekar, Lewandowski and Pullin : Either one regards the approximate operators as effective ones, good only for a semi-classical regime. Or one takes the opposite point of view, that since we want to quantize general relativity non-perturbatively and background-independently, we are forced to adopt a non-standard Hilbert space whose elements are labelled by graphs. Therefore, the $O_{S,\gamma}$ are the *more fundamental building blocks from which everything else must be derived* and as long as a continuum property is regained in the limit of infinitely fine graphs, this is acceptable. This issue will be explored in more depth in future publications.

As we will show below, all that is important about the map $\Phi_{\check{\gamma},\Sigma,X_{\varphi S,\gamma}}$ for quantization purposes is that $X_{\varphi S,\gamma}(\check{\gamma}) = \gamma$ so that the correct quantization of the functions (2.5) is given by

$$\hat{O}_{S,\gamma} := O_{\check{S},\check{\gamma},P_{\check{\gamma}},\Pi_{\check{\gamma}}}(\{\hat{g}_e\}_{e \in E(\gamma)}) \quad (2.7)$$

Namely, we will show below that this operator, interpreted as an operator on the closed subspace \mathcal{H}^γ of the full continuum Hilbert space \mathcal{H} spanned by spin-network functions over γ [8] with dependence on every edge by non-trivial irreducible representations of $SU(2)$, can be written in terms of holonomy operators \hat{h}_e and momentum operators \hat{P}_j^e acting by multiplication and differentiation with respect to h_e respectively where $e \in E(\gamma)$. Then, interpreting \hat{g}_e as the pair (\hat{h}_e, \hat{P}_j^e) there is a natural quantum action of the diffeomorphism group on these operators defined by

$$\hat{U}(\varphi)\hat{g}_e\hat{U}(\varphi)^{-1} = \hat{g}_{\varphi(e)} \text{ and } \hat{U}(\varphi)\hat{O}_{S,\gamma}\hat{U}(\varphi)^{-1} = O_{\check{S},\check{\gamma},P_{\check{\gamma}},\Pi_{\check{\gamma}}}(\hat{g}_{\varphi(e)}) \quad (2.8)$$

so that it is completely diffeomorphism covariantly defined. Moreover, it leaves the subspace $\mathcal{H}_\gamma := \bigoplus_{\gamma' \subset \gamma} \mathcal{H}^{\gamma'}$ invariant (the sum running over subgraphs of γ). The corresponding family of operators $\hat{O}_S := \{\hat{O}_{S,\gamma}\}_{\gamma \in \Gamma}$ may not yet be consistently defined in the sense that $(\hat{O}_{S,\gamma})|_{\mathcal{H}_{\gamma'}} = (\hat{O}_{S,\gamma'})|_{\mathcal{H}_{\gamma'}}$ for any $\gamma' \subset \gamma$. If that is not already the case we replace $\hat{O}_{S,\gamma}$ by $\sum_{\gamma' \subset \gamma} \hat{O}_{S,\gamma'} \hat{P}_{\gamma'}$ (or $\sum_{\gamma' \subset \gamma} \hat{P}_{\gamma'} \hat{O}_{S,\gamma'} \hat{P}_{\gamma'}$ to make it explicitly self-adjoint) where $\hat{P}_{\gamma'}$ is an orthogonal projection onto $\mathcal{H}_{\gamma'}$. From now on we assume that \hat{O}_S has been consistently defined like this and thus we have arrived at a new definition of continuum operators \hat{O} which are diffeomorphism covariantly defined without going through a regularization procedure. This finishes our task associated with problem II.)

In what follows, we will only work with this new kind of operators and drop the label S for O_S again, O will be a general classical function on \mathcal{M} . Then the next question

is : Given Σ, m , for which graph should one construct a semi-classical state in order to arrive at the desired semi-classical interpretation, in particular, what is the size of ϵ ? This is connected with the question of the continuum limit $\epsilon \rightarrow 0$. Certainly, ϵ should be small enough so that the classical error (2.2) is small. But there are also quantum errors : First of all, unless the corresponding quantum operator $\hat{O}_\gamma := O_\gamma(\hat{g}_\epsilon)$ is normal ordered we also have a normal ordering error

$$(\Delta O_\gamma)_{normord}(m) := \left| \frac{\langle \psi_{\gamma,m}, \hat{O}_\gamma \psi_{\gamma,m} \rangle}{O_\gamma(m)} - 1 \right| \quad (2.9)$$

which in applications is usually of the same order as the the quantum fluctuation of \hat{O}_γ given by

$$(\Delta O_\gamma)_{quant}(m) := \langle \psi_{\gamma,m}, \left[\frac{\hat{O}_\gamma}{O_\gamma(m)} - 1 \right]^2 \psi_{\gamma,m} \rangle^{1/2} \quad (2.10)$$

These quantum errors are not yet the quantum errors of our consistently defined continuum operator \hat{O} . However, since when computing expectation values or fluctuations of \hat{O} on cylindrical functions we actually project \hat{O} to one of the \hat{O}_γ we can introduce the following notion of fluctuation of a real valued observable O for which we assume \hat{O}_γ to be self-adjoint :

$$(\Delta O)_{total}(m) := \langle \psi_{\gamma,m}, \left[\frac{\hat{O}}{O(m)} - 1 \right]^2 \psi_{\gamma,m} \rangle^{1/2} = \langle \psi_{\gamma,m}, \left[\frac{\hat{O}_\gamma}{O(m)} - 1 \right]^2 \psi_{\gamma,m} \rangle^{1/2} \quad (2.11)$$

which measures the fluctuation of the substitute operator \hat{O}_γ compared to the exact classical value $O(m)$. It is important to realize that due to cylindrical consistency we could replace \hat{O}_γ in (2.11) by \hat{O} . This quantity can be written in terms of the classical, normal ordering and quantum error as

$$\begin{aligned} (\Delta O)_{total}(m)^2 &= \left[\frac{\langle \hat{O}_\gamma \rangle}{O_\gamma(m)} \right]^2 \left[\frac{O_\gamma(m)}{O(m)} \right]^2 (\Delta \hat{O}_\gamma)_{quant}(m)^2 \\ &\quad + \left| \frac{O_\gamma(m)}{O(m)} \left(\frac{\langle \hat{O}_\gamma \rangle}{O_\gamma(m)} - 1 \right) + \left(\frac{O_\gamma(m)}{O(m)} - 1 \right) \right|^2 \end{aligned} \quad (2.12)$$

Equation (2.12) allows us to derive the following important inequality

$$\begin{aligned} (\Delta O)_{total}(m) &\leq [1 + (\Delta O_\gamma)_{normord}(m)] [1 + (\Delta O)_{class}(m)] (\Delta O_\gamma)_{quant}(m) \\ &\quad + [1 + (\Delta O)_{class}(m)] (\Delta O_\gamma)_{normord}(m) + (\Delta O)_{class}(m) \end{aligned} \quad (2.13)$$

and thus will be small if both $(\Delta O_\gamma)_{quant}(m)$ and ϵ/L are small. If practically possible and physically motivated one should of course normal order the operator \hat{O}_γ . These fluctuations will be the smaller the more elementary observables, say N , O_γ involves *additively* (see first reference of [2]), that is, the more macroscopic O is. This is because of the law of large numbers for fluctuation errors which decreases as $1/\sqrt{N}$. The least macroscopic observables that we can measure are the elementary ones, $O = g_e$ for which we require still good semi-classical behaviour and which we will call the *mesoscopic* ones. We obviously will not be allowed to choose ϵ as small as ℓ_p or lower because then $(\Delta O_\gamma)_{quant}(m)$ will be large since we would probe the geometry at the *microscopic* Planck scale.

These considerations suggest the relation $\ell_p \ll \epsilon \ll L(m)$ or $\ell_p \ll \epsilon \ll L_0$ if we want ϵ to be independent of m . We think of ϵ as a parameter for graphs which we can adjust in such a way as to minimize fluctuations. We will see later in appendix B that minimization fixes ϵ to be of the order of $\epsilon = \ell_p^r L(m)^s$ or $\epsilon = \ell_p^r L_0^s$ where r, s are positive rational numbers adding up to one. Moreover, the right hand side of (2.13) turns out to be (not surprisingly) of the form $(\ell_p/L(m))^\alpha \leq (\ell_p/L_0)^\alpha$ for some positive rational number α . *We see that the continuum limit $\epsilon \rightarrow 0$ and the classical limit $\hbar \rightarrow 0$ are no longer separate limits but happen simultaneously.* This at least restricts our freedom to choose a graph, given Σ, m in order to enforce maximum semi-classical behaviour.

We are then left with the specification of the degrees of freedom $g_{\hat{e}}$ and the maps $\Phi_{\hat{\gamma}, \Sigma, X}$, or equivalently, the g_e . Once we have singled them out, we will choose the states $\psi_{\gamma, m}$ to be *coherent states* for the \hat{g}_e , that is, an overcomplete set of minimal uncertainty states for the g_e , see section 3. Given an embedded graph γ and point $m \in \mathcal{M}$ we simply take the holonomies $h_e(A)$ along its edges as the configuration degrees of freedom where A is the connection datum specified by $m = (A, E)$. As already announced, in order to specify momentum degrees of freedom we have to blow up the label set γ somewhat : We consider a polyhedral decomposition P_γ of the three – manifold Σ such that the graph γ is dual to it. Thus, for each edge e of γ there is a unique, open, face S^e of the polyhedral decomposition which it intersects transversally in an interior point p_e of both e and S^e and whose orientation agrees with that of e . Next, for each $x \in S^e$ we choose a non-self-intersecting path $\pi_e(x)$ within S^e which starts at p_e and ends at x without winding around any point in S^e . The collection of these paths will be denoted by Π_γ . Finally, let a be a fixed parameter of the dimension of a length. Then we define the dimensionless quantity

$$P_j^e(A, E) := -\frac{1}{2a^2} \text{tr}(\tau_j \text{Ad}_{h_e(p_e)} [\int_{S^e} \text{Ad}_{h_{\pi_e(x)}} (*E(x))]) \quad (2.14)$$

where $h_e(p_e)$ is the holonomy along the segment of e starting at the starting point of e and ending at p_e , $*$ denotes the metric independent Hodge dual and $E = E_j \tau_j$ with respect to generators τ_j of $su(2)$ with the properties $\text{tr}(\tau_j \tau_k) = -2\delta_{jk}$, $[\tau_j, \tau_k] = 2\epsilon_{jkl} \tau_l$. Notice that (2.14) is gauge covariant, that is, under local gauge transformations it transforms in the adjoint representation of $SU(2)$ at the starting point of e . It is also background independently and diffeomorphism covariantly defined. Moreover, $P_j^e(m)$ depends linearly on the electric field datum E specified by m . The interpretation of and the motivation for the quantity a with the dimension of a length is not clear at this point but it will become obvious later on when we compute fluctuations where it will be fixed at the order of $L(m)$ or L_0 . Also, tuning a will tune the Gaussian width of the coherent states that we construct.

Together with the holonomies $h_e(m)$ of the connection along edges we obtain the elementary graph degrees of freedom

$$g_{e, \Sigma, P_\gamma, \Pi_\gamma}(m) := (h_e(m), P_j^e(m)) \quad (2.15)$$

which, abusing the notation, are specific functions on \mathcal{M} that depend on $\Sigma, \gamma, P_\gamma, \Pi_\gamma$. It is therefore very surprising at first that they satisfy a Poisson algebra (see the first reference in [3]), derived from the canonical Poisson brackets $\{.,.\}$ between the A, E on \mathcal{M} , which is *completely independent* of $\Sigma, P_\gamma, \Pi_\gamma$, specifically

$$\{h_e, h_{e'}\}(m) = 0$$

$$\begin{aligned}
\{P_j^e, h_{e'}\}(m) &= \frac{\kappa}{a^2} \delta_{e'}^e \frac{\tau_j}{2} h_e(m) \\
\{P_j^e, P_k^{e'}\}(m) &= -\delta^{ee'} \frac{\kappa}{a^2} \epsilon_{jkl} P_l^e(m)
\end{aligned} \tag{2.16}$$

With a little experience in background independent quantum field theory, however, this is not so surprising, it just mirrors the fact that no metric, distances, angles etc. can appear in (2.16).

Let us now forget about the structures $\Sigma, P_\gamma, \Pi_\gamma$ altogether and consider a model graph $\check{\gamma}$ which can be embedded as γ into Σ . Let us also forget the continuum phase space $(\mathcal{M}, \{.,.\})$ and rather consider a model phase space $(\mathcal{M}_{\check{\gamma}}, \{.,.\}_{\check{\gamma}})$ with coordinates $h_{\check{e}} \in SU(2)$, $P_j^{\check{e}} \in \mathbb{R}^3$ which specify $g_{\check{e}} = (h_{\check{e}}, P_j^{\check{e}})$ and the point $m_{\check{\gamma}} := \{g_{\check{e}}\}_{\check{e} \in E(\check{\gamma})}$ where $E(\check{\gamma})$ is the set of directed edges of $\check{\gamma}$. These coordinates enjoy the basic brackets inherited from (2.16)

$$\begin{aligned}
\{h_{\check{e}}, h_{\check{e}'}\}_{\check{\gamma}} &= 0 \\
\{P_j^{\check{e}}, h_{\check{e}'}\}_{\check{\gamma}} &= \frac{\kappa}{a^2} \delta_{\check{e}'}^{\check{e}} \frac{\tau_j}{2} h_{\check{e}} \\
\{P_j^{\check{e}}, P_k^{\check{e}'}\}_{\check{\gamma}} &= -\delta^{\check{e}\check{e}'} \frac{\kappa}{a^2} \epsilon_{jkl} P_l^{\check{e}}
\end{aligned} \tag{2.17}$$

which defines the natural symplectic structure of the $|E(\check{\gamma})|$ fold copy of the cotangent bundle $T^*SU(2)$, displaying $\mathcal{M}_{\check{\gamma}}$ as the cotangent bundle over the $|E(\check{\gamma})|$ fold copy of $SU(2)$.

We now want to construct coherent states for $\mathcal{M}_{\check{\gamma}}$. These will be states on the model Hilbert space (which is of course isomorphic with \mathcal{H}_γ)

$$\mathcal{H}_{\check{\gamma}} := \otimes_{\check{e} \in E(\check{\gamma})} L_2(SU(2), d\mu_H) \tag{2.18}$$

where μ_H is the Haar measure on $SU(2)$, that is, they will functionally depend on the point $h_{\check{\gamma}} := \{h_{\check{e}}\}_{\check{e}}$ of the configuration subspace of $\mathcal{M}_{\check{\gamma}}$. On the other hand, we want them to be peaked at a point $m_{\check{\gamma}}$ of the phase space $\mathcal{M}_{\check{\gamma}}$ and therefore they will carry a label $m_{\check{\gamma}}$. The tensor product structure of the Hilbert space makes it obvious that these states will be of the form of a direct product

$$\psi_{\check{\gamma}, m_{\check{\gamma}}}(h_{\check{\gamma}}) = \prod_{\check{e}} \psi_{g_{\check{e}}}^t(h_{\check{e}}) \tag{2.19}$$

where $\psi_g^t(h)$ is a coherent state on the Hilbert space $L_2(SU(2), d\mu_H)$ with label $g \in T^*SU(2)$ functionally depending on $h \in SU(2)$. These states depend explicitly on the dimensionless *classicality parameter*

$$t = \frac{\ell_p^2}{a^2} \tag{2.20}$$

which naturally arises in quantizing the above Poisson bracket, e.g. $[\hat{P}_j^{\check{e}}, \hat{P}_k^{\check{e}'}] = -it\delta^{\check{e}\check{e}'} \epsilon_{jkl} \hat{P}_l^{\check{e}}$. It is important to see that the Hilbert space (2.18) carries a faithful representation of these commutation relations and the adjointness conditions $(\hat{P}_j^{\check{e}})^\dagger = \hat{P}_j^{\check{e}}$, $(\hat{h}_{\check{e}}^{AB})^\dagger = (\hat{h}_{\check{e}}^{BA})^{-1}$ following from the classical reality (unitarity) respectively if $\hat{h}_{\check{e}}$ acts by multiplication and $\hat{P}_j^{\check{e}}$ is $it/2$ times the right invariant vector field on the copy of $SU(2)$ corresponding to $h_{\check{e}}$ generating left translations in the direction of τ_j in $su(2)$.

More explicitly these states are labelled by points in $SL(2, \mathbb{C})$ which is possible since one can identify $g_{\tilde{e}}$ with the following element of $SL(2, \mathbb{C})$ (we abuse the notation in using the same symbol)

$$g_{\tilde{e}} := \exp(-i\tau_j P_j^{\tilde{e}}/2)h_{\tilde{e}} \quad (2.21)$$

which is actually a diffeomorphism from $T^*SU(2)$ to $SL(2, \mathbb{C})$ (the inverse map being given by polar decomposition). Since the former is a symplectic manifold and the latter a complex manifold it is not surprising that $SL(2, \mathbb{C})$ is actually a Kähler manifold (the complex structure is compatible with the symplectic structure) which suggests to use well known methods from geometric quantization (in particular, heat kernel methods) for the construction of the coherent states. These methods will be described in some detail in section 3, here it is sufficient to contemplate that the situation is actually quite analogous to the harmonic oscillator coherent states which are labelled by a point z in the complex plane, whose real and imaginary part respectively can be interpreted as configuration q and momentum p degree of freedom respectively, and which depend functionally on a point $x \in \mathbb{R}$ in the configuration space of the phase $T^*\mathbb{R}$. The complexification of the real line is the complex plane and the complexification of the group $SU(2)$ is given by $SL(2, \mathbb{C})$. Thus, (2.21) is nothing else than a non-Abelian and exponentiated version of z .

This finishes the construction of a coherent state on the model Hilbert space $\mathcal{H}_{\tilde{\gamma}}$ labelled by a point in a model phase space. We now must make contact with the Hilbert space \mathcal{H} and the phase space \mathcal{M} appropriate to describe a semi-classical situation for a given topology Σ . The formulae (2.14), (2.16) provide the necessary clue : Given a model graph $\tilde{\gamma}$, a three manifold Σ , an embedding X of $\tilde{\gamma}$ into Σ as $\gamma = X(\tilde{\gamma})$, a polyhedral decomposition $P_{\tilde{\gamma}}$ of Σ dual to $\tilde{\gamma}$ and a choice $\Pi_{\tilde{\gamma}}$ of paths within its faces we can construct the concrete maps (2.15) which provide us with a map (we display only its dependence on $\tilde{\gamma}$ but the dependence on the other structures should be kept in mind)

$$\Phi_{\tilde{\gamma}, \Sigma, X} : \mathcal{M} \mapsto \mathcal{M}_{\tilde{\gamma}}; m \mapsto \{g_{\tilde{e}} := g_{X(\tilde{e}), \Sigma, X}(m)\}_{e \in E(\tilde{\gamma})} \quad (2.22)$$

which, according to (2.16), (2.17), is actually a symplectomorphism (more precisely, it leaves the Poisson brackets invariant but it is not invertible and not even necessarily surjective if Σ is not compact since then m needs to obey certain fall-off conditions). The map (2.22) is the final ingredient to obtain a semi-classical state appropriate for a given Σ and phase space point m

$$\psi_{\gamma, m}(h_{\gamma}) := \psi_{\tilde{\gamma}, \Phi_{\tilde{\gamma}, \Sigma, X}(m)}(h_{\tilde{\gamma}}) \quad (2.23)$$

where we have identified \mathcal{H}_{γ} with $\mathcal{H}_{\tilde{\gamma}}$, in particular, h_e with $h_{\tilde{e}}$ whenever $X(\tilde{e}) = e$. Since the model Hilbert space is the same for any Σ , this opens the possibility that one might be able to *describe topology change within canonical quantum general relativity* : If we can define consistently a new representation of the canonical commutation relations based on model graphs rather than embedded ones, then the transition amplitude $\langle \psi_{\tilde{\gamma}, \Phi_{\tilde{\gamma}, \Sigma, X}(m)}, \psi_{\tilde{\gamma}, \Phi_{\tilde{\gamma}, \Sigma', X'}(m')} \rangle$ is well-defined since the scalar product is computed on the common model Hilbert space and not on the embedded one and one and the same model graph can be embedded differently into different Σ . We elaborate more on this idea in future publications [14] where methods from algebraic graph theory will become crucial [15].

By construction, the state (2.23) is sharply peaked at $P_j^{\tilde{e}} = P_j^e(m), h_{\tilde{e}} = h_e(m)$ (these values depend explicitly on $\Sigma, X, P_\gamma, \Pi_\gamma$) and enjoys further important semi-classical properties as we will describe in detail in the next section. They thus are good candidate states to base a semi-classical analysis on.

Let us summarize :

i) *Classical Continuum Phase Space*

For any given three manifold Σ we start from a classical, continuum phase space $(\mathcal{M}, \{.,.\})$ based on Σ with points m and a continuum Poisson algebra \mathcal{O} of classical observables O .

ii) *Classical Discrete Phase Space*

Given a model graph $\tilde{\gamma}$ we have a classical, model phase space $(\mathcal{M}_{\tilde{\gamma}}, \{.,.\}_{\tilde{\gamma}})$ (direct sum of copies of $T^*SU(2)$, one for each edge of $\tilde{\gamma}$) with points $m_{\tilde{\gamma}} = \{g_{\tilde{e}}\}$ and classical model Poisson algebra $\mathcal{O}_{\tilde{\gamma}}$ of observables $O_{\tilde{\gamma}}$ respectively. No reference to a particular manifold Σ is made. For any Σ , a subset of $\mathcal{M}_{\tilde{\gamma}}$ can be obtained as the image of \mathcal{M} based on Σ under a symplectomorphism $\Phi_{\tilde{\gamma},\Sigma,X}$ of the form (2.22).

iii) *Discrete Quantum Hilbert Space*

The model symplectic manifold is easy to quantize and gives a discrete quantum theory given by the model graph Hilbert space $(\mathcal{H}_{\tilde{\gamma}}, \langle .., \rangle_{\tilde{\gamma}})$ (direct product of copies of $L_2(SU(2), d\mu_H)$, one for each edge of $\tilde{\gamma}$) with coherent states $\psi_{\tilde{\gamma},m_{\tilde{\gamma}}}^t$ and model quantum commutator algebra $\hat{\mathcal{O}}_{\tilde{\gamma}}$ of model quantum observables $\hat{O}_{\tilde{\gamma}}$. By construction, the classical limit of $(\mathcal{H}_{\tilde{\gamma}}, \langle .., \rangle_{\tilde{\gamma}}, \hat{\mathcal{O}}_{\tilde{\gamma}})$ gives us back $(\mathcal{M}_{\tilde{\gamma}}, \{.,.\}_{\tilde{\gamma}}, \mathcal{O}_{\tilde{\gamma}})$ which means that for all $m_{\tilde{\gamma}}, O_{\tilde{\gamma}}$ the zeroth order term in \hbar of the quantities

$$\begin{aligned} & \langle \psi_{\tilde{\gamma},m_{\tilde{\gamma}}}^t, \hat{O}_{\tilde{\gamma}} \psi_{\tilde{\gamma},m_{\tilde{\gamma}}}^t \rangle_{\tilde{\gamma}} - O_{\tilde{\gamma}}(m_{\tilde{\gamma}}) \text{ and} \\ & \langle \psi_{\tilde{\gamma},m_{\tilde{\gamma}}}^t, \frac{[\hat{O}_{\tilde{\gamma}}, \hat{O}'_{\tilde{\gamma}}]}{it} \psi_{\tilde{\gamma},m_{\tilde{\gamma}}}^t \rangle_{\tilde{\gamma}} - \{O_{\tilde{\gamma}}, O'_{\tilde{\gamma}}\}_{\tilde{\gamma}}(m_{\tilde{\gamma}}) \end{aligned} \quad (2.24)$$

and of their fluctuations, e.g. $(\Delta O_{\tilde{\gamma}})_{quant}(m_{\tilde{\gamma}})$, vanishes (so-called Ehrenfest property).

iv) *Continuum Quantum Hilbert Space*

Given a model graph $\tilde{\gamma}$ and an embedding X into some Σ we can identify the model Hilbert space $\mathcal{H}_{\tilde{\gamma}}$ with \mathcal{H}_γ where $e = X(\tilde{e})$ and $h_e := h_{\tilde{e}}$. Likewise, given a model operator $\hat{O}_{\tilde{\gamma}}$ on $\mathcal{H}_{\tilde{\gamma}}$ we obtain an operator \hat{O}_γ on \mathcal{H}_γ by substituting multiplication by and differentiation with respect to $h_{\tilde{e}}$ for multiplication by and differentiation with respect to h_e . By this method we obtain diffeomorphism covariant families of operators \hat{O} and we consider only those that are cylindrically consistent. The operators that we have constructed above are of this type and the algebra of these operators will be called $\hat{\mathcal{O}}$.

Given Σ, \mathcal{M} and a classical observable O we produce by the method described above such a consistently defined family of operators $\hat{O} = \{\hat{O}_\gamma\}$ based on classical functions O_γ of the graph degrees of freedom. Among all possible graphs γ , by construction there exists at least one 1-parameter family of graphs γ_ϵ such that $O(m) = \lim_{\epsilon \rightarrow 0} O_{\gamma_\epsilon}(m_{\gamma_\epsilon}(m))$ for any $m \in \mathcal{M}$. Thus, while for a generic graph \hat{O}_γ

will not have a semi-classical interpretation, the operators $\hat{O}_{\gamma_\epsilon}$ do. The point is now that $\mathcal{H}_\gamma \subset \mathcal{H}$, $\langle \cdot, \cdot \rangle_\gamma = \langle \cdot, \cdot \rangle_{|\mathcal{H}_\gamma}$, $\hat{O}_\gamma = \hat{O}_{|\mathcal{H}_\gamma}$. Therefore, since the continuum limit is coupled to the classical limit (no separate continuum limit to be taken) we can make the following definition :

We will say that for given Σ the classical limit of $(\mathcal{H}, \langle \cdot, \cdot \rangle, \hat{O})$ is given by $(\mathcal{M}, \{., .\}, \mathcal{O})$ if for all m, O the zeroth order of the \hbar expansion of the quantities

$$\langle \psi_{\gamma_\epsilon, m}, \hat{O}_{\gamma_\epsilon} \psi_{\gamma_\epsilon, m} \rangle - O(m) \text{ and } \langle \psi_{\gamma_\epsilon, m}, \frac{[\hat{O}_{\gamma_\epsilon}, \hat{O}'_{\gamma_\epsilon}]}{it} \psi_{\gamma_\epsilon, m} \rangle - \{O, O'\}(m) \quad (2.25)$$

and of their fluctuations, e.g. $(\Delta O)_{total}(m)$, vanishes (and that the corrections are in agreement with experiment). Here $\epsilon = \epsilon(\ell_p, L(m))$, $a = a(\ell_p, L(m))$ or $\epsilon = \epsilon(\ell_p, L_0)$, $a = a(\ell_p, L_0)$ take their optimal value as specified above so that they depend only on ℓ_p, Σ, m or $\ell_p, \Sigma, \mathcal{M}$. Notice that in a label for a coherent state we may take a to be m -dependent although non-trivial m -dependence of a would spoil the Poisson brackets (2.16). This is because in the model Poisson brackets (2.17) we can take a to be any function of m without changing them. Alternatively, a is an m -independent parameter to begin with, it just gets fixed (by optimization) at an m -dependent value $a(m)$, thus Poisson brackets are to be evaluated by taking a as m -independent and after computing them one should set it equal to $a(m)$.

We have thus arrived at a closed diagramme of how to pass from a given continuum, classical theory to a continuum, quantum theory and back by taking a route through a discrete regime of both theories. Moreover, we have derived an explicit criterium for how to decide whether canonical quantum general relativity has classical general relativity as its classical limit. The Ehrenfest theorems proved already in [3] and the explicit construction of the operators \hat{O}_γ which guarantees that for given Σ, m such that $\gamma = \gamma_\epsilon$, $\epsilon = \epsilon(\ell_p, m)$ both its classical and quantum error are small indicate that this might actually be the case.

A couple of remarks are in order :

1.)

Our scheme works only if the operators \hat{O}_γ leave the Hilbert space \mathcal{H}_γ invariant. However, the Diffeomorphism and Hamiltonian constraint operators that have been constructed in the literature (e.g. [10, 11]) do not have this property. Either one must modify them in such a way that their cylindrical projections leave the cylindrical subspaces invariant or we have to modify our scheme as follows for such operators : If \hat{O}_γ maps \mathcal{H}_γ into $\mathcal{H}_{\gamma'}$ for some $\gamma' \neq \gamma$ then find the minimal (in the sense of numbers of edges) γ'' such that $\gamma, \gamma' \subset \gamma''$ and replace $\psi_{\gamma_\epsilon, m}$ in (2.25) by $\psi_{\gamma''_\epsilon, m}$. Strictly speaking then, (2.25) does not really test the continuum operator \hat{O} but only its cylindrical projections \hat{O}_γ which is not what we really want but which is the best we can do at the moment.

2.)

All the coherent states that we defined are labelled by a specific γ and although all of them belong to the continuum Hilbert space it would be nicer and it would solve the problem just mentioned in 1.), if we could find a state ψ_m such that its projection onto \mathcal{H}_γ coincides with $\psi_{\gamma, m}$. This could be achieved if our family of states $\psi_{\gamma, m}$ would solve the consistency condition that $\psi_{\gamma', m}$ projected onto \mathcal{H}_γ coincides with $\psi_{\gamma, m}$ for any $\gamma \subset \gamma'$. It is easy to see that this would require (see also [12] for a related observation) a) to make the parameter t in (2.19) a function of the edge e , that is, $t = t(e)$ such that $t(e^{-1}) = t(e)$

and $t(e \circ e') = t(e) + t(e')$ and b) that $g_{e^{-1}} = g_e^{-1}$ and $g_{e \circ e'} = g_e g_{e'}$ which are precisely the defining algebraic relations for a length function and an $SL(2, \mathbb{C})$ holonomy respectively. There is no problem to generalize our scheme and to make t a function of edges. However, our coherent states are constructed in such a way that the operators \hat{P}_j^e, \hat{h}_e are peaked at P_j^e, h_e where $g_e = \exp(-iP_j^e \tau_j) h_e$ and the holonomy property is in conflict with the interpretation $P_j^e = P_j^e(A, E), h_e = h_e(A)$ given in (2.14). One can, of course, turn the argument the other way around and just define a classical complex connection, say $A_b^\mathbb{C}(m) := A_b - ie_b/a$ where e_b^j is the co-triad. Then for given Σ and phase space point m we can define $g_e := h_e(A^\mathbb{C}(m))$ where the latter means the holonomy of the complex connection determined by m along the embedded edge $e = X(\check{e})$. Then, if $\ell_p \ll \epsilon \ll L_0$ the coherent state is peaked at the value

$$h_e \approx h_e(A), P_j^e \approx p_j^e(A, E) := -\frac{1}{2a} \text{tr}(\tau_j [\int_e \text{Ad}_{h_e(A,x)}(e(x))]) \quad (2.26)$$

where $h_e(A, x)$ denotes the holonomy of A along e from the starting point of e until the point $x \in e$. The advantage of this new map $\Phi'_{\check{\gamma}, \Sigma, X}$ is that we need less structure (no polyhedral decomposition, no choices of paths). The disadvantage is that a) (2.26) holds only approximately while (2.23) holds exactly independent of the choice of ϵ and b) there is no closed classical Poisson algebra underlying the functions $h_e(A), p^e(A, E)$: First of all, the electrical field “wants to be smeared in two rather than in one dimension” so the algebra becomes necessarily distributional. Secondly, since e_a^j or any other Lie algebra valued co-vector that can be built from E_j^a is a non-polynomial function of the electric field, the Poisson * algebra actually does not close. It follows that the relation (2.26) between the model graph phase space and the classical phase space does not survive this interpretation of g_e at the level of Poisson brackets. Therefore, the expectation value of the commutator between the model elementary graph operators divided by it (which is non-distributional) cannot be given by the Poisson brackets of the functions on the right hand side of (2.26) (which is distributional) as computed with the symplectic structure of \mathcal{M} .

3.)

The coherent states that we constructed are *kinematical* coherent states, i.e. they are not annihilated by the constraint operators of the theory (in fact, they are not even gauge invariant). One can rightfully ask how such states can possibly be used in the semiclassical analysis. The answer is the following :

First of all, one of our aims is find out whether the Hamiltonian constraint operators constructed in [11] really do have the correct classical limit and obviously it is then not allowed to consider physical coherent states which would be automatically annihilated by them.

Secondly, the gauge and diffeomorphism group are unitarily implemented on the Hilbert space and therefore expectation values and fluctuations of gauge and spatially diffeomorphism invariant operators are in fact gauge and spatially diffeomorphism invariant. This leaves us with the issue of observables which are invariant under the motions of the Hamiltonian constraint which famously cannot be implemented unitarily. The problem is, of course, that we do not know any such observable explicitly so that it is impossible to test the usefulness of kinematical coherent states in full quantum general relativity directly. However, one can study exactly solvable model systems and it turns out that for a large class of such systems [13] including those with a non-polynomial Hamiltonian

constraint and for which the Dirac observables are non-polynomial functions of kinematical observables, the expectation value of these Dirac observables and their commutators divided by \hbar as measured by kinematical coherent states *precisely coincides, to zeroth order in \hbar , with those as measured by dynamical coherent states provided we choose m on the constraint surface of the phase space, no matter in which gauge.* Moreover, the fluctuations of these operators are of the same order in \hbar . Thus, while corrections to the classical theory are not independent of one's choice of kinematical versus dynamical coherent states, one can control the size of their difference and the classical limit itself seems to be unaffected. This is of course no proof that the same will be true in full quantum general relativity but it is a non-trivial consistency check. Of course, since we only check a finite number of commutators in the classical limit $\hbar \rightarrow 0$ we have only control over the infinitesimal dynamics and no a priori control on the finite time evolution of the system. Fortunately, in principle the finite time evolution is not necessarily required in order to compute Dirac observables and this is all we need since we do not have a Hamiltonian but only a Hamiltonian constraint.

Finally, if we consider the GNS representations to be discussed in section 4 with respect to a coherent state labelled by a point on the constraint surface of the phase space then constraint operators will have expectation value zero (or at least very close to if not normal ordered) although their fluctuations do not vanish. Thus, such representations can be considered as *approximately physical representations*, a point of view independently reached also by Ashtekar (they would be exactly physical if the constraint operators would be the zero operators).

4.)

The careful reader may object, as pointed out by Ashtekar and Lewandowski, that in order to make our family of operators cylindrically consistent we must in general introduce the projection operators \hat{P}_γ which have no classical counterpart and thus it is actually no longer clear that the classical limit as defined in (2.25) still gives the desired result. Fortunately, there is no problem : At the optimal value of ϵ, a derived in appendix B we find for the projection \hat{P}_e on spin-network states over a single edge e the expectation value $1 - e^{-\frac{1}{t^{2/3}}}$ where $t \leq \ell_p^2/L_0^2$ is a tiny quantity. Thus the expectation value is $1 + O(t^\infty)$. For a general graph a similar argument holds provided that the projection operators needed in the definition of \hat{O}_γ are finite products of the \hat{P}_e . But this is the case for operators which come from classical functions defined as integrals over regions, surfaces and curves provided that the valence of the vertices of γ is bounded from above, even if γ has a countably infinite number of edges. It should be clear from our exposition that on pathological graphs which lack this boundedness property a good semi-classical behaviour cannot be expected anyway so that this is not a restriction.

3 Coherent States on Cotangent Bundles over Compact Gauge Groups

As explained in the previous section, a first step towards constructing coherent states for quantum general relativity is to construct such states for $SU(2)$. This was achieved in the general setting of compact Lie groups by Hall [16]. To make our presentation more pedagogic, we start by reviewing the situation for the harmonic oscillator in one

dimension. Here the wave function is given by $\psi_z^t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(z-x)^2}{2t}}$, where $z = x_0 - ip_0$ and $t = \frac{\hbar}{m\omega}$, and ψ_z^t is an element of $L^2(\mathbb{R}, dx)$. As is well-known $\psi_z^t(x)$ has a Gaussian shape with peak at x_0 , or, more generally, at $\Re(z)$. Similarly, by transforming $\psi_z^t(x)$ to momentum space one finds that it is peaked there around p_0 , or, again more generally, around $\Im(z)$. Furthermore, the sharpness of the peak in each representation is given by \sqrt{t} . For that reason it is usually stated that coherent states approach the delta function as one considers the limit \hbar and therefore t to zero. What is really meant by this limit, of course, is that the physical quantities characterizing the system under consideration - here, for example, m and ω - are much, much larger than the value of \hbar . In that sense, the classical limit of, e.g, expectation values with respect to these coherent states is given by the zeroth order and semiclassical corrections by the first and higher orders of a power expansion in t . Further important properties of these states are that they are overcomplete, they satisfy the Heisenberg uncertainty relation and they are eigenstates of the annihilation operator: $(\hat{z}\psi_{z_0}^t)(x) = z_0\psi_{z_0}^t(x)$.

To make the transition from the harmonic oscillator to the compact Lie group case, a more technical derivation of the former is helpful. We start with the classical phase space P , coordinatized by (x, p) . By choosing a positive function C on P - the so-called *complexifier* - we change to a complex polarization of P by defining

$$\begin{aligned} z_1 &: = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{x, C\}_n \\ z_2 &: = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{p, C\}_n \end{aligned} \quad (3.1)$$

where $\{\cdot, \cdot\}_n$ stands for a multiple Poisson bracket of order n . The phase space P is now coordinatized by (in general) complex coordinates (z_1, z_2) . In the case of the harmonic oscillator, choosing $C = p^2/2$ results in the usual $z_1 = x - ip$ and $z_2 = p$. One proceeds with quantization by replacing Poisson brackets by commutators divided by $i\hbar$ and by promoting classical functions to operators. For the harmonic oscillator this can be done without problems:

$$\hat{z} = \sum_{n=0}^{\infty} \frac{\hbar^{-n}}{n!} [\hat{x}, \hat{C}]_n = \hat{W}_t \hat{x} \hat{W}_t^{-1} \quad (3.2)$$

with $\hat{W}_t = e^{-\frac{\hat{C}}{t}}$, which for the harmonic oscillator is just the heat kernel as then \hat{C} is, up to factors, the Laplacian Δ on \mathbb{R} . This operator \hat{W}_t also establishes the connection to the *coherent state transform* which is a unitary transformation \hat{U}_t from the position Hilbert space $L^2(\mathbb{R}, dx)$ to the phase space Hilbert space (Segal-Bargmann space) $HL^2(\mathbb{C}, d\mu)$ of holomorphic L^2 -functions over \mathbb{C} . The new measure μ is determined by the unitarity requirement. The transform is defined by

$$(\hat{U}_t f)(z) := \int dx \rho^t(y, x) f(x) \Big|_{y \rightarrow z} =: \langle \bar{\psi}_z^t, f \rangle \quad (3.3)$$

where $\rho^t(x, y) := (\hat{W}_t \delta_y)(x)$ and $y \rightarrow z$ means analytic continuation from y to z . From these relations one can thus deduce the following definition of the harmonic oscillator coherent states:

$$\psi_z^t(x) := (e^{t\Delta/2} \delta_y)(x) \Big|_{y \rightarrow z} \quad (3.4)$$

which one can check to be identical to the form given at the beginning of this section. It should be kept in mind that, while the coherent state transform leads to a Hilbert space over \mathbb{C} , the associated coherent states themselves are still elements of the position Hilbert space over \mathbb{R} .

Hall's idea was to follow this construction for compact Lie groups as closely as possible. Instead of wavefunctions over \mathbb{R} one considers wavefunctions over a compact Lie group G , dx is replaced by Haar measure $d\mu_H$, the complex label $z = x - ip$ by an element of the complexified group $G^{\mathbb{C}}$ that can be written in polar decomposition as $g = Hh$ with $h \in G$ and H positiv definite hermitian, and finally the Laplacian on \mathbb{R} is replaced by the Laplace-Beltrami operator Δ_G on G . Coherent states are then analogously defined by

$$\psi_g^t(h) := (e^{t\Delta_G/2}\delta_{\mu_H,h'})(h) |_{h' \rightarrow g} . \quad (3.5)$$

Using the expression for the delta function on Lie groups one arrives at the explicit expression

$$\psi_g^t(h) = \sum_{\pi} d_{\pi} e^{-\frac{t}{2}\lambda_{\pi}} \chi_{\pi}(gh^{-1}) \quad (3.6)$$

where π is (a representant of) an irreducible representation of G , d_{π} its dimension, λ_{π} the eigenvalue of Δ_G in the representation π , and χ_{π} denotes the trace in that representation. The trace is to be understood in the following sense: One first calculates $\chi_{\pi}(h'h^{-1})$ with $h' \in G$ and then performs the analytical continuation from h' to g . Hall was also able to show that these coherent states form an overcomplete system of states on the Hilbert space $L^2(G, d\mu_H)$.

To use Hall's construction for quantum general relativity we specialize to $G = SU(2)$. Then three additional inputs are required:

- One has to establish the connection between the complex group element parameterizing the coherent state and the coordinates on the classical phase space of general relativity, that is connections and electric fields. This is the analogue of $z = x - ip$ in the case of the harmonic oscillator.
- One has to show that the coherent states so defined have the properties important for a (semi-)classical interpretation. Most importantly, they should be peaked in the configuration, momentum and phase space (Segal-Bargmann) representations around the points in phase space specified by g , respectively. Also, Ehrenfest theorems should hold, that is expectation values of operators and their commutators divided by it should approach the corresponding classical functions and their Poisson brackets in the limit $t \rightarrow 0$.
- Finally, one has to extend Hall's framework from single copies of the gauge group to multiple ones. In the language of quantum general relativity this means the extension from coherent states defined over edges to those defined over graphs [3].

We begin by defining the complexifier for an edge e by $C_e = \frac{1}{2}P_i^e P_j^e \delta^{ij}$ where P_i^e are the momentum variables for general relativity as defined in section 2. The complex holonomies are then given by

$$\begin{aligned} h_e^{\mathbb{C}} := g_e &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \{h_e, C_e\}_n \\ &= e^{-i\tau_j P_j^e / 2} h_e = H_e h_e \end{aligned} \quad (3.7)$$

where H_e is positive definite Hermitian and h_e is an element of $SU(2)$. Notice that g_e now carries information on the connection as well as on the electric field (over the edge e). The heat kernel operator is $\hat{W}_{t,e} := e^{\frac{t}{2}\hat{\Delta}_e}$ with $\hat{\Delta}_e = \frac{1}{4}\delta_{ij}\hat{X}_i^e\hat{X}_j^e$, where the \hat{X}_j^e are the derivative operators coming from the quantization of the P_j^e . The coherent states for a single edge e have now the form

$$\psi_{g_e}^t(h_e) = \sum_{j=0,\frac{1}{2},1,\dots} (2j+1)e^{-tj(j+1)/2}\chi_j(g_e h_e^{-1}) \quad (3.8)$$

where g_e given by (3.7) can be computed, given a point in the classical phase space of general relativity as in the previous section and j runs over $SU(2)$ spin representations. The peakedness or classicality parameter t is given by $t = \frac{l_p^2}{a^2}$ where a has dimension of length. Like the mass and frequency for the harmonic oscillator it characterizes the system under consideration and in practice is fixed as outlined in section 2. Finally we can also define the ‘‘annihilation operator’’

$$\hat{g}_e := e^{\frac{t}{2}\hat{\Delta}_e}\hat{h}_e e^{-\frac{t}{2}\hat{\Delta}_e} = e^{\frac{3t}{8}}e^{-i\hat{P}_j^e\tau_j/2}\hat{h}_e \quad (3.9)$$

Here we have used quotation marks because, although the coherent states are eigenstates of \hat{g}_e , up to now it is not yet clear whether this operator can be used to derive a Fock space structure. Other properties of the coherent states that follow immediately from these definitions are that expectation values of normal ordered polynomials of the \hat{g}_e and their adjoints are given by the corresponding classical functions (without quantum corrections) and that the self-adjoint operators defined by $\hat{x}_e := \frac{1}{2}(\hat{g}_e^\dagger + \hat{g}_e)$ and $\hat{y}_e := \frac{1}{2i}(\hat{g}_e^\dagger - \hat{g}_e)$ saturate the unquenched Heisenberg uncertainty relations. Much more work is required to prove the peakedness properties so essential for a semiclassical interpretation. In the configuration and thus connection representation, e.g., one has to prove that

$$p_g^t(h) := \frac{|\psi_g^t(h)|^2}{\|\psi_g^t\|^2} \quad (3.10)$$

is sharply peaked at $h = h'$ where $g = Hh'$ is the polar decomposition of g , with sharpness being given by the parameter t . More explicitly, one needs to show that an upper bound for $p_g^t(h)$ can be given that (roughly) has the form of a Gaussian times a quantity that decays to 1 exponentially fast with $t \rightarrow 0$. A first look at (3.8) reveals that this is not straightforward as the convergence producing term $e^{-tj(j+1)/2}$ converges very slowly for $t \rightarrow 0$. To circumvent this, we employed in the third reference of [3] the so-called Poisson transformation that transforms the sum in (3.8) into a sum where the exponential contains t in the denominator. This speeds up convergence dramatically and thus allows one to derive the desired results, even though calculations are still very tedious to perform.

By similar methods peakedness was shown in the momentum and phase space representation. In addition, as a further check-up and for illustration purposes, these calculations were also performed numerically with a confirmation of the analytical results, see [3] for some graphics. As a result, the coherent states defined above have been shown to be peaked sharply around a point in the classical phase space (in contrast, e.g., to the so-called weaves [7], which so-far had been used to approximate classical spacetimes, but were peaked only in the momentum configuration). They thus serve as a good starting point to approximate classical spacetimes (remember that quantum states can never reproduce a classical state exactly, due to the uncertainty relations), even more so as we

were able to derive Ehrenfest theorems for them in the fourth reference of [3]. This means that expectation values of arbitrary polynomials in the \hat{h}_e and \hat{p}_e^j with respect to the coherent states approach the corresponding polynomial of h_e and p_e^j on phase space in the limit $t \rightarrow 0$. Also, expectation values of commutators divided by it approach the value of the corresponding Poisson bracket, which means that with respect to the coherent states, infinitesimally, the quantum dynamics reproduces the classical dynamics.

Finally, coherent states over graphs are constructed by simply tensoring together those over the respective edges, that is

$$\psi_{\{g\},\gamma}^t(\{h\}) := \prod_{e \in E(\gamma)} \psi_{g_e}^t(h_e) \quad (3.11)$$

where $E(\gamma)$ is the set of all edges of the graph γ and $\{g\}, \{h\}$ denote the collections $g_{e_1} \dots g_{e_n}, h_{e_1} \dots h_{e_n}$ of the associated group elements, respectively. The peakedness and Ehrenfest properties discussed above for the single-edge case also go through for the case of graphs. This follows from the fact that peakedness on single edges implies peakedness on the whole graph and vice versa, and from the fact that operators on different edges commute. This tensor product structure of the coherent states brings us naturally to the next subject.

4 The Infinite Tensor Product Extension

Quantum field theory on curved spacetimes is best understood if the spacetime is actually flat Minkowski space on the manifold $M = \mathbb{R}^4$. Thus, when one wants to compute the low energy limit of canonical quantum general relativity to show that one gets the standard model (plus corrections) on a background metric one should do this first for the Minkowski background metric. Any classical metric is macroscopically non-degenerate. Since in the Hilbert space that has been constructed for loop quantum gravity (see [2] for a review and [17] for more details) the quantum excitations of the gravitational field are concentrated on the edges of a graph, in order that, say, the expectation values of the volume operator for any macroscopic region is non-vanishing and changes smoothly as we vary the region, the graph must fill the initial value data slice densely enough, the mean separation between vertices of the graph must be much smaller than the size of the region (everything is measured by the three metric, determined by the four metric to be approximated, in this case the Euclidean one). Now \mathbb{R}^4 is spatially non-compact and therefore such a graph must necessarily have an at least *countably infinite* number of edges whose union has *non-compact* range.

The Hilbert spaces in use for loop quantum gravity have a dense subspace consisting of finite linear combinations of so-called cylindrical functions labelled either by a piecewise analytic graph with a *finite* number of edges or by a so-called web, a piecewise smooth graph determined by the union of a *finite* number of smooth curves that intersect in a controlled way [18], albeit possibly a countably infinite number of times in accumulation points of edges and vertices. Moreover, in both cases the edges or curves respectively are contained in compact subsets of the initial data hypersurface. These categories of graphs will be denoted by Γ_0^ω and Γ_0^∞ respectively where $\omega, \infty, 0$ stands for analytic, smooth and compactly supported respectively. Thus, the only way that the current Hilbert spaces can actually produce states depending on a countably infinite graph of non-compact range is

by choosing elements in the closure of these spaces, that is, states that are countably infinite linear combinations of cylindrical functions.

The question is whether it is possible to produce semi-classical states of this form, that is, $\psi = \sum_n z_n \psi_{\gamma_n}$ where γ_n is either a finite piecewise analytic graph or a web, z_n is a complex number and we are summing over the integers. It is easy to see that this is not the case : Minkowski space has the Poincaré group as its symmetry group and thus we will have to construct a state which is at least invariant under (discrete) spatial translations. This forces the γ_n to be translations of γ_0 and $z_n = z_0$. Moreover, the dependence of the state on each of the edges has to be the same and therefore the γ_n have to be mutually disjoint. It follows that the norm of the state is given by

$$\|\psi\|^2 = |z|^2 \left(\left[\sum_n 1 \right] [1 - |\langle 1, \psi_{\gamma_0} \rangle|^2] + \left[\sum_n 1 \right]^2 |\langle 1, \psi_{\gamma_0} \rangle|^2 \right) \quad (4.1)$$

where we assumed without loss of generality that $\|\psi_{\gamma_0}\| = 1$ and we used the diffeomorphism invariance of the measure and 1 is the normalized constant state. By the Schwarz inequality the first term is non-negative and convergent only if $\psi_{\gamma_0} = 1$ while the second is non-negative and convergent only if $\langle 1, \psi_{\gamma_0} \rangle = 0$. Thus the norm diverges unless $z = 0$.

This caveat is the source of its removal : We notice that the formal state $\psi := \prod_n \psi_{\gamma_n}$ really depends on an infinite graph and has unit norm if we formally compute it by $\lim_{N \rightarrow \infty} \|\prod_{n=-N}^N \psi_{\gamma_n}\| = 1$ using disjointness of the γ_n . The only problem is that this state is not any longer in our Hilbert space, it is not the Cauchy limit of any state in the Hilbert space : Defining $\psi_N := \prod_{n=-N}^N \psi_{\gamma_n}$ we find $|\langle \psi_N, \psi_M \rangle| = |\langle 1, \psi_{\gamma_0} \rangle|^{2|N-M|}$ so that ψ_N is not a Cauchy sequence unless $\psi_{\gamma_0} = 1$. However, it turns out that it belongs to the *Infinite Tensor Product (ITP) extension* of the Hilbert space.

To construct this much larger Hilbert space we must first describe the class of embedded graphs that we want to consider. We will consider graphs of the category Γ_σ^ω where σ now stands for countably infinite. More precisely, an element of Γ_σ^ω is the union of a countably infinite number of analytic, mutually disjoint (except possibly for their endpoints) curves called edges of compact or non-compact range which have no accumulation points of edges or vertices. In other words, the restriction of the graph to any compact subset of the hypersurface looks like an element of Γ_0^ω . These are precisely the kinds of graphs that one would consider in the thermodynamic limit of lattice gauge theories and are therefore best suited for our semi-classical considerations since it will be on such graphs that one can write actions, Hamiltonians and the like.

The construction of the ITP of Hilbert spaces is due to von Neumann [19] and already more than sixty years old. We will try to outline briefly some of the mathematical notions involved (see the fifth reference in [3] for a concise summary of the most important definitions and theorems).

Let for the time being I be any index set whose cardinality $|I| = \aleph$ takes values in the set of non-standard numbers (Cantor's alephs). Suppose that for each $e \in I$ we have a Hilbert space \mathcal{H}_e with scalar product $\langle \cdot, \cdot \rangle_e$ and norm $\|\cdot\|_e$. For complex numbers z_e we say that $\prod_{e \in I} z_e$ converges to the number z provided that for each positive number $\delta > 0$ there exists a finite set $I_0(\delta) \subset I$ such that for any other finite J with $I_0(\delta) \subset J \subset I$ it holds that $|\prod_{e \in J} z_e - z| < \delta$. We say that $\prod_{e \in I} z_e$ is *quasi-convergent* if $\prod_{e \in I} |z_e|$ converges. If $\prod_{e \in I} z_e$ is quasi-convergent but not convergent we define $\prod_{e \in I} z_e := 0$. Next we say that for $f_e \in \mathcal{H}_e$ the ITP $\otimes_f := \otimes_e f_e$ is a C_0 vector (and $f = (f_e)$ a C_0 sequence) if

$\| \otimes_f \| := \prod_{e \in I} \|f_e\|_e$ converges to a non-vanishing number. Two C_0 sequences f, f' are said to be strongly resp. weakly equivalent provided that

$$\sum_e | \langle f_e, f'_e \rangle_e - 1 | \text{ resp. } \sum_e \| \langle f_e, f'_e \rangle_e - 1 \| \quad (4.2)$$

converges. The strong and weak equivalence class of f is denoted by $[f]$ and (f) respectively and the set of strong and weak equivalence classes by \mathcal{S} and \mathcal{W} respectively. We define the ITP Hilbert space $\mathcal{H}^\otimes := \otimes_e \mathcal{H}_e$ to be the closed linear span of all C_0 vectors. Likewise we define $\mathcal{H}_{[f]}^\otimes$ or $\mathcal{H}_{(f)}^\otimes$ to be the closed linear spans of only those C_0 vectors which lie in the same strong or weak equivalence class as f . The importance of these notions is that they determine much of the structure of \mathcal{H}^\otimes , namely :

- 1) All the $\mathcal{H}_{[f]}^\otimes$ are isomorphic and mutually orthogonal.
- 2) Every $\mathcal{H}_{(f)}^\otimes$ is the closed direct sum of all the $\mathcal{H}_{[f']}\otimes$ with $[f'] \in \mathcal{S} \cap (f)$.
- 3) The ITP \mathcal{H}^\otimes is the closed direct sum of all the $\mathcal{H}_{(f)}^\otimes$ with $(f) \in \mathcal{W}$.
- 4) Every $\mathcal{H}_{[f]}^\otimes$ has an explicitly known orthonormal von Neumann basis.
- 5) If s, s' are two different strong equivalence classes in the same weak one then there exists a unitary operator on \mathcal{H}^\otimes that maps \mathcal{H}_s^\otimes to $\mathcal{H}_{s'}^\otimes$, otherwise such an operator does not exist, the two Hilbert spaces are unitarily inequivalent subspaces of \mathcal{H}^\otimes .

Notice that two isomorphic Hilbert spaces can always be mapped into each other such that scalar products are preserved (just map some orthonormal bases) but here the question is whether this map can be extended unitarily to all of \mathcal{H}^\otimes . Intuitively then, strong classes within the same weak classes describe the same physics, those in different weak classes describe different physics such as an infinite difference in energy, magnetization, volume etc. See [20] and references therein as well as appendix C for illustrative examples.

Next, given a bounded operator a_e on \mathcal{H}_e (notice that closed unbounded operators have a polar decomposition into an unitary and a self-adjoint piece and that a self-adjoint operator is completely determined by its bounded spectral projections so that restriction to bounded operators is no loss of generality) we can extend it in the natural way to \mathcal{H}^\otimes by defining \hat{a}_e densely on C_0 vectors through $\hat{a}_e \otimes_f = \otimes_{f'}$ with $f'_e = f_e$ for $e' \neq e$ and $f'_e = a_e f_e$. It turns out that the algebra of these extended operators for a given label is automatically a von Neumann algebra for \mathcal{H}^\otimes and we will call the weak closure of all these algebras the von Neumann algebra \mathcal{R}^\otimes of local operators.

Given these notions, the strong equivalence class Hilbert spaces can be characterized further as follows. First of all, for each $s \in \mathcal{S}$ one can find a representant $\Omega^s \in s$ such that $\|\Omega^s\| = 1$. Moreover, one can show that \mathcal{H}_s^\otimes is the closed linear span of those C_0 vectors $\otimes_{f'}$ such that $f'_e = \Omega_e^s$ for all but finitely many e . In other words, the strong equivalence class Hilbert spaces are irreducible subspaces for \mathcal{R}^\otimes , Ω^s is a cyclic vector for \mathcal{H}_s^\otimes on which the local operators annihilate and create local excitations and thus, if I is countable, \mathcal{H}_s^\otimes is actually separable. We see that we make naturally contact with Fock space structures, von Neumann algebras and their factor type classification [21] (modular theory) and algebraic quantum field theory [22]. The algebra of operators on the ITP which are not local do not have an immediate interpretation but it is challenging that they map between different weak equivalence classes and thus change the physics in a drastic way. It is speculative that this might in fact enable us to incorporate dynamical topology change in canonical quantum gravity.

A number of warnings are in order :

- 1) Scalar multiplication is not multi-linear ! That is, if f and $z \cdot f$ are C_0 sequences where

$(z \cdot f)_e = z_e f_e$ for some complex numbers z_e then $\otimes_f = (\prod_e z_e) \otimes_f$ is in general wrong, it is true if and only if $\prod_e z_e$ converges.

2) Unrestricted use of the associative law of tensor products is false ! Let us subdivide the index set I into mutually disjoint index sets $I = \cup_\alpha I_\alpha$ where α runs over some other index set A . One can now form the different ITP $\mathcal{H}'^\otimes = \otimes_\alpha \mathcal{H}_\alpha^\otimes$, $\mathcal{H}_\alpha^\otimes = \otimes_{e \in I_\alpha} \mathcal{H}_e$. Unless the index set A is finite, a generic C_0 vector of \mathcal{H}'^\otimes is orthogonal to all of \mathcal{H}^\otimes . This fact has implications for quantum gravity which we outline below.

After this mathematical digression we now come back to canonical quantum general relativity. In applying the above concepts we arrive at the following surprises :

- i) First of all, we fix an element $\gamma \in \Gamma_\sigma^\omega$ and choose the countably infinite index set $I := E(\gamma)$, the edge set of γ . If $|E(\gamma)|$ is finite then the ITP Hilbert space $\mathcal{H}_\gamma^\otimes := \otimes_{e \in E(\gamma)} \mathcal{H}_e$ is naturally isomorphic with the subspace \mathcal{H}_γ^{AL} of the Ashtekar Lewandowski Hilbert space obtained as the closed linear span of cylinder functions over γ . However, if $|E(\gamma)|$ is truly infinite then a generic C_0 vector of $\mathcal{H}_\gamma^\otimes$ is orthogonal to any possible $\mathcal{H}_{\gamma'}^{AL}$, $\gamma' \in \Gamma_0^\omega$. Thus, even if we fix only one $\gamma \in \Gamma_\sigma^\omega$, the total \mathcal{H}^{AL} is orthogonal to almost every element of $\mathcal{H}_\gamma^\otimes$.
- ii) Does $\mathcal{H}_\gamma^\otimes$ have a measure theoretic interpretation as an L_2 space ? By the Kolmogorov theorem [23] the infinite product of probability measures is well defined and thus one is tempted to identify $\mathcal{H}_\gamma^\otimes = \otimes_e L_2(SU(2), d\mu_H)$ with $\mathcal{H}_\gamma^{AL'} := L_2(\times_e SU(2), \otimes_e d\mu_H)$. However, this cannot be the case, the ITP Hilbert space is non-separable (as soon as $\dim(\mathcal{H}_e) > 1$ for almost all e and $|E(\gamma)| = \infty$) while the latter Hilbert space is separable, in fact, it is the subspace of \mathcal{H}^{AL} consisting of the closed linear span of cylindrical functions over γ' with $\gamma' \in \Gamma_0^\omega \cap E(\gamma)$.
- iii) Yet, there is a relation between $\mathcal{H}_\gamma^\otimes$ and \mathcal{H}^{AL} through a construction called the inductive limit of Hilbert spaces : We can find a directed sequence of elements $\gamma_n \in \Gamma_0^\omega \cap E(\gamma)$, that is, $\gamma_m \subset \gamma_n$ for $m \leq n$, such that γ is its limit in Γ_σ^ω . The subspaces $\mathcal{H}_{\gamma_n}^{AL} \subset \mathcal{H}^{AL}$ are isometric isomorphic with the subspaces of $\mathcal{H}_\gamma^\otimes$ given by the closed linear span of vectors of the form $\psi_{\gamma_n} \otimes [\otimes_{e \in E(\gamma - \gamma_n)} 1]$ where $\psi_{\gamma_n} \in \mathcal{H}_{\gamma_n}^{AL} \equiv \mathcal{H}_{\gamma_n}^\otimes$ which provides the necessary isometric monomorphism to display $\mathcal{H}_\gamma^\otimes$ as the inductive limit of the $\mathcal{H}_{\gamma_n}^{AL}$.
- vi) So far we have looked only at a specific $\gamma \in \Gamma_\sigma^\omega$. We now construct the total Hilbert space

$$\mathcal{H}^\otimes := \overline{\cup_{\gamma \in \Gamma_\sigma^\omega} \mathcal{H}_\gamma^\otimes} \quad (4.3)$$

equipped with the natural scalar product derived in the fifth reference of [3]. This is to be compared with the Ashtekar-Lewandowski Hilbert space

$$\mathcal{H}^{AL} := \overline{\cup_{\gamma \in \Gamma_0^\omega} \mathcal{H}_\gamma^{AL}} = \overline{\cup_{\gamma \in \Gamma_\sigma^\omega} \mathcal{H}_\gamma^{AL'}} \quad (4.4)$$

The identity in the last line enables us to specify the precise sense in which $\mathcal{H}^{AL} \subset \mathcal{H}^\otimes$: For any $\gamma \in \Gamma_\sigma^\omega$ the space $\mathcal{H}_\gamma^{AL'}$ is isometric isomorphic as specified in iii) with the strong equivalence class Hilbert subspace $\mathcal{H}_{\gamma, [1]}^\otimes$ where $1_e = 1$ is the constant function equal to one. Thus, the the Ashtekar Lewandowski Hilbert space describes the local excitations of the Ashtekar-Lewandowski “vacuum” Ω^{AL} with $\Omega_e^{AL} = 1$ for any possible analytic path e .

Notice that both Hilbert spaces are non-separable, but there are two sources of non-separability : the Ashtekar Lewandowski Hilbert space is non-separable because Γ_0^ω has uncountable infinite cardinality. This is also true for the ITP Hilbert since $\Gamma_0^\omega \subset \Gamma_\sigma^\omega$ but it has an additional character of non-separability : even for fixed γ with an infinite number of edges the Hilbert space $\mathcal{H}_\gamma^\otimes$ splits into an uncountably infinite number of mutually orthogonal strong equivalence class Hilbert spaces and $\mathcal{H}_\gamma^{AL'}$ is only one of them.

- v) Recall that spin-network states [8] form a basis for \mathcal{H}^{AL} . The result of iv) states that they are no longer a basis for the ITP. The spin-network basis is in fact the von Neumann basis for the strong equivalence class Hilbert space determined by $[\Omega^{AL}]$ but for the others we need uncountably infinitely many other bases, even for fixed γ . The technical reason for this is that, as remarked above, the unrestricted associativity law fails on the ITP.

We would now like to justify this huge blow up of the original Ashtekar Lewandowski Hilbert space from the point of view of physics. Clearly, there is a blow up only when the initial data hypersurface is non-compact as otherwise $\Gamma_0^\omega = \Gamma_\sigma^\omega$.

- a) Let us fix a three manifold Σ and a graph $\gamma \in \Gamma_\sigma^\omega$ in order to describe semi-classical physics on that graph as outlined in section 2. Given a classical initial data set m we can construct a coherent state $\psi_{\gamma,\Sigma,X,m}$ which in fact is a C_0 vector $\otimes_{f_m}^\gamma$ for $\mathcal{H}_\gamma^\otimes$ of unit norm. This coherent state can be considered as a “vacuum” or “background state” for quantum field theory on the associated spacetime. As remarked above, the corresponding strong equivalence class Hilbert space $\mathcal{H}_{\gamma,[f_m]}^\otimes$ is obtained by acting on the “vacuum” by local operators (where local means that only finitely many edges are affected), resulting in a space isomorphic with the familiar Fock spaces and which is separable. In this sense, the fact that $\mathcal{H}_\gamma^\otimes$ is non-separable, being an uncountably infinite direct sum of strong equivalence class Hilbert spaces, simply accounts for the fact that in quantum gravity *all vacua have to be considered simultaneously, there is no distinguished vacuum as we otherwise would introduce a background dependence into the theory.*
- b) The Fock space structure of the strong equivalence classes immediately suggests to try to identify suitable excitations of $\psi_{\gamma,m}$ as graviton states propagating on a spacetime fluctuating around the classical background determined by m [24]. Also, it is easy to check whether for different solutions of Einstein’s equations the associated strong equivalence classes lie in different weak classes and are thus physically different. For instance, preliminary investigations indicate that Schwarzschild black hole spacetimes with different masses lie in the *same* weak class. Thus, *unitary* black hole evaporation and formation seems not to be excluded from the outset.
- c) From the point of view of $\mathcal{H}_\gamma^{AL'}$ the Minkowski coherent state is an everywhere excited state like a thermal state, the strong classes $[\Omega^{AL}]$ and $[f_m]$ for Minkowski data m are orthogonal and lie in different weak classes. The state Ω^{AL} has no obvious semi-classical interpretation in terms of coherent states for any classical spacetime.
- d) If we look at the construction in the fourth reference of [2] then, provided one looks *only at local* operators in the *bulk* of a non-compact three manifold, we can avoid

the ITP by using the GNS construction. In fact, specifying a GNS state is the same as looking at a specific representant of a strong equivalence class of our ITP Hilbert space, in that sense the construction in the fourth reference in [2] gives only a special subspace of our ITP. More importantly, however, our framework is much more general : First of all, we can incorporate non-local operators such as Hamiltonian constraint operators smeared against test fields of rapid decrease rather than compact support (as required in order to discuss supertranslations) or matter Hamiltonians coupled to gravity important in order to obtain effective matter Hamiltonians of the standard model in a gravitational state fluctuating around Minkowski space. Secondly, we can treat operators that are defined not only in the bulk but also at spatial infinity thus enabling us to consider electromagnetic charges (when coupling matter) and the Poincaré charges. Moreover, our approach provides a unifying and rigorous framework of how to address the infinite volume (thermodynamic) limit of the theory.

- e) Another important kind of operators that is crucial in order to make contact with perturbative quantum gravity and string theory is the graviton annihilation and creation operator labelled by a specific momentum (and helicity) mode and which is therefore a maximally non-local operator. Once we have identified suitable graviton Fock states propagating on a fluctuating quantum metric as defined through the excitations of a unit norm representant of the corresponding strong equivalence class, we should be able, in principle to make contact with perturbative quantum field theory and thus to deliver significant evidence that loop quantum gravity is not just some completely spurious sector of quantum gravity.

5 Applications and Outlook

Given the tools provided in [3, 5] one can start investigating many fascinating physical problems starting from non-perturbative quantum general relativity. We will list a few of them.

- i) *Generalization to Higher Ranks*

In [3] peakedness and Ehrenfest theorems were analytically proved for rank one gauge groups and a generalization of the proof to higher ranks was sketched. It would be important to fill in the details at least for $G = SU(3)$, if not for arbitrary compact gauge groups.

- ii) *Quantum Field Theory on Curved Spacetimes*

As already mentioned in the main text, choosing a coherent state peaked on the initial values of a solution to Einstein's equations one obtains a cyclic state for a Fock-like representation. It would be important to isolate suitable graviton states in such a representation in order to make contact with perturbative quantum gravity and string theory. The same can be done when coupling matter and one could, for instance, study photons propagating on fluctuating quantum spacetimes.

- iii) *Renormalization Group and Diffeomorphism Group*

Given a Fock like structure one can try to set up an analog of the conventional renormalization techniques. One expects that from non-perturbative quantum gravity no

UV singularities arise (see [11]) for hint in that direction). But then one would like to see how they get absorbed as compared to the conventional perturbative approaches and the obvious guess is that it is the diffeomorphism invariance of the theory that lets us get rid of them (since there is no background metric, one cannot tell from a fundamental point of view whether a momentum gets large or small). It should be possible to nail down the precise mechanism for UV-finiteness with our tools.

iv) *Avoidance of Classical Singularity Theorems*

One can plug into the label of a coherent state a solution $m(\tau)$ of Einstein's equations (in some gauge) which becomes singular after some time lapse τ_0 . It would be challenging to see whether the classically singular function remains finite in the sense of expectation values of the corresponding operator at $\tau = \tau_0$. The recent regularity findings obtained within loop quantum cosmology [25] which uses the techniques developed in [11] are an indication that this actually could be the case in the full theory as well.

v) *Corrections to the Standard Model*

One of the major motivations for the construction of coherent states is, of course, the question whether our non-perturbative quantum theory has the correct classical limit. But beyond that one can start computing corrections to the standard model, a development that has just started. See for instance the two first references in [2] discussing Poincaré invariance violating effects that are not yet ruled out to lie in the detectable regime. In order to improve these calculations it would be desirable to have a fast diagonalization computer code for the volume operator [4, 26] since it plays an important role in the quantization of the Hamiltonian (constraints) [11].

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A Construction of Approximate Operators

In this section we construct an approximate area operator for a specific class of surfaces.

Suppose we are given some embedded graph $\gamma = X(\tilde{\gamma})$ (here given as the image under the embedding X of a model graph $\tilde{\gamma}$) and look at the area function $A_S(m) := \int_{Y^{-1}(S)} d^2y \sqrt{\det(Y^*q)}$ of an embedded surface S (here given as the image under the embedding Y of a model two surface \check{S} into Σ) where q denotes the three metric determined by m . We choose a polyhedral decomposition P_γ of Σ dual to γ and a corresponding system of paths Π_γ inside its faces. Our first task is to write down a function of the corresponding graph degrees of freedom $g_e = (h_e, P_j^e)$ that approximates A_S . As compared to the construction in the main text we choose here for simplicity $X_0 = X, \varphi_{\gamma,S} = \text{id}_\Sigma$.

Without specifying exactly the topology of $\tilde{\gamma}, \check{S}$ it is difficult to write down an explicit formula, so let us assume for simplicity that $\tilde{\gamma}$ has the topology of a (possibly infinite) cubic lattice. In this case model edges $\check{e}_I(\check{v})$ can be labelled by a direction index $I = 1, 2, 3$ and a vertex index $\check{v} \in \mathbb{Z}^3$ and we think of them as given by unit intervals along the coordinate axes of \mathbb{R}^3 . Furthermore, we take as P_γ the image under X of the faces $S^I(v)$ (dual to $e_I(v)$) of the model graph given by the translation of $\tilde{\gamma}$ by the vector $(1/2, 1/2, 1/2)$ in \mathbb{R}^3 and for Π_γ we take the image under X of the natural radial paths in each of the faces. Likewise, for definiteness, we assume that \check{S} has the topology of a square so that u, v range over the interval $(-N, N)$ where N is an integer. The embedding $X : \mathbb{R}^3 \mapsto \Sigma; \vec{t} \mapsto X(\vec{t})$ provides a (local) coordinate chart so that we can think of Y as a composition $Y^a(u, v) = X^a(\vec{t}(u, v))$. The coordinates \vec{t}, u, v are taken to be dimensionless. We can then write the area function explicitly as

$$A_S(q) = \int_{[-N, N]^2} dudv \sqrt{[(E_j^a(X(\vec{t}))n_a^I(\vec{t}))_{\vec{t}=\vec{t}(u,v)}n_I(u, v)]^2} \quad (\text{A.1})$$

where $n_a^I(\vec{t}) = \frac{1}{2}\epsilon_{abc}\epsilon^{IJK}X_{,J}^b(\vec{t})X_{,J}^c(\vec{t})$ and $n_I(u, v) = \epsilon_{IJK}t_{,u}^J t_{,v}^K(u, v)$.

Let us now partition \check{S} into the maximal, connected, open pieces \check{S}_k which lie completely within a coordinate cube of \mathbb{R}^3 , let \check{v}_k be the vertex of that cube, choose any $(u_k, v_k) \in \check{S}_k$ and let $\mu_k := \int_{\check{S}_k} dudv$. Then, given any point $m \in \mathcal{M}$ we denote by $P_j^I(v, m) := P_j^{e_I(v)}(m)$ and $n_I(k) := n_I(u_k, v_k)$. With these choices we arrive at the gauge invariant approximate function

$$A_{S,\gamma}(m) := a^2 \sum_k \mu_k \sqrt{[P_j^I(v_k, m)n_I(k)]^2} \quad (\text{A.2})$$

where $v_k = X(\check{v}_k)$ which now depends explicitly on our choices (collectively denoted by γ) as well as the connection and so is a function of m and not only of q_{ab} . By construction, for fixed S (A.2) is a Riemann sum for the integral (A.1) and thus converges to it pointwise in \mathcal{M} in the limit that the sum over k involves infinitely many terms (the model lattice becomes finer and finer within the model surface). Notice that the coefficients $\mu_k, n_I(k)$ and the sum over the \check{S}_k do not depend on X or m and thus (A.2) defines a well-defined function on \mathcal{M} . Moreover, (A.2) is diffeomorphism covariant since the only dependence on X rests in the $P^I(v, m)$ while the $\mu_k, n_I(k), \check{S}_k$ are diffeomorphism invariant. Equation (A.2) thus serves as a suitable departure point for quantization.

According to our general programme we define now the substitute operator

$$\hat{A}_{S,\gamma} = a^2 \sum_k \mu_k \sqrt{[\hat{P}_j^I(v_k) n_I(k)]^2} \quad (\text{A.3})$$

which is to be thought of as an operator on \mathcal{H}_γ since it is already consistently defined on every subgraph of γ . By doing a similar construction for an arbitrary graph and by enforcing cylindrical consistency we in principle (we do not know it explicitly but must construct it graph by graph) arrive at a ‘‘continuum’’ operator \hat{A}_S .

Let us now consider in a specific example for γ, S how the expectation values are computed. Consider the embeddings into $\Sigma := \mathbb{R}^3$

$$X^a(\vec{t}) := \epsilon \delta_I^a t^I, \quad Y^a(u, v) = \epsilon(u, v, \tau u), \quad \vec{t}(u, v) = (u, v, \tau u) \quad (\text{A.4})$$

where $0 \leq \tau \leq 1$ is a fixed parameter and $u, v \in [-N, N]$. Equation (A.4) defines a plane parallel to the t^2 direction and which is tilted against the t^1 direction by an angle α defined by $\tan(\alpha) = \tau$ (for $\alpha \geq \pi/4$ interchange the directions of t^1, t^2). We will not consider the most general case for which it is difficult to write down an explicit formula but rather consider the case that $\tau = 1/M$ where $M \geq 1$ is an integer such that N/M is also a natural number. In this case we are looking at the vertices $\check{v}(n_1, n_2) = (n^I) = (n_1, n_2, n_3(n_1))$, $n_1, n_2 = -N, -N+1, \dots, N-1$ where $n_3(n_1) = -N\tau + [(n_1 + N)\tau]$ (Gauss bracket). The \check{S}_k are simply model unit squares so that $\mu_k = 1$. We easily compute $(n_I(k)) = (-\tau, 0, 1)$ so that (A.3) becomes

$$\hat{A}_{S,\gamma} = a^2 \sum_{n_1, n_2} \sqrt{[\hat{P}_j^3(v(n_1, n_2)) - \tau \hat{P}_j^1(v(n_1, n_2))]^2} \quad (\text{A.5})$$

where $v(n_1, n_2) = X(\check{v}(n_1, n_2))$.

This expression is to be compared with the projection onto \mathcal{H}_γ of the operator of [4]

$$\begin{aligned} \hat{A}'_{S,\gamma} &= \frac{1}{2} \sum_{n_2} \left\{ \sum_{n_1 \neq -N+kM} \sqrt{[J_j^{3,up} - J_j^{3,down}]^2(X(\check{v}(n_1, n_2)))} \right. \\ &\quad \left. + \sum_{n_1 = -N+kM} \sqrt{[J_j^{3,up} - J_j^{3,down} + J_j^{1,left} - J_j^{3,right}]^2(X(\check{v}(n_1, n_2)))} \right\} \quad (\text{A.6}) \end{aligned}$$

where $J_j^e = i l_p^2 / 2 \text{tr}((\tau_j h_e)^T \partial / (\partial h_e))$ and $J^{I,*}(v)_j = J_j^{e_I(v)*}$ where $e_I(v)*$ is the segment of $e_I(v)$ outgoing from the intersection of $e_I(v)$ with S into the direction $* = up, down, left, right$ and $e_I(v) = X(\check{e}_I(\check{v}))$. A simple calculation reveals that for $e = (e^{down})^{-1} \circ e^{up}$ or $e = (e^{left})^{-1} \circ e^{right}$ we have $J_j^{I,*}(v) f(h_e) = a^2 \hat{P}^I(v)_j f(h_e)$ if $* = up, right$ and $J^{I,*}(v) f(h_e) = -a^2 O_{jk}(h_e) \hat{P}^I(v)_k f(h_e)$ if $* = down, left$ where the orthogonal matrix $O_{jk}(h)$ is defined by $\text{Ad}_h(\tau_j) = O_{jk}(h) \tau_k$. It follows that

$$\begin{aligned} \hat{A}'_{S,\gamma} &= \frac{a^2}{2} \sum_{n_2} \left\{ \sum_{n_1 \neq -N+kM} \sqrt{[(\delta_{jk} + O_{jk}(h_{e^3(v(n_1, n_2))})) \hat{P}_k^3(v(n_1, n_2))]^2} \right. \\ &\quad \left. + \sum_{n_1 = -N+kM} \sqrt{[(\delta_{jk} + O_{jk}(h_{e^3(v(n_1, n_2))})) \hat{P}_k^3(v(n_1, n_2)) - (\delta_{jk} + O_{jk}(h_{e^1(v(n_1, n_2))})) \hat{P}_k^1(v(n_1, n_2))]^2} \right\} \quad (\text{A.7}) \end{aligned}$$

Let us test these expressions in a coherent state for flat data $m = (A_a^j = 0, E_j^a = \delta_j^a)$. Notice that then the parameter ϵ is really the edge length determined by m . Then $h_{e^I(v)} = 1, P_j^I(v) = \delta_j^I \epsilon^2$ and the Ehrenfest theorems proved in [3] reveal that the zeroth order in \hbar of the expectation values is given by

$$\begin{aligned} \lim_{t \rightarrow 0} \langle \psi_{\gamma, m}, \hat{A}_{S, \gamma} \psi_{\gamma, m} \rangle &= (2N\epsilon)^2 \sqrt{1 + \tau^2} \\ \lim_{t \rightarrow 0} \langle \psi_{\gamma, m}, \hat{A}'_{S, \gamma} \psi_{\gamma, m} \rangle &= (2N\epsilon)^2 [1 + \tau(\sqrt{2} - 1)] \end{aligned} \quad (\text{A.8})$$

The value in the first line of (A.8) is in fact the exact classical area of the surface while the second line is off (equal or bigger than) that value. *This is the staircase problem.* We get coincident results only for $\tau = 0, 1$ corresponding to $M = \infty, 0$. The relative mistake is zero for $M = 1$, then reaches its maximum of 8% at $M = 2$ and then monotonously decreases as $1/M$ to zero. An error of eight percent is unacceptable which is why we are forced to use \hat{A}_S instead of \hat{A}'_S which is guaranteed to give results of sufficiently high accuracy. Notice that the dual faces of the graph under consideration in this example *are not at all tangential to the surface S* and still we get a good (even exact) result.

Yet, this example was special in the sense that the coefficients $\mu_k, n_I(k)$ were independent of k so that one would have gotten the same result even if there was only one cube needed (replace N by 1 and ϵ by $N\epsilon$) which is due to the fact that in this case (A.2) gives the exact classical value. If the surface would not be a plane, then the coefficients would vary and (A.2) no longer gives the exact classical value. If the surface S wiggles at a scale much lower than ϵ then (A.2) will be a very bad approximation. In that case we can improve the result by increasing the number of edges of the lattice and thus this error is a *finite size effect*. Next, in replacing (A.1) by (A.2) it was crucial that m does not vary too much on an embedded edge of the graph or the dual embedded face. An error due to this is related to the curvature of m and will be called *a curvature effect*.

In what follows we will exclude finite size effects by the assumption that the surfaces that we are interested in are wiggling at most at the scale of the graph. More precisely, the $S_k = X(\tilde{S}_k)$ are allowed not to lie at all inside the dual faces of the embedded graph but the tangent space should be approximately constant over S_k . For instance, we allow for a surface which is parallel to the t^2 direction as above but such that τ jumps between ± 1 from cube to cube (its projection into the t^1, t^3 plane is a zig-zag curve with triangle edge length $\sqrt{2}\epsilon$). However, we do not allow τ to jump within one cube. We are then left with curvature effects to which we turn in the next section.

B Minimization of Fluctuations through Optimization of Scales

In this section we will sketch the computation of fluctuations in our scheme leading to a fixing of the free parameters a, ϵ through optimization.

Given the fact that we are in a gauge field theory context, it is motivated to consider as an elementary set of classical continuum observables the non-Abelian analogues of the electric and magnetic fluxes through embedded surfaces S . At this point we could say that we measure only the quantum fluxes associated with the dual faces of a given graph since this information is enough in order to reconstruct any other classical observable in

the fashion outlined in the previous section A. In that case and after normal ordering there would be no classical or normal ordering error at all. However, one would like to be more general and consider fluxes through arbitrary surfaces and not only elementary ones because one would like to measure these more general observables directly at the quantum level and not only through an indirect procedure using a classical formula. Actually, if the Gauss law $\partial_a E_j^a = 0$ holds and due to the Bianchi identity, we can replace S by any surface S' built entirely from dual faces of the polyhedral decomposition such that $S - S' = \partial R$ is the boundary of a region R (possibly up to some negligible piece at the boundary of S) without changing the classical flux functions. Thus, for the fluxes (on the constraint surface of the Gauss constraint) we could actually keep the discussion at the level of surfaces built from elementary faces which would simplify the discussion, however, we will not do that for the sake of generality and in view of applications to other sets of basic variables which are not fluxes and for which an off-shell approximation might be desirable.

In order to keep the presentation at an elementary level we will replace $SU(2)$ by $U(1)^3$ as otherwise the non-Abelianness blows up the effort to obtain all the estimates by an order of magnitude (see especially the fourth reference in [3]) without changing the end result. We are thus concerned with the following classical (kinematical) observables

$$B_j(S) = \int_S F_j(x) \text{ and } E_j(S) = \int_S (*E_j)(x) \quad (\text{B.1})$$

where F_j is the curvature two-form of A^j . Notice that F_j, E_j are gauge invariant. In the non-Abelian case one would need to choose a point p inside S and a system of paths between p and any other point $x \in S$ and integrate instead of $*E_j(x), F_j(x)$ their image under the adjoint action of $SU(2)$ evaluated at the holonomies along those paths in order to obtain a gauge-covariant result. The non-Abelian case will be discussed in more detail elsewhere.

Using the same notation as in the previous section we have a model graph $\tilde{\gamma}$ of cubic topology and a model surface \tilde{S} of square topology which are embedded into Σ via embeddings X and $Y = X \circ \vec{t}$ respectively so that

$$B_j(S) = \int_{\tilde{S}} dudv [(n_a^I(\vec{t}) B_j^a(X(\vec{t}))_{\vec{t}=\vec{t}(u,v)} n_I(u, v)) \quad (\text{B.2})$$

and similar for $E_j(S)$ where $B_j^a = \frac{1}{2} \epsilon^{abc} F_{bc}^j$. Using the same dual polyhedral decomposition and choices of paths as in the previous section we arrive at the classical substitute functions

$$\begin{aligned} B_{j,\gamma}(S) &= \sum_k \mu_k n_I(u_k, v_k) \frac{h_{\alpha^I(v_k)}^j - (h_{\alpha^I(v_k)}^j)^{-1}}{2i} \\ E_{j,\gamma}(S) &= a^2 \sum_k \mu_k n_I(u_k, v_k) P_j^I(v_k) \end{aligned} \quad (\text{B.3})$$

where $\alpha^I(v) = X(\check{\alpha}_I(\check{v}))$ is the image under X of the model plaquette loop $\check{\alpha}_I(\check{v}) = \check{e}_J(\check{v}) \circ \check{e}_K(\check{v} + b_J)^{-1} \circ \check{e}_J(\check{v} + b_K)^{-1} \circ \check{e}_K(\check{v})^{-1}$ on $\tilde{\gamma}$ in the J, K plane with $\epsilon_{IJK} = 1$ and with vertex at $\check{v} = X^{-1}(v)$. Here, b_I denotes the standard orthonormal basis in \mathbb{R}^3 . Notice

that the path system Π_γ is actually not needed in order to define $P_j^I(v)$ since $U(1)^3$ is Abelian. Due to the relation

$$g_I^j(v) = e^{P_j^I(v)} h_{e_I(v)}^j \quad (\text{B.4})$$

it is possible to replace (B.3) by functions which are linear combinations of products of the $g_I^j(v), (g_I^j(v))^{-1}, \hat{g}_I^j(v), (\hat{g}_I^j(v))^{-1}$. These functions are the classical analogues of the creation operators $\hat{g}_I^j(v), (\hat{g}_I^j(v))^{-1}$ and annihilation operators $(\hat{g}_I^j(v))^\dagger, ((\hat{g}_I^j(v))^{-1})^\dagger$ of section 3 and with the help of these functions it becomes possible to normal order the corresponding operators in the quantum theory. For example, $g_I^j(v) + (g_I^j(v))^{-1} = h_I^j(v)(1 + O((\frac{\epsilon}{a})^4))$ and $\hat{g}_I^j(v)g_I^j(v) - (g_I^j(v))^{-1}(\hat{g}_I^j(v))^{-1} = 4P_j^I(v)(1 + O((\frac{\epsilon}{a})^4))$. By using (finitely many) higher powers of these four functions one can suppress the subleading terms to any desired order in (ϵ/a) .

We can then quantize (B.3) as

$$\begin{aligned} \hat{B}_{j,\gamma}(S) &:= \sum_k \mu_k n_I(u_k, v_k) : \frac{\hat{h}_{\alpha^I(v_k)}^j - (\hat{h}_{\alpha^I(v_k)}^j)^{-1}}{2i} : \\ \hat{E}_{j,\gamma}(S) &= a^2 \sum_k \mu_k n_I(u_k, v_k) : \hat{P}_j^I(v_k) : \end{aligned} \quad (\text{B.5})$$

where the normal ordering symbol is to be understood after substituting the just mentioned approximations up to $(\epsilon/a)^{2n}$ for some desired n . There is then no normal ordering error.

Next, we are concerned with the classical error given by

$$\begin{aligned} E_j(S)(\Delta E_j(S))_{class} &\leq a^2 \sum_k \mu_k |n_I(u_k, v_k)| : P_j^I(v_k) : - P_j^I(v_k) \\ &\quad + |E_j(S) - \sum_k \mu_k n_I(u_k, v_k) E_j(S^I(v_k))| \end{aligned} \quad (\text{B.6})$$

where $S^I(v)$ is the image under X of the surface $\check{S}^I(\check{v})$ dual to $\check{e}_I(\check{v})$ and similar for $(\Delta B_j(S))_{class}$. The normal ordering symbols in (B.6) just mean that one should replace the functions inside the normal ordering symbols by the above mentioned functions of the $g_I^j(v)$. Let us estimate these terms. If we denote by $A_S(E)$ the classical area of S determined by E then the number of terms in the sum \sum_k will be of the order of $A_S(E)/\epsilon^2$ and so the first term can be bounded by $A_S(E)(\tilde{E}_S^j)^n(\epsilon/a)^{2(n-1)}$ where \tilde{E}_S^j is the maximum of $E_j(S')/A_E(S')$ as we vary $S' \subset S$. For the second term we employ the Euler-MacLaurin estimates [27] for the numerical difference between an integral and a Riemann sum. We will use only the coarsest estimate resulting from the identity

$$\int_y^{y+N\epsilon} dx F(x) - \epsilon \left[\frac{F(y) + F(y+N\epsilon)}{2} + \sum_{k=1}^{N-1} F(y+k\epsilon) \right] = \frac{\epsilon^3}{2} \int_0^1 dt \phi_2(t) \left[\sum_{k=0}^{N-1} F''(y+(m+t)\epsilon) \right] \quad (\text{B.7})$$

where $\phi_2(t) = t^2 - t$ is the Bernoulli polynomial of the second degree and F is a twice differentiable function on the interval $[y, y+N\epsilon]$. Using $\phi_2(t) \leq 1/4$ for $0 \leq t \leq 1$ the right hand side of (B.7) can be bounded from above by $\epsilon^2/8 \int_y^{y+N\epsilon} dx |F''(x)|$ and thus involves the *second derivative* of F . Let us introduce the *curvature scale*

$$\frac{1}{L_F^2} := \left| \frac{\frac{\epsilon}{2} \int_0^1 dt \phi_2(t) \left[\sum_{k=0}^{N-1} F''(y+(m+t)\epsilon) \right]}{\int_y^{y+N\epsilon} dx F(x)} \right| \quad (\text{B.8})$$

Using this formula and iterating it since we are dealing with a two-dimensional integral we find that the second term can be bounded by $|E_j(S)|\epsilon^4/L_E^4$ where L_E is a lower bound on the *electric curvature radius* of the phase space point m and thus independent of j, S .

We will now make the assumption that $\epsilon \ll a, L_E$ and that n is sufficiently high so that the first term in (B.6) is sub-leading (since we will relate a to L_E later on, this is a self-consistency assumption). Then we obtain (estimates are in orders, so we neglect numerical factors of order one)

$$(\Delta E_j(S))_{class} \leq \frac{\epsilon^4}{L_E^4} \text{ and } (\Delta B_j(S))_{class} \leq \frac{\epsilon^4}{L_B^4} \quad (\text{B.9})$$

where the magnetic curvature radius will be of the same order as L_E since both types of curvature come from the same four-dimensional curvature tensor.

Finally we are left with the quantum mechanical error. Notice that the operators (B.5) are of the form $\hat{O} = \sum_k \hat{O}_k$ where the \hat{O}_k mutually commute for the electric flux while for the magnetic flux the next neighbour terms are non-commuting. However, the number of these non-commuting terms per vertex is of the same order as the number of the \hat{O}_k and since we are just concerned with orders of magnitude here their contribution to the following estimates gives just a multiplicative numerical factor of order one which we do not display. We can thus assume that all the \hat{O}_k are mutually commuting and obtain for the fluctuations in the coherent state $\psi_{\vec{\gamma}, \Sigma, X, m}$

$$\langle \hat{O} \rangle^2 (\Delta O)(m)_{quantum}^2 = \sum_k [\langle \hat{O}_k^2 \rangle - \langle \hat{O}_k \rangle^2] \leq t \frac{A_E(S)}{\epsilon^2} \quad (\text{B.10})$$

where the Ehrenfest theorems for $\hat{O}_k = \hat{P}_j^I(v), \hat{h}_I^j(v)$ of the third reference in [3] have been used in the estimate. For the total error we therefore find

$$\begin{aligned} (\Delta E_j(S))_{total}(m)^2 &\leq t \frac{a^4 A_E(S)}{\epsilon^2 E_j(S)^2} + \frac{\epsilon^8}{L_E^8} \approx \frac{\ell_p^2 a^2}{\epsilon^2 A_S(E) \bar{E}_S} + \frac{\epsilon^8}{L_E^8} \\ (\Delta B_j(S))_{total}(m)^2 &\leq t \frac{A_E(S)}{\epsilon^2 B_j(S)^2} + \frac{\epsilon^8}{L_B^8} \approx \frac{\ell_p^2 L_B^4}{a^2 \epsilon^2 A_S(E)} + \frac{\epsilon^8}{L_B^8} \end{aligned} \quad (\text{B.11})$$

where $\bar{E}_S = \sqrt{E_j(S)^2}/A_E(S)$ is of order unity and in the last line we have used the approximate identity $B_j(S) \approx A_E(S)/L_B^2$.

In order to fix the parameters ϵ, a we notice that by assumption $A_S(E) \geq \epsilon^2$ ($N = A_S(E)/\epsilon^2$ is the number of contributing faces and gives the law of large numbers $\propto 1/\sqrt{N}$ for the quantum mechanical fluctuations). Thus we can further estimate

$$\begin{aligned} (\Delta E_j(S))(m)^2 &\leq \frac{\ell_p^2 a^2}{\epsilon^4 \bar{E}_S} + \frac{\epsilon^8}{L_E^8} \\ (\Delta B_j(S))(m)^2 &\leq \frac{\ell_p^2 L_B^4}{a^2 \epsilon^4} + \frac{\epsilon^8}{L_B^8} \end{aligned} \quad (\text{B.12})$$

and we require that these two errors be equal (yielding an “unquenched uncertainty relation for the fluxes) and minimal. One can solve $\Delta E = \Delta B$ analytically for a^2 resulting in $a^2 = \alpha x^3 + \sqrt{(\alpha x^3)^2 + L_B^4 \bar{E}_S}$ where $\alpha = \bar{E}_S(1/L_B^8 - 1/L_E^8)/(2\ell_p^2)$ and $x = \epsilon^4$. The remaining optimization problem can then only be solved numerically. We will not treat

the most general case in this paper but make the physical assumption that $L_E \approx L_M \approx L$ so that $a^2 \approx \sqrt{E_S} L^2$. Notice that provided $\epsilon \ll L$ our self-consistency assumption from above is indeed satisfied. The optimization then yields an absolute minimum at $\epsilon^{12} = \ell_p^2 L^{10} / (2\sqrt{E_S})$. Thus, ϵ is locked at a *geometric mean* of the microscopic and macroscopic scale. We verify that indeed $\ell_p \ll \epsilon \ll L$ as long as $\ell_p \ll L$, finishing up our self-consistency check. The total fluctuation becomes up to a numerical factor equal to $(\frac{\ell_p}{LE_S})^{4/3} \leq (\frac{\ell_p}{L_0 E_S})^{4/3}$ where L_0 is the phase space bound mentioned in the main text and depends on \mathcal{M} but not on m . Locking a at L_0 from the outset would lead to qualitatively similar results.

We finish this section with a couple of remarks :

- 1.) Our analysis has optimized only an upper bound, it may well be that one can get even better estimates depending on m .
- 2.) One may wonder why the total fluctuation of our operator should depend on a classical error which then leads to an optimization problem in ϵ which one is not used to from electrodynamics. However, there actually the same effect happens [13] : If one considers in free Maxwell theory the fluctuations of electric and magnetic flux operators then one simply finds infinity. This is due to the fact that in the corresponding Fock representation one has to smear the fields over three dimensional regions rather than two-dimensional surfaces in order to arrive at a well-defined operator. Thus, in order to make the fluctuation calculation well-defined one has to fatten the surfaces by a transversal thickness ϵ (which is a regularization comparable to our replacing continuum functions by discretized objects) and then divides the fattened flux by ϵ . When one now computes the fluctuation of these operators compared to exact classical flux value one finds a similar competition between the quantum mechanical fluctuation blowing up as $\epsilon \rightarrow 0$ and the classical error vanishing as $\epsilon \rightarrow 0$. The optimization thus takes place since one is interested in measuring classical observables quantum mechanically which are smeared by singular smearing functions. Whether this is a sensible thing to do or not is a *physical* question. If one insists to measure fluxes quantum mechanically then one has no choice and fluxes are precisely the kind of objects that are natural in our non-Fock representation.

C The Infinite Spin Chain

In this final appendix we consider a trivial but instructive physical system in order to exemplify infinite tensor product concepts, namely an infinite chain of uncoupled spin degrees of freedom.

Our label set will be the integers $I = \mathbb{Z}$ and for each $n \in \mathbb{Z}$ we have the Hilbert space $\mathcal{H}_n = \mathbb{C}^2$ with standard inner product $\langle f_n, f'_n \rangle_n = \bar{f}_n^+ f_n^{+'} + \bar{f}_n^- f_n^{-'}$. In each Hilbert space we have the standard orthogonal basis of vectors e_n^\pm and spin operators $\sigma_n = \sigma_3$ (Pauli matrix) so that $\sigma_n e_n^\pm = \pm e_n^\pm$ corresponds to spin up/down. We also have ladder operators $\sigma_n^\pm = \frac{1}{2}[\sigma_1 \pm i\sigma_2]$ so that $\sigma_n^\pm e_n^\pm = 0$, $\sigma_n^\pm e_n^\mp = e_n^\pm$. Consider the positive semi-definite, self-adjoint Hamiltonian

$$\hat{H} := \frac{1}{2} \sum_n [1 + \sigma_n] = \sum_n (\sigma_n^-)^\dagger \sigma_n^- \quad (\text{C.1})$$

on the ITP Hilbert space $\mathcal{H}^\otimes = \otimes_n \mathcal{H}_n$ which is non-separable even though each \mathcal{H}_n has finite dimension two.

We will first consider a C_0 vector \otimes_f with $\|f_n\|_n = 1$ and a second one $\otimes_{f'}$ with $f'_n = -f_n$. Are the corresponding C_0 sequences in the same strong (weak) equivalence class? Since $\langle f_n, f'_n \rangle_n = -1$ we see that $\sum_n |\langle f_n, f'_n \rangle_n - 1| = \sum_n 2 = \infty$ but $\sum_n \|\langle f_n, f'_n \rangle_n - 1\| = \sum_n 0 = 0$, thus they are in different strong classes within the same weak class. In fact, the unitary operator \hat{U} on \mathcal{H}^\otimes defined densely on arbitrary C_0 vectors by $\hat{U}\otimes_g = \otimes_{g'}$ with $g'_n = -g_n$ maps the two unit C_0 vectors into each other and thus the strong equivalence class Hilbert spaces built from them will be unitarily equivalent subspaces of the whole ITP. Notice that indeed $\langle \otimes_f, \otimes_{f'} \rangle = \prod_n (-1)^n = 0$ since the product of numbers $z_n = -1$ is only quasi-convergent.

Consider now the unit C_0 vectors $\Omega^\pm := \otimes_{f^\pm}$ with $f_n^\pm := e_n^\pm$ (the total spin up/down state). We have $\sum_n \|\langle f_n^+, f_n^- \rangle_n - 1\| = \sum_n 1 = \infty$, thus they are in different weak classes. It is true that the *non-local operator* $\hat{A} := \otimes_n \sigma_n^+$ maps Ω^- into Ω^+ , however, \hat{A} is not a unitary operator as one easily verifies by computing the norm of $\hat{A}\otimes_f$ with $f_n^+ \neq 0$ for at least one n . In fact, Ω^- is the *ground state* of \hat{H} while Ω^+ is an infinite energy (infinitely excited) state for \hat{H} and thus their strong equivalence class Hilbert spaces describe drastically different physics. It is therefore not to be expected on physical grounds that these representations should be unitarily equivalent.

Finally, it is easy to see that the orthonormal “spin-network” C_0 vector system $\otimes_{\{\alpha_n\}} := \otimes_n e_n^{\alpha_n}$ with $\alpha_n = \pm$ is not a basis: For instance the C_0 vector with tensor product factors $f_n := \frac{1}{\sqrt{2}}[e_n^+ + e_n^-]$ is orthonormal to all of them, a nice example in which one sees that the unrestricted associative law is false.

With the help of simple systems like the infinite spin chain one can study a lot of more phenomena of the ITP such as the occurrence of von Neumann algebra factor types I_∞, II_1, III_s , $s \in (0, 1)$. The von Neumann algebra of local operators \mathcal{R}^\otimes considered in the main text is of that type if we restrict it to the strong equivalence class Hilbert spaces defined by the cyclic vectors

$$\Omega_s := \otimes_n \left[\sqrt{\frac{1+s}{2}} \sigma_n^+ + \sqrt{\frac{1-s}{2}} \sigma_n^- \right] \quad (\text{C.2})$$

of the ITP Hilbert space $\mathcal{H}^\otimes := \otimes_n [\mathbb{C}^2 \otimes \mathbb{C}^2]$ for $s \in \{1\}, \{0\}, (0, 1)$ respectively. These representations can be interpreted as zero, infinite and finite temperature representations respectively, see [20] for details.

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