

On Non-renormalization and OPE in Superconformal Field Theories

G. Arutyunov^{*,**1}, B. Eden^{†2}, E. Sokatchev^{‡ 3}

** Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
Am Mühlenberg 1, D-14476 Golm, Germany*

*‡ Laboratoire d'Annecy-le-Vieux de Physique Théorique⁴ LAPTH, Chemin de Bellevue, B.P. 110,
F-74941 Annecy-le-Vieux, France*

Abstract

The OPE of two $N = 2$ R -symmetry current (short) multiplets is determined by the possible superspace three-point functions that two such multiplets can form with a third, a priori long multiplet. We show that the shortness conditions on the former put strong restrictions on the quantum numbers of the latter. In particular, no anomalous dimension is allowed unless the third supermultiplet is an R -symmetry singlet. This phenomenon should explain many known non-renormalization properties of correlation functions, including the one of four stress-tensor multiplets in $N = 4$ SYM₄.

¹email:agleb@aei-potsdam.mpg.de

²email:burkhard@lapp.in2p3.fr

³email:sokatche@lapp.in2p3.fr

**On leave of absence from Steklov Mathematical Institute, Gubkin str.8, 117966, Moscow, Russia

⁴UMR 5108 associée à l'Université de Savoie

1 Introduction

Conformal field theories in four dimensions have been the subject of intensive investigations since the last few years. This surge of interest has been primarily motivated by the search for confirmation of the celebrated AdS/CFT correspondence [1]-[3]. Various methods have been used to study correlation functions of a special type of CFT operators (called “short” or “analytic”) in $N = 4$ supersymmetric Yang-Mills theory (SYM), which were then compared to their AdS supergravity counterparts. These include calculations in supergravity [4]-[11] and in perturbative SYM [12]-[20], instanton calculations [21], as well as different studies of the general (non-perturbative) properties of CFT correlators [22]-[24]. Most of the results obtained so far concern the so-called “non-renormalization” of correlation functions. The earliest tests for the AdS/CFT correspondence were carried out with the simplest correlators, namely two- or three-point functions of short operators. Their form is predicted by superconformal covariance up to an overall factor (“coupling”). Using various methods it was shown that these correlators are non-renormalized and being extracted from supergravity [4, 5, 11] match their CFT partners [12, 13, 23, 17, 18, 20]. Similar results have been obtained for special classes of n -point correlators called “extremal” [25, 26, 27] and “next-to-extremal” [27]-[30].¹

More elaborate tests involved four-point functions of short operators. These correlators are in general non-trivial because superconformal invariance leaves some functional freedom (see, e.g., [33] for an abstract study of the correlator of four stress-tensor multiplets of $N = 4$ SYM). Nevertheless, their functional form can be further restricted by evoking field-theory dynamical arguments (Intriligator’s insertion procedure [34]). This phenomenon was called “partial non-renormalization” in [35] and it was confirmed by comparing to explicit perturbative [15], instanton [17] and supergravity [10] calculations.

Alternatively, one may explain the special properties of four-point functions of short operators starting from the operator product expansion (OPE). Indeed, a four-point function can be viewed as the convolution of two OPEs (a double OPE). The OPE of two short multiplets may include short as well as long multiplets (see the classification in [36]). The dimension of the short multiplets is protected by superconformal invariance whereas that of the long multiplets may become anomalous due to radiative corrections. This fact was used in [24] to argue that renormalization effects in CFT can be expected to take place whenever a long multiplet can appear in the OPE of two short multiplets. The simplest

¹Note that another special class of so-called “near-extremal” correlators exhibits the interesting property of factorization of the amplitude [31, 32].

example of this phenomenon involves the Konishi multiplet [37]. Its occurrence has been demonstrated and its anomalous dimension has been calculated perturbatively [38, 17, 19]. At the same time, such simple arguments do not explain the absence of anomalous dimension of certain operators. Among them one finds [39, 40] a scalar double trace operator O_{20} arising in the OPE of the lowest dimensional scalars from the stress-tensor (supercurrent) multiplet in $N = 4$ SYM. Recently it was shown [41] that the non-renormalization of O_{20} follows from the partial non-renormalization of [35]. From the free theory one finds several double trace operators in the $\mathbf{20}$ of $SU(4)$ which are in general expected to mix under the renormalization group flow. The mixing problem was addressed in [42], where the perturbative properties of the Konishi multiplet were analyzed and the same conclusion about the O_{20} was again reached at the perturbative level.

The OPE techniques used so far have been based on ordinary space-time. In a supersymmetric theory it is natural to study OPEs in superspace, in order to benefit from the manifest supersymmetry [43]. In Section 2 of this letter we discuss the OPE of two short (Grassmann and harmonic analytic [44]-[47],[22]) multiplets. Its content is determined by the possible three-point functions having two short supermultiplet legs and one general leg. For simplicity we restrict ourselves to the case where the short multiplets are $N = 2$ R -symmetry current multiplets, but our argument can easily be adapted to other types of $N = 2$ or $N = 4$ multiplets. We explain how such three-point functions can be constructed in $N = 2$ harmonic superspace [45, 46].² We show that the combined requirements of shortness (Grassmann analyticity and R symmetry irreducibility, or H-analyticity) at two ends lead to strong restrictions on the supermultiplet at the third, general end. We find that the supermultiplets which are allowed to have anomalous dimension must be singlets under the R -symmetry group. This provides a simple explanation of the surprising absence of anomalous dimension of operators like the aforementioned O_{20} .

Extrapolating the above $N = 2$ considerations to the OPE of two primary operators from the $N = 4$ stress-tensor multiplet, we expect that this OPE can contain superconformal primary operators with a non-vanishing anomalous dimension *only* if it is a singlet of $SU(4)$. In Section 3 we demonstrate that

²Many papers have been devoted to the construction of superconformal two- and three-point functions in superspace. These include the case of $N = 1$ chiral superfields [48, 49] as well as a method valid for all types of $N = 1$ superfields [50, 51]. A construction of analytic $N = 2$ and $N = 4$ two- and three-point functions is presented in [22] (see also [13]). Ref. [52] discusses the $N = 2$ case in the context of three protected (current or stress-tensor) operators. However, no explicit results are available for $N > 1$ in the more general case where one of the operators in the three-point function is generic.

this conclusion is compatible with the known structure of the four-point correlation functions of this type.

The results presented here are still preliminary, in the sense that we only find necessary conditions for the existence of the three-point functions considered. A systematic study of this class of three-point functions is under way and will be presented in a forthcoming publication.

2 Three-point functions involving two short $N = 2$ operators

The simplest example of a short gauge invariant composite operator in $N = 2$ supersymmetry is a bilinear combination of $N = 2$ matter (hypermultiplet) superfields:

$$L^{++}(x, \theta^+, \bar{\theta}^+, u) = \text{Tr}[q^+(x, \theta^+, \bar{\theta}^+, u)]^2 . \quad (1)$$

Off shell it satisfies the so-called Grassmann (G-)analyticity condition [44]-[46]. It means that the superfields in (1) depend only on one half of the Grassmann variables obtained by projecting the $SU(2)$ (R symmetry) index by harmonic variables:

$$\theta^{+\alpha} = u_i^+ \theta^{i\alpha}, \quad \bar{\theta}^{+\dot{\alpha}} = u_i^+ \bar{\theta}^{i\dot{\alpha}} . \quad (2)$$

The charges in (1), (2) correspond to the $U(1)$ subgroup of $SU(2)$.

On shell the superfield L^{++} satisfies an additional condition restricting its harmonic dependence,

$$D^{++}L^{++} = 0 . \quad (3)$$

Here the harmonic derivative

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} - 2i\theta^+ \sigma^\mu \bar{\theta}^+ \frac{\partial}{\partial x^\mu} \quad (4)$$

plays the rôle of the raising operator of $SU(2)$. In this sense eq. (3) can be regarded as the condition for $SU(2)$ irreducibility defining L^{++} as the highest weight state of a triplet irrep of $SU(2)$. An alternative interpretation is obtained by parametrizing the harmonic coset $S^2 \sim SU(2)/U(1)$ by a complex variable and treating eq. (3) as the condition for Cauchy-Riemann analyticity (H-analyticity [47]). The solution of the combined G- and H-analyticity constraints is a *short superfield* (known is the $N = 2$ linear or tensor multiplet [53]) whose bosonic content is as follows:

$$\begin{aligned} L^{++} = & L^{ij}(x) u_i^+ u_j^+ + (\theta^+)^2 M(x) + (\bar{\theta}^+)^2 \bar{M}(x) \\ & + 2i\theta^+ \sigma^\mu \bar{\theta}^+ [J_\mu(x) + \partial_\mu L^{ij}(x) u_i^+ u_j^-] + (\theta^+)^4 \square L^{ij}(x) u_i^- u_j^- . \end{aligned} \quad (5)$$

Note that the vector in (5) has a triplet part $\partial_\mu L^{ij}(x)$ expressed in terms of the scalar, and a singlet part $J_\mu(x)$ which must be conserved,

$$\partial^\mu J_\mu(x) = 0 . \tag{6}$$

Thus, the linear multiplet is also the R -symmetry current multiplet in an $N = 2$ superconformal theory. Superconformal invariance and the above constraints fix the dimension and R weight of the supermultiplet (this is typical for short superconformal multiplets, see [36]). Denoting by

$$\mathcal{D} = (d, s, r, a) \tag{7}$$

the superconformal quantum labels dimension d , spin s , R weight r and $SU(2)$ Dynkin label a (a equals the harmonic $U(1)$ charge), we find

$$\mathcal{D}_{L^{++}} = (2, 0, 0, 2) . \tag{8}$$

Now, in the theory of $N = 2$ hypermultiplets interacting with $N = 2$ SYM one can consider the four-point correlation function

$$\langle L^{++}(1)L^{++}(2)L^{++}(3)L^{++}(4) \rangle . \tag{9}$$

A certain class of such theories (e.g., $N = 4$ SYM written in terms of $N = 2$ superfields) are known to be finite and hence superconformal. Then one can apply the standard approach of (super)conformal OPE and write down the correlator (9) in the form of a double OPE (see, e.g., [54, 55, 56, 57]):

$$\begin{aligned} \langle L^{++}(1)L^{++}(2)L^{++}(3)L^{++}(4) \rangle = & \tag{10} \\ \sum_{\mathcal{D}} \int_{5,5'} \langle L^{++}(1)L^{++}(2)\mathcal{O}_{\mathcal{D}}(5) \rangle \langle \mathcal{O}_{\mathcal{D}}(5)\mathcal{O}_{\mathcal{D}}(5') \rangle^{-1} \langle \mathcal{O}_{\mathcal{D}}(5')L^{++}(3)L^{++}(4) \rangle . \end{aligned}$$

Here $\mathcal{O}_{\mathcal{D}}$ denotes an operator in a superconformal irrep with labels (8). The sum in (10) goes over all possible such irreps.

It is then clear that the spectrum of this OPE as well as the content of the double OPE (10) is determined by the possible superconformal three-point functions. When we say ‘‘possible’’ we have in mind that at points 1 and 2 such a function must satisfy the G- and H-analyticity conditions above. As we shall see shortly, the combination of these conditions with superconformal covariance imposes strong restrictions on the allowed irreps $\mathcal{O}_{\mathcal{D}}(3)$ at point 3 and hence on the content of the OPE (10).

So, our task will be to study the three-point functions

$$G_{\mathcal{D}} = \langle L^{++}(1)L^{++}(2)\mathcal{O}_{\mathcal{D}}(3) \rangle \quad (11)$$

and to find out which irreps \mathcal{D} can appear at point 3. To start with, we remark that because of G-analyticity at points 1 and 2 the function $G_{\mathcal{D}}$ depends on two “half spinor” Grassmann coordinates $\theta_{1,2}^{+\alpha,\dot{\alpha}} = u_{1,2}^+ \theta_{1,2}^{i\alpha,\dot{\alpha}}$ and on one “full spinor” $\theta_3^{i\alpha,\dot{\alpha}}$. At the same time, the superconformal algebra contains two “full spinor” generators, that of Poincaré supersymmetry $Q^{i\alpha,\dot{\alpha}}$ and that of conformal supersymmetry $S^{i\alpha,\dot{\alpha}}$. These generators act on the Grassmann coordinates as shifts (non-linear in the case of S supersymmetry), so the three-point function (11) effectively depends on $1/2+1/2+1-(1+1) = 0$ invariant combinations of Grassmann variables. In other words, given the lowest component (obtained by setting $\theta_1^+ = \theta_2^+ = \theta_3^i = 0$) of the three-point function (11), Q and S supersymmetry allow one to find its *unique completion* to a full superfunction. Another implication of this fact is the impossibility to have R weight at point 3. Indeed, the only superspace coordinates carrying R weight are the Grassmann ones, and the above counting shows that there is no invariant combination of the available θ 's.

Let us first examine the lowest component $G_{\mathcal{D}}(\theta = 0)$. Since the operators at points 1 and 2 are $SU(2)$ triplets, the allowed $SU(2)$ irreps at point 3 are $\underline{3} \times \underline{3} \rightarrow \underline{1} + \underline{3} + \underline{5}$. Using the notation $[2, 2, a]$ for an $SU(2)$ structure with charge ($SU(2)$ Dynkin label) 2 at points 1,2 and charge $a = 0, 2$ or 4 at point 3, we can write:

$$G_{\mathcal{D}}(\theta = 0) = [2, 2, a] \frac{V^{\{\mu_1 \dots \mu_s\}}}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_{13}^2 x_{23}^2)^{\frac{d-s}{2}}} . \quad (12)$$

Here

$$V^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$

is a vector with a homogeneous conformal transformation law (see, e.g., [58]) and $\{\}$ means a symmetrized traceless product.³ The symbol $[2, 2, a]$ denotes the isospin index structure associated to each allowed value of a . Since the $SU(2)$ indices at points 1,2 are projected by harmonics, we find it convenient to introduce an auxiliary harmonic variable at point 3 as well. Using the short-hand notation $(12) \equiv u_1^{+i} u_{2i}^+$ for $SU(2)$ invariant harmonic contractions, we can write down the three possible isospin structures in (12) as follows:

$$[2, 2, 0] = (12)^2 ; \quad [2, 2, 2] = (12)(13)(23) ; \quad [2, 2, 4] = (13)^2(23)^2 . \quad (13)$$

³It is well-known that only symmetric traceless tensors can appear in the OPE of two scalar operators [54].

Next we start the completion of the lowest component (12) to a full superfunction. As explained above, Q and S supersymmetry allow one to do this in a unique way. We find it convenient to first exploit translation invariance and Q supersymmetry and fix a frame in which $x_3 = \theta_3 = 0$. Such a choice is stable under (super)conformal transformations, so we can proceed by just making an S supersymmetry transformation at points 1 and 2. In this letter we shall restrict ourselves to the first non-trivial term in the θ expansion of $G_{\mathcal{D}}$. R weight preservation tells us that these terms must be of the type $\theta\bar{\theta}$. So, we write schematically

$$G_{\mathcal{D}} = [2, 2, a] \frac{V^{\{\mu_1 \dots V^{\mu_s}\}}}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} + 2i \left\{ \theta_1^+ \sigma_\nu \bar{\theta}_1^+ \rho_{11}^{[0,2,a] \nu \{\mu_1 \dots \mu_s\}} + \theta_1^+ \sigma_\nu \bar{\theta}_2^+ \rho_{12}^{[1,1,a] \nu \{\mu_1 \dots \mu_s\}} + (1 \leftrightarrow 2) \right\} + O(\theta^4) \quad (14)$$

In order to determine the vector/tensors $\rho^{\nu\{\mu_1 \dots \mu_s\}}$ we need to make a linearized S supersymmetry transformation at points 1 and 2. The G-analytic coordinates transform as follow [59, 46] (only the right-handed parameter $\bar{\eta}_{i\dot{\alpha}}$ is shown):

$$\begin{aligned} \delta_S x^{\dot{\alpha}\alpha} &= -4i(x^{\alpha\dot{\beta}} \bar{\eta}_{\dot{\beta}}^i u_i^-) \bar{\theta}^{+\dot{\alpha}} \\ \delta_S \theta^{+\alpha} &= x^{\alpha\dot{\beta}} \bar{\eta}_{\dot{\beta}}^i u_i^+ \\ \delta_S \bar{\theta}^{+\dot{\alpha}} &= O(\bar{\theta}^2) \\ \delta_S u_i^+ &= 4i(u_j^+ \bar{\eta}_{\dot{\beta}}^j \bar{\theta}^{+\dot{\beta}}) u_i^- , \quad \delta_S u_i^- = 0 . \end{aligned} \quad (15)$$

Further, the linear multiplet transforms with a fixed weight [46],

$$\delta_S G_{\mathcal{D}} = -8i \bar{\eta}_{\dot{\beta}i} (\bar{\theta}_1^{+\dot{\beta}} u_1^{-i} + \bar{\theta}_2^{+\dot{\beta}} u_2^{-i}) G_{\mathcal{D}} \quad (16)$$

(recall that we have set $x_3 = \theta_3 = 0$, so we need not consider the superconformal transformation at point 3). From (14)–(16) one finds linear equations for the ρ 's. The solution for ρ_{11} is

$$\begin{aligned} \rho_{11}^{[0,2,a] \nu \{\mu_1 \dots \mu_s\}} &= \frac{2}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} \left[\frac{[1, 3, a]}{(12)} \left(-2 \frac{x_1^\nu}{x_1^2}\right) V^{\{\mu_1 \dots V^{\mu_s}\}} + \right. \\ &\left. + \frac{(1^-2) [2, 2, a]}{(12)} \left\{ \left(\left(\frac{d-s}{2} - 2 \right) \frac{x_{12}^\nu}{x_{12}^2} - \frac{d-s}{2} \frac{x_1^\nu}{x_1^2} \right) V^{\{\mu_1 \dots V^{\mu_s}\}} + \frac{s}{2x_1^2} I^{\nu\{\mu_1}(x_1) V^{\mu_2 \dots V^{\mu_s}\}} \right\} \right] \end{aligned} \quad (17)$$

and similarly for the remaining ones. Here $I^{\nu\mu}(x) = \eta^{\nu\mu} - 2x^\nu x^\mu / x^2$ is the inversion tensor; $(1^-2) \equiv u_1^{-i} u_{2i}^+$ and

$$[1, 3, 0] = 0 ; \quad [1, 3, 2] = \frac{1}{2} (12)(23)^2 ; \quad [1, 3, 4] = (13)(23)^3 . \quad (18)$$

So far we have not obtained any restrictions on the allowed values of the dimension and spin at point 3. However, there is one constraint that we have not yet taken into account, namely, H-analyticity (or $SU(2)$ irreducibility) (3) at points 1 and 2. It manifests itself in two ways. Firstly, no harmonic singularities of the type $1/(12)$ are allowed. This is the condition of analyticity on the harmonic sphere $S^2 \sim SU(2)/U(1)$ which is equivalent to having short (polynomial) harmonic dependence and hence $SU(2)$ irreducibility. Secondly, the vector/tensor (17) plays the rôle of the vector component of a linear multiplet (5) considered as a function of x_1, θ_1^+ and u_1^\pm . As such, it may have a triplet part (with respect to the harmonics at point 1) and a singlet part. According to (6), the latter must be conserved.

The implementation of these two constraints depends on the value of the charge a at point 3.

(i) $a = 0$ (singlet operator at point 3). From (13), (17) and (18) we see that there appear no harmonic singularities. Further, in this case (17) can be rewritten as follows:

$$\rho_{11}^{[0,2,0] \nu \{\mu_1 \dots \mu_s\}} = (1^-2)(12) \partial_1^\nu \left\{ \frac{1}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} V^{\{\mu_1 \dots \mu_s\}} \right\} \quad (19)$$

The harmonic factor $(1^-2)(12) = u_1^{-i} u_1^{+j} u_{2i}^+ u_{2j}^+$ corresponds to an $SU(2)$ triplet (the harmonics commute). Thus, the triplet vector/tensor in (19) has the form required by (5). The absence of a singlet vector/tensor means that the constraint (6) does not apply here. The conclusion is that H-analyticity does not imply any new restrictions on the allowed irrep \mathcal{D} at point 3.⁴ The only constraint originates from the unitarity bound

$$d \geq s + 2 \text{ for } s \geq 1 \quad \text{or} \quad d \geq 1 \text{ for } s = 0. \quad (20)$$

We should stress that the superconformal irreps of the type $\mathcal{D} = (d, s, 0, 0)$ allowed in this case are in general *long multiplets*.

(ii) $a = 2$ (triplet operator at point 3). This time we find

$$\begin{aligned} \rho_{11}^{[0,2,2] \nu \{\mu_1 \dots \mu_s\}} &= \frac{1}{2} [(1^-2)(13) + (12)(1^-3)] (23) \partial_1^\nu \left\{ \frac{1}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} V^{\{\mu_1 \dots \mu_s\}} \right\} \\ &- \frac{1}{2} \frac{(23)^2}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} \left\{ (d-s-4) \left(\frac{x_{12}^\nu}{x_{12}^2} - \frac{x_1^\nu}{x_1^2} \right) V^{\{\mu_1 \dots \mu_s\}} + \frac{s}{x_1^2} I^\nu \{\mu_1(x_1) V^{\mu_2} \dots V^{\mu_s}\} \right\} \end{aligned} \quad (21)$$

⁴At least, this is it what we can say by terminating the θ expansion at the level shown in (14). Whether the higher levels bring in some new constraints or not can only be decided after constructing the complete three-point superfunction. This will be done elsewhere.

Once again, there are no harmonic singularities and the triplet part of the vector/tensor (the first line in (21)) is of the form required by (5). However, now the vector/tensor has a singlet part (the second, u_1^\pm -independent line in (21)) which is subject to the conservation condition (6). This is possible only if

$$d = 2 - s \quad (22)$$

or

$$d = s + 4 . \quad (23)$$

The solution (22) is incompatible with the unitarity bound (20) unless

$$d = 2, \quad s = 0 . \quad (24)$$

In fact, this corresponds to having another linear (i.e., short) multiplet at point 3. The second solution (23) is always allowed.

We conclude that the triplet operator at point 3 cannot have an anomalous dimension. In the special case (24) this is explained by the fact that the operator is short, but in the more general case (23) this is a new result.

(iii) $a = 4$ (5-plet operator at point 3). We find:

$$\begin{aligned} \rho_{11}^{[0,2,4] \nu \{ \mu_1 \dots \mu_s \}} &= (1^- 3)(13)(23)^2 \partial_1^\nu \left\{ \frac{1}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} V^{\{ \mu_1 \dots \mu_s \}} \right\} \\ &- \frac{(13)(23)^2}{(12)} \frac{1}{(x_{12}^2)^{2-\frac{d-s}{2}} (x_1^2 x_2^2)^{\frac{d-s}{2}}} \left\{ (d-s-4) \left(\frac{x_{12}^\nu}{x_{12}^2} - \frac{x_1^\nu}{x_1^2} \right) V^{\{ \mu_1 \dots \mu_s \}} \right. \\ &\quad \left. + \frac{s}{x_1^2} I^{\nu \{ \mu_1 (x_1) V^{\mu_2 \dots \mu_s \}} .} \right\} \quad (25) \end{aligned}$$

This time we encounter a harmonic singularity. The two terms in its coefficient are linearly independent, so the only way to remove the harmonic singularity is to set

$$d = 4, \quad s = 0 . \quad (26)$$

Again, we conclude that the 5-plet operator is severely restricted. Moreover, the lowest term (12) becomes singular for the values (26) since $(x^2)^{-2} \sim \delta(x)$ in Euclidean space [61].

3 Four-point functions of the N=4 supercurrent multiplet

Consider now the $N = 4$ case. Here the analogue of the $N = 2$ linear multiplet L^{++} is the supercurrent (stress-tensor) multiplet whose component content can be deduced from $L = \text{tr}W^{\{i}W^{j\}}$, where W^i is the $N = 4$ on-shell superfield carrying the irrep **6** of the R -symmetry group $SU(4)$. A superconformal primary operator generating L is a scalar O^I of dimension 2 transforming in the irrep **20** of $SU(4)$, $I = 1, \dots, 20$.

Our considerations above exploiting the G- and H-analyticity of the three-point functions at points 1 and 2 show in particular that the OPE of two primary operators from the $N = 4$ supercurrent multiplet can contain superconformal primary operators with a non-vanishing anomalous dimension *only* in the singlet of $SU(4)$. Now we shall demonstrate that this conclusion is compatible with the known structure of the four-point correlation functions of the fields O^I .

According to [35] the “quantum” part of this four-point function comprising all possible quantum corrections to the free-field result is given by a *single* function $F(v, Y)$ of conformal cross-ratios, which we conveniently choose to be $v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}$ and $Y = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}$. Under $SU(4)$ the product of two O^I decomposes as

$$\mathbf{20} \times \mathbf{20} = \mathbf{1} + \mathbf{20} + \mathbf{105} + \mathbf{84} + \mathbf{15} + \mathbf{175}. \quad (27)$$

The “quantum” part of the four-point function of the operators O^I projected on the different irreps in (27) is

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_i = \frac{1}{x_{12}^4 x_{34}^4} P_i(v, Y) \frac{vF(v, Y)}{(1-Y)^2}. \quad (28)$$

Here the polynomials $P_i(v, Y)$ are [39, 40, 41]

$$\begin{aligned} P_{\mathbf{1}} &= 1 - Y - \frac{4}{15}v + \frac{1}{6}Y^2 + \frac{2}{15}vY + \frac{1}{60}v^2; & P_{\mathbf{105}} &= v^2; \\ P_{\mathbf{20}} &= -\frac{5}{3}v + \frac{5}{3}Y^2 + \frac{5}{6}vY + \frac{1}{6}v^2; & P_{\mathbf{15}} &= 4Y - 2Y^2 - vY; \\ P_{\mathbf{84}} &= 3v - \frac{3}{2}vY - \frac{1}{2}v^2; & P_{\mathbf{175}} &= vY. \end{aligned}$$

Every irrep i of $SU(4)$ in the OPE of two O^I represents a contribution from an infinite tower of operators $O_{\Delta, l}^i$, where Δ is the conformal dimension of the operator, l is its Lorentz spin. The corresponding contribution to the four-point function can then be represented as an expansion of the type

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_i = \sum_{\Delta, l} a_{\Delta, l}^i \mathcal{H}_{\Delta, l}(x_{1,2,3,4}). \quad (29)$$

Here $\mathcal{H}_{\Delta,l}(x_{1,2,3,4})$ denotes the (canonically normalized) Conformal Partial Wave Amplitude (CPWA) for the exchange of an operator $O_{\Delta,l}^i$ and $a_{\Delta,l}^i$ is a normalization constant. We treat the CPWA as a double series of the type [60, 41]

$$\mathcal{H}_{\Delta,l} = \frac{1}{x_{12}^4 x_{34}^4} v^{\frac{h}{2}} \sum_{n,m=0}^{\infty} c_{nm}^{\Delta,l} v^n Y^m, \quad (30)$$

where the dimension Δ was split into a canonical part Δ_0 and an anomalous part h : $\Delta = \Delta_0 + h$.

Let us assign the grading parameter $T = 2n + m$ to the monomial $v^n Y^m$. As was shown in [41] the monomials in (30) with the lowest value of T have $T = \Delta_0$, where Δ_0 is the canonical (free-field) dimension of the corresponding operator.

Comparing (28) and (29) one finds, within every fractional power $v^{\frac{h}{2}}$, the following compatibility conditions

$$P_i \sum_{\Delta,l} a_{\Delta,l}^j \mathcal{H}_{\Delta,l}(x_{1,2,3,4}) = P_j \sum_{\Delta,l} a_{\Delta,l}^i \mathcal{H}_{\Delta,l}(x_{1,2,3,4}) \quad (31)$$

which hold for all pairs (i, j) of irreps in (27). Here the sums are taken over operators which have the same h . Thus, eqs. (31) imply non-trivial relations between the CPWAs of primary operators belonging to the same supersymmetry multiplet(s) with anomalous dimension h .⁵ Only one of these primary operators is the superconformal primary operator, i.e., it generates under supersymmetry the whole multiplet, while the others are its descendents.

We can now argue that a superconformal primary operator appears only in the singlet of $SU(4)$. Indeed, let us choose in (31) the irrep j to be the singlet. The polynomial P_1 is distinguished from the other P_i 's by the presence of a constant term. Suppose that a superconformal primary operator with a canonical dimension Δ_0 contributes to the OPE and transforms in some irrep i which is not a singlet. Due to the constant in P_1 , the lowest-order monomials on the r.h.s. of (31) would have $T = 2n + m = \Delta_0$. Clearly, all the other P_i 's always raise the T -grading by at least unity. The lowest dimension operator with canonical dimension Δ'_0 in the singlet would have the lowest terms with at least $T = \Delta_0 - 1$ (or lower) to saturate (31). Hence, Δ'_0 is always lower than Δ_0 , and therefore the corresponding operator cannot be a supersymmetry descendent of an operator in the irrep i . This shows that anomalous superconformal primary operators occur only in the singlet.

⁵The theory we consider does not have any additional symmetry except the superconformal one and therefore it is natural to expect that there do not exist two supersymmetry multiplets built upon different superconformal primaries but with the same anomalous dimension considered as a function of the coupling.

The relations (31) may be solved for the normalization constants $a_{\Delta,l}^i$ in terms of the normalization constant of the superconformal primary operator $a_{\Delta_{sp},l_{sp}}^1$:

$$a_{\Delta,l}^i = \lambda_{\Delta,l}^i \cdot a_{\Delta_{sp},l_{sp}}^1. \quad (32)$$

On the other hand, the coefficients $a_{\Delta,l}^i$ are expressed via the normalization constants of the three-point functions involving two CPOs and $O_{\Delta,l}^i$ (we assume that $O_{\Delta,l}^i$ are canonically normalized). Thus, equations (32) and (31) are in fact implied by the existence of the unique superspace structure $\langle L(1)L(2)\mathcal{O} \rangle$, where \mathcal{O} is a (long) superfield whose lowest component is $O_{\Delta,l}^i$.

From this argument we again deduce that superconformal primaries may belong to other irreps of $SU(4)$ only if they have a vanishing anomalous dimension. In the case $h = 0$ the Born (free-field) part of the four-point function should be taken into account which allows the existence of the superconformal primaries in the other irreps of $SU(4)$. A typical example is the operator O^I . Another interesting example is provided by the double-trace operator of dimension 4 in **20** which appears in the OPE of two O^I [39, 40].

4 Conclusions

Summary of the results

The OPE of two $N = 2$ current multiplets is determined by the possible three-point functions that two such multiplets can form with another, a priori long multiplet. The shortness conditions on the former restrict the latter to one of the following superconformal unitary irreps:

$$\begin{aligned} R \text{ symmetry singlet (a=0): } & \quad d \geq s + 2, \quad s \geq 1, \quad r = 0 \\ & \quad d \geq 1, \quad s = 0, \quad r = 0 \\ R \text{ symmetry triplet (a=2): } & \quad d = 2, \quad s = 0, \quad r = 0 \quad (\text{current m-t}) \\ & \quad d = s + 4, \quad s \geq 0, \quad r = 0 \\ R \text{ symmetry 5-plet (a=4): } & \quad d = 4, \quad s = 0, \quad r = 0 \end{aligned} \quad (33)$$

The only case in which an operator with anomalous dimension is allowed in the OPE is that of the singlet. The dimension of the triplet and the 5-plet is protected. This new phenomenon is at the origin of many known non-renormalization theorems.

These necessary conditions for the existence of the three-point function are derived from the first non-trivial term in the θ expansion (14). In a future publication we shall complete the result by examining the full superfunction.

Possible further developments

The generalization of the above simplest example to other $N = 2$ and $N = 4$ analytic (short) multiplets at points 1 and 2 is straightforward and will be given elsewhere.

Another interesting question is what happens if one applies Intriligator's insertion procedure [34] to the double OPE (10). Presumably, this should lead to an additional selection rule for the operators in the OPE. For instance, it is easy to show that analytic multiplets cannot appear in it. This should explain the difference between the general superconformal result of Ref. [33] for the four-point function of $N = 4$ SYM stress-tensor multiplets and the dynamical result of Ref. [35].

We believe that a similar OPE argument may be applied to the extremal and next-to-extremal n -point correlators, providing an easy way to show their non-renormalization. It is also tempting to look for an explanation of the near-extremal factorization phenomenon along the same lines.

Recently non-renormalization properties of certain multitrace operators were discussed [62, 63] by using the twisted formulation of $N = 2$ theories. It would be interesting to understand if this approach is related to ours.

Note added. Before submitting this publication to the e-archive we saw the new paper [64] in which it is argued that long multiplets may appear in the OPE of analytic superfields (cf. Ref. [43]).

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