

Gauge Field Theory Coherent States (GCS) : IV. Infinite Tensor Product and Thermodynamical Limit

T. Thiemann*, O. Winkler†

MPI f. Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, 14476 Golm near Potsdam, Germany

Preprint AEI-2000-030

Abstract

In the canonical approach to Lorentzian Quantum General Relativity in four spacetime dimensions an important step forward has been made by Ashtekar, Isham and Lewandowski some eight years ago through the introduction of a Hilbert space structure which was later proved to be a faithful representation of the canonical commutation and adjointness relations of the quantum field algebra of diffeomorphism invariant gauge field theories by Ashtekar, Lewandowski, Marolf, Mourão and Thiemann.

This Hilbert space, together with its generalization due to Baez and Sawin, is appropriate for semi-classical quantum general relativity if the spacetime is spatially compact. In the spatially non-compact case, however, an extension of the Hilbert space is needed in order to approximate metrics that are macroscopically nowhere degenerate.

For this purpose, in this paper we apply the theory of the Infinite Tensor Product (ITP) of Hilbert Spaces, developed by von Neumann more than sixty years ago, to Quantum General Relativity. The cardinality of the number of tensor product factors can take the value of any possible Cantor aleph, making this mathematical theory well suited to our problem in which a Hilbert space is attached to each edge of an arbitrarily complicated, generally infinite graph.

The new framework opens a Pandora's box full of techniques, appropriate to pose fascinating physical questions such as quantum topology change, semi-classical quantum gravity, effective low energy physics etc. from the universal point of view of the ITP. In particular, the study of photons and gravitons propagating on fluctuating quantum spacetimes is now in reach, the topic of the next paper in this series.

1 Introduction

Quantum General Relativity (QGR) has matured over the past decade to a mathematically well-defined theory of quantum gravity. In contrast to string theory, by definition QGR is a manifestly background independent, diffeomorphism invariant and non-perturbative theory. The obvious advantage is that one will never have to postulate the

*thiemann@aei-potsdam.mpg.de

†winkler@aei-potsdam.mpg.de

existence of a non-perturbative extension of the theory, which in string theory has been called the still unknown M(ystery)-Theory.

The disadvantage of a non-perturbative and background independent formulation is, of course, that one is faced with new and interesting mathematical problems so that one cannot just go ahead and “start calculating scattering amplitudes”: As there is no background around which one could perturb, rather the full metric is fluctuating, one is not doing quantum field theory on a spacetime but only on a differential manifold. Once there is no (Minkowski) metric at our disposal, one loses familiar notions such as causality structure, locality, Poincaré group and so forth, in other words, the theory is not a theory to which the Wightman axioms apply. Therefore, one must build an entirely new mathematical apparatus to treat the resulting quantum field theory which is *drastically different from the Fock space picture to which particle physicists are used to*.

As a consequence, the mathematical formulation of the theory was the main focus of research in the field over the past decade. The main achievements to date are the following (more or less in chronological order) :

i) *Kinematical Framework*

The starting point was the introduction of new field variables [1] for the gravitational field which are better suited to a background independent formulation of the quantum theory than the ones employed until that time. In its original version these variables were complex valued, however, currently their real valued version, considered first in [2] for *classical* Euclidean gravity and later in [3] for *classical* Lorentzian gravity, is preferred because to date it seems that it is only with these variables that one can rigorously define the kinematics and dynamics of Euclidean or Lorentzian *quantum* gravity [4].

These variables are coordinates for the infinite dimensional phase space of an $SU(2)$ gauge theory subject to further constraints besides the Gauss law, that is, a connection and a canonically conjugate electric field. As such, it is very natural to introduce smeared functions of these variables, specifically Wilson loop and electric flux functions. (Notice that one does not need a metric to define these functions, that is, they are background independent). This had been done for ordinary gauge fields already before in [5] and was then reconsidered for gravity (see e.g. [6]).

The next step was the choice of a representation of the canonical commutation relations between the electric and magnetic degrees of freedom. This involves the choice of a suitable space of distributional connections [7] and a faithful measure thereon [8] which, as one can show [9], is σ -additive. The corresponding L_2 Hilbert space and its generalization [10] will be henceforth called the Ashtekar-Isham-Lewandowski-Baez-Sawin (AILBS) Hilbert space. The proof that the AILBS Hilbert space indeed solves the adjointness relations induced by the reality structure of the classical theory as well as the canonical commutation relations induced by the symplectic structure of the classical theory can be found in [11]. Independently, a second representation of the canonical commutation relations, called the loop representation, had been advocated (see e.g. [12] and especially [13] and references therein) but both representations were shown to be unitarily equivalent in [14] (see also [15] for a different method of proof).

This is then the first major achievement : The theory is based on a rigorously defined kinematical framework.

ii) *Geometrical Operators*

The second major achievement concerns the spectra of positive semi-definite, self-adjoint geometrical operators measuring lengths [16], areas [17, 18] and volumes [17, 19, 20, 21, 12] of curves, surfaces and regions in spacetime. These spectra

are pure point (discrete) and imply a discrete Planck scale structure. It should be pointed out that the discreteness is, in contrast to other approaches to quantum gravity, not put in by hand but it is a *prediction* !

iii) *Regularization- and Renormalization Techniques*

The third major achievement is that there is a new regularization and renormalization technique [22, 23] for diffeomorphism covariant, density-one-valued operators at our disposal which was successfully tested in model theories [24]. This technique can be applied, in particular, to the standard model coupled to gravity [25, 26] and to the Poincaré generators at spatial infinity [27]. In particular, it works for *Lorentzian* gravity while all earlier proposals could at best work in the Euclidean context only (see, e.g. [13] and references therein). The algebra of important operators of the resulting quantum field theories was shown to be consistent [28]. Most surprisingly, these operators are *UV and IR finite* ! Notice that, at least as far as these operators are concerned, this result is stronger than the believed but unproved finiteness of scattering amplitudes order by order in perturbation theory of the five critical string theories, in a sense we claim that the perturbation series converges. The absence of the divergences that usually plague interacting quantum fields propagating on a Minkowski background can be understood intuitively from the diffeomorphism invariance of the theory : “short and long distances are gauge equivalent”. We will elaborate more on this point in future publications.

iv) *Spin Foam Models*

After the construction of the densely defined Hamiltonian constraint operator of [22, 23], a formal, Euclidean functional integral was constructed in [29] and gave rise to the so-called spin foam models (a spin foam is a history of a graph with faces as the history of edges) [30]. Spin foam models are in close connection with causal spin-network evolutions [31], state sum models [32] and topological quantum field theory, in particular BF theory [33]. To date most results are at a formal level and for the Euclidean version of the theory only but the programme is exciting since it may restore manifest four-dimensional diffeomorphism invariance which in the Hamiltonian formulation is somewhat hidden.

- v) Finally, the fifth major achievement is the existence of a rigorous and satisfactory framework [34, 35, 36, 37, 38, 39, 40] for the quantum statistical description of black holes which reproduces the Bekenstein-Hawking Entropy-Area relation and applies, in particular, to physical Schwarzschild black holes while stringy black holes so far are under control only for extremal charged black holes.

Summarizing, the work of the past decade has now culminated in a promising starting point for a quantum theory of the gravitational field plus matter and the stage is set to pose and answer physical questions.

The most basic and most important question that one should ask is : *Does the theory have classical general relativity as its classical limit ?* Notice that even if the answer is negative, the existence of a consistent, interacting, diffeomorphism invariant quantum field theory in four dimensions is already a quite non-trivial result. However, we can claim to have a satisfactory quantum theory of Einstein’s theory only if the answer is positive.

In order to address this question with a mathematically well-defined procedure we have developed in [41, 42] a theory of coherent states for the matter content of the standard model (with possible supersymmetric extensions) coupled to gravity. These states are labelled by classical solutions to the field equations and have the property that a) the expectation values of densely defined field operators with respect to these states take the

value prescribed by the classical solution and b) they saturate the Heisenberg uncertainty bound without quenching.

The way this has been achieved so far is the following : The degrees of freedom of, say, the gravitational field, are labelled by piecewise analytic (smooth) graphs (webs) composed of a *finite* number of edges (paths) only. For each such graph one finds a subspace of the AILBS Hilbert space which is the *finite* tensor product of mutually isomorphic Hilbert spaces, one for each edge (path) of the graph (web). The closure of finite linear combinations of vectors from these subspaces labelled by graphs (webs), which turn out to be mutually orthogonal, forms the AILBS Hilbert space. What has been done in [41, 42] is to develop a theory of coherent states for each of these Hilbert spaces labelled by a finite graph γ . More precisely, one constructs coherent states ψ_e^s for each of the Hilbert spaces labelled by a single edge e of γ and the classicality parameter s ($s \rightarrow 0$ is the classical limit) and then the coherent state for the whole graph γ is simply the tensor product of those for each of its edges.

This framework is sufficient if the initial data hypersurface Σ is compact since one can describe the quantum metric as precisely as one wishes in terms of finite graphs by taking the graph to be finer and finer, filling Σ more and more densely. However, if Σ is non-compact, say of the topology of \mathbb{R}^3 as required for Minkowski space or the Kruskal extension of the Schwarzschild spacetime which in turn are the most important spacetimes if we want to make contact with the low energy physics of the standard model, scattering theory, Hawking radiation and thus the semiclassical approximation of quantum gravity by the theory of ordinary Quantum Field Theory on (curved) backgrounds, then the above framework is insufficient. What one needs in this case is an infinite graph no matter how coarse the graph is, that is, no matter whether the lattice spacing is 1mm or of the order of the Planck length, in order to fill Σ everywhere we need an infinite graph, no region of Σ of infinite volume must be empty if we wish to approximate a non-degenerate metric as all the classical metrics are.

One may think that one can get away by taking an infinite superposition of states labelled by mutually different finite graphs. However, such states have infinite norm with respect to the AILBS scalar product as the following simple example shows : Namely, let γ_∞ be a cubic lattice, an infinite graph filling all of $\Sigma := \mathbb{R}^3$ as densely as we wish and construct the state $\psi^s := \sum_e z_e T_e$ where the sum runs over all edges of γ_∞ , z_e are complex coefficients and T_e is some linear combination of spin-network states over e . Obviously, this state is an infinite linear combination of states over finite graphs. Then, because of homogeneity, this state produces the correct classical limit, corresponding to, say, Minkowski space, for each of the holonomy operators \hat{h}_e at most if $z_e = z$ is independent of e and $T_e(h_e) = T(h_e)$ is the same linear combination of spin-network states for each e . But then the norm of the state is formally $|z|^2 \|T\|^2 \sum_e 1 = \infty$ and badly diverges.

On the other hand, we will show that one can give meaning to states of the form $\psi_{\gamma_\infty}^s := \otimes_e \psi_e^s$ where γ_∞ is an infinite graph and if one defines the inner product to be the product of the inner products of the tensor product factors, then $\|\psi_{\gamma_\infty}^s\| = 1$ while the semiclassical behaviour with respect to every possible operator over γ_∞ is preserved and identical to the one for finite tensor products. Notice that in our case the dimension of the Hilbert space over each edge is countably infinite. If γ_∞ is countably infinite, the case to which we restrict in this paper, then the direct sum Hilbert space is still separable. But even if the Hilbert space over each edge would be only two-dimensional then the countably infinite tensor product Hilbert space is non-separable !

This article is organized as follows :

In section two we recall the basic kinematical structure of canonical Quantum General Relativity.

In section three we list the essential properties of our family of coherent states for finite tensor products as needed for the purpose of the present paper.

In section four we give an account of von Neumann's theory of the Infinite Tensor Product [43] for the general case, in particular the occurrence of von Neumann algebras of different factor types induced by the operator algebras on each tensor product factor.

Section five contains the new results of this paper. We apply the general ITP theory to our situation focussing on general and abstract properties only. We extend the quantum kinematical framework of Ashtekar, Isham and Lewandowski to piecewise analytical, infinite graphs, connect it with the (semi)classical analysis for canonical quantum field theories over non-compact initial data hypersurfaces and finally discuss the transfer of dynamical results as obtained earlier for finite graphs.

In particular, for any possible solution of the Einstein field equations we are able to identify an element of the ITP Hilbert space, a so-called C_0 -vector Ω in von Neumann's terminology, which in the theory of quantum fields propagating on curved background spacetimes, plays the role of the vacuum or ground state and which can be constructed purely in terms of our coherent states. Perturbations of this vacuum, which in von Neumann's terminology lie in the subspace of the ITP Hilbert space generated by the strong equivalence class of the C_0 -vector Ω , can naturally be identified with the usual Fock states of QFT on the curved background that Ω approximates. This opens the possibility to make contact with the usual perturbation theory defined in terms of Fock states.

In fact, in [74] we show that it is possible to map a precisely defined subspace of the ITP Hilbert space for Einstein-Maxwell theory, to the Fock space defined in terms of, say, n -Photon states propagating on Minkowski spacetime up to corrections due to pure quantum gravity effects caused by the fluctuating nature of the quantum metric and which one hopes to measure in experiment. More precisely, this subspace is generated by the operator algebra of the Maxwell field acting on a C_0 -vector Ω of the Einstein-Maxwell ITP Hilbert space and which is cyclic for that subspace. This vector Ω is a minimal uncertainty vector for Einstein-Maxwell theory approximating the Minkowski metric and vanishing electromagnetic field respectively. It should be noted, however, that all the states so constructed are states of the fully interacting Einstein-Maxwell theory and not only of the free Maxwell theory propagating on Minkowski space (an example of a free quantum field theory on a fixed curved background). The two sets of states so constructed are in a one-to-one and onto correspondence, leading to expectation values for physical operators which coincide to lowest order in the Planck length. As far as quantum gravity corrections are concerned, however, these states are physically very different, the states of the interacting theory give rise to the so-called γ -ray-burst effect [44] which is just one way to measure the Poincaré non-invariance of the present state of our universe at the fundamental level. In [45] we will explicitly compute the size of this effect from first principles by a down-to-the-ground-computation, thereby significantly improving the results of [46].

2 Kinematical Structure of Diffeomorphism Invariant Quantum Gauge Theories

In this section we will recall the main ingredients of the mathematical formulation of (Lorentzian) diffeomorphism invariant classical and quantum field theories of connections with local degrees of freedom in any dimension and for any compact gauge group. See [41, 11] and references therein for more details.

2.1 Classical Theory

Let G be a compact gauge group, Σ a D -dimensional manifold and consider a principal G -bundle with connection over Σ . Let us denote the pull-back (by local sections) to Σ of the connection by A_a^i where $a, b, c, \dots = 1, \dots, D$ denote tensorial indices and $i, j, k, \dots = 1, \dots, \dim(G)$ denote indices for the Lie algebra of G . Likewise, consider a vector bundle of electric fields, whose projection to Σ is a Lie algebra valued vector density of weight one. We will denote the set of generators of the rank $N - 1$ Lie algebra of G by τ_i which are normalized according to $\text{tr}(\tau_i \tau_j) = -N \delta_{ij}$ and $[\tau_i, \tau_j] = 2f_{ij}{}^k \tau_k$ defines the structure constants of $\text{Lie}(G)$.

Let F_i^a be a Lie algebra valued vector density test field of weight one and let f_a^i be a Lie algebra valued covector test field. We consider the smeared quantities

$$F(A) := \int_{\Sigma} d^D x F_i^a A_a^i \quad \text{and} \quad E(f) := \int_{\Sigma} d^D x E_i^a f_a^i \quad (2.1)$$

While both objects are diffeomorphism covariant, only the latter is gauge covariant, one reason to introduce the singular smearing discussed below. The choice of the space of pairs of test fields $(F, f) \in \tilde{\mathcal{S}}$ depends on the boundary conditions on the space of connections and electric fields which in turn depends on the topology of Σ and will not be specified in what follows.

Consider the set M of all pairs of smooth functions (A, E) on Σ such that (2.1) is well defined for any $(F, f) \in \mathcal{S}$. We define a topology on M through the globally defined metric :

$$d_{\rho, \sigma}[(A, E), (A', E')] \quad (2.2)$$

$$:= \sqrt{-\frac{1}{N} \int_{\Sigma} d^D x [\sqrt{\det(\rho)} \rho^{ab} \text{tr}([A_a - A'_a][A_b - A'_b]) + \frac{[\sigma_{ab} \text{tr}([E^a - E'^a][E^b - E'^b])]}{\sqrt{\det(\sigma)}}]}$$

where ρ_{ab}, σ_{ab} are fiducial metrics on Σ of everywhere Euclidean signature. Their fall-off behaviour has to be suited to the boundary conditions of the fields A, E at spatial infinity (if Σ is spatially non-compact). Notice that the metric (2.2) on M is gauge invariant. It can be used in the usual way to equip M with the structure of a smooth, infinite dimensional differential manifold modelled on a Banach (in fact Hilbert) space \mathcal{E} where $\mathcal{S} \times \mathcal{S} \subset \mathcal{E}$. (It is the weighted Sobolev space $H_{0, \rho}^2 \times H_{0, \sigma^{-1}}^2$ in the notation of [48]).

Finally, we equip M with the structure of an infinite dimensional symplectic manifold through the following strong (in the sense of [49]) symplectic structure

$$\Omega((f, F), (f', F'))_m := \int_{\Sigma} d^D x [F_i^a f_a^{i'} - F_i^{a'} f_a^i](x) \quad (2.3)$$

for any $(f, F), (f', F') \in \mathcal{E}$. We have abused the notation by identifying the tangent space to M at m with \mathcal{E} . To see that Ω is a strong symplectic structure one uses standard Banach space techniques. Computing the Hamiltonian vector fields (with respect to Ω) of the functions $E(f), F(A)$ we obtain the following elementary Poisson brackets

$$\{E(f), E(f')\} = \{F(A), F'(A)\} = 0, \quad \{E(f), F(A)\} = F(f) \quad (2.4)$$

As a first step towards quantization of the symplectic manifold (M, Ω) one must choose a polarization. As usual in gauge theories, we will use connections as the configuration variables and electric fields as canonically conjugate momenta. As a second step one must decide on a complete set of coordinates of M which are to become the elementary quantum operators. The analysis just outlined suggests to use the coordinates $E(f), F(A)$.

However, the well-known immediate problem is that these coordinates are not gauge covariant. Thus, we proceed as follows :

The idea is to construct the theory from smaller building blocks, labelled by graphs embedded into Σ . In the literature, two sets of graphs, labelling the so-called cylindrical functions, have been proposed : the set of *finite* piecewise analytical graphs Γ_0^ω in [8] and in [10, 50] the restriction Γ_0^∞ to so-called “webs” of the set of *all* piecewise smooth graphs Γ^∞ . (We do not discuss here a third alternative, the set of finite piecewise linear graphs [51]). Here we call a graph γ finite if its sets of oriented edges e and vertices v respectively, denoted by $E(\gamma)$ and $V(\gamma)$ respectively, have finite cardinality. A web is a special kind of a piecewise smooth graph which may not be finite but which can be obtained as the union of a *finite* number of smooth curves with *finite range* (the diffeomorphic image in Σ of a closed interval in \mathbb{R}) and such that its vertex set has a finite number of accumulation points. (In fact, this is the essential difference between Γ_0^ω and Γ_0^∞ since a graph generated by a finite number of analytical curves is a piecewise analytical, finite graph which cannot have any accumulation points). There are some additional restrictions on the common intersections of the curves in a web which we do not need to explain here, see [10] for all details. It is not difficult to prove that both $\Gamma_0^\omega, \Gamma_0^\infty$ are closed under forming *finite* numbers of intersections and unions.

In this paper we are going to extend the framework to *truly infinite* graphs. That is, a priori, we do not impose any finiteness restriction neither on the number of edges or vertices of a graph nor on the range of its edges. Various extensions are possible. A simple possibility is the set Γ^ω of piecewise analytic graphs with possibly a countably infinite number of edges. Such graphs can still have accumulation points of edges and vertices (e.g. the graph which looks like a ladder in a two-plane whose spokes are mutually parallel and come arbitrarily close to each other). An even simpler choice is the set Γ_σ^ω of piecewise analytic, σ -finite graphs which can be considered in locally compact manifolds Σ (every point has a compact neighbourhood) which, of course, is satisfied for any finite-dimensional manifold that we have in mind here. They are characterized by the fact that $\gamma \cup U \in \Gamma_0^\omega$, i.e. the restriction of γ to any compact set is a piecewise analytic finite graph whose number of edges is uniformly bounded. More precisely :

Definition 2.1 *Let Σ be a locally compact manifold. A graph $\gamma \in \Gamma_\sigma^\omega$ is said to be a piecewise analytic, σ -finite graph, if for each compact subset $U \subset \Sigma$ the restriction of the graph is a finite graph, $\gamma \cup U \in \Gamma_0^\omega$. Moreover, for any compact cover \mathcal{U} of Σ the set $\{|E(\gamma \cap U)|; U \in \mathcal{U}\}$ is bounded.*

Clearly, truly infinite piecewise analytic graphs exist only if Σ is not compact and in this case Γ_0^ω is a proper subset of Γ^ω . In order to obtain maximally nice graphs we will make the further restriction that Σ is paracompact, see section 5.1.

The next simple choice is the set Γ^∞ of all piecewise smooth graphs with possibly a countable number of edges and possibly a countable number of accumulation points. More properly, we should call them the set of *infinite webs*, that is, the web γ is allowed to be generated by a countably infinite number of smooth curves such that for each accumulation point p_i , $i = 1, \dots, N \leq \infty$ there exists a neighbourhood U_i such that the U_i are mutually disjoint and such that γ restricted to U_i is an element of Γ_0^∞ . It is a non-trivial task to decide whether any of the three sets $\Gamma^\omega, \Gamma_\sigma^\omega, \Gamma^\infty$ are closed under taking finite unions and we will do this in this paper only for Γ_σ^ω , leaving the remaining cases for future publications.

Finally, we could consider Γ , the set of *all* piecewise smooth, oriented graphs γ embedded into Σ . That is, we do not impose any restriction on the cardinality of the sets $E(\gamma), V(\gamma)$, or on the nature of the accumulation points. This set is trivially closed under

arbitrary unions but it is beyond present analytical control, furthermore, it is not clear whether Γ and Γ^∞ are really different and to analyze these questions is beyond the scope of the present paper, too.

Suffice it to say that for the purposes that we have in mind, to take the classical limit, it is sufficient to work with the set Γ_σ^ω that is technically much easier to handle. Thus, from now on we will assume that $\gamma \in \Gamma_\sigma^\omega$, the typical graph that we will need in our applications and that is good to have in mind as an example is a regular cubic lattice in \mathbb{R}^3 .

Let γ be a graph and e an edge of γ . We denote by $h_e(A)$ the holonomy of A along e and say that a function f on \mathcal{A} is cylindrical with respect to γ if there exists a function f_γ on $G^{|E(\gamma)|}$ such that $f = p_\gamma^* f_\gamma = f_\gamma \circ p_\gamma$ where $p_\gamma(A) = \{h_e(A)\}_{e \in E(\gamma)}$. The set of functions cylindrical over γ is denoted by Cyl_γ . Holonomies are invariant under reparameterizations of the edge and in this article we assume that the edges are always analyticity preserving diffeomorphic images from $[0, 1]$ to a one-dimensional submanifold of Σ if it has compact range and from $[0, 1)$, $(0, 1]$, $(0, 1)$ if it has semi-finite or infinite range. Gauge transformations are functions $g : \Sigma \mapsto G; x \mapsto g(x)$ and they act on holonomies as $h_e \mapsto g(e(0))h_e g(e(1))^{-1}$ where in the (semi)finite case $e(0)$ or $e(1)$ or both are not points in Σ and we simply set $g(e(0)) = 1$ or $g(e(1)) = 1$, which is justified by the the boundary conditions, restricting gauge transformations to be trivial at spatial infinity.

Next, given a graph γ we choose a polyhedral decomposition P_γ of Σ dual to γ . The precise definition of a dual polyhedral decomposition can be found in [47] but for the purposes of the present paper it is sufficient to know that P_γ assigns to each edge e of γ an open ‘‘face’’ S_e (a polyhedron of codimension one embedded into Σ) with the following properties :

- (1) the surfaces S_e are mutually non-intersecting,
- (2) only the edge e intersects S_e , the intersection is transversal and consists only of one point which is an interior point of both e and S_e ,
- (3) S_e carries the normal orientation which agrees with the orientation of e .

Furthermore, we choose a system Π_γ of paths $\rho_e(x) \subset S_e$, $x \in S_e$, $e \in E(\gamma)$ connecting the intersection point $p_e = e \cap S_e$ with x . The paths vary smoothly with x and the triples $(\gamma, P_\gamma, \Pi_\gamma)$ have the property that if γ, γ' are diffeomorphic, so are $P_\gamma, P_{\gamma'}$ and $\Pi_\gamma, \Pi_{\gamma'}$.

With these structures we define the following function on (M, Ω)

$$P_i^e(A, E) := -\frac{1}{N} \text{tr}(\tau_i h_e(0, 1/2)) \left[\int_{S_e} h_{\rho_e(x)} * E(x) h_{\rho_e(x)}^{-1} \right] h_e(0, 1/2)^{-1} \quad (2.5)$$

where $h_e(s, t)$ denotes the holonomy of A along e between the parameter values $s < t$, $*$ denotes the Hodge dual, that is, $*E$ is a $(D - 1)$ -form on Σ , $E^a := E_i^a \tau_i$ and we have chosen a parameterization of e such that $p_e = e(1/2)$.

Notice that in contrast to similar variables used earlier in the literature the function P_i^e is *gauge covariant*. Namely, under gauge transformations it transforms as $P^e \mapsto g(e(0))P^e g(e(0))^{-1}$, the price to pay being that P^e depends on both A and E and not only on E . The idea is therefore to use the variables h_e, P_i^e for all possible graphs γ as the coordinates of M .

The problem with the functions $h_e(A)$ and $P_i^e(A, E)$ on M is that they are not differentiable on M , that is, Dh_e, DP_i^e are nowhere bounded operators on \mathcal{E} as one can easily see. The reason for this is, of course, that these are functions on M which are not properly smeared with functions from $\tilde{\mathcal{S}}$, rather they are smeared with distributional test functions with support on e or S_e respectively. Nevertheless, one would like to base the quantization of the theory on these functions as basic variables because of their gauge and diffeomorphism covariance. Indeed, under diffeomorphisms $h_e \mapsto h_{\varphi^{-1}(e)}, P_i^e \mapsto P^{\varphi^{-1}(e)}$

where we abuse notation since P^e depends also on S_e, ρ_e , see [47] for more details. We proceed as follows.

Definition 2.2 *By \bar{M}_γ we denote the direct product $[G \times \text{Lie}(G)]^{|E(\gamma)|}$. The subset of \bar{M}_γ of pairs $(h_e(A), P_i^e(A, E))_{e \in E(\gamma)}$ as (A, E) varies over M will be denoted by $(\bar{M}_\gamma)|_M$. We have a corresponding map $p_\gamma : M \mapsto \bar{M}_\gamma$ which maps M onto $(\bar{M}_\gamma)|_M$.*

Notice that the set $(\bar{M}_\gamma)|_M$ is, in general, a proper subset of \bar{M}_γ , depending on the boundary conditions on (A, E) , the topology of Σ and the “size” of e, S_e . For instance, in the limit of $e, S_e \rightarrow e \cap S_e$ but holding the number of edges fixed, $(\bar{M}_\gamma)|_M$ will consist of only one point in \bar{M}_γ . This follows from the smoothness of the (A, E) .

We equip a subset M_γ of \bar{M}_γ with the structure of a differentiable manifold modelled on the Banach space $\mathcal{E}_\gamma = \mathbb{R}^{2 \dim(G)|E(\gamma)|}$ by using the natural direct product manifold structure of $[G \times \text{Lie}(G)]^{|E(\gamma)|}$. While \bar{M}_γ is a kind of distributional phase space, M_γ has suitable regularity properties similar to (2.2).

In order to proceed and to give M_γ a symplectic structure *derived from* (M, Ω) one must regularize the elementary functions h_e, P_i^e by writing them as limits (in which the regulator vanishes) of functions which can be expressed in terms of the $F(A), E(f)$. Then one can compute their Poisson brackets with respect to the symplectic structure Ω at finite regulator and then take the limit pointwise on M . The result is the following well-defined strong symplectic structure Ω_γ on M_γ .

$$\begin{aligned} \{h_e, h_{e'}\}_\gamma &= 0 \\ \{P_i^e, h_{e'}\}_\gamma &= \delta_{e'}^e \frac{\tau_i}{2} h_e \\ \{P_i^e, P_j^{e'}\}_\gamma &= -\delta^{ee'} f_{ij}{}^k P_k^e \end{aligned} \quad (2.6)$$

Since Ω_γ is obviously block diagonal, each block standing for one copy of $G \times \text{Lie}(G)$, to check that Ω_γ is non-degenerate and closed reduces to doing it for each factor together with an appeal to well-known Hilbert space techniques to establish that Ω_γ is a surjection of \mathcal{E}_γ . This is done in [47] where it is shown that each copy is isomorphic with the cotangent bundle T^*G equipped with the symplectic structure (2.6) (choose $e = e'$ and delete the label e).

Now that we have managed to assign to each graph γ a symplectic manifold $(M_\gamma, \Omega_\gamma)$ we can quantize it by using geometric quantization. This can be done in a well-defined way because the relations (2.6) show that the corresponding operators are non-distributional. This is therefore a clean starting point for the regularization of any operator of quantum gauge field theory which can always be written in terms of the \hat{h}_e, \hat{P}^e , $e \in E(\gamma)$ if we apply this operator to a function which depends only on the h_e , $e \in E(\gamma)$.

The question is what $(M_\gamma, \Omega_\gamma)$ has to do with (M, Ω) . In [47] it is shown that there exists a partial order \prec on the set of triples $(\gamma, P_\gamma, \Pi_\gamma)$ and one can form a generalized projective limit M_∞ of the M_γ (in particular, $\gamma \prec \gamma'$ means $\gamma \subset \gamma'$). Moreover, the family of symplectic structures Ω_γ is self-consistent in the sense that if $(\gamma, P_\gamma, \Pi_\gamma) \prec (\gamma', P_{\gamma'}, \Pi_{\gamma'})$ then $p_{\gamma'\gamma}^* \{f, g\}_\gamma = \{p_{\gamma'\gamma}^* f, p_{\gamma'\gamma}^* g\}_{\gamma'}$ for any $f, g \in C^\infty(M_\gamma)$ and $p_{\gamma'\gamma} : M_{\gamma'} \mapsto M_\gamma$ is a natural projection.

Now, via the maps p_γ of definition 2.2 we can identify M with a subset of M_∞ . Moreover, in [47] it is shown that there is a generalized projective sequence $(\gamma_n, P_{\gamma_n}, \Pi_{\gamma_n})$ such that $\lim_{n \rightarrow \infty} p_{\gamma_n}^* \Omega_{\gamma_n} = \Omega$ pointwise in M . This displays (M, Ω) as embedded into a generalized projective limit of the $(M_\gamma, \Omega_\gamma)$, intuitively speaking, as γ fills all of Σ , we recover (M, Ω) from the $(M_\gamma, \Omega_\gamma)$. On non-compact manifolds Σ this is possible only if the label set Γ_σ^ω contains infinite graphs.

It follows that quantization of (M, Ω) , and conversely taking the classical limit, can be studied purely in terms of M_γ, Ω_γ for *all* γ . The quantum kinematical framework for this will be given in the next subsection.

2.2 Quantum Theory

At this point there is a clash with the previous subsection because the quantum kinematical structure has so far been defined only for the *finite* category of graphs Γ_0^ω . We thus have to extend this framework which we will do in section 5.1. However, as the structure from Γ_0^ω can be nicely embedded into the more general context, we will repeat here the relevant notions for finite, piecewise analytical graphs γ .

Let us denote the set of all smooth connections by \mathcal{A} . This is our classical configuration space and we will choose for its coordinates the holonomies $h_e(A)$, $e \in \gamma$, $\gamma \in \Gamma_0^\omega$. \mathcal{A} is naturally equipped with a metric topology induced by (2.2).

Recall the notion of a function cylindrical over a graph from the previous subsection. A particularly useful set of cylindrical functions are the so-called spin-network functions [52, 53, 14] which so far have been introduced only for Γ_0^ω , in fact, it is unclear whether one can define spin-network functions for all elements of Γ_0^ω , see [50] for a discussion. As we will see in section 5, the spin-network bases proves to be of modest practical use in the context of Γ_σ^ω only, to be replaced by what we will call *von Neumann bases based on C_0 -vectors*. To see what the problem is, we anyway have to introduce them here.

A spin-network function is labelled by a graph $\gamma \in \Gamma_0^\omega$, a set of non-trivial irreducible representations $\vec{\pi} = \{\pi_e\}_{e \in E(\gamma)}$ (choose from each equivalence class of equivalent representations once and for all a fixed representant), one for each edge of γ , and a set $\vec{c} = \{c_v\}_{v \in V(\gamma)}$ of contraction matrices, one for each vertex of γ , which contract the indices of the tensor product $\otimes_{e \in E(\gamma)} \pi_e(h_e)$ in such a way that the resulting function is gauge invariant. We denote spin-network functions as T_I where $I = \{\gamma, \vec{\pi}, \vec{c}\}$ is a compound label. One can show that these functions are linearly independent. From now on we denote by $\tilde{\Phi}_\gamma$ finite linear combinations of spin-network functions over γ , by Φ_γ the finite linear combinations of elements from any possible $\tilde{\Phi}_{\gamma'}$, $\gamma' \subset \gamma$ a subgraph of γ and by Φ the finite linear combinations of spin-network functions from an arbitrary finite collection of graphs. Clearly $\tilde{\Phi}_\gamma$ is a subspace of Φ_γ which by itself is a proper subspace of the set Cyl_γ^∞ of smooth cylindrical functions over γ . To express this distinction we will say that functions in $\tilde{\Phi}_\gamma$ are labelled by ‘‘coloured graphs’’ γ while functions in Φ_γ are labelled simply by graphs γ , abusing the notation by using the same symbol γ .

The set Φ of finite linear combinations of spin-network functions forms an Abelian $*$ algebra of functions on \mathcal{A} . By completing it with respect to the sup-norm topology it becomes an Abelian C^* algebra \mathcal{B} (here the compactness of G is crucial). The spectrum $\overline{\mathcal{A}}$ of this algebra, that is, the set of all algebraic homomorphisms $\mathcal{B} \mapsto \mathbb{C}$ is called the quantum configuration space. This space is equipped with the Gel’fand topology, that is, the space of continuous functions $C^0(\overline{\mathcal{A}})$ on $\overline{\mathcal{A}}$ is given by the Gel’fand transforms of elements of \mathcal{B} . Recall that the Gel’fand transform is given by $\hat{f}(\overline{A}) := \overline{A}(f) \forall \overline{A} \in \overline{\mathcal{A}}$. It is a general result that $\overline{\mathcal{A}}$ with this topology is a compact Hausdorff space. Obviously, the elements of \mathcal{A} are contained in $\overline{\mathcal{A}}$ and one can show that \mathcal{A} is even dense [46]. Generic elements of $\overline{\mathcal{A}}$ are, however, distributional.

The idea is now to construct a Hilbert space consisting of square integrable functions on $\overline{\mathcal{A}}$ with respect to some measure μ . Recall that one can define a measure on a locally compact Hausdorff space by prescribing a positive linear functional χ_μ on the space of continuous functions thereon. The particular measure we choose is given by $\chi_{\mu_0}(\hat{T}_I) = 1$ if $I = \{\{p\}, \vec{0}, \vec{1}\}$ and $\chi_{\mu_0}(\hat{T}_I) = 0$ otherwise. Here p is any point in Σ , 0 denotes the

trivial representation and 1 the trivial contraction matrix. In other words, (Gel'fand transforms of) spin-network functions play the same role for μ_0 as Wick-polynomials do for Gaussian measures and like those they form an orthonormal basis in the Hilbert space $\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu_0)$ obtained by completing their finite linear span Φ .

An equivalent definition of $\overline{\mathcal{A}}, \mu_0$ is as follows :

$\overline{\mathcal{A}}$ is in one to one correspondence, via the surjective map H defined below, with the set $\overline{\mathcal{A}}' := \text{Hom}(\mathcal{X}, G)$ of homomorphisms from the groupoid \mathcal{X} of composable, holonomically independent, analytical paths into the gauge group. The correspondence is explicitly given by $\overline{\mathcal{A}} \ni \bar{A} \mapsto H_{\bar{A}} \in \text{Hom}(\mathcal{X}, G)$ where $\mathcal{X} \ni e \mapsto H_{\bar{A}}(e) := \bar{A}(h_e) = \tilde{h}_e(\bar{A}) \in G$ and \tilde{h}_e is the Gel'fand transform of the function $\mathcal{A} \ni A \mapsto h_e(A) \in G$. Consider now the restriction of \mathcal{X} to \mathcal{X}_γ , the groupoid of composable edges of the graph γ . One can then show that the projective limit of the corresponding *cylindrical sets* $\overline{\mathcal{A}}'_\gamma := \text{Hom}(\mathcal{X}_\gamma, G)$ coincides with $\overline{\mathcal{A}}'$. Moreover, we have $\{\{H(e)\}_{e \in E(\gamma)}; H \in \overline{\mathcal{A}}'\} = \{\{H_{\bar{A}}(e)\}_{e \in E(\gamma)}; \bar{A} \in \overline{\mathcal{A}}\} = G^{|E(\gamma)|}$. Let now $f \in \mathcal{B}$ be a function cylindrical over γ then

$$\chi_{\mu_0}(\tilde{f}) = \int_{\overline{\mathcal{A}}} d\mu_0(\bar{A}) \tilde{f}(\bar{A}) = \int_{G^{|E(\gamma)|}} \otimes_{e \in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e \in E(\gamma)})$$

where μ_H is the Haar measure on G . As usual, \mathcal{A} turns out to be contained in a measurable subset of $\overline{\mathcal{A}}$ which has measure zero with respect to μ_0 . It turns out that it is this definition of the measure which can be extended to the category of infinite graphs.

Let, as before, Φ_γ be the finite linear span of spin-network functions over γ or any of its subgraphs and \mathcal{H}_γ its completion with respect to μ_0 . Clearly, \mathcal{H} itself is the completion of the finite linear span Φ of vectors from the mutually orthogonal $\tilde{\Phi}_\gamma$. Our basic coordinates of M_γ are promoted to operators on \mathcal{H} with dense domain Φ . As h_e is group-valued and P^e is real-valued we must check that the adjointness relations coming from these reality conditions as well as the Poisson brackets (2.6) are implemented on our \mathcal{H} . This turns out to be precisely the case if we choose \hat{h}_e to be a multiplication operator and $\hat{P}_j^e = i\hbar\kappa X_j^e/2$ where κ is the gravitational constant, $X_j^e = X_j(h_e)$ and $X^j(h)$, $h \in G$ is the vector field on G generating left translations into the $j - th$ coordinate direction of $Lie(G) \equiv T_h(G)$ (the tangent space of G at h can be identified with the Lie algebra of G) and κ is the coupling constant of the theory. For details see [11, 41].

The question is now whether all of this structure can be extended to the infinite analytic category. In particular, in what sense does a spin-network function converge, what is the sup-norm for a function which is a finite linear combination of infinite products of holonomy functions etc. Obviously, at this point one must invoke the theory of the Infinite Tensor Product. We therefore have to postpone the answer to these questions to section 5.

3 Gauge Field Theory Coherent States

For a rather general idea of how to obtain coherent states for arbitrary canonically quantized quantum (field) theories and quantum gauge field theories in particular, see [41] which is based on [55]. Here we will stick with the heat kernel family initialized by the mathematician Brian Hall [56] who proved that the associated Segal-Bargmann space is unitarily equivalent with the usual L_2 space. These results were extended to the Hilbert spaces underlying cylindrical functions of section 2.2 in [57]. However, the semiclassical properties of these states were only later analyzed in [42].

3.1 Compact Group Coherent States

We will begin with only one copy of G and consider the space of square integrable functions over G with respect to the Haar measure $d\mu_H$, that is, the Hilbert space $\mathcal{H}_G = L_2(G, d\mu_H)$. Let s be a positive real number, π a (once and for all fixed, arbitrary representant from its equivalence class) irreducible representation, χ_π its character and d_π its dimension. Let Δ be the Laplacian on G , then it is well known that the \dim_π^2 functions π_{AB} are eigenfunctions of $-\Delta$ with eigenvalue $\lambda_\pi \geq 0$ which vanishes if and only if π is the trivial representation.

Let $h \in G$ denote an element of G and $g \in G_{\mathbb{C}}$ an element of its complexification (for instance, if $G = SU(2)$ then $G_{\mathbb{C}} = SL(2, \mathbb{C})$). Then the (non-normalized) coherent state ψ_g^s at classicality parameter s and phase space point g (the reason for this notation will be explained shortly) is defined by

$$\psi_g^s(h) := \sum_{\pi} d_{\pi} e^{-s\lambda_{\pi}/2} \chi_{\pi}(gh^{-1}) = (e^{s\Delta/2} \delta_{h'})(h)|_{h' \rightarrow g} \quad (3.1)$$

that is, it is given by heat kernel evolution with time parameter s of the δ -distribution on G followed by analytic continuation.

On \mathcal{H}_G we introduce multiplication and derivative operators on the dense domain $\mathcal{D} := C^\infty(G)$ by

$$(\hat{h}_{AB}f)(h) := h_{AB}f(h) \text{ and } (\hat{p}_j f)(h) = \frac{is}{2}(X_j f)(h) \quad (3.2)$$

where h_{AB} denote the matrix elements of the defining representation of G and $i, j, k, \dots = 1, 2, \dots, \dim(G)$ and $X_j(h) = \text{tr}([\tau_j h]^T \partial / \partial h)$ denotes the generator of right translations on G into the j 'th coordinate direction of $Lie(G)$, the Lie algebra of G . We choose a basis τ_j in $Lie(G)$ with respect to which $\text{tr}(\tau_j \tau_k) = -N \delta_{jk}$, $[\tau_j, \tau_k] = 2f_{jk}{}^l \tau_l$ where $N - 1$ is the rank of G . The operators (3.2) enjoy the canonical commutation relations

$$[\hat{h}_{AB}, \hat{h}_{CD}] = 0, [\hat{p}_j, \hat{h}_{AB}] = \frac{is}{2}(\tau_j \hat{h})_{AB}, [\hat{p}_j, \hat{p}_k] = -is f_{jk}{}^l \hat{p}_l \quad (3.3)$$

mirroring the classical Poisson brackets

$$\{h_{AB}, h_{CD}\} = 0, \{p_j, h_{AB}\} = \frac{1}{2}(\tau_j h)_{AB}, \{p_j, p_k\} = -f_{jk}{}^l p_l \quad (3.4)$$

on the phase space T^*G , the cotangent bundle over G , where s plays the role of Planck's constant. It is easy to check that the CCR of (3.3) and the adjointness relations coming from $\bar{p}_j = p_j$, $\bar{h}_{AB} = f_{AB}(h)$ are faithfully implemented on \mathcal{H}_G . Here, f_{AB} depends on the group, e.g. $f_{AB}(h) = (h^{-1})_{BA}$ for $G = SU(N)$, and \hat{p}_j is essentially self-adjoint with core \mathcal{D} .

We now consider G as a subgroup of some unitary group so that the τ_j are antihermitean. We then identify $G_{\mathbb{C}}$ with T^*G by the diffeomorphism

$$\phi : T^*G \mapsto G_{\mathbb{C}}; (h, p) \mapsto g := e^{-ip^j \tau_j / 2} h =: Hh \quad (3.5)$$

where the inverse is simply given by the unique polar decomposition of $g \in G_{\mathbb{C}}$. One can show that the symplectic structure (3.4) is compatible with the complex structure of $G_{\mathbb{C}}$, displaying the complex manifold $G_{\mathbb{C}}$ as a Kähler manifold.

Next we define on \mathcal{D} the annihilation and creation operators

$$\hat{g}_{AB} := e^{s\Delta/2} \hat{h}_{AB} e^{-s\Delta/2} \text{ and } (\hat{g}_{AB})^\dagger := e^{-s\Delta/2} f(\hat{h})_{AB} e^{s\Delta/2} \quad (3.6)$$

Then, as one can show, $\hat{g} = e^{Ns/4} e^{-i\hat{p}_j \tau_j} \hat{h}$ so that the operator \hat{g} qualifies as a quantization of the polar decomposition of g .

To call these operators annihilation and creation operators is justified by the following list of properties with respect to the coherent states (3.1).

i) *Eigenstate Property*

The states (3.1) are simultaneous eigenstates of the operators \hat{g}_{AB} with eigenvalue g_{AB} .

$$\hat{g}_{AB}\psi_g^s = g_{AB}\psi_g^s \quad (3.7)$$

ii) *Expectation Value Property*

From this it follows immediately that the expectation values of the operators (3.6) with respect to the states (3.1) exactly equal their classical ones as prescribed by the phase space point g .

$$\frac{\langle \psi_g^s, \hat{g}_{AB}\psi_g^s \rangle}{\|\psi_g^s\|^2} = g_{AB}, \quad \frac{\langle \psi_g^s, (\hat{g}_{AB})^\dagger \psi_g^s \rangle}{\|\psi_g^s\|^2} = \overline{g_{AB}} \quad (3.8)$$

iii) *Saturation of the Heisenberg Uncertainty Bound*

Consider the self-adjoint operators $\hat{x}_{AB} = (\hat{g}_{AB} + [\hat{g}_{AB}]^\dagger)/2$, $\hat{y}_{AB} = (\hat{g}_{AB} - [\hat{g}_{AB}]^\dagger)/(2i)$ and their classical counterparts analogously built from g_{AB} . Then these operators saturate the Heisenberg uncertainty obstruction bound, moreover, the coherent states are *unquenched* for \hat{x}, \hat{y} .

$$\langle [\hat{x}_{AB} - x_{AB}]^2 \rangle_g^s = \langle [\hat{y}_{AB} - y_{AB}]^2 \rangle_g^s = \frac{|\langle [\hat{x}_{AB}, \hat{y}_{AB}] \rangle_g^s|^2}{2} \quad (3.9)$$

where $\langle . \rangle_g^s$ denotes the expectation value with respect to ψ_g^s . Thus they are minimal uncertainty states.

iv) *Completeness and Segal-Bargmann Hilbert Space*

There exists a measure ν_s on $G_{\mathbf{C}}$ and a unitary map

$$\hat{U}_s : \mathcal{H}_G \mapsto \mathcal{H}_{G_{\mathbf{C}}} := \text{Hol}(G_{\mathbf{C}}) \cap L_2(G_{\mathbf{C}}, d\nu_s); f(h) \mapsto (\hat{U}_s f)(g) := (e^{s\Delta/2} f)(h)_{h \rightarrow g} \quad (3.10)$$

between \mathcal{H}_G and the space of ν_s -square integrable, holomorphic functions, the Segal-Bargmann space. Moreover, the coherent states satisfy the overcompleteness relation

$$1_{\mathcal{H}_G} = \int_{G_{\mathbf{C}}} d\nu_s(g) \hat{P}_{\psi_g^s} \quad (3.11)$$

where \hat{P}_f denotes the projection onto the one-dimensional subspace of \mathcal{H}_G spanned by the element f .

v) *Peakedness Properties*

As usual, semiclassical behaviour of the system is most conveniently studied in terms of $\mathcal{H}_{G_{\mathbf{C}}}$ because wave functions depend on phase space rather than on configuration space only. For instance, we have the peakedness property of the overlap function

$$\frac{|\langle \psi_g^s, \psi_{g'}^s \rangle|^2}{\|\psi_g^s\|^2 \|\psi_{g'}^s\|^2} = e^{-\frac{F(p,p') + G(\theta,\theta')}{s}} (1 - K_s(g, g')) \quad (3.12)$$

where $g = e^{-ip_j\tau_j/2} e^{\theta_j\tau_j}$ (and similar for g') is the polar decomposition of g . K_s is a positive function, uniformly bounded by a constant K'_s independent of g, g' that approaches zero exponentially fast as $s \rightarrow 0$. F, G are positive definite functions which take the value zero if and only if $p_j = p'_j$ and $\theta_j = \theta'_j$, moreover for small $p'_j - p_j, \theta'_j - \theta_j$ we have $F(p, p') \approx (p'_j - p_j)^2$, $G(\theta, \theta') \approx (\theta'_j - \theta_j)^2$ which shows that these states generalize the familiar $T^*\mathbb{R}$ coherent states to the non-linear setting of T^*G . It can be shown [58, 42] that (3.12) is the probability density, with respect to

the Liouville measure on T^*G , to find the system at the phase space point g' if it is in the state ψ_g^s and that density equals unity at $g = g'$ and is otherwise strongly, Gaussian suppressed as $s \rightarrow 0$ with width \sqrt{s} . Similar peakedness properties can be established in the configuration or momentum representation.

vi) *Ehrenfest Theorems*

The expectation value property holds for the operators \hat{g}_{AB} and \hat{g}_{AB}^\dagger at any value of s . For the remaining operators one can show

$$\lim_{s \rightarrow 0} \langle \hat{h}_{AB} \rangle_g^s = h_{AB} \text{ and } \lim_{s \rightarrow 0} \langle \hat{p}_j \rangle_g^s = p_j \quad (3.13)$$

where $g = e^{-ip_j \tau_j / 2} h$ and the convergence is exponentially fast. The result (3.13) extends to arbitrary polynomials of \hat{h}_{AB}, \hat{p}_j and even to non-polynomial, non-analytic functions of \hat{p}_j of the type that occur in quantum gravity, most importantly the volume operator mentioned in the introduction.

These beautiful properties of the states introduced by Hall will be extended to the gauge field theory case in the next subsection.

3.2 Graph Coherent States

Let $\gamma \in \Gamma_0^\omega$ be a piecewise analytic, finite graph, that is, with a finite number of edges $e \in E(\gamma)$. For each edge e we introduce the functions h_e, P_j^e on (M, ω) as in subsection (2.1). Furthermore, we introduce the dimensionless quantities

$$p_j^e := \frac{P_j^e}{a^{n_D}} \text{ and } s := \frac{\hbar \kappa}{a^{n_D}} \quad (3.14)$$

Here $n_D = n'_D$ if $n'_D \neq 0$ and $n_D = 1$ otherwise where $n'_D = D - 3$ for Yang-Mills theory and $n'_D = D - 1$ for general relativity. Furthermore, if $n'_D \neq 0$ then a is some fixed, arbitrary parameter of the dimension of a length (e.g. $a = 1\text{cm}$), if $n'_D = 0$ then a is dimensionfree and $\hbar \kappa$ is the Feinstruktur constant. Then the Poisson bracket relations of (2.6) become

$$\begin{aligned} [\hat{h}_e, \hat{h}_{e'}]_\gamma &= 0 \\ [\hat{p}_j^e, \hat{h}_{e'}]_\gamma &= i s \delta_{e'}^{T_j} \hat{h}_e \\ [\hat{p}_i^e, \hat{p}_j^{e'}]_\gamma &= -i s \delta^{ee'} f_{ij}{}^k \hat{p}_k^{e'} \end{aligned} \quad (3.15)$$

where the notation $[\cdot, \cdot]_\gamma$ indicates that all operators are restricted to the subspace \mathcal{H}_γ of \mathcal{H} . It is trivial to see that these relations classically carry over from the category Γ_0^ω to the category Γ_σ^ω .

We can now introduce the graph coherent states

$$\psi_{\gamma, \vec{g}}^s(\vec{h}) := \prod_{e \in E(\gamma)} \psi_{g_e}^s(h_e) \quad (3.16)$$

which are obviously neither gauge invariant nor diffeomorphism invariant. In [41] it was indicated how to obtain diffeomorphism invariant coherent states from those in (3.16) and in [42] the same was done in order to obtain gauge invariant ones, employing the group averaging technique [11]. Since at the moment we are interested in issues related to the classical limit of the theory, in particular, whether we obtain in the classical limit the classical Einstein equations in an appropriate sense, we will not use those invariant states for two reasons :

1) In order to check the correctness of the classical limit we must verify, in particular, whether the quantum constraint algebra of the quantum theory becomes the Dirac algebra in the classical limit. However, one cannot check an algebra on its kernel, see [28] for a discussion.

2) As far as the gauge – and diffeomorphism constraint are concerned, it is perfectly fine to work with non-invariant coherent states because the corresponding gauge groups are represented as unitarily on the Hilbert space. This implies that expectation values of gauge – and diffeomorphism invariant operators are automatically also gauge – and diffeomorphism invariant and so qualify as expectation values of the reduced theory. Famously, the time reparameterizations associated with the Hamiltonian constraint of the theory cannot be unitarily represented and so the argument just given does not carry over to operators commuting with the Hamiltonian constraint. Presumably, the Hamiltonian constraint cannot be exponentiated at all and one will then have to work with its infinitesimal version. To pass then to the reduced theory one would need to work with coherent states that are annihilated by the Hamiltonian constraint (trivial representation of the “would be time reparameterization group”).

The coherent states (3.16) then form a valid starting point for addressing semiclassical questions in the case that Σ is compact, say, in some cosmological situations. To cover the case that Σ is asymptotically flat we must blow up the framework and pass to the Infinite Tensor Product.

4 The Abstract Infinite Tensor Product

*Bei Systemen mit N Teilchen ist der Hilbertraum das Tensorprodukt von den N Hilberträumen der einzelnen Teilchen.
Das unendliche Tensorprodukt öffnet die Tür zu den mathematischen Feinheiten der Feldtheorie.*

(Walter Thirring)

The Infinite Tensor Product (ITP) of Hilbert spaces is a standard construction in statistical physics (through the thermodynamic (or infinite volume) limit) as well as in Operator Theory (von Neumann Algebras). In fact, the first examples of von Neumann algebras which are not of factor type I_n, I_∞ (isomorphic to an algebra of bounded operators on a (separable) Hilbert space) have been constructed by using the ITP.

On the other hand, since the concept of separable (Fock) Hilbert spaces plays such a dominant role in high energy physics, presumably many theoretical physicists belonging to that community have never come across the concept of the Infinite Tensor Product (ITP) of Hilbert spaces which produces a non-separable Hilbert space in general. In fact, let us quote from Streater&Wightman, [59] p. 86, 87 in that respect :

“...It is sometimes argued that in quantum field theory one is dealing with a system of an infinite number of degrees of freedom and so must use a non-separable Hilbert space.... Our next task is to explain why this is wrong, or at best is grossly misleading....All these arguments make it clear that there is no evidence that separable Hilbert spaces are not the natural state spaces for quantum field theory....”

Because of this, we have decided to include here a rough account of the most important concepts associated with the abstract Infinite Tensor Product. As it will become clear shortly, the ITP decomposes into an uncountable direct sum of Hilbert spaces which in most applications are separable. Each of these tiny subspaces of the complete ITP are isomorphic with the usual Fock spaces of quantum field theory on Minkowski space (or some other background). Presumably, the fact that one can do with separable Hilbert spaces in ordinary QFT is directly related to the fact that one *fixes the background* since this fixes the vacuum. The necessity to deal with the full ITP in quantum gravity could therefore be based on the fact that, in a sense, one *has to consider all possible backgrounds at once* ! More precisely, the metric cannot be fixed to equal a given background but becomes itself a fluctuating quantum operator.

We follow the beautiful and comprehensive exposition by von Neumann [43] who invented the Infinite Tensor Product (ITP) more than sixty years ago already. The reader is recommended to consult this work for more details.

4.1 Definition of the Infinite Tensor Product of Hilbert Spaces

Let \mathcal{I} be some set of indices α . We will not restrict the cardinality $|\mathcal{I}|$, rather for the sake of maximal generality we will allow $|\mathcal{I}|$ to take any possible value in the set of Cantor's Alephs [60]. The cardinality of the countably infinite sets is given by the non-standard number \aleph . Then the cardinality of any other infinite set can be written as a function of \aleph (usually exponentials (of exponentials of..) \aleph), e.g. the set \mathbb{R} has the cardinality 2^{\aleph} . The mathematical justification for this amount of generality is because, following von Neumann [43],

“...while the theory of enumerably infinite direct products $\otimes_{n=1}^{\infty} \mathcal{H}_n$ presents essentially new features, when compared with that of the finite $\otimes_{n=1}^N \mathcal{H}_n$, the passage from $\otimes_{n=1}^{\infty} \mathcal{H}_n$

to the general $\otimes_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha$ presents no further difficulties...., the generalizations of the direct product lead to higher set-theoretical powers (G. Cantor's "Alephs"), and to no measure problems at all."

Definition 4.1 Let $\{z_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of complex numbers. The infinite product

$$\prod_{\alpha \in \mathcal{I}} z_\alpha \quad (4.1)$$

is said to converge to the number $z \in \mathbb{C}$

$$\Leftrightarrow \forall \delta > 0 \exists \mathcal{I}_0(\delta) \subset \mathcal{I}, |\mathcal{I}_0(\delta)| < \infty \ni |z - \prod_{\alpha \in \mathcal{J}} z_\alpha| < \delta \forall \mathcal{I}_0(\delta) \subset \mathcal{J} \subset \mathcal{I}, |\mathcal{J}| < \infty.$$

From the definition it is also straightforward to prove that if $\prod_\alpha z_\alpha, \prod_\alpha z'_\alpha$ converge to z, z' respectively then $\prod_\alpha z_\alpha z'_\alpha$ converges to zz' .

Recall that a series $\sum_\alpha z_\alpha$ converges if and only if it converges absolutely which in turn is the case if and only if $z_\alpha = 0$ for all but countably infinitely many $\alpha \in \mathcal{I}$. The following theorem gives a useful convergence criterion for infinite products.

Theorem 4.1

1)

Let $\rho_\alpha \geq 0$.

i) If $\exists \alpha_0 \in \mathcal{I} \ni \rho_{\alpha_0} = 0$ then $\prod_\alpha \rho_\alpha = 0$.

ii) If $\rho_\alpha > 0 \forall \alpha$ then $\prod_\alpha \rho_\alpha$ converges if and only if $\sum_\alpha \max(\rho_\alpha - 1, 0)$ converges.

iii) If $\rho_\alpha > 0 \forall \alpha$ then $\prod_\alpha \rho_\alpha$ converges to $\rho > 0$ if and only if $\sum_\alpha |\rho_\alpha - 1|$ converges.

2)

Let $z_\alpha = \rho_\alpha e^{i\varphi_\alpha} \in \mathbb{C}$ where $\rho_\alpha = |z_\alpha|, \varphi_\alpha \in [-\pi, \pi]$. Then $\prod_\alpha z_\alpha$ converges if and only if

i) either $\prod_\alpha \rho_\alpha$ converges to zero in which case $\prod_\alpha z_\alpha = 0$,

ii) or $\prod_\alpha \rho_\alpha$ converges to $\rho > 0$ and $\sum_\alpha |\varphi_\alpha|$ converges in which case $\prod_\alpha z_\alpha = \rho e^{i \sum_\alpha \varphi_\alpha}$.

In contrast to the case of an infinite series, absolute convergence of an infinite product does not imply convergence, the phases of the factors could fluctuate too wildly. This motivates the following definition.

Definition 4.2 Let $z_\alpha \in \mathbb{C}$. We say that $\prod_\alpha z_\alpha$ is quasi-convergent if $\prod_\alpha |z_\alpha|$ converges. In this case we define the value of $\prod_\alpha z_\alpha$ to equal $\prod_\alpha z_\alpha$ if $\prod_\alpha z_\alpha$ is even convergent and to equal zero otherwise.

This definition assigns a value to the infinite product of numbers which converge absolutely but not necessarily non-absolutely. As a corollary of theorem 4.1 we have

Corollary 4.1 Quasi-convergence of $\prod_\alpha z_\alpha$ to a non-vanishing value is equivalent with convergence to the same value. A necessary and sufficient criterion is that $z_\alpha \neq 0 \forall \alpha$ and that $\sum_\alpha |z_\alpha - 1|$ converges.

After having defined convergence for infinite products of complex numbers we are ready to turn to the ITP of Hilbert spaces.

Definition 4.3 Let $\mathcal{H}_\alpha, \alpha \in \mathcal{I}$ be an arbitrary collection of Hilbert spaces. For a sequence $f := \{f_\alpha\}_{\alpha \in \mathcal{I}}, f_\alpha \in \mathcal{H}_\alpha$ the object

$$\otimes_f := \otimes_\alpha f_\alpha \quad (4.2)$$

is called a C -vector provided that $\prod_\alpha \|f_\alpha\|_\alpha$ converges, where $\|\cdot\|_\alpha$ denotes the Hilbert norm of \mathcal{H}_α . The set of C -vectors will be called V_C .

The following property holds for C -vectors, enabling us to compute their inner products.

Lemma 4.1 For two C -vectors $\otimes_f = \otimes_\alpha f_\alpha, \otimes_g = \otimes_\alpha g_\alpha$ the inner product

$$\langle \otimes_f, \otimes_g \rangle := \prod_\alpha \langle f_\alpha, g_\alpha \rangle_\alpha \quad (4.3)$$

is a quasi-convergent product of the individual inner products $\langle f_\alpha, g_\alpha \rangle_\alpha$ on \mathcal{H}_α .

There are C -vectors \otimes_f such that $\prod_\alpha \|f_\alpha\|_\alpha = 0$ although $\|f_\alpha\|_\alpha > 0 \forall \alpha$. Thus, it is conceivable that it happens that $\langle \Phi_f, \Phi_g \rangle \neq 0$ for some C -vector Φ_g . If that would be the case, the Schwarz inequality would be violated for the inner product (4.3) on C -vectors. That this is not the case is the content of the following lemma.

Lemma 4.2 Let \otimes_f be a C -sequence with $\prod_\alpha \|f_\alpha\|_\alpha = 0$. Then $\langle \otimes_f, \otimes_g \rangle = 0$ for any C -vector \otimes_g .

To distinguish trivial C -vectors from non-trivial ones we define

Definition 4.4 A sequence (f_α) defines a C_0 -vector $\otimes_f = \otimes_\alpha f_\alpha$ iff

$$\sum_\alpha | \|f_\alpha\|_\alpha - 1 | \quad (4.4)$$

converges. The set of C_0 -vectors will be denoted by V_0 .

It is easy to prove by means of theorem 4.1 that every C_0 -vector is a C -vector but only those C -vectors are C_0 -vectors for which $\langle \otimes_f, \cdot \rangle$, considered as a linear functional on C -vectors, does not equal zero which by lemma 4.2 implies, in particular, that $\prod_\alpha \|f_\alpha\|_\alpha \neq 0$. It follows that the norm of a C_0 -vector does not vanish, as the following lemma reveals.

Lemma 4.3 For any complex numbers, the convergence of one of $\sum_\alpha | |z_\alpha| - 1 |$, $\sum_\alpha | |z_\alpha|^2 - 1 |$ implies the convergence of the other.

Thus, since by definition of a C_0 -vector \otimes_f and theorem 4.11)iii) $z_\alpha = \|f_\alpha\|_\alpha$ satisfies the assumption of lemma 4.3, by that lemma and again theorem 4.11)iii) in the opposite direction we find that $\| \otimes_f \| > 0$.

Obviously we will construct the ITP Hilbert space from the linear span of C_0 -vectors (we can ignore the C -vectors which are not C_0 -vectors by lemma 4.2). For this it will be useful to know how the Hilbert space decomposes into orthogonal subspaces. The following definition serves this purpose.

Definition 4.5 If \otimes_f is a C_0 -vector, we will call the sequence $f = \{f_\alpha\}$ a C_0 -sequence. We will call two C_0 -sequences f, g strongly equivalent, denoted $f \approx g$, provided that

$$\sum_\alpha | \langle f_\alpha, g_\alpha \rangle_\alpha - 1 | \quad (4.5)$$

converges.

Lemma 4.4 Strong equivalence of C_0 -sequences is an equivalence relation (reflexive, symmetric, transitive).

This lemma motivates the following definition.

Definition 4.6 The strong equivalence class of a C_0 sequence f will be denoted by $[f]$. The set of strong equivalence classes of C_0 -sequences will be called \mathcal{S} .

The subsequent theorem justifies the notion of strong equivalence.

Theorem 4.2 i) If $f^0 \in [f] \neq [g] \ni g^0$ then $\langle \otimes_{f^0}, \otimes_{g^0} \rangle = 0$.

ii) If $f^0, g^0 \in [f]$ then $\langle \otimes_{f^0}, \otimes_{g^0} \rangle = 0$ if and only if there exists $\alpha \in \mathcal{I}$ such that $\langle f_\alpha, g_\alpha \rangle_\alpha = 0$.

So, C_0 -vectors from different strong equivalence classes are always orthogonal and those from the same strong equivalence class are orthogonal if and only if they are orthogonal in at least one tensor product factor.

The following theorem gives two useful criteria for strong equivalence.

Theorem 4.3 i) $[f] = [g]$ if and only if $\sum_\alpha \|f_\alpha^0 - g_\alpha^0\|_\alpha^2$ and $\sum_\alpha |\mathfrak{S}(\langle f_\alpha^0, g_\alpha^0 \rangle_\alpha)|$ converge for some $f^0 \in [f], g^0 \in [g]$.

ii) If $f_\alpha = g_\alpha$ for all but finitely many α then $f \approx g$.

Obviously, it will be convenient to choose a representant $f^0 \in [f]$ which is normalized in each tensor product factor. This is always possible.

Lemma 4.5 For each $[f] \in \mathcal{S}$ there exists $f^0 \approx f$ such that $\|f_\alpha^0\|_\alpha = 1$ for all $\alpha \in \mathcal{I}$.

The next lemma reveals that caution is due when trying to extend multilinearity from the finite to the infinite tensor product.

Lemma 4.6 Let $\prod_\alpha z_\alpha$ be quasi-convergent. Then

i) If f is a C -sequence, so is $z \cdot f$ with $(z \cdot f)_\alpha := z_\alpha f_\alpha$.

ii) If moreover $\sum_\alpha \|z_\alpha\| - 1$ converges and f is a C_0 sequence, so is $z \cdot f$.

iii) The product formula

$$\otimes_{z \cdot f} = \left[\prod_\alpha z_\alpha \right] \otimes_f \quad (4.6)$$

fails to hold only if 1) $\prod_\alpha z_\alpha$ is not convergent and 2) $\langle \otimes_f, \cdot \rangle \neq 0$ considered as a linear functional on C -vectors. In that case, $\{z_\alpha\}, f$ satisfy the assumptions of ii), moreover $z_\alpha \neq 0 \forall \alpha$.

iv) If $\{z_\alpha\}, f$ satisfy the assumptions of ii) then $[z \cdot f] = [f]$ iff $\sum_\alpha |z_\alpha - 1|$ converges. If even $z_\alpha \neq 0 \forall \alpha$, the latter condition implies convergence of $\prod_\alpha z_\alpha$.

An important conclusion that we can draw from this lemma is the following. If (4.6) fails then, by iii), $f, z \cdot f$ are both C_0 -sequences while $\prod_\alpha z_\alpha$ is only quasi-convergent. Thus, both $\otimes_f, \otimes_{z \cdot f} \neq 0$ while $\prod_\alpha z_\alpha = 0$ by definition 4.2. Thus, $[\prod_\alpha z_\alpha] \otimes_f = 0 \neq \otimes_{z \cdot f}$.

Next, since, also by iii), $z_\alpha \neq 0 \forall \alpha$ we have from corollary 4.1 that $\sum_\alpha |z_\alpha - 1|$ cannot be convergent as otherwise $\prod_\alpha z_\alpha$ would be convergent which cannot be the case as $\prod_\alpha z_\alpha$ is only quasi-convergent. Thus, by iv) $f, z \cdot f$ lie in different strong equivalence classes and therefore by theorem 4.2 $\langle \otimes_f, \otimes_{z \cdot f} \rangle = 0$.

Definition 4.7 By \mathcal{H}_C we denote the completion of the complex vector space of finite linear combinations of elements from V_C equipped with the sesquilinear form $\langle \cdot, \cdot \rangle$ obtained by extending (4.3) from V_C to \mathcal{H}_C by sesquilinearity.

Notice that for C -vectors which are not C_0 -vectors we have $\otimes_f = 0$ as an element of \mathcal{H}_C .

Lemma 4.7 $\langle \xi, \xi \rangle \geq 0 \forall \xi \in \mathcal{H}_C$ and we define $\|\xi\|^2 = \langle \xi, \xi \rangle$. In particular, $\langle \cdot, \cdot \rangle$ satisfies the Schwarz inequality and $\|\xi\| = 0$ if and only if $\xi = 0$.

We can now give the definition of the ITP.

Definition 4.8 We will denote by

$$\mathcal{H}^\otimes := \otimes_\alpha \mathcal{H}_\alpha \quad (4.7)$$

the Cauchy-completion of the pre-Hilbert space \mathcal{H}_C . It is called the complete ITP of the \mathcal{H}_α .

To analyze the structure \mathcal{H}^\otimes in more detail, the strong equivalence classes provide the basic tool.

Definition 4.9 For a strong equivalence class $[f] \in \mathcal{S}$ we define the closed subspace of \mathcal{H}^\otimes

$$\mathcal{H}_{[f]} := \overline{\left\{ \sum_{k=1}^N z_k \otimes f^k; z_k \in \mathbb{C}, f^k \in [f], N < \infty \right\}} \quad (4.8)$$

by the closure of the finite linear combinations of $\otimes_{f'}$'s with $f' \in [f]$. It is called the $[f]$ -adic incomplete ITP of the \mathcal{H}_α 's.

Notice that we could absorb the z_k in (4.8) into one of the f_α^k . Now we have the fundamental theorem which splits \mathcal{H}^\otimes into simpler pieces.

Theorem 4.4 The complete ITP decomposes as the direct sum over strong equivalence classes $[f]$ of $[f]$ -adic ITP's,

$$\mathcal{H}^\otimes = \overline{\otimes_{[f] \in \mathcal{S}} \mathcal{H}_{[f]}} \quad (4.9)$$

Also each $[f]$ -adic ITP can be given a simple description.

Lemma 4.8 For a given $[f] \in \mathcal{S}$, fix any $f^0 \in [f]$. By lemma 4.5 we can choose an f^0 with $\|f_\alpha^0\|_\alpha = 1$. Then $\mathcal{H}_{[f]}$ is the closure of the vector space of finite linear combinations of $\otimes_{f'}$'s where $f' \in [f]$ and $f'_\alpha = f_\alpha^0$ for all but finitely many $\alpha \in \mathcal{I}$.

It is easy to provide a complete orthonormal basis for an $[f]$ -adic ITP if we know one in each \mathcal{H}_α .

Lemma 4.9 Let $f^0 \in [f] \in \mathcal{S}$, $\|f_\alpha^0\|_\alpha = 1 \forall \alpha$. Let $d_\alpha = \dim(\mathcal{H}_\alpha)$ (takes the value of some higher Cantor aleph if \mathcal{H}_α is not separable). Let \mathcal{J}_α , $0 \in \mathcal{J}_\alpha \forall \alpha \in \mathcal{I}$ be a set of indices of cardinality d_α and choose a complete orthonormal basis e_α^β , $\beta \in \mathcal{J}_\alpha$ such that $e_\alpha^0 = f_\alpha^0$.

Consider the set \mathcal{F} of functions

$$\beta : \mathcal{I} \mapsto \times_\alpha \mathcal{J}_\alpha; \alpha \mapsto \{\beta(\alpha)\}_{\alpha \in \mathcal{I}} \quad (4.10)$$

such that 1) $\beta(\alpha) \in \mathcal{J}_\alpha$ and 2) $\beta(\alpha) \neq 0$ for finitely many α only. Let

$$\otimes_{e^\beta} := \otimes_\alpha e_\alpha^{\beta(\alpha)} \quad (4.11)$$

Then $e^\beta \in [f]$ and the set of C_0 -vectors $\{\otimes_{e^\beta}; \beta \in \mathcal{F}\}$ forms a complete orthonormal basis of $\mathcal{H}_{[f]}$, called a von Neumann basis.

The following corollary establishes that the $[f]$ -adic ITP's are mutually isomorphic.

Corollary 4.2 Each $[f]$ -adic ITP is unitarily equivalent to the Hilbert space $\mathcal{H}_{\mathcal{F}} = L_2(\mathcal{F}, d\nu_0)$ of square summable functions on \mathcal{F} , $\hat{\xi} : \mathcal{F} \mapsto \mathbb{C}; \beta \mapsto \hat{\xi}(\beta)$, where ν_0 is the counting measure. The unitary map is given by

$$\hat{U}_{[f]} : \mathcal{H}_{\mathcal{F}} \mapsto \mathcal{H}_{[f]}; \hat{\xi} \mapsto \sum_{\beta \in \mathcal{F}} \hat{\xi}(\beta) \otimes_{e^\beta} \quad (4.12)$$

The inverse map is given by

$$\hat{U}_{[f]}^{-1} : \mathcal{H}_{[f]} \mapsto \mathcal{H}_{\mathcal{F}}; \xi \mapsto \hat{\xi}(\beta) = \langle \otimes_{e^\beta}, \xi \rangle \quad (4.13)$$

In particular, since each $[f]$ -adic ITP has a complete orthonormal basis labelled by \mathcal{F} and since the ITP is the direct sum of (the mutually isomorphic) $[f]$ -adic ITP's we have $\dim(\mathcal{H}^\otimes) = |\mathcal{F}| \cdot |\mathcal{S}|$ where the appearing cardinalities will be aleph-valued in general (already in the simplest non-trivial case $\dim(\mathcal{H}_\alpha) = 2$, $\mathcal{I} = \mathbb{N}$).

Notice that the index set \mathcal{I} is not required to have any ordering structure, thus we have identities of the form $\otimes_\alpha f_\alpha = f_{\alpha_0} \otimes [\otimes_{\alpha \neq \alpha_0} f_\alpha]$, these are just different notations for the same object. However, is important to realize that the associative law generically does not hold for the ITP. By this we mean the following :

Let us decompose \mathcal{I} into mutually disjoint index sets \mathcal{I}_l with $l \in \mathcal{L}$ then we can form the following Hilbert spaces : $\mathcal{H}^\otimes = \otimes_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha$ and $\mathcal{H}^{\otimes'} := \otimes_{l \in \mathcal{L}} \mathcal{H}_l$ where $\mathcal{H}_l := \otimes_{\alpha \in \mathcal{I}_l} \mathcal{H}_\alpha$. The C_0 -vectors of \mathcal{H}^\otimes are given by $\otimes_f = \otimes_{\alpha \in \mathcal{I}} f_\alpha$ while the C_0 -vectors of $\mathcal{H}^{\otimes'}$ are given by $\otimes'_f = \otimes_{l \in \mathcal{L}} f'_l$ where $f'_l \in \mathcal{H}_l$ is a (Cauchy limit of a) finite linear combination of vectors of the form $\otimes^l_f = \otimes_{\alpha \in \mathcal{I}_l} f_\alpha$. Inner products between C_0 -vectors are computed as $\langle \otimes_f, \otimes_g \rangle = \prod_\alpha \langle f_\alpha, g_\alpha \rangle_\alpha$ and $\langle \otimes'_f, \otimes'_g \rangle = \prod_\alpha \langle f_l, g_l \rangle_l$ respectively where $\langle \otimes^l_f, \otimes^l_g \rangle_l = \prod_{\alpha \in \mathcal{I}_l} \langle f_\alpha, g_\alpha \rangle_\alpha$.

It is easy to see that if $f = \{f_\alpha\}_{\alpha \in \mathcal{I}}$ is a C_0 -sequence for \mathcal{H}^\otimes then $f' = \{f'_l := \otimes^l_f\}_{l \in \mathcal{L}}$ is a C_0 -sequence for $\mathcal{H}^{\otimes'}$. However, the obvious map between C_0 -sequences given by

$$C : f \mapsto f' \quad (4.14)$$

in general does not preserve the decomposition into strong equivalence classes of \mathcal{H}^\otimes and $\mathcal{H}^{\otimes'}$ respectively. We will give a few examples to illustrate this point.

i)

Let $\mathcal{I} = \mathcal{L} = \mathbb{N}$ and $\mathcal{I}_l = \{2l-1, 2l\}$ so that $\mathcal{I} = \cup_{l=1}^\infty \mathcal{I}_l$. Consider the following two C_0 -sequences : f_α , $\alpha \in \mathbb{N}$ is just some normal vector in \mathcal{H}_α , that is, $\|f_\alpha\|_\alpha = 1$ and $g_\alpha = -f_\alpha$. Then certainly their strong equivalence classes with respect to \mathcal{H}^\otimes are different, $[f] \neq [g]$ since $|\langle f_\alpha, g_\alpha \rangle - 1| = 2$ so that (4.5) blows up. On the other hand we have $f'_l = \otimes^l_f = f_{2l-1} \otimes f_{2l} = [-f_{2l-1}] \otimes [-f_{2l}] = \otimes^l_g = g'_l$. Thus, trivially $[f']' = [g']'$ where the prime at the bracket indicates that the class is with respect to $\mathcal{H}^{\otimes'}$.

ii)

Even multiplication by complex numbers is problematic : Take the same index sets as in i) and consider the complex numbers $z_\alpha = -1$. Then $\prod_\alpha z_\alpha$ is quasi-convergent but not convergent and therefore by definition $\prod_\alpha z_\alpha = 0$. Our map (4.14) now sends $z \cdot f$ to $z' \cdot f'$ with $z'_l = z_{2l-1} z_{2l}$. Now $z'_l = 1$ and thus $\prod_l z'_l$ is convergent to 1. It follows that $\otimes_{z \cdot f} \neq [\prod_\alpha z_\alpha] \otimes_f = 0$ but $\otimes'_{z' \cdot f'} = [\prod_l z'_l] \otimes'_{f'} = \otimes'_{f'}$, in particular, $[f] \neq [z \cdot f]$ but $[f']' = [z' \cdot f']'$.

iii)

Our map is certainly not invertible : Consider, for the same index sets as in i), the vector

$$f'_l := \frac{1}{\sqrt{2}} [e_{2l-1}^1 \otimes e_{2l}^1 + e_{2l-1}^2 \otimes e_{2l}^2] \quad (4.15)$$

where we assume that \mathcal{H}_α is at least two-dimensional and we choose two orthonormal vectors e_α^j , $j = 1, 2$ for each α . Then $\|f'_l\|_l = 1$ and f' is a C_0 -sequence for $\mathcal{H}^{\otimes'}$. However, we cannot write f' as a finite linear combination of C_0 -vectors of \mathcal{H}^\otimes : Any attempt to use the distributive law and to write it as a linear combination of C_0 -vectors for \mathcal{H}^\otimes of the form $\otimes_l [e_{2l-1}^{j_l} \otimes e_{2l}^{j_l}]$ with $j_l \in \{1, 2\}$ fails because all of these vectors are orthogonal (with respect to $\mathcal{H}^{\otimes'}$) to $\otimes'_{f'}$:

$$\langle \otimes_l [e_{2l-1}^{j_l} \otimes e_{2l}^{j_l}], \otimes_l f'_l \rangle = \prod_l \frac{1}{\sqrt{2}} = 0 \quad (4.16)$$

It is plausible and one can indeed show that these complications do not arise if $|\mathcal{L}| < \infty$.

4.2 Von Neumann Algebras on the Infinite Tensor Product

The set of von Neumann algebras that one can define on the Infinite Tensor Product Hilbert space is of a surprisingly rich structure. In fact, every possible type of von Neumann's factors ($I_\infty, II_1, II_\infty, III_0, III_1, III_\lambda$; $\lambda \in (0, 1)$) can be realized on the ITP. Physically, one will start from the local operators that "come from the various \mathcal{H}_α ". However, there are many more operators which are not local and which are well-defined on the ITP. All the algebras that we consider are assumed to be unital.

Definition 4.10 *We denote by $\mathcal{B}(\mathcal{H}_\alpha)$ the set of bounded operators on \mathcal{H}_α and by $\mathcal{B}^\otimes := \mathcal{B}(\mathcal{H}^\otimes)$ the set of bounded operators on the ITP \mathcal{H}^\otimes .*

The restriction to bounded operators is not a severe one since every unbounded operator can be written (up to domain questions) as a linear combination of self-adjoint ones and those are known if we know their spectral projections which are bounded operators.

An operator on one of the tensor product factors is not a priori defined on the ITP. The following lemma embeds $\mathcal{B}(\mathcal{H}_\alpha)$ into \mathcal{B}^\otimes .

Lemma 4.10 *Let $\alpha_0 \in \mathcal{I}$ and $A_{\alpha_0} \in \mathcal{B}(\mathcal{H}_{\alpha_0})$. Then there exists a unique operator $\hat{A}_{\alpha_0} \in \mathcal{B}^\otimes$ such that for any C-sequence f*

$$\hat{A}_{\alpha_0} \otimes_f = \otimes_{f'} \text{ where } f'_\alpha = \begin{cases} f_\alpha & : \alpha \neq \alpha_0 \\ A_{\alpha_0} f_{\alpha_0} & : \alpha = \alpha_0 \end{cases} \quad (4.17)$$

We will use the notation

$$\hat{A}_{\alpha_0} \otimes_f = [A_{\alpha_0} f_{\alpha_0}] \otimes [\otimes_{\alpha \neq \alpha_0} f_\alpha] \quad (4.18)$$

This lemma gives rise to the following definition.

Definition 4.11 *We denote by \mathcal{B}_α the extension of $\mathcal{B}(\mathcal{H}_\alpha)$ to the ITP, that is,*

$$\mathcal{B}_\alpha = \{\hat{A}_\alpha; A_\alpha \in \mathcal{B}(\mathcal{H}_\alpha)\} \quad (4.19)$$

Obviously $\mathcal{B}_\alpha \subset \mathcal{B}^\otimes$.

It is not difficult to prove that $A_\alpha \leftrightarrow \hat{A}_\alpha$ is in fact a $*$ algebra isomorphism. The algebras $\mathcal{B}^\otimes, \mathcal{B}(\mathcal{H}_\alpha)$ are C^* algebras by definition. Recall that, on the other hand, a von Neumann algebra over a Hilbert space is a weakly (equivalently strongly) closed sub- $*$ algebra of the algebra of bounded operators on that Hilbert space.

Lemma 4.11 *For all $\alpha \in \mathcal{I}$, the algebra \mathcal{B}_α is a von Neumann algebra (v.N.a.) over \mathcal{H}^\otimes .*

The idea of proof is quite simple : One writes $\mathcal{B}^\otimes = \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_{\bar{\alpha}})$ where $\bar{\alpha} = \mathcal{I} - \alpha$. Next, it is almost obvious that \mathcal{B}_α coincides with $\mathcal{B}'_{\bar{\alpha}} = \{\hat{B} \in \mathcal{B}^\otimes; [\hat{A}, \hat{B}] = 0 \forall \hat{A} \in \mathcal{B}_{\bar{\alpha}}\}$, the commutant of $\mathcal{B}_{\bar{\alpha}}$. Then an appeal to the bicommutant (or von Neumann density) theorem [61] finishes the proof.

Actually, the correspondence of lemma 4.10 extends to von Neumann algebras $\mathcal{R}(\mathcal{H}_\alpha) \subset \mathcal{B}(\mathcal{H}_\alpha)$ as we state in the subsequent theorem.

Theorem 4.5 *The one to one correspondence $\mathcal{B}(\mathcal{H}_\alpha) \ni A_\alpha \leftrightarrow \hat{A}_\alpha \in \mathcal{B}_\alpha$ extends to a $*$ isomorphism between von Neumann algebras $\mathcal{B}(\mathcal{H}_\alpha) \supset \mathcal{R}(\mathcal{H}_\alpha) \leftrightarrow \mathcal{R}_\alpha = \{\hat{A}_\alpha; A_\alpha \in \mathcal{R}(\mathcal{H}_\alpha)\}$*

The largest von Neumann algebra on \mathcal{H}^\otimes that we can construct from the algebras $\mathcal{B}(\mathcal{H}_\alpha)$ is the following one.

Definition 4.12 By \mathcal{R}^\otimes we denote the smallest v.N.a. that contains all the \mathcal{B}_α , that is, the weak closure of the set

$$\cup_{\alpha \in \mathcal{I}} \mathcal{B}_\alpha \quad (4.20)$$

It turns out that not surprisingly \mathcal{R}^\otimes is a proper subalgebra of \mathcal{B}^\otimes unless $|\mathcal{I}| < \infty$. Physically, the indices α label local degrees of freedom and therefore the elements of \mathcal{B}_α correspond to local operators of a quantum field theory. Thus the algebra \mathcal{R}^\otimes is the algebra of local observables represented on the ITP \mathcal{H}^\otimes . The remainder $\mathcal{B}^\otimes - \mathcal{R}^\otimes$ can therefore be identified with a set of non-local operators. Thus, while the algebra \mathcal{R}^\otimes is rather important from the point of view of local (or algebraic) quantum field theory [62] it is the remainder which offers challenging possibilities in the sense that it could be the universal home for operators that map a given physical system to a drastically different one. Examples for this could be the change of energy by an infinite amount or *topology change* of the underlying spacetime manifold. We will come back to this point in section 5. These issues should be particularly important for quantum general relativity since there all the (Dirac) observables are supposed to be non-local.

In any case we should investigate the subalgebra \mathcal{R}^\otimes in more detail. To that end, recall from lemma 4.6 that the equation $\otimes_{z \cdot f} = [\prod_\alpha z_\alpha] \otimes_f$ is false only if both $f, z \cdot f$ are C_0 -vectors, $z_\alpha \neq 0 \forall \alpha$ but $\prod_\alpha z_\alpha$ is only quasi-convergent. This fact gives rise to the next definition.

Definition 4.13 Two C_0 -sequences f, g are said to be weakly equivalent, denoted by $f \sim g$, provided that there are complex numbers z_α such that $z \cdot f$ and g are strongly equivalent, that is, $z \cdot f \approx g$.

Important facts about weak equivalence are contained in the following lemma which also contains a necessary and sufficient criterion.

Lemma 4.12

- i) Definition 4.13 remains unchanged if we restrict to complex numbers with $|z_\alpha| = 1$.
- ii) Weak equivalence is an equivalence relation (reflexive, symmetric, transitive).
- iii) $f \sim g$ if and only if

$$\sum_\alpha | | \langle f_\alpha, g_\alpha \rangle | - 1 | \quad (4.21)$$

converges.

Comparing with definition 4.5 we see that the “only” difference between strong and weak equivalence is the additional modulus for $\langle f_\alpha, g_\alpha \rangle$ in (4.21).

Definition 4.14

- i) For a C_0 -sequence f its weak equivalence class is denoted by (f) . The set of weak equivalence classes is denoted by \mathcal{W} .
- ii) For given $(f) \in \mathcal{W}$ we denote by $\mathcal{H}_{(f)}$ the closure of the set of finite linear combinations of $\otimes_{f'}$ ’s where $f' \in (f)$.

Obviously, weak equivalence is weaker than strong equivalence. Thus, each $(f) \in \mathcal{W}$ decomposes into mutually disjoint $[f'] \in \mathcal{S}$, $f' \in (f)$. It follows from this and the mutual orthogonality of the $\mathcal{H}_{[f']}$ ’s (theorem 4.2) that we may write

$$\mathcal{H}_{(f)} = \overline{\oplus_{[f'] \in \mathcal{S} \cap (f)} \mathcal{H}_{[f']}} \quad (4.22)$$

Lemma 4.13 *i) For every sequence of complex numbers $\{z_\alpha\}_\alpha$ such that $|z_\alpha| = 1 \forall \alpha$ there exists a unique, unitary operator \hat{U}_z , densely defined on (finite linear combinations of) C_0 -vectors f such that $\hat{U}_z \otimes_f = \otimes_{z.f}$.*

ii) Given $s \in \mathcal{S}$, $w \in \mathcal{W}$ respectively, denote by \hat{P}_s, \hat{P}_w respectively the projection operators from \mathcal{H}^\otimes onto the closed subspaces $\mathcal{H}_s, \mathcal{H}_w$ respectively. Then :

a) $[\hat{U}_z, \hat{P}_w] = 0$,

b) $[\hat{U}_z, \hat{P}_s] = 0$ if and only if $\prod_\alpha z_\alpha$ converges to z , $|z| = 1$ in which case $\hat{U}_z = z1_{\mathcal{H}^\otimes}$ and

c) if $[\hat{U}_z, \hat{P}_s] \neq 0$ then $\hat{U}_z \mathcal{H}_s = \mathcal{H}_{s'}$ where $s \neq s' \in \mathcal{S}$, that is, \hat{U}_z maps different s -adic ITP subspaces onto each other which are thus unitarily equivalent.

The following theorem describes much of the structure of \mathcal{R}^\otimes .

Theorem 4.6

i) An operator $\hat{A} \in \mathcal{B}^\otimes$ belongs actually to \mathcal{R}^\otimes if and only if it commutes with all the \hat{U}_z, \hat{P}_s of lemma 4.13. In particular, the elements of \mathcal{R}^\otimes leave all the $\mathcal{H}_s, s \in \mathcal{S}$ invariant.

ii) For each $w \in \mathcal{W}$, fix once and for all an element $s_w \in \mathcal{S} \cap w$. Suppose that we are given a family of bounded operators \hat{A}_w on \mathcal{H}_{s_w} for each $w \in \mathcal{W}$. Then there exists an operator $\hat{A} \in \mathcal{R}^\otimes$ such that its restriction \hat{A}_{s_w} to \mathcal{H}_{s_w} coincides with \hat{A}_w , provided that the set of non-negative numbers $\{||\hat{A}_w||; w \in \mathcal{W}\}$ is bounded. In that case, \hat{A}_w is actually unique.

iii) The norm of the operator \hat{A} of ii) is given by

$$||\hat{A}|| = \sup \{||\hat{A}_s||; s \in \mathcal{S}\} = \sup \{||\hat{A}_w||; w \in \mathcal{W}\} \quad (4.23)$$

This theorem tells us the following about \mathcal{R}^\otimes :

1)

As $\hat{P}_w = \bigoplus_{s \in \mathcal{S} \cap w} \hat{P}_s$, item i) reveals that each $\mathcal{H}_w, w \in \mathcal{W}$ is an invariant subspace for any element $\hat{A} \in \mathcal{R}^\otimes$, it is “block diagonal” with respect to \mathcal{H}^\otimes where the blocks correspond to the $\mathcal{H}_w, w \in \mathcal{W}$. Within each of these blocks, \hat{A} is further reduced by each $\mathcal{H}_s, s \in \mathcal{S} \cap w$. Moreover, since \hat{U}_z commutes with \hat{A} and we obtain any $\mathcal{H}_s, s \in \mathcal{S} \cap w$ by mapping \mathcal{H}_{s_w} of theorem 4.6 with \hat{U}_z , knowlegde of \hat{A} on \mathcal{H}_{s_w} is sufficient to determine it all over \mathcal{H}_w .

2)

Item ii) tells us that certainly not every element of \mathcal{B}^\otimes lies in \mathcal{R}^\otimes , actually it is easy to construct bounded operators, e.g. the $\hat{U}_z \in \mathcal{B}^\otimes$, which do not lie in \mathcal{R}^\otimes .

Finally we determine the cardinality of the set $\mathcal{S} \cap w$.

Lemma 4.14

i) If $|\mathcal{I}| < \aleph = \aleph$ then $\mathcal{S} = \mathcal{W}$, $|\mathcal{S}| = 1$ and $\mathcal{H}^\otimes = \mathcal{H}_w = \mathcal{H}_s$.

ii) If $|\mathcal{I}| \geq \aleph$ then $|\mathcal{S} \cap w| = 2^{|\mathcal{I}|}$.

iii) If the number of α 's such that $\dim(\mathcal{H}_\alpha) \geq 2$ is finite, then $|\mathcal{W}| = 1$. Otherwise, $|\mathcal{W}| \geq 2^\aleph$.

To investigate the structure of \mathcal{R}^\otimes further we need to recall some of the notions from the theory of von Neumann algebras, e.g. [62].

Definition 4.15

i)

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a v.N.a. over the Hilbert space \mathcal{H} . The commutant of \mathcal{R} , denoted by \mathcal{R}' is the set of operators in $\mathcal{B}(\mathcal{H})$ that commute with all elements of \mathcal{R} . For a v.N.a. we have $\mathcal{R}'' = \mathcal{R}$. $\mathcal{Z}(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}'$ is called the center of \mathcal{R} . The v.N.a. is called a factor if $\mathcal{Z}(\mathcal{R}) = \{\lambda 1_{\mathcal{H}}; \lambda \in \mathbb{C}\}$, that is, the center consists only of the scalars.

ii)

Let $\hat{P}, \hat{Q} \in \mathcal{R}$ be projections. We say that

- a) \hat{Q} is a subprojection of \hat{P} , denoted $\hat{Q} \leq \hat{P}$, iff $\hat{Q}\mathcal{H} \subset \hat{P}\mathcal{H}$.
b) \hat{Q}, \hat{P} are equivalent, denoted $\hat{Q} \sim \hat{P}$, iff there exists a partial isometry [64] with initial subspace $\hat{P}\mathcal{H}$ and final subspace $\hat{Q}\mathcal{H}$.
c) $\hat{P} \neq 0$ is a minimal projection if there is no proper subprojection $\hat{Q} \neq 0$ of \hat{P} .
d) $\hat{P} \neq 0$ is an infinite projection if there is a proper subprojection $\hat{Q} \neq 0$ of \hat{P} to which it is equivalent.

iii) Let \mathcal{R} be a factor. Then we call \mathcal{R} of type

I : if \mathcal{R} contains a minimal projection. If $1_{\mathcal{H}}$ is an infinite projection, then the type is I_{∞} otherwise it is I_n where $n = \dim(\mathcal{H})$.

III : every non-zero projection of \mathcal{R} is infinite. A further systematic classification of type III factors is due to Connes, see e.g. [65] and references therein. One distinguishes between type III_0 (the Krieger factor [66]), type III_{λ} , $\lambda \in (0, 1)$ (the Powers factor [67]) and III_1 (the factor of Araki and Woods [68]).

II : if \mathcal{R} is neither of type I nor of type III. If $1_{\mathcal{H}}$ is an infinite projection, then \mathcal{R} is called type II_{∞} otherwise type II_1 .

One can show that factors of type I are isomorphic to algebras of bounded operators on some Hilbert space. Factors of type II_{∞} are generated by operators of the form $A_1 \otimes 1_{\mathcal{H}_2}, 1_{\mathcal{H}_1} \otimes A_2$ acting on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ where A_1 belongs to a factor of type I_{∞} over \mathcal{H}_1 and A_2 to one of type II_1 over \mathcal{H}_2 . For factors of type I and II it is possible to introduce a dimension function for projections, that is, a positive definite function $\dim(\hat{P}) \geq 0$ vanishing only if $\hat{P} = 0$, uniquely determined by the two properties that

- 1) $\dim(\hat{P} + \hat{Q}) = \dim(\hat{P}) + \dim(\hat{Q})$ if $\hat{P} \perp \hat{Q}$ and
- 2) $\dim(\hat{P}) = \dim(\hat{Q})$ if $\hat{P} \sim \hat{Q}$.

The range of that function is $0, 1, 2, \dots, n$ for type I_n , $0, 1, 2, \dots, \infty$ for type I_{∞} , $[0, 1]$ for type II_1 and $[0, \infty]$ for type II_{∞} . For type III a dimension function can be introduced but it takes only the values $0, \infty$ and therefore cannot be used to obtain the finer subdivision of type III factors outlined above for which the use of modular (or Tomita-Takesaki) theory and the Connes invariant is necessary (a self-contained exposition aimed at mathematical physicists can be found in [69]).

We close this section by mentioning that the more unfamiliar factors of type II and III are not only of academic interest. In fact, they appear already in systems as simple as the infinite spin chain (see the second reference of [43]). If one represents the abstract CCR C^* -algebra of spin 1/2 operators $\hat{\sigma}_l^j$; $j = 1, 2, 3$; $l = 1, 2, \dots$, via the GNS theorem [61], for a state ω_s ($s \in [0, 1]$) for which we get GNS data $(\Omega_s^{\infty}, \mathcal{H}_s^{\infty}, \pi_s^{\infty})$ where the cyclic vector is

$$\Omega_s^{\infty} = \otimes_{l=1}^{\infty} \Omega_s, \quad \Omega_s = \left[\sqrt{\frac{1+s}{2}} e_1 \otimes e_1 + \sqrt{\frac{1-s}{2}} e_2 \otimes e_2 \right],$$

the Hilbert space is the ITP $\mathcal{H}_s^{\infty} = \otimes_{l=1}^{\infty} [\mathbb{C}^2 \otimes \mathbb{C}^2]$ corresponding to the index set \mathcal{I} of pairs $\alpha = (l, \tau)$, $\tau = 1, 2$, and the representation is $\pi_s^{\infty}(\hat{\sigma}_l^j)$ acting only on the Hilbert \mathbb{C}^2 (with standard orthonormal basis e_1, e_2) corresponding to $\alpha = (l, 1)$, then upon weak closure a factor of type I_{∞} or II_1 or III_s results for $s = 1$ or $s = 0$ or $s \in (0, 1)$. The physical interpretation of the parameter s is that Ω_s is the GNS datum for the mixed state

$$\omega_s(A) = \frac{\text{tr}(Ae^{\beta\sigma^3})}{\text{tr}(e^{\beta\sigma^3})}$$

with $s = \text{th}(\beta)$ on $\mathcal{H}_s = \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$, thus type I_{∞} , II_1 and III_s respectively means zero, infinite or finite temperature respectively.

Finally, the type of local algebras $\mathcal{R}(\mathcal{O})$ appearing in algebraic quantum field theory is the unique hyperfinite factor of type III₁ for diamond regions \mathcal{O} [62] (intersections of past and future light cones in the obvious way; this result can be extended to arbitrary \mathcal{O} in case that the theory has a scaling limit (short distance conformal invariance) [63]). Here a v.N.a. is said to be hyperfinite if it is the inductive limit of finite dimensional algebras. This brings to the next topic.

4.3 Inductive Limits of Hilbert Spaces and von Neuman Algebras

For the applications that we have in mind, specifically quantum gravity and quantum gauge theory coupled to gravity, the framework of the Infinite Tensor Product is not general enough for the following reason. Recall from section 2 that the degrees of freedom of these field theories are labelled by graphs. Moreover, given a graph γ the degrees of freedom associated with it are labelled by the edges of that graph. Thus, it seems that we are in position to apply the theory outlined in sections 4.1 and 4.2 by choosing $\mathcal{I} = E(\gamma)$. While that is indeed true for the given graph γ , rather than working with a fixed, infinite graph γ we are working with all of them because we do not have a lattice gauge field theory but a continuum one. So we actually get an uncountably infinite family of ITP's. That would not pose any problems if we could treat each of them independently, however, this is not the case, e.g. not if a graph is contained in a bigger one. The inductive limit construction is well suited to handle this problem.

Definition 4.16

0)

Let \prec be a partial order (that is, a reflexive, antisymmetric and transitive relation) on the index set \mathcal{L} . The index set is said to be directed if for any l, l' there exists l'' such that $l \prec l''$ and $l' \prec l''$.

1i)

Let $\{\mathcal{R}_l\}_{l \in \mathcal{L}}$ be a family of C^* algebras (v.N.a.'s) labelled by a directed index set \mathcal{L} . Suppose that for all l, l' with $l \prec l'$ there is a $*$ monomorphism (injective homomorphism) $F_{ll'} : \mathcal{R}_l \mapsto \mathcal{R}_{l'}$ satisfying

$$1) F_{ll'}(1_{\mathcal{R}_l}) = 1_{\mathcal{R}_{l'}} \text{ and}$$

$$2) F_{ll'} \circ F_{l'l''} = F_{ll''} \text{ for any } l \prec l' \prec l''.$$

Then the pair of families $\{\mathcal{R}_l, F_{ll'}\}$ is called a directed system of C^* algebras (v.N.a.'s).

1ii)

Let $\{\mathcal{R}_l\}_{l \in \mathcal{L}}$ be a family of C^* algebras (v.N.a.'s) labelled by a directed index set \mathcal{L} . A C^* algebra (v.N.a.) \mathcal{R} is said to be the C^* (W^*) inductive limit of the \mathcal{R}_l provided there exist $*$ monomorphisms $F_l : \mathcal{R}_l \mapsto \mathcal{R}$ such that

$$1) F_l(1_{\mathcal{R}_l}) = 1_{\mathcal{R}} \text{ and}$$

$$2) \cup_l F_l(\mathcal{R}_l) \text{ is uniformly (weakly) dense in } \mathcal{R}.$$

2i)

Let $\{\mathcal{H}_l\}_{l \in \mathcal{L}}$ be a family of Hilbert spaces labelled by a directed index set \mathcal{L} . Suppose that for all l, l' with $l \prec l'$ there is an isometric monomorphism $\hat{U}_{ll'} : \mathcal{H}_l \mapsto \mathcal{H}_{l'}$ such that $\hat{U}_{ll'} \hat{U}_{l'l''} = \hat{U}_{ll''}$ for any $l \prec l' \prec l''$.

Then the pair of families $\{\mathcal{H}_l, \hat{U}_{ll'}\}$ is called a directed system of Hilbert spaces.

2ii)

Let $\{\mathcal{H}_l\}_{l \in \mathcal{L}}$ be a family of Hilbert spaces labelled by a directed index set \mathcal{L} . A Hilbert space \mathcal{H} is said to be the inductive limit of the \mathcal{H}_l provided there exist isometric monomorphisms $\hat{U}_l : \mathcal{H}_l \mapsto \mathcal{H}$ such that $\cup_l \hat{U}_l \mathcal{H}_l$ is dense in \mathcal{H} .

3i)

Given a directed system of Hilbert spaces \mathcal{H}_l , suppose that we are given a family of operators $\hat{A}_l \in \mathcal{R}_l \subset \mathcal{B}(\mathcal{H}_l)$ such that

1) $\sup\{\|\hat{A}_l\|; l \in \mathcal{L}\} < \infty$ and

2) there exists $l_0 \in \mathcal{L}$ so that $\hat{U}_{l'}\hat{A}_l = \hat{A}_{l'}\hat{U}_{l'}$ for any $l_0 \prec l \prec l'$.

Then the family is called a directed system of operators.

3ii)

Given an inductive limit \mathcal{H} of Hilbert spaces \mathcal{H}_l together with a family of operators $\hat{A}_l \in \mathcal{R}_l \subset \mathcal{B}(\mathcal{H}_l)$, an operator $\hat{A} \in \mathcal{R} \subset \mathcal{B}(\mathcal{H})$ is called the inductive limit of the \hat{A}_l provided that there exists $l_0 \in \mathcal{L}$ so that $\hat{U}_l\hat{A}_l = \hat{A}\hat{U}_l$ for any $l_0 \prec l$.

The connection between ni) and nii), $n = 1, 2, 3$ is made through the following theorem.

Theorem 4.7

1)

Given a directed system of C^* algebras (v.N.a.'s) $\{\mathcal{R}_l, F_{l'}\}$ there exists a unique (up to $*$ isomorphisms) C^* (W^*) inductive limit \mathcal{R} of the \mathcal{R}_l where the corresponding $*$ monomorphisms F_l satisfy the compatibility condition $F_{l'} \circ F_l = F_l$.

2)

Given a directed system of Hilbert spaces $\{\mathcal{H}_l, \hat{U}_{l'}\}$ there exists a unique (up to unitarity) inductive limit \mathcal{H} of the \mathcal{H}_l where the corresponding isometric monomorphisms \hat{U}_l satisfy the compatibility condition $\hat{U}_{l'}\hat{U}_l = \hat{U}_l$.

3)

Given a directed system of operators $\{\hat{A}_l\}$ on a directed system of Hilbert spaces \mathcal{H}_l , there exists a unique (up to unitarity) inductive limit operator \hat{A} on the inductive limit Hilbert space \mathcal{H} .

The proof of this theorem can be found in the second volume of the first reference of [61]. Notice that inductive and projective limits (as used, e.g. in [8, 9]) are essentially identical, just that the projective limit employs “projections downwards” a chain in the directed system while the inductive limit employs “embeddings upwards” the chain.

The importance of the inductive limit for our purposes lies in the following construction. Suppose we are given an index set \mathcal{I} and consider the set \mathcal{L} of all possible subsets of \mathcal{I} (notice that we allow the cardinality of $l \in \mathcal{L}$ to be infinite). Then \mathcal{L} is a directed set where the partial order \prec is given by the inclusion relation \subset . For each $l \in \mathcal{L}$ we can form the Infinite Tensor Product \mathcal{H}_l^\otimes of the \mathcal{H}_α , $\alpha \in l$ and the corresponding von Neumann algebra \mathcal{R}_l^\otimes . Moreover, we have for $l \prec l'$ the obvious $*$ monomorphism $F_{l'}$ assigning to $\hat{A}_l \in \mathcal{R}_l$ the operator $\hat{A}_l \otimes [\otimes_{\alpha \in l' - l} 1_{\mathcal{H}_\alpha}] \in \mathcal{R}_{l'}$. Finally, choose for each $\alpha \in \mathcal{I}$ a fixed standard unit vector $\Omega_\alpha \in \mathcal{H}_\alpha$, then for $l \prec l'$ we have isometric monomorphisms $\hat{U}_{l'}$ mapping $\xi_l \in \mathcal{H}_l$ to $\xi_l \otimes [\otimes_{\alpha \in l' - l} \Omega_\alpha]$. It is easy to see that $F_{l'}, \hat{U}_{l'}$ satisfy the requirements of definition 4.14 and so we can form the inductive limit von Neumann algebra $\mathcal{R}_\infty^\otimes$ and inductive limit Hilbert space $\mathcal{H}_\infty^\otimes$ respectively which are the universal objects from which our various “lattice” algebras \mathcal{R}_l and Hilbert spaces \mathcal{H}_l respectively can be obtained by theorem 4.7.

5 Infinite Tensor Products and Continuum Quantum Gauge Theories

We will now apply the machinery of section 4 to quantum gauge field theories on globally hyperbolic, spatially non-compact manifolds along the lines suggested by the exposition of section 2 and make contact with the semi-classical analysis machinery in connection with the coherent states as outlined in section 3. We proceed in several steps.

5.1 Kinematical Framework

In this subsection we carefully carry over the Ashtekar-Isham-Lewandowski kinematical framework developed for the finite analytical category to the infinite analytical one.

5.1.1 Properties of Infinite Graphs

Notice that in order that $\gamma \in \Gamma_\sigma^\omega$ has an infinite number of edges, Σ must not be compact by the very definition of compactness.

Next, while Γ_σ^ω contains graphs with an infinite number of edges, the number of these edges is at most countably infinite if Σ is paracompact as we assume here as otherwise integration theory cannot be employed. To see this, notice that since a finite dimensional manifold Σ is locally compact we can apply the theorem in [70] chapter I, paragraph 9 which says that a (connected) locally compact space is paracompact if and only if it is the countable union of compact sets. Assume now that γ has an uncountably infinite number of edges and let U_n , $n = 1, 2, \dots$ be a countable compact cover of Σ . We conclude that at least one of the U_n must contain an uncountably infinite number of edges of γ because γ has an uncountable number of edges and the countable union of countable sets is countable. But this cannot happen if γ is piecewise analytic and σ -finite by definition.

We conclude that each element $\gamma \in \Gamma_\sigma^\omega$ is of a rather controllable form with at most a countable number of edges and vertices and no accumulation points as it would happen for webs. They thus resemble maximally the lattices that one is used to from lattice gauge theory and this is the form of graphs which are clearly most suitable for semiclassical analysis and the continuum limit. (The typical element of Γ_0^∞ has at least one accumulation point of vertices and on such graphs one will certainly not approximate actions, Hamiltonians and the like).

Moreover, we have the following basic lemma and this is where analyticity comes in.

Lemma 5.1 *The set Γ_σ^ω is a directed set by inclusion.*

Proof of Lemma 5.1 :

Notice that if γ, γ' are two piecewise analytical, σ -finite graphs then $\gamma'' := \gamma \cup \gamma'$ is also piecewise analytic. We claim that it is also σ -finite. Suppose this was not the case. Then, either a) there exists a compact subset $U \subset \Sigma$ such that $\gamma'' \cap U$ is an infinite graph or b) there exists a compact cover \mathcal{U} such that the set $\{|E(\gamma'' \cap U)|; U \in \mathcal{U}\}$ is unbounded.

As for case a), we know that $\gamma \cap U, \gamma' \cap U$ are both finite graphs with finite number of edges e, e' respectively. Since $\gamma'' \cap U = [\gamma \cap U] \cup [\gamma' \cap U]$ the only way that $\gamma'' \cap U$ can possibly be infinite is that there is at least one edge e of γ and one edge e' of γ' such that $e \cup e'$ intersect each other in an infinite number of isolated points. (The possibility that they overlap in a finite segment is excluded by analyticity as they would be analytic extensions of each other in this case and thus would make up a single analytical curve). However, two analytical curves that coincide in an infinite number of points are analytical extensions of each other. Thus, case a) cannot occur.

As for case b), we find compact sets U_n labelled by natural numbers n such that $\gamma'' \cap U_n$ has at least n edges. However, we know that there are natural numbers N, N' such that $|E(\gamma \cap U_n)| < N, |E(\gamma' \cap U_n)| < N'$ for all n . It follows that U_∞ has the property of case a) which we excluded already. Thus, case b) can also not occur.

□

That Γ_σ^ω is a directed set is of paramount importance for inductive limit constructions.

5.1.2 Quantum Configuration Space

Recall that in section 2.2 the quantum configuration space $\overline{\mathcal{A}}$ arose as the Gel'fand spectrum of the Abelian C^* algebra generated by finite linear combinations of functions of smooth connections, restricted to *finite* graphs (cylindrical functions) and completed in the supremum norm. It is natural to ask whether we can extend this construction to functions of smooth connections restricted to *infinite* graphs and to see if the size of the quantum configuration space is changed. The following simple example reveals that a naive transcription of this method is problematic :

Take $\Sigma = \mathbb{R}^3$, $G = SU(2)$ and let γ be the x -axis split into the countably infinite number of intervals e of equal unit length. Thus, γ is a piecewise analytic, σ -finite graph. Let us consider the following function of smooth connections

$$A \mapsto f(A) := \prod_e [k\chi_j(h_e(A))] \quad (5.1)$$

where k is a constant, $\chi_j(h) = \text{tr}(\pi_j(h))$ is the character of the spin j representation and the convergence of (5.1) is meant in the sense of definition 4.1. By definition, the sup-norm of that function is $\|f\| = \sup_{A \in \mathcal{A}} \prod_e |k\chi_j(h_e(A))|$. Now the zero connection is certainly an element of \mathcal{A} , so $\|f\| \geq \sup_{A \in \mathcal{A}} \prod_e |k(2j+1)|$ and this infinite product converges to 0 if $|k| < 1/(2j+1)$, to 1 if $|k| = 1/(2j+1)$ and diverges otherwise. Now it is easy to see that for any $h \in SU(2)$ we have in fact $|\chi_j(h)| \leq 2j+1$ and equality is reached for $h = 1$ so that indeed $\|f\| = \sup_{A \in \mathcal{A}} \prod_e |k(2j+1)|$. It follows that in the only case that the norm is finite and non-vanishing, we have that $f(A)$ is non-vanishing iff $\sum_e |k\chi_j(h_e(A)) - 1|$ converges which means that for each $\epsilon > 0$ the set of e 's such that $|k\chi_j(h_e(A)) - 1| \geq \epsilon$ is finite. In other words, $f(A)$ is almost given by $\prod_e \delta_{h_e(A), 1}$, an infinite product over Kronecker δ 's rather than δ -distributions and it is almost granted that its support is of measure zero for every reasonable measure even if we extend \mathcal{A} to $\overline{\mathcal{A}}$. We will prove shortly that this is indeed the case with respect to the Ashtekar-Lewandowski measure which turns out to be extendible to our context. Thus, we face the problem that the (Gel'fand transforms of) functions of finite sup-norm (and thus all the elements of the Abelian C^* algebra) seem to be supported on measure zero subsets of interesting measures on the resulting spectrum.

On the other hand, it is physically plausible that the quantum configuration space as obtained from Γ_0^ω should not change when we extend to Γ_σ^ω . The reason is that, by the very definition of σ -finiteness, if we consider a function depending on the infinite number of degrees of freedom labelled by the edges of $\gamma \in \Gamma_\sigma^\omega$ but restrict its dependence to a finite number of degrees of freedom by "freezing" all degrees of freedom labelled by $\gamma - [\gamma \cap U]$ for any compact set $U \subset \Sigma$ then we get a function cylindrical over $\gamma \cap U \subset \Gamma_0^\omega$ whose behaviour is certainly not different from the ones considered in section 2.2. In other words, functions over γ satisfy a locality property.

Thus, rather than deriving the spectrum $\overline{\mathcal{A}}$ as the Gel'fand spectrum arising from an Abelian C^* algebra of cylindrical functions over truly infinite graphs it is the characterization of the Ashtekar-Isham spectrum derived in [7, 8] for finite graphs which we *simply extend* to the infinite category ! This works as follows :

We need the set W_0^ω that one obtains as the union of a *finite* number of, not necessarily compactly supported *analytical* paths. Since analytical paths of non-compact range can intersect each other in an infinite number of isolated points and since generic elements of Γ_σ^ω cannot be obtained as the union of a finite number of paths we see that we have the proper inclusions $\Gamma_0^\omega \subset W_0^\omega \subset \Gamma_\sigma^\omega$. The set W_0^ω is trivially directed by inclusion, in a sense it is very similar to the set Γ_0^∞ . In fact, if one would blow up the neighbourhood of the source of a tassel [10] by an infinite amount then one gets, apart from the difference between smooth and analytic paths, precisely the kind of objects that lie in W_0^ω . For this reason, we will call them *analytical webs*. Notice that in contrast to smooth webs the

paths that determine an analytical web are obviously holonomically independent because 1) they cannot overlap each other in a finite segment due to analyticity, they can only intersect each other in a possibly infinite number of isolated points and 2) because they always have a (non-overlapped) segment in the bulk of Σ where no fall-off conditions on A restrict the range of the holonomy along that segment.

Let now \mathcal{A} be the classical configuration space of section 2.1 where appropriate fall-off conditions at spatial infinity are obeyed. Then the holonomy of $A \in \mathcal{A}$ along an analytic path of infinite range is in fact well-defined precisely due to the fall-off conditions on A at spatial infinity. As in the context of Γ_0^ω we can now consider the $*$ algebra of cylindrical functions of $A \in \mathcal{A}$ which are simply finite linear combinations of functions of the form $f(A) = f_w(\{h_e(A)\}_{e \in w})$ where $w \in W_0^\omega$ denotes the analytical web and f_w is a complex valued function on $G^{|w|}$, $|w|$ the number of paths that determine w . Now the complications that we observed above in the context of cylindrical functions over $\gamma \in \Gamma_\sigma^\omega$ are out of the way because the cylindrical functions for webs depend on a finite number of arguments only. We can therefore complete the $*$ algebra in the sup-norm just as in section 2.2, obtain a C^* algebra and can follow exactly the same steps reviewed there for Γ_0^ω to arrive at the Ashtekar-Isham spectrum $\overline{\mathcal{A}}$ as the Gel'fand spectrum of that algebra. Finally, by following exactly the same proofs as in [7, 8, 50] we find $\overline{\mathcal{A}}$ to be in one to one correspondence with the set of *all* homomorphisms from the groupoid \mathcal{X} of (composable) analytic paths in Σ into the gauge group G . The isomorphism is the same as the one from [7, 8], that is,

$$\overline{\mathcal{A}} \ni \bar{A} \mapsto H_{\bar{A}} \in \text{Hom}(\mathcal{X}, G); (H_{\bar{A}}(e))_{mn} := \bar{A}((h_e)_{mn}) = (\hat{h}_e)_{mn}(\bar{A}) \quad (5.2)$$

Here m, n are the indices of the matrix elements of the defining representation of G and \wedge denotes the Gel'fand transform. Notice that in contrast to [71] it was not necessary to consider the one point compactification $\hat{\Sigma}$ of Σ . In fact, we refrained from doing that because we now can consider the paths $e \in \mathcal{X}$ that determine an analytical web w also as possible edges of a truly infinite graph $\gamma \in \Gamma_\sigma^\omega$. Clearly, considering the one point compactification $\hat{\Sigma}$ with an embedded generic element of Γ_σ^ω results in a highly singular object and therefore we do not have the luxury to do this.

In summary, essentially we do not change the spectrum $\overline{\mathcal{A}}$ as compared to [7, 8] except that the correspondence (5.2) is now extended to paths with non-compact range and therefore all the properties of $\overline{\mathcal{A}}$ derived in the literature are preserved.

One could ask whether there is a more fundamental reason for this choice, trying to define, as in the finite category, an Abelian C^* algebra of cylindrical functions depending on an infinite graph. This meets mathematical difficulties which are once more related to the fact that the associative law does not hold in general for the ITP and boils down to saying that one cannot really build an algebra of cylindrical functions over infinite graphs, only a vector space. We thus just adopt the above point of view with respect to definition of $\overline{\mathcal{A}}$. However, an outline of these difficulties will be given in the subsequent digression since it is instructive and gives rise to some natural definitions.

A natural way to proceed with infinite graphs comes from the observation that the set Γ_0^ω is a subset of Γ_σ^ω which consists of compactly supported graphs. This observation motivates the following definition.

Definition 5.1 *Let $\gamma \in \Gamma_\sigma^\omega$.*

i)

A function f on \mathcal{A} is said to be a C function (not to be confused with the C vectors of section 4.1) over γ with values in $\mathbb{C} \cup \{\infty\}$ provided that for each $e \in E(\gamma)$ there exist functions f_e on \mathcal{A} of the form $f_e(A) = F_e(h_e(A))$, where F_e is a complex valued function

on G , such that

$$f(A) = \prod_{e \in E(\gamma)} f_e(A) \quad (5.3)$$

and convergence is defined as in definition 4.1 where we set $f(A) = \infty$ if $\prod_e |f_e(A)| = \infty$ irrespective of the phases of the $f_e(A)$.

ii)

A function f on \mathcal{A} is said to be cylindrical over γ if it is a finite linear combination of C functions over γ . The set of cylindrical functions over γ is denoted by Cyl_γ .

iii)

A function f on \mathcal{A} is said to be cylindrical if it is a finite linear combination of cylindrical functions over some graphs γ . The set of cylindrical functions is denoted Cyl .

iv)

An element $0 \neq f = \sum_{n=1}^N z_n \prod_{e \in E(\gamma)} f_e^{(n)} \in \text{Cyl}_\gamma$, $z_n \in \mathbb{C}$ is said to be σ -bounded if and only if

$$\|f\|_\gamma := \sup_{U \subset \Sigma} \sup_{A \in \mathcal{A}} \left| \sum_{n=1}^N z_n \prod_{e \in U \neq \emptyset} f_e^{(n)}(A) \right| \quad (5.4)$$

is finite where U runs over all compact subsets of Σ . For $f = 0$ we set $\|f\| = 0$. Notice that the argument of the modulus in (5.4) is a cylindrical function in the sense of section 2.2. We will denote the set of σ -bounded, cylindrical functions by Cyl_γ^b . Notice that Cyl_γ^b is not empty precisely due to the usual boundary conditions on smooth connections \mathcal{A} for non-compact Σ .

The norm (5.4) assigns a finite value to functions f even if there is $A \in \mathcal{A}$ such that $f(A) = \infty$ which corresponds to our motivation to take over the structure from Γ_0^ω .

Lemma 5.2 *The space of σ -bounded cylindrical functions over γ forms a $*$ algebra with the C^* property.*

Proof of Lemma 5.1 :

That Cyl_γ^b is closed under linear combination, multiplication by scalars and factor-wise complex conjugation is obvious. Suppose now that $f = \sum_{m=1}^M u_m \prod_{e \in E(\gamma)} f_e^{(m)}$, $g = \sum_{n=1}^N v_n \prod_{e \in E(\gamma)} g_e^{(n)}$ are given and we define

$$fg := \sum_{m,n} u_m v_n \prod_{e \in E(\gamma)} f_e^{(m)} g_e^{(n)} \quad (5.5)$$

simply by factorwise multiplication. Then

$$\begin{aligned} \|fg\|_\gamma &= \sup_{A,U} \left| \sum_{m,n} u_m v_n \prod_{e \in U \neq \emptyset} f_e^{(m)}(A) g_e^{(n)}(A) \right| \\ &= \sup_{A,U} \left[\left| \sum_{m=1}^M u_m \prod_{e \in U \neq \emptyset} f_e^{(m)}(A) \right| \left| \sum_{n=1}^N v_n \prod_{e \in U \neq \emptyset} g_e^{(n)}(A) \right| \right] \\ &\leq \left[\sup_{A,U} \left| \sum_{m=1}^M u_m \prod_{e \in U \neq \emptyset} f_e^{(m)}(A) \right| \right] \left[\sup_{A,U} \left| \sum_{n=1}^N v_n \prod_{e \in U \neq \emptyset} g_e^{(n)}(A) \right| \right] \\ &= \|f\|_\gamma \|g\|_\gamma \end{aligned} \quad (5.6)$$

is also bounded. The C^* property follows from $|\overline{f_U(A)}| = |f_U(A)|$ and $\sup_{U,A} |f_U(A)|^2 = (\sup_{U,A} |f_U(A)|)^2$.

□

So far we have considered only one cylindrical algebra Cyl_γ^b . Can we consider the algebra Cyl^b of finite linear combinations of elements of Cyl_γ^b for some γ 's? As we have

shown in lemma 5.1, Γ_σ^ω is a directed set so that for any finite collection $\gamma_1, \dots, \gamma_n$ there exists a γ containing each of them. However, it may no longer be true that a given $f_k \in \text{Cyl}_{\gamma_k}^b$, $k = 1, \dots, n$ can be written as a finite linear combination of C functions over γ , in fact, this will almost never be the case. Thus, while linear combinations pose no problem, products do as we then can no longer multiply factor-wise without having to consider infinite linear combinations of C functions. In other words, as soon as we allow linear combinations of functions cylindrical over different infinite graphs, we end up having no algebra any more, products are ill-defined. The only exception is that for each of $\gamma_1, \dots, \gamma_n$ only a finite number of edges have to be decomposed into a finite number of segments each of which is an edge of γ . In that case, each of f_k can be considered already as a function in Cyl_γ^b so that nothing new is gained. Thus, the only way to proceed along the lines of [7, 8] is therefore to consider all the Cyl_γ^b separately.

Once this is agreed on, the remainder is now standard. We complete the $*$ algebra Cyl_γ^b in the norm (5.4) and obtain an Abelian C^* algebra \mathcal{B}_γ which now depends on γ , in contrast to section 2.2. By the Gel'fand theorem we obtain the spectrum $\overline{\mathcal{A}}_\gamma$ of this algebra and \mathcal{B}_γ is, via the Gel'fand transform $f \mapsto \hat{f}$, isometrically isomorphic to the algebra of continuous functions $C^0(\overline{\mathcal{A}}_\gamma)$ over the compact Hausdorff space $\overline{\mathcal{A}}_\gamma$. But now we meet the next difficulty and this finishes our trial to proceed this way : Namely, the set \mathcal{A} is now no longer a subset of $\overline{\mathcal{A}}_\gamma$. Namely, let $A_0 \in \mathcal{A} \cap \overline{\mathcal{A}}_\gamma$ then we have from isometricity

$$\|f\| = \|\hat{f}\| = \sup_{\bar{A} \in \overline{\mathcal{A}}_\gamma} |\hat{f}(\bar{A})| = \sup_{\bar{A} \in \overline{\mathcal{A}}_\gamma} |\bar{A}(f)| \geq |A_0(f)| = |f(A_0)| \quad (5.7)$$

which from the definition (5.4) can be true only if A_0 has compact support. However, we are precisely interested in (distributional) connections which are supported everywhere in Σ as this corresponds to the intended physical application in connection with the classical limit for non-compact Σ . There is no claim that one could not introduce a different C^* norm on cylindrical functions which would lead to the desired distributional extension of \mathcal{A} but there seems to be no obvious, natural candidate as the above discussion reveals. We leave the question on the existence of such a norm for future research. This terminates our digression.

We thus will not use the norm (5.4) any further but simply consider the vector space Cyl of arbitrary cylindrical functions of \mathcal{A} without any convergence requirements, to begin with. As we will see, a subset of this space, extended to distributional connections, is dense in the Hilbert space which we are going to construct and although it is not an algebra, inner products *can be computed* even if we have linear combinations of functions over different infinite graphs.

This extension works as follows : Since every cylindrical function is a finite linear combination of C functions over some γ we can also extend any $f \in \text{Cyl}$ to a function \hat{f} on $\overline{\mathcal{A}}$ simply by the pull-back of the Gel'fand transform on C functions

$$\hat{f} := \prod_{e \in E(\gamma)} \hat{f}_e \text{ where } \hat{f}_e(\bar{A}) = F_e(\bar{A}(h_e)) = F_e(\hat{h}_e(\bar{A})) = (\wedge^* f_e)(\bar{A}) \quad (5.8)$$

extended by linearity. The notation means that \hat{f} is the Gel'fand transform of $f = \prod_e f_e$, $f_e = F_e \circ h_e$ extended from finite to infinite graphs. We will continue to call the extensions \hat{f} cylindrical functions.

Although a general $\hat{f} \in \text{Cyl}$ will take an infinite value on almost every point $\bar{A} \in \overline{\mathcal{A}}$ it is still possible to equip Cyl with a topology which is weaker than the Hilbert space topology that we are going to construct, moreover, the Hilbert space measure is such

that these infinite values are integrable. This is important in order to have a framework for solving quantum constraints via (analogs of) Gel'fand triples [11]. we postpone the definition of this topology to subsection 5.2.1.

5.1.3 Measure and Hilbert Space

Consider for a moment the set \mathcal{C}^ω of all possible oriented, analytic curves in Σ . Clearly, at most countable collections of elements $e \in \mathcal{C}_\sigma^\omega$ constitute an element $\gamma \in \mathcal{G}_\sigma^\omega$ through their union if that union is σ -finite. The idea is now to construct the Infinite Tensor Product Hilbert spaces $\mathcal{H}_\gamma^\otimes$ associated with the Hilbert spaces \mathcal{H}_e , $e \in E(\gamma)$ of section 3.2, that is,

$$\mathcal{H}_\gamma^\otimes := \otimes_{e \in E(\gamma)} \mathcal{H}_e \quad (5.9)$$

Using the notation of section 4.3 we would have index sets $\mathcal{I} = \mathcal{C}^\omega$ and the set of arbitrary index subsets \mathcal{L} (or power set) of \mathcal{I} of which $\Gamma_\sigma^\omega \subset \mathcal{L}$ is a proper subset.

The reader may now wonder why we do not use the full power of the Infinite Tensor Product of being able to deal with index sets of arbitrary cardinality and rather stick with Γ_σ^ω . Indeed, an interesting observation is now the following : Consider instead of \mathcal{L} the slightly smaller set \mathcal{P} of arbitrary subsets C of \mathcal{C}^ω (not necessarily elements of Γ_σ^ω) such that no element $e \in C$ can be written as a composition of elements of $C - \{e\}$ and their inverses. Then we say $C \prec C'$ if every element $e \in C$ can be written as a composition of elements $e' \in C'$ and their inverses which gives also \mathcal{P} a partial order. For $C \prec C'$ we define $C \cup C' = C'$. Recall that a subset $P \subset \mathcal{P}$ is called a chain if all elements $C \in P$ are in relation \prec . Given a chain P , consider the element $C_P := \cup_{C \in P} C$ which is an element of \mathcal{P} (not necessarily of P), moreover, $C \prec C_P \forall C \in P$. In other words, every chain in \mathcal{P} has an upper bound in \mathcal{P} and by the lemma of Zorn we obtain that \mathcal{P} has a maximal element C_∞ , that is, $C \prec C_\infty$ for all $C \in \mathcal{P}$. Certainly, there are infinitely many such maximal elements each of which we will call a “supergraph”. By construction, every element $e \in C_\infty$ is not composition of elements of $C_\infty - \{e\}$ and thus they are holonomically independent. (This construction can obviously be repeated for the smooth category of curves as well). Notice that the existence of C_∞ , while of theoretical interest since it allows us to construct the universal ITP $\mathcal{H}_\infty^\otimes := \otimes_{e \in C_\infty} \mathcal{H}_e$, universal in the sense that every possible piecewise analytic graph γ can be written as composition of elements of C_∞ , it is practically so far of modest interest only because 1) no one knows how to describe C_∞ explicitly and 2) even if one knew C_∞ explicitly, given $\gamma \in \Gamma^\omega$, every edge e of γ would generically decompose into an infinite number of segments each of which is an element of $E(C_\infty)$. Thus, even a very simple function from the point of view of γ would look very complicated from the point of view of C_∞ . In particular, as we have seen already in section 4.1, the associative law fails for the ITP and it will in general happen that a function on an incomplete ITP associated with some γ cannot be written as an element of the universal ITP. We are therefore forced to work with all the $\mathcal{H}_\gamma^\otimes$ simultaneously rather than with the single universal object $\mathcal{H}_\infty^\otimes$ only.

However, the supergraph γ_∞ allows us to give a simple proof of the existence of a σ -additive, faithful, Borel measure on $\overline{\mathcal{A}}$ with respect to which we can compute arbitrary inner products of cylindrical functions. This is a simple corollary of the Kolmogorov theorem for the case of an uncountably infinite tensor product of probability measures [72] and works as follows in the present context :

The supergraph $C_\infty \in \mathcal{P}$ is a generating system of holonomically independent analytic curves for every element $P \in \mathcal{P}$, in particular, for every element $\gamma \in \Gamma_\sigma^\omega$. Each element \bar{A} of the Ashtekar-Isham space $\overline{\mathcal{A}}$ of generalized connections assigns to each curve $e \in C_\infty$ an element $\bar{A}(h_e) = \hat{h}_e(\bar{A})$ of G and as \bar{A} varies, this map is onto (except if e is just a point in which case $\bar{A}(h_e) = 1_G$). Given $P \in \mathcal{P}$ we consider the σ -algebra \mathcal{M}_P generated by

preimages of Borel subsets of $G^{|P|}$ under the map $p_P : \overline{\mathcal{A}} \mapsto G^{|P|}; \bar{A} \mapsto \{\bar{A}(h_e)\}_{e \in P}$ where $|P|$ denotes the cardinality of the set P . Consider the σ -algebra \mathcal{M} generated by all the \mathcal{M}_P displaying $(\overline{\mathcal{A}}, \mathcal{M})$ as a measurable space. We say that a function f is measurable if it is of the form $f = F \circ p_P$ for some $P \in \mathcal{P}$ and some function F on $G^{|P|}$. A measure on $\overline{\mathcal{A}}$ can now be defined on measurable functions by

$$\mu_0(f) := \int_{G^{|P|}} \otimes_{e \in P} d\mu_H(h_e) F(\{h_e\}_{e \in P}) \quad (5.10)$$

where μ_H is the Haar measure on G . The normalization, right – and left invariance and the invariance under inversion display this measure as a consistently defined measure on measurable functions, the proof is completely analogous to the one displayed in [8] so that we can omit it here. Notice, however, that in contrast to [8] we allow $|P| = \infty$. Being consistently defined, the measure qualifies as one to have a σ -additive extension to \mathcal{M} . Rather than using the existence theorem of [9] we simply write it down :

$$\mu_0(\cdot) := \int_{G^{|\mathcal{C}_\infty|}} \otimes_{e \in \mathcal{C}_\infty} d\mu_H(h_e)(\cdot) \quad (5.11)$$

We will call it the *extended Ashtekar-Lewandowski measure*. Again, the remarkable properties of the Haar measure on G reveal that (5.11) reduces to (5.10) for $f = F \circ p_P$ and theorem 12.1 in [72] guarantees that (5.11) has the required properties. Thus, although we do not know the object \mathcal{C}_∞ , its mere existence can be used to define μ_0 . Henceforth we will denote $\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu_0)$.

A more explicit construction of that Hilbert space is as follows and this provides a simple way to embed the kinematical framework of section 2.2 for finite piecewise analytical graphs into the context of Γ_σ^ω of infinite piecewise analytical σ -finite graphs.

Given $\gamma \in \Gamma_\sigma^\omega$ we can use the inductive limit construction of section 4.3 to obtain $\mathcal{H}_\gamma^\otimes$ for infinite γ from the Hilbert spaces \mathcal{H}_γ constructed in [8] for finite γ . Notice that we get this way a genuine extension of the so-called Ashtekar-Lewandowski Hilbert space

$$\mathcal{H}_{AL} := \overline{\bigcup_{\gamma \in \Gamma_0^\omega} \mathcal{H}_\gamma} \quad (5.12)$$

as we will see in a moment. In fact, the Hilbert space that we will construct is defined by

$$\mathcal{H}^\otimes := \overline{\bigcup_{\gamma \in \Gamma_\sigma^\omega} \mathcal{H}_\gamma^\otimes} \quad (5.13)$$

Let us then proceed to the explicit construction. Recall that \mathcal{H}_γ is the completion, with respect to the Ashtekar-Lewandowski measure μ_0 of section 2.2, of the space of cylindrical functions Cyl_γ over $\gamma \in \Gamma_0^\omega$. Since Cyl_γ can be replaced by the finite linear combinations of (non-coloured) spin-network functions over γ we see that \mathcal{H}_γ can be equivalently described as the closure of the finite linear combinations of C_0 -vectors of the *finite* tensor product

$$\mathcal{H}_\gamma^\otimes = \otimes_{e \in E(\gamma)} \mathcal{H}_e \quad (5.14)$$

where each \mathcal{H}_e is isometric isomorphic with $L_2(G, d\mu_H)$ where μ_H is the Haar measure. Thus, $\mathcal{H}_\gamma = \mathcal{H}_\gamma^\otimes$ for $\gamma \in \Gamma_0^\omega$.

Indeed, as it is immediately obvious from the cylindrical consistency of the measure μ_0 , it reduces on \mathcal{H}_γ precisely to the tensor product Haar measure, corresponding to one copy of G for each $e \in \gamma$ and this is precisely the original definition of the Ashtekar-Lewandowski measure in terms of its cylindrical projections given in [8].

Let now $\gamma \in \Gamma_\sigma^\omega$ be given, then we find a sequence of elements $\gamma_n \in \Gamma_0^\omega$ such that $\gamma_n \subset \gamma$ and $\gamma_n \subset \gamma_{n+1}$ for each $n = 1, 2, \dots$, moreover $\bigcup_{n=1}^\infty \gamma_n = \gamma$. By means of the isometric monomorphisms defined on C_0 -vectors for $m \leq n$

$$\hat{U}_{\gamma_m \gamma_n} : \mathcal{H}_{\gamma_m}^\otimes \mapsto \mathcal{H}_{\gamma_n}^\otimes; \otimes_{e \in E(\gamma_m)} f_e \mapsto [\otimes_{e \in E(\gamma_m)} f_e] \otimes [\otimes_{e \in E(\gamma_n - \gamma_m)} 1] \quad (5.15)$$

and extended by linearity, where $1(A) = 1$ is the unit function, we display the system of Hilbert spaces $\mathcal{H}_\gamma \simeq \mathcal{H}_\gamma^\otimes$ as a directed system. By theorem 4.7 the unique inductive limit of the \mathcal{H}_{γ_n} exists and can be identified with the Infinite Tensor Product for each $\gamma \in \Gamma_\sigma^\omega$

$$\mathcal{H}_\gamma^\otimes = \otimes_{e \in E(\gamma)} \mathcal{H}_e \quad (5.16)$$

and indeed the required isometric isomorphisms are given by

$$\hat{U}_{\gamma_n} : \mathcal{H}_{\gamma_n}^\otimes \mapsto \mathcal{H}_\gamma^\otimes; \otimes_{e \in E(\gamma_n)} f_e \mapsto [\otimes_{e \in E(\gamma_n)} f_e] \otimes [\otimes_{e \in E(\gamma - \gamma_n)} 1] \quad (5.17)$$

Thus, for a truly infinite graph γ the Hilbert space $\mathcal{H}_\gamma^\otimes$ is hyperfinite, that is, it is the inductive limit of the finite dimensional Hilbert spaces $\mathcal{H}_{\gamma_n}^\otimes$.

Several remarks are in order :

- A) From the point of view of $\mathcal{H}_\gamma^\otimes$ the vectors of $\mathcal{H}_{\gamma_n} \simeq \mathcal{H}_{\gamma_n}^\otimes$ lie in the strong equivalence class of the C_0 -sequence $f^0 = \{f_e^0\}_{e \in E(\gamma)}$ where $f_e^0 = 1$ for each e . This is an immediate consequence of lemma 4.8. It follows that the Hilbert space (5.12) is just a tiny subspace of the Hilbert space (5.13) since every vector over γ which is not in the strong equivalence class of f^0 is orthogonal to all of the \mathcal{H}_{γ_n} and there are uncountably infinitely many different strong equivalence classes even for fixed γ as follows from lemma 4.14 since $|E(\gamma)| = \aleph$. To see this in more detail, notice that if a generic element $\xi \in \mathcal{H}_\gamma^\otimes$ would be a Cauchy sequence of elements $\xi_n \in \mathcal{H}_{AL}$ then for any $\epsilon > 0$ we would find $n_0(\epsilon)$ such that $\|\xi - \xi_n\| < \epsilon \forall n > n_0(\epsilon)$. Now each ξ_n can be chosen to be in some \mathcal{H}_{γ_n} with $\Gamma_0^\omega \ni \gamma_n \subset \gamma$ since any vector in the completion of \mathcal{H}_{AL} can be approximated by vectors of that form and since any vector depending on a coloured graph which is not contained in γ is automatically orthogonal to ξ . However, if we choose, e.g., ξ to be a linear combination of C_0 -vectors each of which lies in a different strong equivalence class Hilbert space than the vector f^0 above then we get the contradiction $\|\xi\|^2 < \|\xi_n\|^2 + \|\xi\|^2 < \epsilon \forall n > n_0(\epsilon)$.
- B) While the Ashtekar-Lewandowski Hilbert space is just a tiny subspace of \mathcal{H}^\otimes in (5.5), the Ashtekar-Lewandowski measure is still the appropriate measure to use in our extended context.

Indeed, it has been identified already as the σ -additive extension of the cylindrically defined measure of [8] to the projective (or inductive) limit of arbitrarily large and complicated, but finite piecewise analytic graphs in [9]. Therefore, it could be used to date only in order to integrate special functions depending on an infinite number degrees of freedom (i.e. depending on infinite graphs) : Namely those which can be written as infinite *sums* of functions each of which depends only on a finite graph, an exception being [73] where some sort of infinite volume limit has been taken. One contribution of the present paper is to show that the measure can be used to integrate more general functions depending on an infinite number of degrees of freedom : namely those which are infinite *products* of functions each of which depends only on a finite graph.

- C) In the context of finite graphs we can (even for non-gauge-invariant states) write down a complete orthonormal (with respect to the Ashtekar-Lewandowski measure μ_0) basis, the so-called spin-network basis of section 2.2. It is frequently stressed that the Ashtekar-Lewandowski measure can then be dispensed with by just requiring these functions to be orthogonal and to check that a positive definite sesquilinear form results in this way [12, 15]. Adopting this point of view, given arbitrary functions in \mathcal{H}_{AL} one can explicitly compute their inner products by writing them in terms of spin-network functions and using sesquilinearity. *This is no longer possible*

in the context of the Infinite Tensor Product (spatially non-compact Σ), here the Ashtekar-Lewandowski measure is the only way to calculate inner products ! To see this we just need to display *one* simple example :

Consider an infinite graph γ with a countable number of analytic edges e (say a cubic lattice in $\Sigma = \mathbb{R}^3$). Consider the C_0 -sequence $f := \{f_e\}$ where (from now on we drop the bar in \bar{A} for a distributional connection and we write h_e instead of the Gel'fand transform \hat{h}_e)

$$f_e(A) := f^0(h_e(A)) := \frac{1 + \chi_j(h_e(A))}{\sqrt{2}} \quad (5.18)$$

where χ_j is again the character in the spin $j > 0$ representation of $SU(2)$. Using the extended Ashtekar Lewandowski measure (5.11), which on this infinite graph just reduces to $d\mu_{0\gamma} = \otimes_{e \in E(\gamma)} d\mu_H(h_e)$ we immediately verify that the norm of the C_0 -vector \otimes_f equals unity. Suppose now we wanted to use only the knowledge that the set of functions

$$\prod_{k=1}^n \chi_j(h_{e_k}) \quad (5.19)$$

for finite n and mutually distinct $e_1, \dots, e_n \in E(\gamma)$ are mutually orthogonal spin-network functions. Then, in order to compute the norm of \otimes_f we would need to decompose this vector into the latter set of functions which at least formally can be done using the distributive law over and over again. However, it is easy to see that each of these infinite number of terms comes with the coefficient $(1/\sqrt{2})^\infty$ and so our attempt to compute the norm would result in the ill-defined expression $0 \cdot \infty$. More precisely, this ill-defined result is due to the fact that the inner product between the vectors (5.18) and (5.19) for $n \rightarrow \infty$ equals zero.

Concluding, in the ITP or Γ_σ^ω category the spin-network functions no longer provide a basis (a related observation has been made independently already in [10, 50] in the context of webs or Γ_0^∞), simply because, even for a single $\gamma \in \Gamma_\sigma^\omega$, the orthonormal set of functions given by

$$A \mapsto \prod_e [\sqrt{2j_e + 1} \pi_{j_e}(h_e(A))_{m_e n_e}] \quad (5.20)$$

where $e \in E(\gamma)$; $2j_e = 0, 1, 2, \dots$; $m_e, n_e = -j_e, -j_e + 1, \dots, j_e$ and which from experience with spin-network functions one might think to provide a basis, *is not complete* ! For instance, the unit C_0 -vector

$$A \mapsto \prod_e \chi_j(h_e(A)) \quad (5.21)$$

is orthogonal to all of them for any $j > 0$, even if we choose $j_e = j$ for all e since $|\langle \sqrt{2j+1} \pi_{jmn}, \chi_j \rangle_{L_2(SU(2), d\mu_H)}| \leq 1/\sqrt{2j+1} < 1$. The ITP Hilbert space has many more orthogonal directions than one is used to due to its non-separability. A complete orthonormal bases on a single $\gamma \in \Gamma_\sigma^\omega$ is not given by spin-network functions but rather by a von Neumann basis defined in lemma 4.9 and corollary 4.2, one for each $[f]$ -adic Infinite Tensor Product subspace of \mathcal{H}_γ . The only $[f]$ -adic ITP that has indeed a spin-network basis is the one given by the trivial strong equivalence class $[f^0]$ where for any given $\gamma \in \Gamma_\sigma^\omega$ we have $f_e^0 = 1$ for each $e \in E(\gamma)$.

Our treatment is still incomplete because, while we can compute inner products between finite linear combinations of C_0 vectors over a single γ , nothing has so far been said about inner products between finite linear combinations of C_0 vectors over different γ 's and this

is what we need if we wish to glue together the $\mathcal{H}_\gamma^\otimes$ as displayed in (5.13). The idea is, of course, to use the inductive limit construction once again, however, as far as inner products are concerned we have to go somewhat beyond von Neumann's theory which tells us only how to compute inner products between finite linear combinations of C_0 vectors over the same γ .

A concrete and natural definition can be given employing the extended Ashtekar-Lewandowski measure. Let us derive it, proceeding formally to begin with :

Let $\gamma, \gamma' \in \Gamma_\sigma^\omega$ and let $f_\gamma, g_{\gamma'}$ respectively be C_0 -sequences over γ, γ' respectively. Consider the graph $\gamma'' := \gamma \cup \gamma'$. The idea is to define the inner product between the corresponding C_0 -vectors $\otimes_{f_\gamma}, \otimes_{g_{\gamma'}}$ by

$$\begin{aligned} \langle \otimes_{f_\gamma}, \otimes_{g_{\gamma'}} \rangle &:= \int_{G^{|C_\infty|}} [\otimes_{e'' \in C_\infty} d\mu_H(h_{e''})] \overline{[\otimes_{f_\gamma}]} [\otimes_{g_{\gamma'}}] \\ &= \int_{G^{|E(\gamma'')|}} [\otimes_{e'' \in E(\gamma'')} d\mu_H(h_{e''})] \overline{[\otimes_{f_\gamma}]} [\otimes_{g_{\gamma'}}] \end{aligned} \quad (5.22)$$

where the second equality follows from cylindrical consistency. The problem, that by now we are already used to with these infinite tensor products, is that the associative law does not hold. In other words, the ITP

$$\mathcal{H}_{\gamma''}^\otimes := \otimes_{e'' \in E(\gamma'')} \mathcal{H}_{e''} \quad (5.23)$$

is in general quite different from the subdivisions

$$\begin{aligned} &(\otimes_{e \in E(\gamma)} [\otimes_{e'' \in E(\gamma'') \cap e} \mathcal{H}_{e''}]) \otimes (\otimes_{e'' \in E(\gamma'') - \gamma} \mathcal{H}_{e''}) \text{ and} \\ &(\otimes_{e' \in E(\gamma')} [\otimes_{e'' \in E(\gamma'') \cap e'} \mathcal{H}_{e''}]) \otimes (\otimes_{e'' \in E(\gamma'') - \gamma'} \mathcal{H}_{e''}) \end{aligned} \quad (5.24)$$

to which $f_\gamma, g_{\gamma'}$ belong respectively. This is precisely the problem outlined at the end of section 4.1 : The correspondence with the notation there is that $\mathcal{I} = E(\gamma'')$, $\mathcal{L} = E(\gamma) \cap \{\gamma'' - \gamma\}$, $\mathcal{I}_l = E(\gamma'') \cap e$ for $l = e$ and $\mathcal{I}_l = E(\gamma'') - \gamma$ for $l = \gamma'' - \gamma$ and similar for γ' . Here we have identified \mathcal{H}_e with $\otimes_{e'' \in E(\gamma'') \cap e} \mathcal{H}_{e''}$ and similar for e' . Notice that in general $|E(\gamma'') \cap e|, |E(\gamma'') \cap e'| = \infty$, an example being given by two graphs consisting of a single edge only, $\gamma = e, \gamma' = e'$, which however both have non-compact range and intersect each other an infinite number of times in isolated points. This is not excluded by piecewise analyticity since there is no accumulation point of intersection points (take, e.g. $e = (x, 0)$ and $e' = (x, \sin(x))$ in \mathbb{R}^2).

Step I)

In order to proceed, we subdivide γ'' into the mutually disjoint sets $\gamma^* := \gamma \cap \gamma', \bar{\gamma} = \gamma'' - \gamma, \bar{\gamma}' = \gamma'' - \gamma'$. Then we formally embed \otimes_{f_γ} into $\mathcal{H}_{\gamma''}$ by identifying it with $(\otimes_{f_\gamma}) \otimes (\otimes_{e'' \in E(\gamma'') \cap \bar{\gamma}} 1)$ and similarly we identify $\otimes_{g_{\gamma'}}$ with $(\otimes_{g_{\gamma'}}) \otimes (\otimes_{e'' \in E(\gamma'') \cap \bar{\gamma}'} 1)$. Clearly, we will now perform first the easy integrals corresponding to the tensor products factors of the unit function. In order to do this, for given $e \in E(\gamma)$ we recall that we can write $f_e(h_e) = \sum_\pi f_{e\pi}^{mn} \pi_{mn}(h_e)$ by the Peter& Weyl theorem where the sum is over a complete set of equivalence classes of irreducible representations of G , π_{mn} denotes the matrix elements of a group element in the representation π and $f_{e\pi}^{mn}$ are the Fourier coefficients of f_e . Now suppose that $e'' \subset e$ and that $e'' \not\subset \gamma'$, that is, $e'' \in E(\gamma'') \cap (e \cap \bar{\gamma}')$. Then we can write $f_e(h_e) = \sum_\pi f_{e\pi}^{mn} \pi_{mn}(h_e^{(1)} h_{e''} h_e^{(2)})$ where $h(1)_e, h_e^{(2)}$ depend on $e - e''$. In other words, we can consider it as a function F of $h_{e''}$ only and as far as the integral over $h_{e''}$ is concerned it reduces to evaluating $\langle F, 1 \rangle_{L_2(G, d\mu_H)} = \overline{f_{e\pi_0}} = \langle f_e, 1 \rangle_{\mathcal{H}_e}$. It follows that for any $e \in E(\gamma)$ which is not fully overlapped by edges of $E(\gamma')$ we can replace f_e by $\langle f_e, 1 \rangle$ (we drop the index at the inner product). Likewise, for any $e' \in E(\gamma)$ which is not fully overlapped by edges of $E(\gamma)$ we can replace $g_{e'}$ by $\langle 1, g_{e'} \rangle$. This is the result

of performing the integral over all $h_{e''}$ with $e'' \in E(\gamma'') \cap [\tilde{\gamma} \cup \tilde{\gamma}'] = E(\gamma'') \cap [\gamma'' - \gamma^*]$. It remains to perform the integral over the edges of $E(\gamma'') \cap \gamma^*$.

Step II)

Now notice that from $\otimes_{e \in E(\gamma)} f_e$ only those factors are left corresponding to edges e fully overlapped by edges of $E(\gamma')$ and from $\otimes_{e' \in E(\gamma')} g_{e'}$ only those factors are left corresponding to edges e' fully overlapped by edges of $E(\gamma)$. Let us denote the corresponding subsets by $E(\gamma)|_{\gamma^*} \subset E(\gamma)$, $E(\gamma')|_{\gamma^*} \subset E(\gamma')$. The union of both sets of edges is contained in γ^* . Suppose now that $e \in E(\gamma)|_{\gamma^*}$ is overlapped by a collection of edges e' of $E(\gamma')$, that is, there is a countable number of edges e'_{10}, \dots, e'_{11} of $E(\gamma')$ so that the endpoint of one is the starting point of the next, such that e is contained in their union and such that it is not any more contained if we remove e'_{10} or e'_{11} from the collection. It follows that e'_{10}, \dots, e'_{11} are analytical continuations of each other.

Step III)

Let us first focus on e'_{10} . Now either, A) e'_{10} is also contained in e or, B) it is not. In case B), if there are no other edges of $E(\gamma)$ overlapping the remaining segment of e'_{10} not contained in e then the edge e'_{10} does not appear any more in $E(\gamma')|_{\gamma^*}$. By the same argument as in Step I), if we now perform the integral over any $h_{e''}$ with e'' contained in $e - e'_{10}$ then we can replace $\overline{f_e}$ by $\langle f_e, 1 \rangle$ and that factor also drops out of the integral. Thus, we can focus on the case that $e'_{10} \in E(\gamma')|_{\gamma^*}$, that is, there are such other edges e_0, \dots, e_1 of $E(\gamma)$ where e_0 is adjacent to e , an endpoint of one is the starting point of the next and if e_1 is removed, the collection e, e_0, \dots, e_1 no longer overlaps e'_{10} . We see that e, e_0, \dots, e_1 are analytical continuations of each other. Now either, A) e_1 is also contained in e'_{10} or B), it is not. In case B), if there are no other edges of $E(\gamma')$ overlapping the remaining segment of e_1 not contained in e'_{10} then e_1 does not belong to $E(\gamma)|_{\gamma^*}$ and so as in Step I) we can replace $g_{e'_{10}}$ by $\langle 1, g_{e'_{10}} \rangle$ and so that factor drops out of the integral. However, then as just explained also $\overline{f_e}$ drops out of the integral. Thus, we may assume that $e_1 \in E(\gamma)|_{\gamma^*}$ and there are new edges e'_{20}, \dots, e'_{21} with e'_{21} adjacent to e'_{20} , the endpoint of one is the starting point of the next and such that e_1 is no longer overlapped if we remove e'_{20} .

Let us now rename $e \circ e_0 \circ \dots \circ e_1$ by e and the collection $e'_{20}, \dots, e'_{21}, e'_{10}, \dots, e'_{11}$ by e'_{10}, \dots, e'_{11} . Then we are in the same situation as in the beginning of Step III). Iterating, we conclude that either we end up with case A) or with case B) but that e'_{10} is no longer overlapped. In case B) the whole chain collapses like a cardhouse and we can replace $\overline{f_e}$ by $\langle f_e, 1 \rangle$. In case A) we see that we found a maximal analytical continuation of the original e , into the direction of its starting point, by other edges of $E(\gamma)$ all of which are overlapped by edges of $E(\gamma')$ and those edges are also contained in that maximal analytical continuation contained in γ .

Step IV)

Now we focus on e'_{11} and proceed completely analogously. The end result is that $\overline{f_e}$ can be replaced by $\langle f_e, 1 \rangle$ unless there exists a maximal bothsided maximal analytic continuation \tilde{e} of e by edges of $E(\gamma)$ completely overlapped by edges of $E(\gamma')$ and those edges of $E(\gamma')$ are also completely overlapped by \tilde{e} .

Step V)

As the argument is completely symmetric with respect to γ, γ' we conclude that the remaining integral depends only on the graph $\tilde{\gamma}$ consisting of analytical edges \tilde{e} which can be written simultaneously as compositions of edges of $E(\gamma)$ alone and edges of $E(\gamma')$ alone. For all other edges $e \in E(\gamma) - \tilde{\gamma}$ we can replace $\overline{f_e}$ by $\langle f_e, 1 \rangle$ and for all other edges $e' \in E(\gamma') - \tilde{\gamma}$ we can replace $g_{e'}$ by $\langle 1, g_{e'} \rangle$. We are thus left with

$$\langle \otimes_{f_\gamma}, \otimes_{g_{\gamma'}} \rangle \tag{5.25}$$

$$= \left[\prod_{e \in E(\gamma) - \tilde{\gamma}} \langle f_e, 1 \rangle \right] \left[\prod_{e' \in E(\gamma') - \tilde{\gamma}} \langle 1, g_{e'} \rangle \right] \left[\prod_{\tilde{e} \in E(\tilde{\gamma})} \langle [\otimes_{e \in E(\gamma) \cap \tilde{e}} f_e], [\otimes_{e' \in E(\gamma') \cap \tilde{e}} g_{e'}] \rangle \right]$$

Step VI)

It remains to compute the inner product labelled by \tilde{e} in (5.25). For each $\tilde{e} \in \tilde{\gamma}$ consider its unique breakup into segments $e'' \in E(\gamma'')$ defined by the breakpoints given by the union of the endpoints of the $E(\gamma) \ni e \subset \tilde{e}$ and $E(\gamma') \ni e' \subset \tilde{e}$ respectively. Then the last inner product in (5.25) is defined by

$$\begin{aligned} & \langle \otimes_{e \in E(\gamma) \cap \tilde{e}} f_e, \otimes_{e' \in E(\gamma') \cap \tilde{e}} g_{e'} \rangle \\ & := \int_{G^{|\tilde{e}|}} [\otimes_{e'' \subset \tilde{e}} d\mu_H(h_{e''})] \overline{\left[\prod_{e \in E(\gamma) \cap \tilde{e}} f_e \left(\prod_{e'' \subset e} h_{e''} \right) \right]} \left[\prod_{e' \in E(\gamma') \cap \tilde{e}} g_{e'} \left(\prod_{e'' \subset e'} h_{e''} \right) \right] \end{aligned} \quad (5.26)$$

where we have symbolically written the holonomies along the edges e, e' respectively as products of holonomies along the e'' . The integral (5.26) is already well-defined if the number of $e'' \subset \tilde{e}$ is finite, if not, then we proceed as follows :

Since γ, γ' are both σ -finite graphs, \tilde{e} must be an infinite curve in Σ with either A) one or B) both ends at infinity, otherwise there would be an accumulation point. If only one endpoint is at infinity, choose the other point as the starting point of \tilde{e} . If both endpoints are at infinity, choose an arbitrary breakpoint p on \tilde{e} and choose it as the startpoint of the the resulting semi-infinite curves, that is, $\tilde{e} = ([\tilde{e}^{(1)}]^{-1} \circ [\tilde{e}^{(2)}])^{\pm 1}$ is a choice of orientation of \tilde{e} . Since γ, γ' are both σ -finite graphs, the number of $e'' \subset \tilde{e}$ is at most countable and we can label them by integers which are increasing into the direction of the orientation of \tilde{e} in case A) and of $\tilde{e}^{(1,2)}$ respectively in case B), that is, $\tilde{e} = e''_1 \circ e''_2 \circ \dots$ and $\tilde{e}^{(1,2)} = e''_1 \circ e''_2 \circ \dots$ respectively. The integral is then defined by performing the integrals over the $h_{e''_n}$ in case A) and over the pairs $h_{e''_n}, h_{e''_{n+1}}$ in case B) in both cases in the order of increasing n . It is easy to see that the prescription in case B) is independent of the choice of breakpoint p because the two integrals differ by a change of the order of a finite number of integrations which is irrelevant by properties of the measure μ_0 and the compactness of G . Namely, all appearing functions are certainly absolutely integrable in any order and the assertion follows from Fubini's theorem.

Steps I)-VI) provide the motivation for the following definition.

Definition 5.2 *Let $\gamma, \gamma' \in \Gamma_\sigma^\omega$ and let $f_\gamma, g_{\gamma'}$ respectively be C_0 -sequences over γ, γ' respectively. Let $\gamma'' = \gamma \cup \gamma'$ and $\tilde{\gamma} \subset \gamma \cap \gamma'$ be the piecewise analytic, σ -finite graph consisting of analytic edges \tilde{e} which can be written simultaneously as the (countable) composition of edges of $E(\gamma)$ alone and of edges $E(\gamma')$ alone. For $\tilde{e} \in \tilde{\gamma}$ we define*

$$\begin{aligned} & \langle \otimes_{f_\gamma}, \otimes_{g_{\gamma'}} \rangle_{\tilde{e}} \\ & := \int_{G^{|\tilde{e}|}} [\otimes_{e'' \in E(\gamma'') \cap \tilde{e}} d\mu_H(h_{e''})] \overline{\left[\prod_{e \in E(\gamma) \cap \tilde{e}} f_e \left(\prod_{e'' \subset e} h_{e''} \right) \right]} \left[\prod_{e' \in E(\gamma') \cap \tilde{e}} g_{e'} \left(\prod_{e'' \subset e'} h_{e''} \right) \right] \end{aligned} \quad (5.27)$$

where e, e' have been written as their decompositions over γ'' and the order of integrations is defined in step VI) above. Then the scalar product between the C_0 vectors over γ, γ' is defined by

$$\begin{aligned} & \langle \otimes_{f_\gamma}, \otimes_{g_{\gamma'}} \rangle \\ & := \left[\prod_{e \in E(\gamma) - \tilde{\gamma}} \langle f_e, 1 \rangle \right] \left[\prod_{e' \in E(\gamma') - \tilde{\gamma}} \langle 1, g_{e'} \rangle \right] \left[\prod_{\tilde{e} \in E(\tilde{\gamma})} \langle \otimes_{f_\gamma}, \otimes_{g_{\gamma'}} \rangle_{\tilde{e}} \right] \end{aligned} \quad (5.28)$$

where the separate convergence of the infinite products in the square brackets is in the sense of definition 4.1.

In order to define a scalar product on finite linear combinations of C_0 vectors over different γ 's we extend definition 5.2 by sesquilinearity. Notice that the definition reduces to the scalar product on \mathcal{H}_γ if both vectors are finite linear combinations of C_0 vectors over γ .

Of course, in order to serve as a scalar product we must check that the scalar product is positive definite. However, this is obvious from the explicit measure theoretic expression (5.22) and can be verified by direct means as well.

Definition 5.3 *The pre-Hilbert space of finite linear combinations of C_0 vectors over graphs $\gamma \in \Gamma_\sigma^\omega$ completed in the scalar product (5.28) defines the Hilbert space \mathcal{H}^\otimes (5.13).*

Why do we choose the Hilbert space \mathcal{H} of definition 5.3 as our quantum mechanical starting point ? The reason is the same as in the case of the original Hilbert space \mathcal{H}_{AL} in which finite linear combinations of cylindrical functions over finite graphs were dense : the basic operators of the theory are still the same local operators as in section 2.2. They can be realized as operators on the infinite tensor product following the operator extension procedure of lemma 4.10 in section 4.2. Therefore, canonical commutation relations and adjointness relations are completely unchanged compared to the finite category.

5.2 Inductive Limit Structure

In the previous subsection we showed that any $\mathcal{H}_\gamma^\otimes$ for $\gamma \in \Gamma_\sigma^\omega$ can be obtained as the inductive limit of a sequence of Hilbert spaces $\mathcal{H}_{\gamma_n}^\otimes$ where $\gamma_n \in \Gamma_0^\omega$. It is therefore natural to ask whether not all of \mathcal{H}^\otimes arises in turn as the inductive limit of the $\mathcal{H}_\gamma^\otimes$ for $\gamma \in \Gamma_\sigma^\omega$.

The answer turns out to be negative, however, there is an inductive substructure which we now describe.

Notice that if either 1) any of $\gamma - \tilde{\gamma}, \gamma' - \tilde{\gamma}$ is an infinite graph or 2) any $\tilde{e} \in \tilde{\gamma}$ is a composition of an infinite number of edges of γ or γ' , or 3) the number of those \tilde{e} , which are compositions of more than one edge of γ or γ' respectively, is infinite then almost always the expression (5.28) will vanish, simply again because the associative law does not hold on the ITP. It follows that if $\gamma \subset \gamma'$ but one of the three cases 1), 2), 3) just listed applies, a generic function $f_\gamma \in \mathcal{H}_\gamma^\otimes$ cannot be written as a linear combination of functions $f_{\gamma'} \in \mathcal{H}_{\gamma'}^\otimes$.

This implies that, although Γ_σ^ω is a set directed by inclusion, we *cannot* simply define \mathcal{H}^\otimes as the inductive limit of the $\mathcal{H}_\gamma^\otimes$. To see this, notice that given $\gamma, \gamma' \in \Gamma_\sigma^\omega$ with $\gamma \subset \gamma'$ there is only one natural candidate for a unitary map $\hat{U}_{\gamma\gamma'} : \mathcal{H}_\gamma \mapsto \mathcal{H}_{\gamma'}$: For any $e \in E(\gamma)$, find its breakup $e = e_1^{m_1} \circ \dots \circ e_N^{m_N}$, $N \leq \infty$ into edges of γ' where $n_k = \pm 1$. We then consider the function $p_{\gamma\gamma'} : \overline{\mathcal{A}}_\gamma \mapsto \overline{\mathcal{A}}_{\gamma'}$; $h_e \mapsto h_{e_1^{m_1}} \dots h_{e_N^{m_N}}$ and then define

$$\hat{U}_{\gamma\gamma'} f_\gamma := [p_{\gamma\gamma'}^* f_\gamma] \otimes [\otimes_{e' \in E(\gamma') - \gamma} 1] \quad (5.29)$$

This map is unitary when considered as a map from $\mathcal{H}_\gamma^\otimes$ into \mathcal{H}^\otimes , with \mathcal{H}^\otimes as defined in the previous section, since the extended Ashtekar Lewandowski measure is consistently defined. However, the right hand side of (5.29) will for a generic element $f_\gamma \in \mathcal{H}_\gamma^\otimes$ simply not define an element of $\mathcal{H}_{\gamma'}^\otimes$ for the reason already explained.

This state of affairs is in sharp contrast with the situation for the category Γ_0^ω where the Hilbert space could indeed be written as the inductive limit of the various $\mathcal{H}_\gamma \equiv \mathcal{H}_\gamma^\otimes$. For the category Γ_σ^ω the only way to define the Hilbert space structure is through (5.28).

The inductive limit still has a limited application in the following sense : First, we define a new partial order \sqsubset on Γ_σ^ω , motivated by the conditions 1), 2) and 3) at the beginning of this subsection.

Definition 5.4 For $\gamma, \gamma' \in \Gamma_\sigma^\omega$ we define $\gamma \sqsubset \gamma'$ if and only if 1) $\gamma \subset \gamma'$ and 2) there exist disjoint (up to common vertices) unions $\gamma = \tilde{\gamma} \cup \gamma_1$, $\gamma' = \tilde{\gamma} \cup \gamma'_1$ where $\tilde{\gamma} \in \Gamma_\sigma^\omega$ and $\gamma_1, \gamma'_1 \in \Gamma_0^\omega$.

It is important to notice that condition and 2) is not equivalent with 2') that $\gamma' - \gamma \in \Gamma_0^\omega$. This is because $\gamma \subset \gamma'$ only means that every edge $e \in E(\gamma)$ is a countable composition of edges $e' \in E(\gamma')$ and 2') then does not exclude the existence of either a) at least one edge of γ which is a countably infinite composition of edges of γ' or b) an infinite number of edges which are composed of at least two edges of γ' . Both possibilities a) and b) are excluded by condition 2) which, in addition, implies 2').

Lemma 5.3 The relation \sqsubset of definition 5.4 is a partial order.

Proof of Lemma 5.3 :

Only transitivity is nontrivial to prove. If $\gamma \sqsubset \gamma' \sqsubset \gamma''$ then first of all $\gamma \subset \gamma' \subset \gamma''$ so that $\gamma \subset \gamma''$. Secondly, if $\gamma = \tilde{\gamma} \cup \gamma_1, \gamma' = \tilde{\gamma} \cup \gamma'_1 = \tilde{\gamma}' \cup \gamma'_2, \gamma'' = \tilde{\gamma}' \cup \gamma''_1$ are the corresponding disjoint unions with $\tilde{\gamma}, \tilde{\gamma}' \in \Gamma_\sigma^\omega$ and $\gamma_1, \gamma'_1, \gamma'_2, \gamma''_1 \in \Gamma_0^\omega$ then we may define $\tilde{\gamma}'' := \gamma' - (\gamma'_1 \cup \gamma'_2) = \tilde{\gamma} - \gamma'_2 = \tilde{\gamma}' - \gamma'_1$ which is obviously a subgraph of both $\tilde{\gamma}, \tilde{\gamma}'$ and an element of Γ_σ^ω since $\gamma'_1 \cup \gamma'_2 \in \Gamma_0^\omega$ (recall that Γ_0^ω is closed under unions due to piecewise analyticity and compact support of all its edges). It follows that there exist $\gamma_2, \gamma''_2 \in \Gamma_0^\omega$ such that $\gamma = \tilde{\gamma}'' \cup \gamma_2$ and $\gamma'' = \tilde{\gamma}'' \cup \gamma''_2$ are disjoint unions.

□

It is easy to see that Γ_σ^ω equipped with this partial order is not a directed set. This motivates to construct directed subsets.

Definition 5.5 Two graphs $\gamma, \gamma' \in \Gamma_\sigma^\omega$ are said to be finitely related, $\gamma \sim \gamma'$, provided that $\gamma, \gamma' \sqsubset \gamma \cup \gamma'$.

Lemma 5.4 Finite relatedness is an equivalence relation.

Proof of Lemma 5.4 :

Reflexivity and symmetry are trivial to check. To see transitivity notice that $\gamma, \gamma' \sqsubset \gamma \cup \gamma'$ implies the existence of $\tilde{\gamma}, \tilde{\gamma}' \in \Gamma_\sigma^\omega$ and of $\gamma_1, \gamma'_1, \gamma''_1, \gamma''_2 \in \Gamma_0^\omega$ such that we obtain disjoint unions $\gamma = \tilde{\gamma} \cup \gamma_1, \gamma' = \tilde{\gamma}' \cup \gamma'_1, \gamma \cup \gamma' = \tilde{\gamma} \cup \gamma''_1 = \tilde{\gamma}' \cup \gamma''_2$. The last equality demonstrates that we may write the disjoint union $\gamma \cup \gamma' = \tilde{\gamma}'' \cup (\gamma''_1 \cup \gamma''_2)$ with $\Gamma_\sigma^\omega \ni \tilde{\gamma}'' = \gamma \cup \gamma' - (\gamma''_1 \cup \gamma''_2) = \tilde{\gamma} - \gamma''_2 = \tilde{\gamma}' - \gamma''_1$. This in turn implies that we may actually write also disjoint unions $\gamma = \tilde{\gamma}'' \cup \gamma_2, \gamma' = \tilde{\gamma}'' \cup \gamma'_2$ for some γ_2, γ'_2 . In other words, $\gamma \sim \gamma'$ guarantees property 2) of definition 5.4. Transitivity now follows from the transitivity part of the proof of lemma 5.3.

□

We conclude that Γ_σ^ω decomposes into equivalence classes (γ_0) , called *clusters* and labelled by representants γ_0 , called *sources*. Now, by construction, each cluster is directed by \sqsubset . Moreover, since also by construction for any $\gamma, \gamma' \in (\gamma_0)$ the three conditions 1), 2) and 3) observed at the beginning of this subsection are not met, we find, in particular, that the operator (5.29) for $\gamma \sqsubset \gamma'$ is now indeed a unitary operator which obviously satisfies the consistency condition $\hat{U}_{\gamma'\gamma''} \hat{U}_{\gamma\gamma'} = \hat{U}_{\gamma\gamma''}$ for $\gamma \sqsubset \gamma' \sqsubset \gamma''$. The general results of section 4.3 now reveal the existence of the inductive limit Hilbert space $\mathcal{H}_{(\gamma_0)}^\otimes$ for any cluster (γ_0) and corresponding unitarities $\hat{U}_\gamma : \mathcal{H}_\gamma^\otimes \mapsto \mathcal{H}_{(\gamma_0)}^\otimes$ for any $\gamma \in (\gamma_0)$ such that $\hat{U}_\gamma = \hat{U}_{\gamma'} \hat{U}_{\gamma\gamma'}$.

It would be a very pretty result if one could establish that the Hilbert spaces $\mathcal{H}_{(\gamma_0)}$ corresponding to different clusters are mutually orthogonal with respect to (5.28). But this is certainly not the case, just take any $\gamma \in (\gamma_0) \neq (\gamma'_0) \ni \gamma'$ and consider the C_0 -vectors $f_\gamma := \otimes_{e \in E(\gamma)} 1$ and $f_{\gamma'} := \otimes_{e' \in E(\gamma')} 1$. Then trivially $\langle \otimes_{f_\gamma}, \otimes_{f_{\gamma'}} \rangle = 1$.

Denote by $\mathcal{C}_\sigma^\omega$ the set of clusters in Γ_σ^ω . Then we have the following equivalent definition of the full Hilbert space

$$\mathcal{H}^\otimes = \overline{\bigcup_{(\gamma_0) \in \mathcal{C}_\sigma^\omega} \mathcal{H}_{(\gamma_0)}^\otimes} \quad (5.30)$$

which displays it as a kind of cluster decomposition. The decomposition is, however, not a direct sum decomposition. Obviously, the cluster Hilbert spaces $\mathcal{H}_{(\gamma_0)}^\otimes$ are mutually isomorphic and therefore in particular isomorphic with the original Ashtekar Lewandowski Hilbert space $\mathcal{H}_{AL} \equiv \mathcal{H}_{(\emptyset)}^\otimes$ based on Γ_0^ω obtained by choosing as the source the empty graph. The fact that the cluster Hilbert spaces can be written as inductive limits is then not any more surprising because they are isomorphic with \mathcal{H}_{AL} of which we knew already that it is an inductive limit.

5.2.1 Rigging Triple Structures

Finally we can equip the space Cyl with a topology in analogy with the one defined in [28], the difference coming from the fact that we do not know an explicit orthonormal basis for \mathcal{H} .

Definition 5.6 *Choose for any $\gamma \in \Gamma_\sigma^\omega$ once and for all a von Neumann basis $T_{\gamma s \beta}$ over γ where $s \in \mathcal{S}_\gamma$ runs through the strong equivalence classes in \mathcal{H}_γ and β through the set of functions \mathcal{F}_γ defined in (4.10). Let $f \in \text{Cyl}$ be a cylindrical function.*

i)

The family of Fourier semi-norms of f is defined by

$$|||f|||_\gamma := \sum_{s\beta} | \langle T_{s\beta}, f \rangle | \quad (5.31)$$

where the inner product in (5.31) is defined by (5.28). Notice that indeed $|||f + g|||_\gamma \leq |||f|||_\gamma + |||g|||_\gamma$, $|||zf|||_\gamma = |z| |||f|||_\gamma$ for all $\gamma \in \Gamma_\sigma^\omega$, $z \in \mathbb{C}$, $f, g \in \text{Cyl}$. Obviously the family separates the points of Cyl since $|||f|||_\gamma < \infty$ for all γ implies $f \in \mathcal{H}$ and $|||f|||_\gamma = 0$ for all γ implies $||f|| = 0$, that is, f is the zero C_0 sequence and so $f = 0$, the zero C function in Cyl .

ii)

Consider the subspace Φ of Cyl consisting of elements which are finite linear combinations of C functions f with the property that

$$|||f||| := \sup_\gamma |||f|||_\gamma < \infty \quad (5.32)$$

iii)

Item i) displays Φ as a locally convex vector space. Upon equipping it with the natural topology (the weakest topology such that all the $|||\cdot|||_\gamma$ and addition are continuous) it becomes a topological vector space Φ .

An alternative choice for a topology for Φ , upon which it would become a normed (but not necessarily complete, Banach) topological vector space, is given by the norm (5.32). The natural topology defined in iii) is not generated by a countable set of seminorms, therefore it is not metrizable [64] and (upon completion) cannot be a Fréchet space. On the other hand, the norm $|||\cdot|||$ certainly defines a metric. The two topologies are therefore not equivalent, clearly the natural topology is weaker than the norm topology.

Lemma 5.5 *We have $\Phi \subset \mathcal{H}$ and Φ is dense in \mathcal{H} .*

Proof of Lemma 5.5 :

To see this, consider an arbitrary element $f = \sum_{n=1}^N f_{\gamma_n}$ where f_{γ_n} is a C function over γ_n in Φ and $N < \infty$. Therefore, $|||f_{\gamma_n}|||_{\gamma} < \infty$ for any γ , $n = 1, \dots, N$. In particular, $|||f_{\gamma_n}|||_{\gamma_n} < \infty$ for any γ . Thus,

$$|||f_{\gamma_n}|||_{\gamma_n} = \sum_{s,\beta} | \langle T_{\gamma s\beta}, f_{\gamma_n} \rangle | < \infty$$

and therefore

$$||f_{\gamma_n}||^2 = \sum_{s,\beta} | \langle T_{\gamma s\beta}, f_{\gamma_n} \rangle |^2 < (|||f_{\gamma_n}|||_{\gamma_n})^2 < \infty$$

from which we see that f_{γ_n} is a C_0 vector. It follows that f is a finite linear combinations of C_0 vectors and thus an element of \mathcal{H} . As the finite linear combinations of C_0 vectors form a dense subset of \mathcal{H} which we just showed to be contained in Φ , we conclude that Φ is dense in \mathcal{H} .

□

With the natural topology on Φ we are equipped with the rigging triple $\Phi \subset \mathcal{H} \subset \Phi'$ and can take over the framework of [11] to solve constraints also in the context of the ITP.

5.3 Contact with Semiclassical Analysis

In this subsection we will make contact with the semi-classical states of section 3.

Let $\gamma \in \Gamma_{\sigma}^{\omega}$ be an infinite graph, filling all of Σ arbitrarily densely (in the absence of a background metric by this we mean simply that for an arbitrary choice of neighbourhoods of each point of Σ , γ can be chosen to intersect all of them). Suppose we are given a solution of the classical field equations (say the Einstein equations in the absence of matter or the Einstein-Yang-Mills equations in the presence of matter, in the latter case G is the direct product of the gravitational $SU(2)$ with the $SU(3) \times SU(2) \times U(1)$ of the standard model), that is, for each time slice Σ_t , $t \in \mathbb{R}$ we have an initial data set $(A_t^0(x), E_t^0(x))$, $x \in \Sigma$ satisfying the field equations and in particular the constraint equations. Moreover, we will have to choose a certain gauge to write down the solution explicitly. Then, with the techniques of section 3, for each edge $e \in E(\gamma)$ and given classicality parameter s we obtain a normalized coherent state

$$\xi_{g_e^t(A^0, E^0)}^s := \frac{\psi_{g_e^t(A^0, E^0)}^s}{||\psi_{g_e^t(A^0, E^0)}^s||_{\mathcal{H}_e}} \quad (5.33)$$

where $g_e^t((A^0, E^0)) = \exp(-i\tau_j P_j^e(E_t^0, A_t^0)/(2a^2))h_e(A_t^0)$ for pure general relativity. Finally, we consider the C_0 -vector over γ given by

$$\xi_{\gamma, (A_t^0, E_t^0)}^s := \otimes_{e \in E(\gamma)} \xi_{g_e^t(A^0, E^0)}^s \quad (5.34)$$

These states comprise a preferred set of *coherent states over the infinite graph* γ and provide the basic tool with which to address the following list of fascinating physical problems :

- i) Given one and the same graph γ and classicality parameter s , when are the strong and weak equivalence classes of the states (5.34) equal to each other ? What is the physical significance of strong and weak equivalence anyway ? From experience with model systems one expects that two different weak equivalence classes correspond to drastically different physical situations such as an infinite difference in ground state energies or topologically different situations while the general analysis of section 4.1

(lemma 4.14) states that two incomplete ITP's corresponding to different strong equivalence classes within the same weak one are unitarily equivalent.

Of course, different topological situations can be described within the same complete ITP only if we get rid off the embedding spacetime that one classically started with. One way to do this would be roughly as follows :

Consider $\gamma \in \Gamma_\sigma^\omega$ not as an embedded graph but merely as a countable, combinatorial one. A countable combinatorial graph is simply a countable collection of edges (which are analytic curves when embedded into any given Σ), and vertices together with its connectivity relations, that is, information telling us at which vertices a given edge ends. Now recall that the spectrum $\text{spec}_\Sigma(\hat{O})$ of important operators \hat{O} of the theory such as the area operator (see [20]), as obtained on the Hilbert spaces corresponding to graphs embedded in a concrete Σ , depends on the topology of Σ and it should be true that a complete set of operators encodes full information about the topology of Σ via the range of their respective spectra. We can now define the *universal operator* \hat{O} acting on Hilbert spaces over combinatorial graphs by a new kind of *summing over topologies*, namely, one allows the spectrum of \hat{O} to take all possible values, that is, $\text{spec}(\hat{O}) = \cup_\Sigma \text{spec}_\Sigma(\hat{O})$. One would then say that a given closed subspace of the Hilbert space, carrying a representation of the operator algebra describes a concrete topology of Σ provided the spectra of the operators \hat{O} restricted to that subspace are compatible with th spectra $\text{spec}_\Sigma(\hat{O})$.

Now the Infinite Tensor Product Hilbert space as obtained from combinatorial graphs space comes in as follows. It is expected that closed $[f]$ -adic subspaces of that ITP corresponding to different topologies of Σ in the way just described also correspond to strong equivalence classes within different weak equivalence classes. Now while elementary operators of the theory will leave these subspaces invariant, there are densely defined operators on the complete ITP which mediate between the two. Thus, the ITP might be used to describe *topology change* in Quantum General Relativity and would then wipe away one of the main criticisms directed towards the whole programme.

A related interesting question is, whether classical states (C_0 -vectors) corresponding to Minkowski and Kruskal spacetime respectively are orthogonal, more generally, whether one can superimpose classical states corresponding to globally different spacetimes within the same strong equivalence class. Interestingly, all this can be analyzed by performing relatively straightforward calculations of the type outlined in [41, 42].

- ii) Given one and the same graph γ and solution (A_t^0, E_t^0) , are the C_0 vectors (5.34) for different values of s in the same weak equivalence class for non-compact Σ ? Since the parameter s plays a role similar to a mass parameter in free scalar field theory on Minkowski space one might expect this not to be the case as the Fock representations over Minkowski space with different mass are not unitarily equivalent. Indeed, it is easy to see that for Minkowski space in the gauge $A_a^j = 0, E_j^a = \delta_j^a$ we have $\langle \xi_e^s, \xi_e^{s'} \rangle = | \langle \xi_e^s, \xi_e^{s'} \rangle | = q < 1$ for $s \neq s'$ so that we obtain different weak equivalence classes.
- iii) Given one and the same (A_t^0, E_t^0) and s , what happens under refinements of the graph γ ? Again, since under refinements of an infinite graph we perform an infinite change on the graph, from expression (5.28) we expect the corresponding C_0 vectors to be orthogonal. This turns out to be correct.

iv) Given a C_0 vector ξ of the type (5.34) we know from lemma 4.8 that any other vector in the $[\xi]$ -adic incomplete ITP can be obtained as a (Cauchy sequence of) linear combinations of C_0 vectors each of which differs from ξ in only a finite number of entries $e \in E(\gamma)$. Now consider a deformation of a classical solution (A_t^0, E_t^0) , say the Minkowski metric plus a graviton or photon. A plane wave graviton is everywhere excited over Σ , that is, differs everywhere significantly from the Minkowski background, and therefore will not lie in the closure of the states just described. In the case of the electromagnetic field this is expected because plane wave solutions have infinite energy. But the anyway more physical graviton wave packets, although also everywhere excited, are Gaussian damped and thus have a chance to lie in that $[\xi]$ -adic strong equivalence class of Minkowski space (or any other background). We expect that there is a unitary map between the usual Fock space description of gravitons and the coherent state Gaussian wave packet gravitons of the present framework. If that turns out to be correct, we can also describe Einstein-Maxwell-Theory this way and consider photons propagating on quantum spacetimes. These issues will be examined in [74].

The same analysis can, of course, be performed on any background and this is the way we will try to describe the Hawking effect in this approach [81].

v) Related to this is the question if we can recover the spectacular results of Quantum Field Theory on Curved Backgrounds anyway [76]. The philosophy of that approach is that if the backreaction of matter to geometry can be safely neglected then treating the metric as a given, classical background should be a good approximation to the physics of the system. Recently, [77] there has been a quantum leap in this field of research due to a precise formulation of the microlocal spectrum condition on arbitrary, globally hyperbolic but not necessarily stationary backgrounds.

It is to be expected, and an important consistency check, that to zeroth order in the Planck length the full quantum gravity calculation should agree with the predictions of Quantum Field Theory on curved backgrounds, in other words, QFT on curved backgrounds is a semi-classical limit of quantum gravity. The way to check this expectation is of course the following : The total Hilbert space of the system matter plus geometry is the tensor product of the Hilbert spaces for the matter sector and the gravity sector respectively. Given a classical background metric, we will choose states ψ_{total} which are tensor products of arbitrary states ψ_{matter} from the matter Hilbert space with one fixed state ψ_{grav}^0 from the gravity Hilbert space, namely the coherent C_0 vector for the metric to be approximated, symbolically $\psi_{total} = \psi_{matter} \otimes \psi_{grav}^0$. The matter Hamiltonian operator of, say, bosonic matter coupled to quantum gravity is roughly a linear combination of operators of the form $\hat{H}_{total} = \hat{A}_{matter} \hat{A}_{grav}$. Thus we find for the matrix elements of that operator

$$\langle \psi_{total}, \hat{H}_{total} \psi'_{total} \rangle_{total} = \langle \psi_{matter}, \hat{A}_{matter} \psi'_{matter} \rangle_{matter} \langle \psi_{grav}^0, \hat{A}_{grav} \psi_{grav}^0 \rangle_{grav}$$

which shows that we obtain an *effective matter Hamiltonian* given by $\hat{H}_{total}^{eff} = \hat{A}_{matter} \langle \psi_{grav}^0, \hat{A}_{grav} \psi_{grav}^0 \rangle_{grav}$ which by the properties of the operator of [25] is *finite* ! The corrections to the classical background metric are of course contained in the difference $\langle \psi_{grav}^0, \hat{A}_{grav} \psi_{grav}^0 \rangle_{grav} - A_{grav}^{class}$ where A_{grav}^{class} is the classical limit of the gravity operator evaluated for the given classical background. This quantity is certainly at least of order ℓ_p . However, this is not the only correction to Quantum Field Theory on curved backgrounds. A second correction comes from the fact that our theory is non-perturbative which means, in particular, that the matter Hilbert space is not the usual perturbative Fock space. Therefore it is not at all obvious

that the spectrum of the operator \hat{A}_{matter} on the non-perturbative Hilbert space coincides with the spectrum of the usual matter Hamiltonian on the usual perturbative Fock space. The spectra better agree, at least modulo corrections of order at least ℓ_p , in order that we can claim to have quantized a theory which has general relativity plus standard matter as the classical limit.

- vi) A first application of this procedure to discover new physical effects due to quantum gravity is the exact treatment of the so-called *γ -ray-burst effect* [44] which we will do in [45].

To date the exact astrophysical explanation or source for high energetic γ -photons (up to TeV !) is unclear but what is important for us is that these photons were created billions of years ago, they can come from distances comparable to the Hubble radius. The idea is that these photons on their way to us constantly are influenced by the vacuum fluctuations of the gravitational field and although the influence is very, very, very small, it can accumulate due to the long travelling time of the photons. Now the higher energetic the photon, the more it should probe the small scale discreteness of quantum geometry and we thus expect an energy-dependent dispersion law. The dispersion law being energy and therefore (Minkowski) frame dependent, it violates Poincaré invariance. The effect therefore cannot come from any perturbative theory (interacting QFT on Minkowski space, perturbative quantum (super)gravity, perturbative string theory) all of whose S -matrix elements or n -point functions are by definition (or (Wightman) axiom) Poincaré invariant. For observational purposes it is convenient that the intensity peak of the burst can have a time width as small as of the order of 1ms. The idea is then to calibrate the detector to detect events at energies $E_2 > E_1$ at times $t_2 > t_1$ due to the energy dependence of the speed of light which according to [44] is speculated to be of the form $c(E)/c(0) = 1 - k(E/E_{eff})^\alpha$. Here k is a coefficient of the order of 1, E_{eff} is the effective quantum gravity scale and α is a power which is hopefully of the order 1 for the effect to be detectable. For $\alpha = 1$ one finds $t_2 - t_1 = k[(E_2 - E_1)/E_{eff}] [L/c(0)]$ where L is the distance of the source inside a galaxy and so can be determined from its redshift. Inserting the numbers for a burst which is a billion lightyears away and $E_2 - E_1 = 1\text{TeV}$ we get $t_2 - t_1$ of the order of a second (!) if we set $E_{eff} = m_p$ which is large enough compared to the width of the signal. Thus, for $\alpha = 1$ the effect could be indeed observable, say by a Čerenkov observatory [78] (but not for $\alpha = 2$), at least in principle, however, experimentally it is a highly non-trivial task to take into account all possible errors (dark matter, gravitational lensing, dust, atmosphere, ...) and to make sure that the measured intensities really came from the same burst.

Our aim in [45] will be to compute k and α , or more generally, the precise dispersion law, *exactly* along the lines outlined in item v). It is important to realize, that the effect is an inevitable theoretical prediction of quantum general relativity in the present formulation due to the *Heisenberg Uncertainty Obstruction*. Namely, the quantum metric depends on magnetic (connection A) and electric (conjugate momentum E) degrees of freedom which upon quantization become *noncommutative* operators as we have seen in section 2.2 and therefore cannot be simultaneously diagonalized. Thus, the best we can do is to write down a best approximation eigenstate of the metric operator, that is, a *coherent state* which saturates the Heisenberg uncertainty bound. As we showed in [41, 42], our states of section 3 have precisely these semiclassical properties. However, while a best approximation state, it is not an exact eigenstate and thus cannot be Poincaré invariant.

It is also important to see that our analysis is more ambitious than the pioneering work [46] for three reasons : First of all, instead of coherent states only weaves [80] were used, however, these approximate only half of the degrees of freedom and are more similar to momentum eigenstates than semiclassical states. Secondly, the matter field was treated classically and one was computing only the dispersion law coming from the changed d'Alembert operator, an option which we also have. Thirdly, in contrast to our coherent states, the weave with the assumed semi-classical behaviour was not proved to exist as a normalizable state of the Hilbert space.

- vii) The results of [22, 23, 24, 25, 26, 27, 28] are a small indication that quantum gravity plus quantum matter combine to a *finite* quantum field theory. An elementary particle physicist who computes Feynman diagrammes and has to renormalize divergent quantities all the time will rightfully ask what happened to the ultraviolet divergencies of his everyday life. A short answer seems to be, that in a diffeomorphism invariant, background independent theory there is no room for UV divergencies since there is no difference between “large” and “small” distances, the renormalization group gets “absorbed” into the diffeomorphism group. While plausible, to the best of our knowledge nobody has so far investigated these speculations in detail. This will be the topic of [75].
- viii) The result of [22, 25] shows that the geometry and matter Hamiltonian (constraints) are densely defined operators on the *unextended* Ashtekar-Lewandowski Hilbert space. How does the situation change with the huge extension of the Hilbert space performed in section 5.1 ? The answer, investigated in detail in [82] is that, not unexpectedly, these operators are not densely defined on all of \mathcal{H} , however, they are on *physical* $[f]$ -adic Hilbert spaces for f a coherent C_0 -vector. Here we call a coherent C_0 -vector physical if it is labelled by classical field configuration that obeys the fall-off conditions at spatial infinity. To see how this roughly comes about suppose we have a coherent C_0 vector $\otimes_f = \otimes_e f_e$ over some infinite graph γ . The Hamiltonian constraint operators applied to \otimes_f are of the form $\hat{H} \otimes_f = [\sum_v \hat{H}_v] \otimes_f$ where the sum is over all vertices of γ and \hat{H}_v influences only those f_e for which its edge e is incident at v , that is, it is a local operator. Our first observation is that therefore $\hat{H}_v \otimes_f \in \mathcal{H}_{[f]}$ as guaranteed again by lemma 4.8. So $\hat{H} \otimes_f$ is a countably infinite sum of vectors in $\mathcal{H}_{[f]}$ and the question is whether it is convergent, that is, whether $\|\hat{H} \otimes_f\|^2 < \infty$. Suppose now that $f_e = \xi_{g_e^s(A^0, E^0)}$ and \otimes_f is a coherent C_0 vector peaked at A_t^0, E_t^0 . Then we have by the Ehrenfest theorem of [42]

$$\|\hat{H} \otimes_f\|^2 = \langle \otimes_f, \hat{H}^\dagger \hat{H} \otimes_f \rangle = \left| \sum_v H_v(A_t^0, E_t^0) \right|^2 [1 + O(s)]$$

where $H_v(A_t^0, E_t^0)$ is by construction a discretization of an integral over a small region in Σ_t of the classical Hamiltonian density $H(A_t^0, E_t^0)$ and the sum over vertices is a Riemann sum for the integral $\int_{\Sigma_t} d^3x H(A_t^0, E_t^0)$. It follows that the norm exists if and only if the field configuration A_t^0, E_t^0 satisfies the fall-off conditions, that is, *if it is a point in the classical phase space* (no constraints or field equations yet being imposed). Next, again from lemma 4.8 we see that for such coherent C_0 sequences f the Hamiltonian constraint is densely defined on the $[f]$ -adic Hilbert space because a dense domain is given by the vector space of finite linear combinations of C_0 -vectors differing from \otimes_f in at most a finite number of entries f_e and the convergence proof just outlined certainly goes through since the finite number of changes made affect a finite number of vertices only and these are of d^3x measure zero in the limit of infinitely fine graphs.

This result is rather pretty because it tells us that the classical theory still has some effect on the quantum theory, not every $[\cdot]$ -adic incomplete ITP carries a representation of the operator algebra. The set of C_0 vectors whose $[\cdot]$ -adic ITP's do carry a representation includes the physical coherent C_0 vectors but excludes the non-physical ones. Since coherent states form an overcomplete basis on the complete ITP, this statement seems to give a rather complete classification of $[\cdot]$ -adic ITP's carrying a representation.

However, there are also other $[\cdot]$ -adic ITP's which are not of that form : An example is provided by the Ashtekar-Lewandowski state ω_{AL} which on the complete ITP \mathcal{H}^\otimes is given by the GNS cyclic vector $\Omega_{AL} = \otimes_{e \in C_\infty} 1$ where as before C_∞ denotes a supergraph. Now it is easy to check, using the overcompleteness of Hall's coherent states, that $1 = 1_e = \int_{G^x} d\nu_s(g_e) \psi_{g_e}^s$ where ν_t is Hall's measure [56], that is, the Ashtekar-Lewandowski C_0 -vector is an infinite-fold superposition of *all* coherent states formally given by the (kind of functional integral)

$$\Omega_{AL} = \int [\otimes_e d\nu_s(g_e)] \otimes_e \psi_{g_e}^s$$

and so includes non-physical coherent C_0 -vectors which, however, come with the appropriate weight enabling it to carry a representation of the observable algebra. This way to write Ω_{AL} makes it obvious that it is not peaked on a particular metric at all although it is annihilated by all momentum operators which might lead one to assume that Ω_{AL} approximates the zero metric ! This is clearly not the case.

Another way to write Ω_{AL} is $\Omega_{AL} = \lim_{s \rightarrow \infty} \otimes_{e \in C_\infty} \psi_{g_e}^s$ which displays it as a C_0 -vector, but only in the *anticlassical limit* $s \rightarrow \infty$! Both ways to write Ω_{AL} reaffirm one more time the impression that in the spatially non-compact case the $[1]$ -adic incomplete ITP, or, in other words, the original, unextended Ashtekar-Lewandowski Hilbert space is a *pure quantum representation* of the observable algebra which seems to have no obvious semi-classical correspondence. All the solutions found in [23] are (diffeomorphism invariant versions of) states in that Hilbert space and thus have presumably no (semi)classical relevance, as speculated by many, it is indeed correct that not only infinite linear combinations but indeed *infinite products* are necessary to catch the physically relevant sector of the complete ITP. This could also explain the discrepancy with respect to the number of degrees of freedom in 2+1 gravity pointed out in [24] : There the $[1]$ -adic incomplete ITP was used and gave rise to an infinite number of physical states each of which describes zero volume almost everywhere. As in that case Σ was assumed to be compact, the $[1]$ -adic incomplete ITP coincides in fact with \mathcal{H}^\otimes , however, the true physical states are only obtained as infinite linear combinations of zero volume states which presumably builds up an $[f]$ -adic incomplete ITP describing an everywhere non-degenerate metric. We will come back to this in a future publication [79].

- ix) A heuristic method to use the coherent states in order to derive a Hamiltonian constraint operator with the correct classical limit is as follows : We choose a point $(A_{t=0}^0(x), E_{t=0}^0(x))$, $x \in \Sigma$ on the constraint surface of the phase space of, say, general relativity in some gauge (that is, a field configuration on Σ_0) and obtain its trajectory under the Einstein evolution to first order in the time parameter t as $A_t^0(x) = A_0^0(x) + t\{H_0(N), A_0^0(x)\}$ (that is, a field configuration on Σ_t) and likewise for $E_t(x)$. Here, $H_0(N)$ is the Hamiltonian constraint on Σ_0 . We thus obtain coherent states as in (5.34) over any $\gamma \in \Gamma_\sigma^\omega$ and now define a family of operators $\hat{H}_\gamma(N)$, each of which is densely defined on the $[\cdot]$ -adic incomplete ITP

corresponding to the strong equivalence class of the C_0 -sequence associated with the coherent C_0 -vector (5.34) by the definition

$$\hat{H}_\gamma(N)\xi_{\gamma,(A_0^0,E_0^0)}^s := \frac{1}{i}\left(\frac{d}{dt}\right)_{t=0}\xi_{\gamma,(A_t^0,E_t^0)}^s \quad (5.35)$$

Notice that time evolution preserves the kinematical constraints of the phase space, in particular, the fall-off conditions and so the strong equivalence classes of the C_0 sequences defined by the C_0 -vectors $\xi_{\gamma,(A_0^0,E_0^0)}^s, \xi_{\gamma,(A_t^0,E_t^0)}^s$ are equal to each other. Since the map $t \mapsto (A_t^0, E_t^0)$ is smooth on the continuum phase space M , it induces a smooth map on the subset $\bar{M}_\gamma|_M$ of the graph phase space of section 2.1. Therefore, $t \mapsto \xi_{\gamma,(A_t^0,E_t^0)}^s$ is strongly continuous since $\|\xi_{\gamma,(A_t^0,E_t^0)}^s - \xi_{\gamma,(A_0^0,E_0^0)}^s\|$ depends smoothly on $\bar{M}_\gamma|_M$. If we could verify also that $\|\xi_{\gamma,(A_t^0,E_t^0)}^s\| = \|\xi_{\gamma,(A_0^0,E_0^0)}^s\|$ up to terms of order t^2 then time evolution would be given to first order in t by a one-parameter group of unitary operators and the existence of $\hat{H}_\gamma(N)$ would follow from Stone's theorem [64]. However, due to the complicated classical constraint algebra $\{H(N), H(M)\} = \int d^3x(MN_{,a} - M_{,a}N)q^{ab}V_b \neq 0$, where V_b is the infinitesimal generator of the vector constraint, the quantum evolution is better not unitary in order that it has the correct classical limit. Therefore, Stone's theorem will not apply and if we interpret (5.35) as a strong limit then it might be ill-defined.

- x) A great surprise of quantum theory was the resolution of a classical paradoxon, the explanation for the stability of atoms. According to classical electrodynamics the electrons orbiting the nucleus should emit radiation and fall into it after a finite time. The discreteness of the bound state energy spectrum bounded from below prevents this from happening and displays a mechanism for the avoidance of a classical *singularity*. It is an interesting speculation that something like this could also happen in quantum gravity, that the classical singularities predicted by the singularity theorems of Hawking and Penrose [83] are actually absent in quantum gravity, providing a resolution of the information paradox. This question can also be naturally addressed within the framework of coherent states : given a classical black hole spacetime with its singularity, say the Kruskal spacetime, we could compute the expectation value, with respect to a coherent state for that black hole spacetime, of an operator whose classical counterpart becomes singular there. If the singularity is quantum mechanically absent, then the operator should be bounded from above and the expectation value should be finite. From the point of view of the Bargmann-Segal Hilbert space, the coherent state is peaked at the singular spacetime but there is a non-zero probability to be away from it just in the right way to be square integrable. This is in analogy with the eigenstates of the electron energy operator of the hydrogenium atom whose probability density at the origin is finite and which are also square integrable (notice that coherent states are approximate eigenstates of any operator). More generally, one would like to treat quantum black holes with the new semi-classical input provided by coherent states which *come out of the quantum theory* and are not a purely classical input such as classically encoding the presence of an isolated horizon into the topology of Σ , thus inducing corresponding boundary conditions, in quantum general relativity [35, 36, 37, 38, 39, 40] or the identification of classical supergravity black hole solutions with D-brane configurations protected against quantum corrections due to the BPS nature of the corresponding states in string theory (see, e.g. [84] and references therein). These and related questions will be examined in [81].
- xi) The coherent state framework of [41, 42] so far is worked out in full detail only for the compact groups of rank one and direct products of those. As argued there, the

extension to groups of higher rank should be straightforward given the strategy for $U(1)$, $SU(2)$ but it is yet a lot of work. It would be important to give the full details at least for the physically important case of $SU(3)$. The analysis will be given in [85].

- xii) The exposition in section 4.2 underlines the relevance of von Neumann algebras and operator theory for the Infinite Tensor Product. This provides a pretty interface with the methods of Algebraic Quantum Field Theory. In particular, it would be interesting to work out the Tomita Takesaki Theory for the appearing operator algebras as it was done for scalar field theory on Minkowski space for diamond regions (the Bisognano Wichmann theorem) where the challenge in our context is that we have only a spacetime background topology and differentiable structure but not a spacetime background metric. Modular theory is the basic tool to determine the precise type of the type III factors which from experience with scalar field theory seem to be the most relevant types of factors in quantum field theory and will be studied in [86].

At this point the careful reader will wonder how we can apply the theory outlined in section (4.2) which was geared only at bounded operators. However, our basic operators are \hat{h}_e, \hat{p}_i^e , $e \in E(\gamma)$ and the latter is unbounded although essentially self-adjoint on \mathcal{H}_e , a property which trivially extends to the ITP. This mismatch can be cured by considering instead of \hat{p}_i^e the Weyl kind of operator

$$\hat{H}_e := e^{\tau_j \hat{p}_j^e / 2} \quad (5.36)$$

which takes values in the set of group valued bounded operators and transforms as $\hat{H}_e \mapsto \text{Ad}_{g_e(0)} \hat{H}_e$ under gauge transformations. Its boundedness, for instance for $G = SU(2)$, is evident from the formula

$$\hat{H}_e = \frac{e^{is\Delta_e/2} \hat{h}_e e^{-is\Delta_e/2} (\hat{h}_e)^{-1}}{e^{is}} \quad (5.37)$$

which can be proved from a similar formula in the first reference of [42] by analytical continuation of the classicality parameter s . Here, Δ_e is the Laplacian on the copy of G corresponding to h_e . Formula (5.37) displays \hat{H}_e as a product of four bounded operators. Taking the operator adjoint of (5.37) one finds that the operators \hat{H}_e, \hat{h}_e obey the kind of Weyl algebra

$$\hat{H}_e \hat{h}_e = e^{-2is} \epsilon (\hat{h}_{e^{-1}} \hat{H}_{e^{-1}})^T \quad (5.38)$$

where $(\cdot)^T$ denotes transposition, ϵ is the skew symmetric spinor of second rank of unit determinant and $\hat{H}_{e^{-1}} = (\hat{H}_e)^{-1} = ((\hat{H}_e)^\dagger)^T$ (matrix and operator inverse but only operator adjoint) clarifies the adjointness relations. Notice that one could also consider the objects $\hat{H}_e^j := e^{i\hat{p}_j^e}$ which satisfy the simpler Weyl algebra

$$\hat{H}_e^j \hat{h}_e = e^{-s\tau_j/2} \hat{h}_e \hat{H}_e^j \epsilon^{-1}$$

but the \hat{H}_e^j do not transform covariantly under gauge transformations.

Although it is slightly inconvenient to work with \hat{H}_e in place of \hat{p}_e^j since physical operators are more easily expressed in terms of the latter, it can be done and will be useful to prove abstract theorems. In that respect it is worthwhile mentioning that one could also try to work directly with the unbounded operators but the general theory for this does not yet exist due to complications associated with the fact that

domains do not interact with the linear structure, efforts have, to the best of our knowledge, so far been restricted to the discussion of essential self-adjointness of (infinite) sums of operators restricted to one $[f]$ -adic Hilbert space, where $[f]$ is a strong equivalence class [87].

- xiii) The present framework could also be employed to make contact with the so-called spin foam models [29, 30, 31, 32, 33]. These are a class of state sum models including the one that one obtains by studying the transition amplitudes associated with the Hamiltonian constraint constructed in [22] and which since then have attracted a large amount of researchers. The procedure will be to exploit the fact that coherent states are the most convenient (over)complete bases to construct a formal Feynman path integral (see e.g. [88] and references therein). Certainly, a lot of work will be necessary to make that formal path integral rigorous but coherent states provide a definite starting point. The coherent state path integral should then be equivalent to a spin foam model which can be considered as a path integral using “momentum eigenstate bases”. These issues will be worked out in [89].
- xiv) Finally, the coherent states that we have constructed are coherent over a fixed graph only, they are *pure states*. However, our techniques readily combine with the random lattice approach developed in [90] to produce *mixed coherent states*, that is, trace class operators on \mathcal{H} (in physical terms : density matrices). We can outline some of the ideas already here : Given a density parameter λ , an infinite volume cut-off parameter r and a spatial metric q_{ab} , to be approximated by a mixed coherent state $\hat{\rho}_{A^0, E^0}^{s\lambda r}$, we choose a number of $1 \ll N^r < \infty$ points at random in Σ^r where Σ^r is a compact subset of Σ which tends to Σ as $r \rightarrow \infty$. This is done in such a way that the density of points as measured by q_{ab} is roughly constant and equal to λ . More precisely, a region $R \subset \Sigma^r$ is macroscopic if its volume satisfies $V_R(q) = \int_R d^3x \sqrt{\det(q(x))} \gg \ell_p^3$. Then we find approximately $N_R(q) = \lambda V_R(q)$ points inside this region (it will be convenient to choose $\lambda = 1/a^3$ where a is the length parameter of equation (3.14)). We will also set $V^r := V_{\Sigma^r}(q)$. It follows that

$$d\mu_q^r(x) := \frac{\sqrt{\det(q)(x)}}{V^r} d^3x \quad (5.39)$$

is a probability measure on Σ_r (the necessity for the cut-off r is evident). The probability to find the $N^r = \lambda V^r$ points p_k in the infinitesimal volumes $\sqrt{\det(q)(p_k)} d^3p_k$ is given by $\prod_{k=1}^{N^r} d\mu_q^r(p_k)$. For each such random distribution of points we can construct a four-valent lattice $\gamma_{p_1, \dots, p_{N^r}}^q$ by the generalized Dirichlet-Voronoi construction [91] which depends on q . Automatically, also a dual lattice is generated which we can use for the polyhedral decomposition of Σ^r dual to $\gamma_{p_1, \dots, p_{N^r}}^q$ and which goes into the definition of the momenta p_j^e , $e \in E(\gamma_{p_1, \dots, p_{N^r}}^q)$. For this lattice, let $\hat{P}_{\gamma_{p_1, \dots, p_{N^r}}^q}^{s\lambda r}(A_0, E_0)$ be the one-dimensional projector on the coherent C_0 -vector $\xi_{\gamma_{p_1, \dots, p_{N^r}}^q, (A_0, E_0)}^s$ as in (5.34). Then, the task is to show that for the following operator, (which is trace class at $r < \infty$),

$$\hat{\rho}_{A^0, E^0}^{s\lambda r} := \int_{(\Sigma^r)^{N^r}} \prod_{k=1}^{N^r} d\mu_q^r(x_k) \hat{P}_{\gamma_{p_1, \dots, p_{N^r}}^q}^{s\lambda r}(A_0, E_0) \quad (5.40)$$

the limit $r \rightarrow \infty$ exists as a trace class operator on \mathcal{H} which can presumably be proved by invoking inductive limit methods. That (5.40) is trace class at finite r follows from

$$\text{tr}[\hat{P}_{\gamma_{p_1, \dots, p_{N^r}}^q}^{s\lambda r}(A_0, E_0)] = \|\xi_{\gamma_{p_1, \dots, p_{N^r}}^q, (A_0, E_0)}^s\|^2 = 1$$

by construction so that actually $\text{tr}(\hat{\rho}_{A_0, E_0}^{s\lambda r}) = 1$ for all r as it should be for a mixed state. This is a strong indication that the limit within the trace class ideal of the set of bounded operators exists. Practically, it might even be unnecessary to actually perform the limit as long as one measures only local operators : if the surfaces and paths with respect to which the operator is smeared lie within Σ_{r_0} then the measurement should be the same for all $r > r_0$.

This state is an average over a huge class of graphs and should have an improved semi-classical behaviour as compared to the pure ones. Notice that it is here that the possibility to compute inner products between C_0 vectors over different graphs becomes important. The details of this construction will appear in [92].

As the above list reveals there exists a plethora of fascinating and challenging open question and a huge programme is to be performed. In particular, the formalism is expectedly rather complicated as far as computations are concerned. The development of approximation techniques and error estimates as outlined in [42] will become important. The coherent states together with the Infinite Tensor Product beautifully combine three main research streams in general relativity :

- A) Quantum Gravity, since these are states of a quantum theory of general relativity,
- B) Classical Mathematical General Relativity, since the states are labelled by classical solutions of Einstein's equations and
- C) Numerical Relativity, since the computations will need the help of *supercomputers*, the stage is prepared for *numerical canonical quantum general relativity*. In fact, since the graphs that we are using are not too different from the grids employed in numerical general relativity and lattice gauge theory, some codes in classical numerical relativity or lattice gauge theory might be easily adaptable to our purposes, although many new codes have to be written as well, for instance a fast diagonalization code for the volume operator.

Remark :

In [93] the authors observe that the quantum fluctuations for the holonomy operator of a macroscopic loop, being the product of a large number of holonomies along “plaquettes” or elementary loops, are always large and it seems that there is no state that can approximate such holonomy operators. First of all, this “problem” is not tied to, say, lattice gauge theories but applies to any theory in which operators that are products of a large (or infinite) number of elementary operators play a role. Next, while the observation is certainly correct, given a large loop α on a graph γ we can trade it for a single plaquette loop β while keeping the number of holonomically independent loops constant. With this relabelling of our degrees of freedom over γ the loop α is now elementary and we can write down a coherent state which approximates it arbitrarily well. From the point of view of the ITP, while the coherent states with either α or β considered as elementary are defined over the same γ , they correspond to *different* regroupings of Hilbert spaces labelled by edges in the infinite tensor product. Thus, we see once more that the observation of [93] is directly related to the fact that the associative law is generally wrong for the infinite tensor product of Hilbert spaces.

5.4 Dynamical Framework

So far our discussion has not touched the question whether it is possible to construct coherent states which are at the same time physical, that is, annihilated by the constraint operators in an appropriate (generalized) sense. At least with respect to the gauge – and diffeomorphism constraint one might think that the answer should be given by the group averaging proposal applied in [11] to finite graphs. This section is intended to point out

in which sense this can be carried over to infinite graphs. We first consider the averaging of general functions and after that averaging of coherent states.

5.4.1 Gauge Group Averaging

The following trivial example demonstrates that the group averaging proposal requires due modification in the ITP context already at the level of the Gauss constraint :

Recall that the group \mathcal{G} of local (generalized, i.e. without continuity requirements) gauge transformations $g : \Sigma \mapsto G; x \mapsto g(x)$ is unitarily represented on \mathcal{H} by extending its action on C_0 -vectors over γ

$$\hat{U}(g) \otimes_f = \otimes_{e \in E(\gamma)} f_e(g(e(0)) h_e g(e(1))^{-1}) \quad (5.41)$$

to a dense subset of \mathcal{H} by linearity and to all of \mathcal{H} by continuity for any g .

Consider once more for γ the x -axis in \mathbb{R}^3 subdivided into unit intervals $e_n = [n - 1, n]$, $n \in \mathbf{Z}$. On this graph we can introduce the non-gauge invariant C_0 -vector

$$\chi_\pi := \prod_n \chi_\pi(h_{e_n}) \quad (5.42)$$

where each edge carries the same irreducible representation π . Group averaging this vector with respect to the Gauss constraint means to compute the infinite dimensional integral

$$\eta_G \cdot \chi_j := \prod_n \int_G d\mu_H(g(n)) \delta(g(-\infty), 1) \delta(g(\infty), 1) \prod_m \chi_\pi(g(m-1) h_{e_m} g(m)^{-1}) \quad (5.43)$$

where the δ distributions are due to the boundary condition that $g(\pm\infty) = 1$. We consider (5.43) as the limit as $N \rightarrow \infty$ of

$$\eta_G^N \cdot \chi_j := \prod_{n=-N}^N \int_G d\mu_H(g(n)) \delta(g(-N), 1) \delta(g(N), 1) \prod_{m=-N+1}^N \chi_\pi(g(m-1) h_{e_m} g(m)^{-1}) \quad (5.44)$$

which can be readily computed and gives

$$\eta_G^N \cdot \chi_j = \frac{\chi_j(h_{e_{-N+1} \circ \dots \circ e_N})}{d_\pi^{2N-1}} \quad (5.45)$$

Thus the norm of this vector is $\|\eta_G^N \cdot \chi_j\| = 1/d_\pi^{2N-1}$ and so (5.43) vanishes unless $d_\pi = 1$.

In order to cure this we must obviously factor out the power of d_π . This can be achieved by requiring that group averaging should produce a norm one vector from a norm one vector, that is, we propose (see [42])

$$\chi_G \cdot f = \frac{\prod_{v \in V(\gamma)} \int_G d\mu_H(g_v) \prod_{e \in E(\gamma)} f_e(g(e(0)) h_e g(e(1))^{-1})}{\|\prod_{v \in V(\gamma)} \int_G d\mu_H(g_v) \prod_{e \in E(\gamma)} f_e(g(e(0)) h_e g(e(1))^{-1})\|} \quad (5.46)$$

where one makes sense of the formally zero numerator and denominator through a limiting procedure as outlined above. (One does not need to check that the result is independent of the way one performs the limit, if one gets different gauge invariant answers one simply gets different gauge invariant vectors which is all that we want from the group averaging machinery anyway for the case of the gauge group since, due to its finiteness, we can still use the extended Ashtekar-Lewandowski measure on group averaged states). This makes group averaging a non-(anti)linear procedure. It means, in particular, that we produce completely new Infinite Tensor Product Hilbert Spaces. Namely, in the case of

the example the procedure (5.46) gives us the gauge invariant vector $\Xi_j = \chi_j(h_e)$ where e is the x -axis, the prototype of a tangle [27]. So in this case the original graph γ with its countable number of edges has collapsed to a graph with a single edge, a finite tensor product Hilbert space. Following definition 5.2 to compute ITP inner products for C_0 vectors over different graphs we see that the scalar product between χ_j and Ξ_j vanishes, the averaged and unaveraged vectors are orthogonal to each other.

That this happens is not an accident but generic : Consider a graph γ which is the union of an infinite number of mutually disjoint, finite graphs γ_n , $n = 1, 2, \dots$. Then a C_0 vector over γ is of the form

$$f = \otimes_n [\otimes_{e \in E(\gamma_n)} f_e], \quad \|f_e\| = 1$$

and defines an element of $\mathcal{H}_\gamma = \otimes_{e \in E(\gamma)} \mathcal{H}_e$. Group averaging evidently turns this C_0 -vector into a new C_0 -vector of the form

$$\eta_G \cdot f = \otimes_n f_{\gamma_n}, \quad \|f_{\gamma_n}\| = 1$$

which now is an element of $\mathcal{H}'_\gamma = \otimes_n \mathcal{H}_{\gamma_n}$. This once more demonstrates the source of the trouble : the associative law does not hold on the ITP and the latter vector simply cannot be written, in general, as a linear combinations of vectors of the former Hilbert space.

5.4.2 Diffeomorphism Group Averaging

Next we turn to group averaging with respect to the diffeomorphism constraint. Recall [11, 28] that this is done by relying explicitly on the spin-network basis. This is necessary because only if a function cylindrical over a graph γ depends on each of its edges through *non-trivial* irreducible representations does group averaging over the diffeomorphism group produce a well-defined distribution, the complication being due to the infinite volume of the diffeomorphism group with respect to the ‘‘counting measure’’ which produces a singularity each time we sum over diffeomorphisms which do not modify the graph on which a function depends. Another complication associated with so-called graph symmetries can be satisfactorily dealt with, see [28] for details.

However, as we have seen in section 5.1.3 spin-network functions do not provide a basis in the ITP context. It follows that not all functions of the ITP Hilbert space can be group averaged with respect to the diffeomorphism group.

More precisely, recall that the group $\text{Diff}(\Sigma)$ of analyticity preserving diffeomorphisms of Σ is unitarily represented on \mathcal{H} by extending its action on C_0 -vectors over γ

$$\hat{U}(\varphi) \otimes_f = \otimes_{e \in E(\gamma)} f_e(h_{\varphi(e)}) \quad (5.47)$$

to a dense subset of \mathcal{H} by linearity and to all of \mathcal{H} by continuity for any $\varphi \in \text{Diff}(\Sigma)$. Given a C_0 sequence f we define its orbit $\{f\}$ to be the set of C_0 sequences given by

$$\{f\} = \{f'; \exists \varphi \in \text{Diff}(\Sigma) \ni \otimes_{f'} = \hat{U}(\varphi) \otimes_f\} \quad (5.48)$$

The following C_0 sequences lie in the range of the group average map.

Definition 5.7 *A C_0 sequence f is called a spin-network C_0 sequence over γ if and only if $\langle f, f_e \rangle = 0$ for all $e \in E(\gamma)$. A spin-network C_0 sequence is called finite if its graph symmetry group is finite. For finite spin-network C_0 sequences we define the group average of its associated C_0 -vector with respect to the diffeomorphism group by*

$$\eta_{\text{Diff}} \cdot \otimes_f := [\otimes_f] := \sum_{f' \in \{f\}} \otimes_{f'} \quad (5.49)$$

where we have assumed that the graph symmetry group of γ is trivial, otherwise we modify the procedure as in [28] or [50]. The object (5.49) lies in Φ' , the topological dual of Φ as follows from results of [50].

Graphs with infinite graph symmetry group are excluded from the domain of the average map, similar as in [50]. Notice that for the typical graphs that we have in mind (e.g. cubic lattices) the graph symmetry group is in fact infinite due to the infinite number of translations which leave the graph invariant, but in order to cure this it is enough to replace a single edge by a kink. With this problem out of the way, this defines η_{Diff} on finite spin-network C_0 vectors over typical lattices and can be extended by linearity to finite linear combinations of those. That this indeed defines a linear operation is granted due to our treatment of graph symmetries.

5.4.3 Averaging of Coherent States

The interesting question is, of course, whether the coherent states that we defined are in the domain of the average map.

A) Gauge group averaging.

Returning to the example graph already discussed in equation (5.42) above, consider the (non-normalized) coherent state over the graph with $2N$ adjacent unit intervals as edges symmetrically around the origin along the x -axis, that is,

$$\psi_{g_N}^s(A) := \otimes_{n=-N+1}^N \psi_{g_{e_n}}^s(h_{e_n}(A)) \quad (5.50)$$

where ψ_g^s was defined in (3.1). Under a gauge transformation, represented by the unitary operator $\hat{U}(g)$, the tensor product factor with label n is transformed into

$$\psi_{g_{e_n}}^s(g(e_n(0))h_{e_n}g(e_n(1))^{-1}) = \psi_{g_{g(e(0))^{-1}g_{e_n}g(e_n(1))}}^s(h_{e_n})$$

and integrating over $g(e_n(1))$, $n = -N + 1, \dots, N - 1$ with the Haar measure produces the state

$$\Psi_{g_N}^s(A) := \psi_{g_{e_N}}^{2Ns}(h_{e_N}(A)) \quad (5.51)$$

where $e_N = e_{-N+1} \circ \dots \circ e_N$ and $g_{e_N} = g_{e_{-N+1}} \dots g_{e_N}$, $h_{e_N} = h_{e_{-N+1}} \dots h_{e_N}$. In other words, the finite number of integrations produce a coherent state with the correct dependence on h_e, g_e , however, the classicality parameter gets augmented from s to $2Ns$ which in the limit $N \rightarrow \infty$, of course, does not show any classical behaviour any longer. Thus, in order to produce a gauge invariant coherent state from a non-gauge invariant one on the ITP by group averaging not only do we have to go through a limiting procedure as $N \rightarrow \infty$ as already discussed above with an associated ‘‘renormalization’’ of the norms of the vectors before and after averaging, but also one has to rescale the classicality parameter s appropriately.

Thus, gauge group averaging becomes very difficult to perform if the graph γ has at least one infinite connected component. At the opposite extreme are the infinite cluster graphs which are infinite graphs obtained by the infinite disjoint union of finite graphs, called clusters. Obviously, each of these finite graphs can be gauge group averaged (and renormalized) individually. In particular, if all clusters are diffeomorphic then group averaging reduces to the infinite repetition of one averaging for functions over one finite graph. An example to keep in mind is a cubic lattice in which we remove some edges to obtain disjoint finite cubic sub-lattices.

B) Diffeomorphism group averaging.
Recall (see the first reference in [42])

$$\sqrt{\frac{\sinh(p^e)}{2\sqrt{\pi}p^e} s^{3/2}(1 - K_s)} \leq \langle 1, \xi_{g_e}^s \rangle = 1/||\psi_{g_e}^s|| \leq \sqrt{\frac{\sinh(p^e)}{2\sqrt{\pi}p^e} s^{3/2}(1 + K'_s)}$$

where the constants K_s, K'_s tend to zero exponentially fast with $s \rightarrow 0$. Since p^e is bounded, tending to zero for ever and ever finer lattice at least for classical configurations we see that for sufficiently fine lattices at given (small) s we have not only $|\langle 1, \xi_{g_e}^s \rangle| < 1$ as granted by the Schwarz inequality but moreover that there exist numbers $0 < q, q' < 1$ with $q < c_e := |\langle 1, \xi_{g_e}^s \rangle| \leq q'$ for all e for sufficiently fine lattices which is precisely the application that we are aiming at.

Splitting $\xi_{g_e}^s = \delta\xi_{g_e}^s + c_e \cdot 1$ we may want to write for given $\gamma \in \Gamma_\sigma^\omega$ the state

$$\xi_{\gamma g_\gamma}^s := \otimes_{e \in E(\gamma)} \xi_{g_e}^s \quad (5.52)$$

as

$$\xi_{\gamma g_\gamma}^s = \sum_{N=0}^{\infty} \sum_{\{e_1, \dots, e_N\} \subset E(\gamma)} \left[\prod_{k=1}^N c_{e_k} \right] [\otimes_{e \in E(\gamma) - \{e_1, \dots, e_N\}} \delta\xi_{g_e}^s] \quad (5.53)$$

or as

$$\xi_{\gamma g_\gamma}^s = \sum_{N=0}^{\infty} \sum_{\{e_1, \dots, e_N\} \subset E(\gamma)} \left[\prod_{e \in E(\gamma) - \{e_1, \dots, e_N\}} c_e \right] [\otimes_{k=1}^N \delta\xi_{g_{e_k}}^s] \quad (5.54)$$

However, both attempts fail since in (5.53) all appearing vectors have zero norm (in fact $||\delta\xi_{g_e}^s||^2 = 1 - c_e^2 < 1 - q^2 < 1$) and in (5.54) all coefficients vanish identically. Thus, the vector (5.52) does not lie in the domain of the average map.

A substitute for averaging and to deal with the diffeomorphism group is to work with representatives, i.e. from each diffeomorphism class $\{f\}$ we choose an element $f_{\{f\}}^0$. Thus $f^0 : \{f\} \mapsto f_{\{f\}}^0$ is a choice function, its existence being granted by the lemma of choice. We specify this choice function further by choosing from each graph diffeomorphism class $\{\gamma\}$ a representative $\gamma_{\{\gamma\}}^0$. Given a function f , let γ_f be the minimal graph on which it depends non-trivially. Then $f_{\{f\}}^0$ can be chosen to depend on $\gamma_{\{\gamma_f\}}^0$. If γ_f has graph symmetries then this prescription does not yet fix $f_{\{f\}}^0$ uniquely and we must further choose from one of the $\hat{U}(\varphi_n) f_{\{f\}}^0$ where φ_n is a symmetry of $\gamma_{\{\gamma_f\}}^0$. A kind of group averaging map is now defined by $\eta_{Diff} \circ f := f_{\{f\}}^0$ which obviously satisfies $\eta_{Diff} \circ \hat{U}(\varphi) = \eta_{Diff}$ and the inner product on these “solutions to the diffeomorphism constraint” is just the usual inner product between representatives. This makes the whole proposal unfortunately very choice dependent and thus less attractive. Notice, however, that diffeomorphism invariant operators which are defined on the kinematical Hilbert space obviously keep their adjointness properties.

A different way to deal with diffeomorphism invariance is by gauge fixing (alternatively, one has to construct gauge and diffeomorphism invariant coherent states from scratch) : Given a collection $g_\gamma = \{g_e\}_{e \in E(\gamma)}$, a local gauge transformation $g \in \mathcal{G}$ and a diffeomorphism $\varphi \in \text{Diff}(\Sigma)$ we define $g_\gamma^g := \{g_e^g\}_{e \in E(\gamma)}$ with $g_e^g := g(e(0))^{-1} g_e g(e(1))$ and $g_\gamma^\varphi := \{g_e^\varphi\}_{e \in E(\gamma)}$ with $g_e^\varphi := g_{\varphi^{-1}(e)}$. It is then easy to see, using unitarity (invariance of norms) that

$$\hat{U}(g) \psi_{\gamma g_\gamma}^s = \psi_{\gamma g_\gamma^g}^s \quad \text{and} \quad \hat{U}(\varphi) \psi_{\gamma g_\gamma}^s = \psi_{\varphi(\gamma) g_{\varphi(\gamma)}^\varphi}^s \quad (5.55)$$

Given classical initial value data (A^0, E^0) in a certain gauge the $g_\gamma = g_\gamma((A^0, E^0))$ are fixed and we require that $g_\gamma^g = g_\gamma = g_\gamma^\varphi$ for all γ which (generically) trivializes the residual gauge freedom to $g = 1, \varphi = \text{id}$.

Thus, as far as the gauge and diffeomorphism constraints are concerned, we can fix a gauge to take care of gauge and diffeomorphism invariance. The issue lies much harder with respect to the Hamiltonian constraint because its action [22] is much more complicated than the action of the kinematical constraints, and almost no Hamiltonian invariant observables are known with respect to which one would need to construct the invariant coherent states. Fortunately, there are certain “simple” solutions to the Hamiltonian constraint [22] corresponding to states whose underlying graph is out of the range of graphs that the Hamiltonian constraint produces. If we build (non-distributional) coherent states on such graphs, then they lie in the kernel of the Hamiltonian constraint in the sense of generalized eigenvectors with eigenvalue zero. Thus, at least for these simple solutions, together with fixing of gauge and diffeomorphism freedom, we can incorporate the quantum dynamics of general relativity.

All these observations reveal that group averaging non-gauge and/or non-diffeomorphism invariant coherent states over the gauge or diffeomorphism group is a non-trivial task, at least not if Σ is non-compact and applied naively without some sort of renormalization leads to meaningless results. More work is needed in order to construct rigorous solutions to all constraints which at the same time behave semi-classically. However, at the moment we are not so much interested in obtaining semi-classical solutions to all constraints. Rather, besides the applications already mentioned in section 5.3, it is of paramount importance to test the consistency of a quantum representation of the classical constraint algebra and the verification of its correct classical limit [82]. In order to do this one obviously *must not* have semi-classical states which solve the constraints.

Acknowledgements

O. W. thanks the “Studienstiftung des Deutschen Volkes” for financial support.

References

- [1] A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244, Phys. Rev. **D36** (1987) 1587
- [2] A. Ashtekar, in “Mathematics and General Relativity”, American Mathematical Society, Providence, Rhode Island, 1987
- [3] F. Barbero, Phys. Rev. **D51** (1995) 5507
F. Barbero, Phys. Rev. **D51** (1995) 5498
- [4] T. Thiemann, Physics Letters B **380** (1996) 257-264, gr-qc/960688
- [5] R. Gambini, A. Trias, Phys. Rev. **D22** (1980) 1380
C. Di Bartolo, F. Nori, R. Gambini, A. Trias, Lett. Nuov. Cim. **38** (1983) 497
R. Gambini, A. Trias, Nucl. Phys. **B278** (1986) 436
- [6] C. Rovelli, L. Smolin, Nucl. Phys. **B331** (1990) 80
- [7] A. Ashtekar and C.J. Isham, Class. Quantum Grav. **9** (1992) 1433
- [8] A. Ashtekar, J. Lewandowski, “Representation theory of analytic holonomy C^* algebras”, in “Knots and Quantum Gravity”, J. Baez (ed), Oxford University Press, Oxford, 1994
A. Ashtekar, J. Lewandowski, Journ. Geo. Physics **17** (1995) 191
A. Ashtekar and J. Lewandowski, J. Math. Phys. **36**, (1995) 2170
- [9] D. Marolf and J. M. Mourão, Commun. Math. Phys. **170** (1995) 583-606

- [10] J. Baez, S. Sawin, “Functional Integration on Spaces of Connections”, q-alg/9507023
- J. Baez, S. Sawin, “Diffeomorphism Invariant Spin-Network States”, q-alg/9708005
- [11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, Journ. Math. Phys. **36** (1995) 6456-6493, gr-qc/9504018
- [12] R. De Pietri, C. Rovelli, Phys. Rev. **D54** (1996) 2664
- [13] R. Gambini, J. Pullin, “Loops, Knots, Gauge Theories and Quantum Gravity”, Cambridge University Press, Cambridge, 1996
- [14] T. Thiemann, Journ. Math. Phys. **39** (1998) 1236-48, hep-th/9601105
- [15] R. De Pietri, Class. Quantum Grav. **14** (1997) 53
- [16] T. Thiemann, Journ. Math. Phys. **39** (1998) 3372-92, gr-qc/9606092
- [17] C. Rovelli, L. Smolin, Nucl. Phys. B **442** (1995) 593; Erratum : Nucl. Phys. B **456** (1995) 734
- [18] A. Ashtekar, J. Lewandowski, Class. Quantum Grav. **14** A55-81 (1997)
- [19] R. Loll, Nucl. Phys. **B460** (1996) 143
R. Loll, Class. Quantum Grav. **14** (1997) 1725
R. Loll, Nucl. Phys. **B500** (1997) 405
- [20] A. Ashtekar, J. Lewandowski, Adv. Theo. Math. Phys. **1** (1997) 388-429
- [21] T. Thiemann, Journ. Math. Phys. **39** (1998) 3347-71, gr-qc/9606091
- [22] T. Thiemann, Class. Quantum Grav. **15** (1998) 839-73, gr-qc/9606089
- [23] T. Thiemann, Class. Quantum Grav. **15** (1998) 875-905, gr-qc/9606090
- [24] T. Thiemann, Class. Quantum Grav. **15** (1998) 1249-80, gr-qc/9705018
- [25] T. Thiemann, Class. Quantum Grav. **15** (1998) 1281-1314, gr-qc/9705019
- [26] T. Thiemann, Class. Quantum Grav. **15** (1998) 1487-1512, gr-qc/9705021
- [27] T. Thiemann, Class. Quantum Grav. **15** (1998) 1463-85, gr-qc/9705020
- [28] T. Thiemann, Class. Quantum Gravity **15** (1998) 1207-47, gr-qc/9705017
- [29] M. Reisenberger, C. Rovelli, Phys. Rev. **D56** (1997) 3490-3508
- [30] J. Baez, Class. Quantum Grav. **15** (1998) 1827-58
- [31] F. Markopoulou, L. Smolin, Phys. Rev. **D58** (1998) 084032
- [32] J. W. Barret, L. Crane, Class. Quantum Grav. **14** (1997) 2113
- [33] L. Freidel, K. Krasnov, Adv. Theor. Math. Phys. **2** (1999) 1183
- [34] K. Krasnov, Gen. Rel. Grav. **30** (1998) 53-68
- [35] A. Ashtekar, C. Beetle, S. Fairhurst, Class. Quantum Grav. **17** (2000) 1317
- [36] A. Ashtekar, A. Corichi, K. Krasnov, “Isolated horizons : The classical phase space”, gr-qc/9905089
- [37] A. Ashtekar, C. Beetle, S. Fairhurst, Class. Quantum Grav. **16** (1999) L1-L7
- [38] A. Ashtekar, K. Krasnov, “Quantum Geometry and Black Holes, gr-qc/9804039
- [39] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Phys. Rev. Lett. **80**:904-907, 1998
- [40] A. Ashtekar, “Interface of General Relativity, Quantum Physics and Statistical Mechanics : Some Recent Developments”, gr-qc/9910101
- [41] T. Thiemann, “Gauge Field Theory Coherent States (GCS) : I. General Properties”, hep-th/0005233
- [42] T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : II. Peakedness Properties”, hep-th/0005237
T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : III. Ehrenfest Theorems”, hep-th/0005234
T. Thiemann, “Gauge Field Theory Coherent States (GCS) : V. Extension to Higgs Fields and Fermions”, to appear
- [43] J. von Neumann, Comp. Math. **6** (1938) 1-77
W. Thirring, “Lehrbuch der Mathematischen Physik”, vol. 4, Springer-Verlag, Wien, 1994

- [44] G. Amelino-Camelia et al, “Tests of quantum gravity from observations of γ -ray bursts”, *Nature* **393** (1998) 763
- [45] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : VII. The Non-Perturbative γ -Ray Burst Effect”, to appear
- [46] R. Gambini, J. Pullin, “Non-standard optics from quantum spacetime, *Phys. Rev. D* **59** (1999) 124021
- [47] T. Thiemann, “Quantum Spin Dynamics (QSD) : VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories”, hep-th/0005232
- [48] E. Hebey, “Sobolev Spaces on Riemannian Manifolds”, *Lecture Notes in Mathematics* **1635**, Springer Verlag, Berlin, 1996
- [49] S. Lang, “Differential Manifolds”, Addison Wesley, Reading, Massachusetts, 1972
- [50] J. Lewandowski, T. Thiemann, *Class. Quantum Grav.* **16** (1999) 2299-2322, gr-qc/9901015
- [51] J.-A. Zapata, *Journ. Math. Phys.* **38** (1997) 5663
J.-A. Zapata, *Gen. Rel. Grav.* **30** (1998) 1229
- [52] C. Rovelli, L. Smolin, “Spin networks and quantum gravity” pre-print CGPG-95/4-1, *Phys. Rev. D* **52** (1995) 5743
- [53] J. Baez, *Adv. Math.* **117** (1996) 253-272
- [54] A. Rendall, *Class. Quantum Grav.* **10** (1993) 605
A. Rendall, “Adjointness Relations as a Criterion for Choosing an Inner Product”, gr-qc/9403001
- [55] T. Thiemann, *Class. Quantum Gravity* **13** (1996) 1383-1403, gr-qc/9511057
T. Thiemann, *Acta Cosmologica* **21** (1995), 145-167, gr-qc/9511049
T. Thiemann, in : “Symplectic Singularities and Geometry of Gauge Fields”, Banach Center Publications, **39** 389-403 (1996), hep-th/9511122
- [56] B. C. Hall, *Journ. Funct. Analysis* **122** (1994) 103
- [57] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, *Journ. Funct. Analysis* **135** (1996) 519-551, gr-qc/9412014
- [58] B. C. Hall, *Comm. Math. Phys.* **184** (1997) 233
- [59] R. F. Streater, A. S. Wightman, “PCT, Spin and Statistics, and all that”, Benjamin, New York, 1964
- [60] M. D. Potter, “Mengentheorie”, ch. 5, Spektrum Akademischer Verlag, Berlin, 1990
P. J. Cameron, “Sets, Logic and Categories”, Springer Verlag, Berlin, 1999
- [61] R. V. Kadison, J. R. Ringrose, “Fundamentals of the Theory of Operator Algebras”, vol. 1 and 2, Academic Press, London, 1983
O. Bratteli, D. W. Robinson, “Operator Algebras and Quantum Statistical Mechanics”, vol. 1 and 2, Springer Verlag, Berlin, 1997
- [62] R. Haag, “Local Quantum Physics”, 2nd ed., Springer Verlag, Berlin, 1996
- [63] K. Fredenhagen, *Comm. Math. Phys.* **97** (1985) 79
- [64] M. Reed, B. Simon “Methods of Modern Mathematical Physics”, vol. 1 and 2, Academic Press, New York, 1975
- [65] A. Connes, “Noncommutative Geometry”, Academic Press, 1994
- [66] W. Krieger, *Math. Ann.* **223** (1976) 19
- [67] R. T. Powers, *Ann. Math.* **86** (1967) 138
- [68] H. Araki, E.J. Woods, *Publ. Res. Inst. Math. Sci. Ser.* **A4** (1968) 51
- [69] H. Baumgärtel, M. Wollenberg, “Causal Nets of Operator Algebras”, Akademie Verlag, Berlin, 1992
- [70] N. Bourbaki, “General Topology”, vol. 1 and 2, Hermann, Paris, 1966
- [71] M. Arnsdorf, R. S. Garcia, *Class. Quantum Grav.* **16** (1999) 3405
- [72] Y. Yamasaki, “Measures on Infinite Dimensional Spaces”, World Scientific, Singapore, 1985

- [73] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, *Journ. Math. Phys.* **38** (1997) 5453-82, hep-th/9605128
- [74] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : VI. Photons and Gravitons Propagating on Quantum Spacetimes”, to appear
- [75] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : VIII. Perturbation Theory Results from Non-Perturbative Coherent States”, in preparation
H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : IX. Renormalization Group and Diffeomorphism Invariance”, in preparation
- [76] S. A. Fulling, “Aspects of Quantum Field Theory in Curved Space-Time”, Cambridge University Press, Cambridge, 1989
- [77] M. J. Radzikowski, “The Hadamard Condition and Kay’s Conjecture in (Axiomatic) Quantum Field Theory on Curved Space-Time”, PhD Thesis, Princeton University, October 1992
R. Brunetti, K. Fredenhagen, *Comm. Math. Phys.* **180** (1996) 312
W. Junker, *Rev. Math. Phys.* **8** (1996) 1091
- [78] S. D. Biller et al, *Phys. Rev. Lett.* **83** (1999) 2108
- [79] O. Winkler, “Infinite Tensor Product and Metric Non-Degeneracy Condition in 2+1 Gravity”, in preparation
- [80] A. Ashtekar, C. Rovelli, L. Smolin, *Phys. Rev. Lett.* **69** (1992) 237
- [81] H. Sahlmann, T. Thiemann, O. Winkler “Gauge Field Theory Coherent States (GCS) : X. Quantum Black Holes, Hawking Effect and Singularity Theorems”, in preparation
- [82] T. Thiemann, “Quantum Spin Dynamics (QSD) : VIII. The Classical Limit”, to appear
- [83] S.W. Hawking, G.F.R. Ellis, “The large scale structure of space-time”, Cambridge University Press, Cambridge, 1989
- [84] A. Sen, “An Introduction to Non-perturbative String Theory”, hep-th/9802051
- [85] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : XI. Coherent States for Groups of Higher Rank”, in preparation
- [86] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : XII. Modular Theory and von Neumann Algebra Factor Types”, in preparation
- [87] M. C. Reed, *Journ. Funct. Analysis* **5** (1970) 94
- [88] J. R. Klauder, B.-S. Skagerstam, “Coherent States”, World Scientific, Singapore, 1985
- [89] H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : XIII. Coherent State Path Integral for Quantum General Relativity”, in preparation
- [90] L. Bombelli, “Statistical Lorentzian Geometry and the Closeness of Lorentzian Manifolds”, gr-qc/0002053
A. Ashtekar, L. Bombelli, “Weaves and Random Lattices”, in preparation
- [91] C. Itzykson, J.-M. Drouffe, “Statistical Field Theory”, vol. 2, Cambridge University Press, Cambridge, 1997
- [92] A. Ashtekar, L. Bombelli, H. Sahlmann, T. Thiemann, O. Winkler, “Gauge Field Theory Coherent States (GCS) : XIV. Random Coherent States for Quantum General Relativity”, in preparation
- [93] M. Varadarajan, J.-A. Zapata, “A proposal for Analyzing the Classical Limit of Kinematic Loop Quantum Gravity”, gr-qc/0001040