

QUIESCENT COSMOLOGICAL SINGULARITIES

LARS ANDERSSON¹ AND ALAN D. RENDALL²

ABSTRACT. The most detailed existing proposal for the structure of spacetime singularities originates in the work of Belinskii, Khalatnikov and Lifshitz. We show rigorously the correctness of this proposal in the case of analytic solutions of the Einstein equations coupled to a scalar field or stiff fluid. More specifically, we prove the existence of a family of spacetimes depending on the same number of free functions as the general solution which have the asymptotics suggested by the Belinskii-Khalatnikov-Lifshitz proposal near their singularities. In these spacetimes a neighbourhood of the singularity can be covered by a Gaussian coordinate system in which the singularity is simultaneous and the evolution at different spatial points decouples.

CONTENTS

1. Introduction	1
2. The main results	3
3. Framework of the proofs	9
4. Fuchsian systems	11
5. Setting up the reduced equations	14
6. Curvature estimates	21
7. The constraints	31
8. Discussion	35
References	37

1. INTRODUCTION

The singularity theorems of Penrose and Hawking are among the best known theoretical results in general relativity. They guarantee the existence of spacetime singularities under rather general circumstances but say little about the structure of the singularities they predict. In the literature there are heuristic approaches to describing the structure of singularities, notably that of Belinskii, Khalatnikov and Lifshitz (BKL), described in [14], [4], [6] and their references. The BKL work indicates that generic singularities are oscillatory and therefore, in a certain sense, complicated. This complexity may explain why it has not been possible to determine the structure of the singularities by rigorous mathematical arguments.

Date: January 10, 2004 *File:* quiescent12a.tex.

According to the BKL analysis, the presence of oscillatory behaviour in solutions of the Einstein equations coupled to some matter fields is to a great extent independent of the details of the matter content. There are, however, exceptions. It was pointed out by Belinskii and Khalatnikov[5] that a massless scalar field can change the situation dramatically, producing singularities without oscillations. A massless scalar field is closely related to a stiff fluid, i.e. a perfect fluid with pressure equal to energy density, as will be explained in more detail below. Barrow [3] exploited the singularity structure of solutions of the Einstein equations coupled to a stiff fluid for a description of the early universe he called ‘quiescent cosmology’. We will refer to singularities where oscillatory behaviour is absent due to the matter content of spacetime as quiescent singularities.

Recently the Einstein equations coupled to a scalar field have once again been a source of interest, this time in the context of string cosmology. A formal low energy limit of string theory gives rise to the Einstein equations coupled to various matter fields. Under simplifying assumptions the collection of matter fields can be reduced to a single scalar field, the dilaton. The field equations are then equivalent to the standard Einstein-scalar field equations. (Note, however, that the metric occurring in this formulation of the equations is not the physical metric.) The structure of the singularity in these models plays a role in the so-called pre-big bang scenario. For more information on these matters the reader is referred to the work of Buonanno, Damour and Veneziano[7].

In view of the above facts, the Einstein-scalar field equations and, more generally, the Einstein-stiff fluid equations represent an opportunity to prove something about the structure of spacetime singularities in a context simpler than that encountered in the case of the vacuum Einstein equations or the Einstein equations coupled to a perfect fluid with a softer equation of state. In this paper we take this opportunity and prove the existence of a family of solutions of the Einstein-scalar field equations whose singularities can be described in detail and are quiescent. These spacetimes are very general in the sense that no symmetry is assumed and they depend on as many free functions as the general solution of the Einstein-scalar field equations. They have an initial singularity near which they can be approximated by solutions of a simpler system of differential equations, the velocity dominated system. Like the full system, it consists of constraints and evolution equations. The evolution equations contain no spatial derivatives and are thus a system of ordinary differential equations. This is an expression of the idea of BKL that the evolution at different spatial points decouples near the singularity.

The structure of the paper is as follows. In the second section we recall the Einstein-scalar field and Einstein-stiff fluid equations and define the corresponding velocity dominated systems. This allows the main theorems to be stated. They assert the existence of a unique solution of the Einstein-scalar field equations or Einstein-stiff fluid equations asymptotic to a given solution of the velocity dominated system. The proofs of the existence and

uniqueness theorems are described in the third section. In the fourth section the main analytical tool used in these proofs, the theory of Fuchsian systems, is presented. The algebraic machinery needed for the application of the Fuchsian theory is set up in the fifth section. This provides the basis for the estimates of spatial curvature and other important quantities in the section which follows. The seventh section treats relevant aspects of the constraints. The paper concludes with a discussion of what can be learned from the results of the paper and what generalizations are desirable.

Throughout the paper the scalar field and stiff fluid cases are treated in parallel. These are independent except in section 7 where the propagation of constraints for the scalar field is deduced from the corresponding statement for the stiff fluid. Hence concentrating on the scalar field case on a first reading would give a good idea of the main features of the proofs.

2. THE MAIN RESULTS

Let ${}^{(4)}g_{\alpha\beta}$ be a Lorentz metric on a four-dimensional manifold M which is diffeomorphic to $(0, T) \times S$ for a three-dimensional manifold S . Let a point of M be denoted by (t, x) , where $t \in (0, T)$ and $x \in S$. It will be important in the following to express the geometrical quantities of interest in terms of a local frame $\{e_a\}$ on S . Let $\{\theta^a\}$ denote the coframe dual to $\{e_a\}$. Throughout the paper lower case Latin indices refer to components in this frame, except where other conventions are introduced explicitly. Suppose that the metric takes the form:

$$(2.1) \quad -dt^2 + g_{ab}(t)\theta^a \otimes \theta^b$$

where $g_{ab}(t)$ denotes the one-parameter family of Riemannian metrics on S defined by the metrics induced on the hypersurfaces $t = \text{constant}$ by the metric ${}^{(4)}g_{\alpha\beta}$. A function t such that the metric takes the form (2.1) is called a Gaussian time coordinate. In this case, the second fundamental form of a hypersurface $t = \text{constant}$ is given by $k_{ab} = -\frac{1}{2}\partial_t g_{ab}$.

2.1. The Einstein-matter equations. The Einstein field equations coupled to matter can be written in the following equivalent 3 + 1 form. The constraints are:

$$(2.2a) \quad R - k_{ab}k^{ab} + (\text{tr}k)^2 = 16\pi\rho$$

$$(2.2b) \quad \nabla^a k_{ab} - e_b(\text{tr}k) = 8\pi j_b$$

The evolution equations are:

$$(2.3a) \quad \partial_t g_{ab} = -2k_{ab}$$

$$(2.3b) \quad \partial_t k^a_b = R^a_b + (\text{tr}k)k^a_b - 8\pi(S^a_b - \frac{1}{2}\delta^a_b \text{tr}S) - 4\pi\rho\delta^a_b$$

Here R is the scalar curvature of g_{ab} and R_{ab} its Ricci tensor. The quantities ρ , j_a and S_{ab} are projections of the energy-momentum tensor. Their explicit forms in the cases of interest in this paper will be given below.

The energy-momentum tensor of a scalar field is given by

$$(2.4) \quad T_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(\nabla_\gamma\phi\nabla^\gamma\phi)^{(4)}g_{\alpha\beta}$$

The Einstein equations can be written in the equivalent form ${}^{(4)}R_{\alpha\beta} = 8\pi\nabla_\alpha\phi\nabla_\beta\phi$, where ${}^{(4)}R_{\alpha\beta}$ is the Ricci tensor of ${}^{(4)}g_{\alpha\beta}$. We have $\rho = T_{00}$, $j_a = -T_{0a}$ and $S_{ab} = T_{ab}$ so that in the case of a scalar field it follows from (2.4) that:

$$(2.5a) \quad \rho = \frac{1}{2}[(\partial_t\phi)^2 + g^{ab}e_a(\phi)e_b(\phi)]$$

$$(2.5b) \quad j_b = -\partial_t\phi e_b(\phi)$$

$$(2.5c) \quad S_{ab} = e_a(\phi)e_b(\phi) + \frac{1}{2}[(\partial_t\phi)^2 - g^{cd}e_c(\phi)e_d(\phi)]g_{ab}$$

Note that it follows from the Einstein-scalar field equations as a consequence of the Bianchi identity that ϕ satisfies the wave equation ${}^{(4)}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0$. This has the 3 + 1 form:

$$(2.6) \quad -\partial_t^2\phi + (\text{tr}k)\partial_t\phi + \Delta\phi = 0$$

The constraints and evolution equations are together equivalent to the full Einstein-scalar field equations. In the following we will work with the 3 + 1 formulation of the equations rather than the four-dimensional formulation.

A stiff fluid is a perfect fluid with pressure equal to energy density. As we will see, it is closely related to the scalar field. The energy-momentum tensor of a stiff fluid is

$$(2.7) \quad T_{\alpha\beta} = \mu(2u_\alpha u_\beta + {}^{(4)}g_{\alpha\beta})$$

where μ is the energy density of the fluid in a comoving frame and u^α is the four-velocity. The Euler equations are obtained by substituting this expression into the equation $\nabla_\alpha T^{\alpha\beta} = 0$. The relation between the scalar field and the stiff fluid is as follows. Given a solution of the Einstein equations coupled to a scalar field, where the gradient of the scalar field ϕ is everywhere timelike, define $\mu = -(1/2)\nabla_\alpha\phi\nabla^\alpha\phi$ and $u^\alpha = \pm(-\nabla_\beta\phi\nabla^\beta\phi)^{-1/2}\nabla^\alpha\phi$. Here the sign is chosen so that u^α is future pointing, and so can be interpreted as the four-velocity of a fluid. Then the energy-momentum tensor defined by (2.7) is equal to the energy-momentum tensor of the scalar field. Since the latter is divergence-free we see that the fluid variables just defined together with the original metric define a solution of the Einstein equations coupled to a stiff fluid. In the spacetimes of interest in the following, the gradient of ϕ is always timelike near the singularity, so that the condition on the gradient is not a restriction in that situation.

The matter terms needed for the Einstein equations are given in the stiff fluid case by

$$(2.8a) \quad \rho = \mu(1 + 2|u|^2)$$

$$(2.8b) \quad j_b = 2\mu(1 + |u|^2)^{1/2}u_b$$

$$(2.8c) \quad S_{ab} = \mu(2u_a u_b + g_{ab})$$

Here $|u|^2 = g_{ab}u^a u^b$. The Euler equations can be written in the following 3+1 form:

$$(2.9a) \quad \begin{aligned} \partial_t \mu - 2(\text{tr}k)\mu &= -2|u|^2 \partial_t \mu - 4\mu u^a \partial_t u_a - 2\mu k_{ab} u^a u^b \\ &+ 2(\text{tr}k)\mu |u|^2 - 2e_a(\mu)(1 + |u|^2)^{1/2} u^a - \mu(1 + |u|^2)^{-1/2} u^a \nabla_a u_b u^b \\ &- 2\mu(1 + |u|^2)^{1/2} \nabla_a u^a \end{aligned}$$

$$(2.9b) \quad \begin{aligned} \partial_t u_a + (\text{tr}k)u_a &= -\mu^{-1}[\partial_t \mu - 2(\text{tr}k)\mu]u_a + (1 + |u|^2)^{-1} u^b u_a \partial_t u_b \\ &- (1 + |u|^2)^{-1/2} [(u_a u^b + (1/2)\delta_a^b)\mu^{-1} e_b(\mu) + (\nabla_c u^c u_a + u^c \nabla_c u_a)] \end{aligned}$$

2.2. The velocity dominated system. We will prove the existence of a large class of solutions of the Einstein-scalar field equations and Einstein-stiff fluid equations whose singularities we can describe in great detail. These singularities are of the type known as velocity dominated. (Cf. [9] and [11].) This means that near the singularity the solution can be approximated by a solution of a simpler system, the velocity dominated system. Like the 3+1 version of the full equations it consists of constraints and evolution equations. Solutions of the velocity dominated system will always be written with a left superscript zero and the convention is adopted that the indices of all quantities with this superscript are moved with the velocity dominated metric ${}^0g_{cd}$. The velocity dominated equations will now be written out explicitly. The constraints are:

$$(2.10a) \quad -{}^0k_{ab} {}^0k^{ab} + (\text{tr}{}^0k)^2 = 16\pi {}^0\rho$$

$$(2.10b) \quad \nabla^a ({}^0k_{ab}) - e_b(\text{tr}{}^0k) = 8\pi {}^0j_b$$

The evolution equations are:

$$(2.11a) \quad \partial_t {}^0g_{ab} = -2{}^0k_{ab}$$

$$(2.11b) \quad \partial_t {}^0k^a_b = (\text{tr}{}^0k) {}^0k^a_b - 8\pi ({}^0S^a_b - \frac{1}{2}\delta^a_b \text{tr}{}^0S) - 4\pi {}^0\rho \delta^a_b$$

In these equations the matter terms are not identical to those in the full equations but have been obtained from those by discarding certain terms.

In the case of a scalar field the (truncated) energy-momentum tensor components are given by

$$(2.12a) \quad {}^0\rho = \frac{1}{2}(\partial_t {}^0\phi)^2$$

$$(2.12b) \quad {}^0j_b = -\partial_t {}^0\phi e_b({}^0\phi)$$

$$(2.12c) \quad {}^0S_{ab} = \frac{1}{2}(\partial_t {}^0\phi)^2({}^0g_{ab})$$

The scalar field satisfies the equation

$$-\partial_t^2({}^0\phi) + (\text{tr}^0k)\partial_t {}^0\phi = 0$$

It is important to note that the velocity dominated evolution equations are ordinary differential equations. However the constraints still include partial differential equations.

In the stiff fluid case the (truncated) energy-momentum tensor components are

$$(2.13a) \quad {}^0\rho = {}^0\mu$$

$$(2.13b) \quad {}^0j_b = 2{}^0\mu {}^0u_b$$

$$(2.13c) \quad {}^0S_{ab} = {}^0\mu {}^0g_{ab}$$

The velocity dominated Euler equations are

$$(2.14a) \quad \partial_t {}^0\mu - 2(\text{tr}^0k){}^0\mu = 0$$

$$(2.14b) \quad \partial_t {}^0u_a + (\text{tr}^0k){}^0u_a = -(1/2){}^0\mu^{-1}e_a({}^0\mu)$$

In contrast to the scalar field case, this is not a system of ordinary differential equations. However it has a hierarchical ODE structure in the sense that if the ODE for ${}^0\mu$ is solved and the result substituted into the other equations an ODE system for the 0u_a results.

Substituting the expressions for the truncated energy-momentum tensor into the velocity dominated system shows that the matter terms in the velocity dominated evolution equation for ${}^0k_{ab}$ cancel both for the scalar field and the stiff fluid, leaving

$$\partial_t {}^0k^a_b = (\text{tr}^0k){}^0k^a_b$$

Taking the trace of this equation gives $\partial_t(\text{tr}^0k) = (\text{tr}^0k)^2$. This has the general solution $\text{tr}^0k = (C - t)^{-1}$. If we wish an initial singularity, as signalled by the blow-up of tr^0k , to occur at $t = 0$ then $\text{tr}^0k = -t^{-1}$. Going back to the equation for ${}^0k^a_b$ we see that $t({}^0k^a_b)$ is independent of time. Thus all components of the mixed form of the second fundamental form are proportional to t^{-1} .

At any given spatial point we can simultaneously diagonalize ${}^0g_{ab}$ and ${}^0k_{ab}$ by a suitable choice of frame. The matrix of components of the metric in this frame is diagonal with the diagonal elements being proportional to powers of t . This form of the metric is that originally used by BKL. Its disadvantage is that in general this frame cannot be chosen to depend smoothly

on the spatial point. (There are difficulties when there are changes in the multiplicity of the eigenvalues of ${}^0k^a_b$.) This is one reason why a different formulation is used in this paper.

The velocity dominated matter equations can be solved exactly to give

$$(2.15) \quad {}^0\phi(t, x) = A(x) \log t + B(x)$$

for given functions A and B on S and

$$(2.16) \quad {}^0\mu(t, x) = A^2(x)t^{-2}$$

$$(2.17) \quad {}^0u_a(t, x) = t \log t (A(x))^{-1} e_a(A(x)) + t B_a(x)$$

for given quantities $A(x)$ and $B_a(x)$ on S .

2.3. Statement of the main theorems. The main theorems can now be stated.

Theorem 2.1. *Let S be a three-dimensional analytic manifold and let $({}^0g_{ab}(t), {}^0k_{ab}(t), {}^0\phi(t))$ be a C^ω solution of the velocity dominated Einstein-scalar field equations on $S \times (0, \infty)$ such that $\text{tr}^0k = -1$ and each eigenvalue λ of $-t^0k^a_b$ is positive. Then there exists an open neighbourhood U of $S \times \{0\}$ in $S \times [0, \infty)$ and a unique C^ω solution $(g_{ab}(t), k_{ab}(t), \phi(t))$ of the Einstein-scalar field equations on $U \cap (S \times (0, \infty))$ such that for each compact subset $K \subset S$ there are positive real numbers ζ, β, α^a_b , with $\zeta < \beta < \alpha^a_b$, for which the following estimates hold uniformly on K :*

1. ${}^0g^{ac}g_{cb} = \delta^a_b + o(t^{\alpha^a_b})$
2. $k^a_b = {}^0k^a_b + o(t^{-1+\alpha^a_b})$
3. $\phi = {}^0\phi + o(t^\beta)$
4. $\partial_t \phi = \partial_t {}^0\phi + o(t^{-1+\beta})$
5. ${}^0g^{ac}e_f(g_{cb}) = o(t^{\alpha^a_b - \zeta})$
6. $e_a(\phi) = e_a({}^0\phi) + o(t^{\beta - \zeta})$

Note that the condition on tr^0k can always be arranged by means of a time translation and that the condition on the eigenvalues of ${}^0k^a_b$ is satisfied provided it holds for a single value of $t > 0$. The positivity condition on the eigenvalues together with the velocity dominated Hamiltonian constraint imply that A^2 must be strictly positive in the velocity dominated solution. Thus vacuum solutions are ruled out by the hypotheses of this theorem. If an analogous analysis were done for the Einstein equations coupled to other matter models, for instance a perfect fluid with equation of state $p = k\rho$, $k < 1$, then in many cases, including that of the fluid just mentioned, the matter would make no contribution to the velocity dominated Hamiltonian constraint, so that it would not be possible to prove an analogous theorem. This reflects the fact that for those matter models an oscillatory approach to the singularity is predicted by the BKL analysis. The scalar field is an exception, as is the stiff fluid which will be discussed next.

Theorem 2.2. *Let S be a three-dimensional analytic manifold and let $({}^0g_{ab}(t), {}^0k_{ab}(t), {}^0\mu(t), {}^0u_a)$ be a C^ω solution of the velocity dominated Einstein-stiff fluid equations on $S \times (0, \infty)$ such that $\text{tr}^0k = -1$ and each eigenvalue λ of $-t^0k^a_b$ is positive. Then there exists an open neighbourhood U of $S \times \{0\}$ in $S \times [0, \infty)$ and a unique C^ω solution $(g_{ab}(t), k_{ab}(t), \mu(t), u_a(t))$ of the Einstein-stiff fluid equations on $U \cap (S \times (0, \infty))$ such that for each compact subset $K \subset S$ there are positive real numbers $\zeta, \beta_1, \beta_2, \alpha^a_b$, with $\zeta < \beta_2 < \beta_1 < \alpha^a_b$, for which the following estimates hold uniformly on K :*

1. ${}^0g^{ac}g_{cb} = \delta^a_b + o(t^{\alpha^a_b})$
2. $k^a_b = {}^0k^a_b + o(t^{-1+\alpha^a_b})$
3. $\mu = {}^0\mu + o(t^{-2+\beta_1})$
4. $u_a = {}^0u_a + o(t^{1+\beta_2})$
5. ${}^0g^{ac}e_f(g_{cb}) = o(t^{\alpha^a_b - \zeta})$

The interest of these theorems depends very much on what information is available on constructing solutions of the velocity dominated system. Suppose that a solution of the velocity dominated constraints is given for some $t = t_0 > 0$. The velocity dominated evolution equations constitute a system of ordinary differential equations which can be solved with these initial data. It follows from the remarks above that the solution exists globally on the interval $(0, \infty)$. If we define

$$(2.18) \quad {}^0C = -{}^0k_{ab}{}^0k^{ab} + (\text{tr}^0k)^2 - 16\pi^0\rho$$

$$(2.19) \quad {}^0C_b = \nabla^a({}^0k_{ab}) - e_b(\text{tr}^0k) - 8\pi^0j_b$$

then the velocity dominated evolution equations imply that:

$$(2.20) \quad \partial_t {}^0C + 2t^{-1}({}^0C) = 0$$

$$(2.21) \quad \partial_t {}^0C_a + t^{-1}({}^0C_a) = \frac{1}{2}e_a({}^0C)$$

To prove this it is necessary to use the following equations for the matter quantities, which can be derived from the velocity dominated matter equations in both the scalar field and stiff fluid cases.

$$(2.22) \quad \partial_t {}^0\rho = 2(\text{tr}^0k)^0\rho$$

$$(2.23) \quad \partial_t {}^0j_a = (\text{tr}^0k)^0j_a - e_a({}^0\rho)$$

Since 0C vanishes at $t = t_0$ the evolution equation for 0C implies that it vanishes everywhere. Then the evolution equation for 0C_a , together with the fact that it vanishes for $t = t_0$ implies that 0C_a vanishes everywhere. To sum up, if the velocity dominated constraints are satisfied at some time and the velocity dominated evolution equations are satisfied everywhere then the velocity dominated constraints are satisfied everywhere. Thus in order to have a parametrization of the general solution of the velocity dominated system, it is enough to obtain a parametrization of solutions of the velocity dominated constraints. The latter question will be treated in section 7.

3. FRAMEWORK OF THE PROOFS

In this section the proofs of Theorems 2.1 and 2.2 are outlined. Only the general logical structure of the proof is explained here and the hard technical parts of the argument are left to later sections. These results will be referred to as required in this section.

The first step is to make a suitable ansatz for the desired solution. This essentially means giving names to the remainder terms occurring in the statements of the main theorems. Assume that a velocity dominated solution is given as in those statements. Then a solution is sought in the form:

$$(3.1a) \quad g_{ab} = {}^0g_{ab} + {}^0g_{ac}t^{\alpha^c}{}_b\gamma^c_b$$

$$(3.1b) \quad k_{ab} = g_{ac}({}^0k^c_b + t^{-1+\alpha^c}{}_b\kappa^c_b)$$

In the following the summation convention applies only to repeated tensor indices and not to non-tensorial quantities like α^a_b . Thus in the above equations there is a summation on the index c but none on the index b . Matter fields are sought in the form:

$$(3.2) \quad \phi = {}^0\phi + t^\beta\psi$$

and

$$(3.3a) \quad \mu = {}^0\mu + t^{-2+\beta_1}\nu$$

$$(3.3b) \quad u_a = {}^0u_a + t^{1+\beta_2}v_a$$

respectively. The Einstein-scalar field equations (2.2), (2.3), (2.5) and (2.6) can be rewritten as equations for γ^a_b , κ^a_b and ψ . Similarly the Einstein-stiff fluid equations can be written as equations for γ^a_b , κ^a_b , ν and v_a . This system of equations (for either choice of matter model) will be called the **first reduced system**. Since γ^a_b and κ^a_b are mixed tensors, there is no direct way to express the fact that they originated from symmetric tensors. Instead it must be shown that when the first reduced system is solved with suitable asymptotic conditions as $t \rightarrow 0$ then the quantities g_{ab} and k_{ab} defined by the above equations are in fact symmetric as a consequence of the differential equations and the initial conditions. When allowing non-symmetric tensors g_{ab} and k_{ab} , we need to establish some conventions in order to make the definition of the first reduced system unambiguous. Firstly, define g^{bc} as the unique tensor which satisfies $g_{ab}g^{bc} = \delta_a^c$. Next, use the convention that indices on tensors are lowered by contraction with the second index of g_{ab} and raised with the first index of g^{bc} . This maintains the usual properties of index manipulations in the case of symmetric g_{ab} as far as possible. The covariant derivatives in the equations are expressed in terms of the connection coefficients in the frame $\{e_a\}$ in an unambiguous way. The definition of the connection coefficients is extended to the case of a non-symmetric tensor g_{ab}

by fixing the order of indices according to

$$(3.4) \quad g_{cd}\Gamma_{ab}^d = \frac{1}{2}(e_a(g_{bc}) + e_b(g_{ac}) - e_c(g_{ab}) + \gamma_{ab}^d g_{cd} - \gamma_{ac}^d g_{bd} - \gamma_{bc}^d g_{ad})$$

where $\gamma_{ab}^c = \theta^c([e_a, e_b])$ are the structure functions of the frame.

Finally, in order to define the Ricci tensor in the evolution equation for k_{ab} we define R_{ab} to be the Ricci tensor of the symmetric part ${}^S g_{ab} = (1/2)(g_{ab} + g_{ba})$ of g_{ab} . In Lemma 5.2 it is shown that given a solution of the velocity dominated system as in the statement of one of the main theorems 2.1–2.2, any solution of the first reduced system which satisfies points 1–6 of Theorem 2.1, in the scalar field case, and points 1–5 of Theorem 2.2, in the stiff fluid case, gives rise to symmetric tensors g_{ab} and k_{ab} , and thus to a solution of the Einstein-matter evolution equations. It then follows from Lemma 7.1 and the remarks following it that the Einstein-matter constraints are also satisfied. It follows that to prove the main theorems it is enough to prove the existence and uniqueness of solutions of the first reduced system of the form given in the main theorems.

The existence theorem for the first reduced system will be proved using the theory of Fuchsian systems. Since the form of this theory we use in the following concerns a system of first order equations it is not immediately applicable, due to the occurrence of second order derivatives of the metric and scalar field. It is necessary to introduce some suitable new variables representing spatial derivatives of the basic variables. Define $\lambda_{bc}^a = t^\zeta e_c(\gamma_b^a)$ and, in the scalar field case, $\omega_a = t^\zeta e_a(\psi)$ and $\chi = t\partial_t\psi + \beta\psi$, where ζ and β are positive constants. The evolution equations satisfied by these quantities are given explicitly in (5.13b) and (5.15). With the help of the new variables these equations together with the first reduced system can be written as a first order system. Call the result the **second reduced system**. It is easy to show, using the evolution equations for differences like $\gamma_{bc}^a - t^\zeta e_c(\gamma_b^a)$ which follow from the second reduced system, that the first and second reduced systems give rise to the same sets of solutions under the assumptions of the main theorems, together with corresponding assumptions on the new variables. Thus it suffices to solve the second reduced system, which is of first order.

If it can be shown that the second reduced system is Fuchsian, then the main theorems follow from Theorem 4.2. In fact it is enough to show that the restriction of the system to a neighbourhood of an arbitrary point of S is Fuchsian. For the local solutions thus obtained can be pieced together to get a global solution. Moreover asymptotic estimates as in the statements of the main theorems follow from corresponding local statements, since a compact subset of S can be covered by finitely many of the local neighbourhoods.

In sections 5 and 6 it is shown that the second reduced system is Fuchsian on some neighbourhood of each point of S for a suitable choice of the constants α_b^a , β , β_1 , β_2 and ζ depending on the given velocity dominated solution. This requires a detailed analysis of the degree of singularity of

all terms in the second reduced system, in particular that of the Ricci tensor. The result of the latter is Lemma 6.3. This is the hardest part of the proof. Due the above considerations, it is clear that the main theorems follow directly from Theorem 4.2.

4. FUCHSIAN SYSTEMS

The proofs of Theorems 2.1 and 2.2 rely on a result of Kichenassamy and Rendall[13] on Fuchsian systems which uses a method going back to Baouendi and Goulaouic[2]. The result of [13] will now be recalled. It concerns a system of the form:

$$(4.1) \quad t \frac{\partial u}{\partial t} + A(x)u = f(t, x, u, u_x)$$

Here $u(t, x)$ is a function on an open subset of $\mathbf{R} \times \mathbf{R}^n$ with values in \mathbf{R}^k and $A(x)$ is a C^ω matrix-valued function. The derivatives of u with respect to the x variables are denoted by u_x . The function f is defined on $(0, T_0] \times U_1 \times U_2$, where U_1 is an open subset of \mathbf{R}^n and U_2 is an open subset of \mathbf{R}^{k+nk} , and takes values in \mathbf{R}^k . We assume that $A(x)$ is defined on U_1 .

In this and later sections it will be useful to have some terminology for comparing the sizes of certain expressions.

Definition 4.1. Let $F(t, x, p)$, $G(t, x, p)$ be functions on $(0, T_0] \times U_1 \times U_2$, where U_1, U_2 are open subsets of \mathbf{R}^n and \mathbf{R}^N respectively. Then we will say that

$$F \preceq G$$

if for every compact $K \subset U_1 \times U_2$, there is a constant C such that

$$|F(t, x, p)| \leq C|G(t, x, p)| \quad \text{for } t \in (0, t_0], (x, p) \in K.$$

In the particular case that G is just a function of t we will often use the familiar notation $F = O(G(t))$ to replace $F \preceq G$. The notation $F = o(G(t))$ will also be used to indicate that F/G tends to zero uniformly on compact subsets of $U_1 \times U_2$ as $t \rightarrow 0$.

In the theorem on Fuchsian systems the function f is supposed to be regular in a sense which will now be explained. To do this we need the notion of a function which is continuous in t and analytic in other (complex) variables. This means by definition that it should be a continuous function of all variables, that the first order partial derivatives with respect to all variables other than t should exist and be continuous, and that the Cauchy-Riemann equations should be satisfied in these variables. For further remarks on this concept see [15]. Assume that there is an open subset \tilde{U} of \mathbf{C}^{n+k+nk} whose intersection with the real section is equal to $U_1 \times U_2$ and a function \tilde{f} on $(0, T_0] \times \tilde{U}$ continuous in t and analytic in the remaining arguments whose restriction to $(0, T_0] \times \mathbf{R}^{n+k+nk}$ is equal to f . The function f is called **regular** if it has an analytic continuation \tilde{f} of the kind just described, and if there is some $\theta > 0$ such that \tilde{f} and its first derivatives with respect to the arguments u and u_x are $O(t^\theta)$ as $t \rightarrow 0$, in the sense introduced above.

For a matrix A with entries A^a_b let $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$ (operator norm) and $\|A\|_\infty = \max_{a,b} |A^a_b|$ (maximum norm). Since these two norms are equivalent the operator norm could be replaced by the maximum norm in the statement of the theorem which follows and in Lemma 4.3 below. However in the proof of that lemma below the use of the operator norm is important.

Theorem 4.2. *Suppose that the function f is regular, $A(x)$ has an analytic continuation to an open set \tilde{U}_1 whose intersection with the real section is U_1 , and there is a constant C such that $\|\sigma^{A(x)}\| \leq C$ for $x \in \tilde{U}_1$ and $0 < \sigma < 1$. Then the equation (4.1) has a unique solution u defined near $t = 0$ which is continuous in t and analytic in x and tends to zero as $t \rightarrow 0$. If \tilde{f} is analytic for $t > 0$ then this solution is also analytic in t for $t > 0$.*

Remark 4.1. Under the hypotheses of the theorem spatial derivatives of any order of u are also $o(1)$ as $t \rightarrow 0$. This follows directly from the proof of the theorem in [13].

Remark 4.2. If the coefficients of the equation depend analytically on a parameter and are suitably regular, then the solution depends analytically on the parameter. It suffices to treat the parameter as an additional spatial variable.

The statement of the theorem is not identical to that given in [13] but the proof is just the same. We can write $f(t, x, u, u_x) = t^\theta g(t, x, u, u_x)$ for some bounded function g . By replacing t as time variable by t^θ , it can be assumed without loss of generality that $\theta = 1$. Then the iteration used in the proof given in [13] converges to the desired solution under the regularity hypothesis we have made on f .

For the applications in this paper an extension of this result which applies to equations slightly more general than (4.1) will be required. These have the form

$$(4.2) \quad t \frac{\partial u}{\partial t} + A(x)u = f(t, x, u, u_x) + g(t, x, u) t \frac{\partial u}{\partial t}$$

If we have an equation of this form where f and g are regular then an analogous existence and uniqueness result holds. For we can rewrite (4.2) in the form

$$(4.3) \quad t \frac{\partial u}{\partial t} + A(x)u = [I - (I - g(t, x, u))^{-1}]A(x)u + (I - g(t, x, u))^{-1}f(t, x, u, u_x)$$

If we call the right hand side of this equation $h(t, x, u, u_x)$ then it satisfies the conditions required of $f(t, x, u, u_x)$ in Theorem 4.2. In other words h is regular. For if $g(t, x, u)$ is $O(t^\theta)$ then $(I - g(t, x, u))^{-1}$ is $O(1)$ and $[I - (I - g(t, x, u))^{-1}]$ is $O(t^\theta)$.

In the following it will be necessary to verify the regularity hypothesis for some particular systems, and some general remarks which can be used to simplify this task will now be made. In these systems we always have

$$(4.4) \quad f(t, x, u, u_x) = \sum_{i=1}^m t_i F_i(x, v(t, x), u, u_x)$$

where the F_i are analytic functions on an open subset V of $\mathbf{R}^{n+l+k+nk}$ which includes $U_1 \times \{0\} \times U_2$, t_1, \dots, t_m are some functions of t and $v(t, x)$ is a given function with values in \mathbf{R}^l . The t_i are continuous functions on $(0, T_0]$ which tend to zero as $t \rightarrow 0$ as least as fast as some positive power of t and are analytic for $t > 0$. The function $v(t, x)$ is the restriction of an analytic function on $(0, T_0] \times \tilde{U}_1$ to real values of its arguments. Each component of v tends to zero as $t \rightarrow 0$, uniformly on \tilde{U}_1 . These properties ensure that f is regular. For the functions F_i have analytic continuations to some open neighbourhood \tilde{V} of V in $\mathbf{C}^{n+l+k+nk}$.

In the examples we will meet the functions t_i are either positive powers of t or positive powers of t times positive powers of $\log t$. The role of $v(t, x)$ is played by the functions t_j , the components of the velocity dominated metric ${}^0g_{ab}$, their spatial derivatives of first and second order, and the components of the inverse metric multiplied by suitable powers of t so that the product vanishes in the limit $t \rightarrow 0$. We saw in the last section that ${}^0g_{ab}$ can be written in the form t^{2K} which, for each fixed t , is an entire function of K . Thus $t^{2K(x)}$ is analytic on any region where $K(x)$ is analytic. The velocity dominated metric and its derivatives all tend to zero uniformly on compact sets as $t \rightarrow 0$ while the same is true of the inverse metric multiplied by a suitable power of t .

Next a criterion will be given which allows the hypothesis on the matrix A in Theorem 4.2 to be checked in many cases.

Lemma 4.3. *Let $A(x)$ be a $k \times k$ matrix-valued continuous function defined on a compact subset of \mathbf{R}^n . If there is a constant α such that, for each eigenvalue λ of $A(x)$ at any point of the given compact set, $\text{Re}\lambda > \alpha$ then there is a constant C such that the estimate $\|t^{A(x)}\| \leq Ct^\alpha$ holds for t small and positive and x in the compact set.*

Proof The general case can be reduced to the case $\alpha = 0$ by the following computation:

$$(4.5) \quad \|t^{A-\alpha I}\| \leq \|t^A\| t^{-\alpha}$$

In the rest of the proof only the case $\alpha = 0$ will be considered. Without the parameter dependence the result could easily be proved by reducing the matrix to Jordan canonical form. The difficulty with a parameter is that the reduction to canonical form is in general not a continuous process. At least it can be concluded from the continuity properties of the eigenvalues that there is a $\beta > 0$ such that all eigenvalues satisfy $\text{Re}\lambda > \beta$. Let $s = \log t$. Then the problem is to show that for a fixed s_0 there is a constant

$C > 0$ such that $\|e^{-sA}\| \leq C$ for all $s > s_0$. By scaling t we may suppose without loss of generality that $s_0 = 0$. For each x we can conclude by reduction to canonical form that there exists a value s_x of s such that the inequality $\|e^{-s_x A(x)}\| < e^{-\beta s_x/2}$ holds. By continuity of the exponential function there is an open neighbourhood U_x of x where this continues to hold for the given value of s_x . Let $C_x = \sup\{\|e^{-sA(y)}\| : s \in [0, s_x], y \in U_x\}$. It follows that for any $s \in [0, \infty)$ and any $y \in U_x$ we have

$$(4.6) \quad \|e^{-sA(y)}\| \leq C_x \|e^{-s_x A(y)}\|^{[s/s_x]} \leq C_x e^{-\beta s_x [s/s_x]/2} \leq C_x$$

By compactness, it is possible to pass to a subcover consisting of a finite number of the sets U_x and letting C be the maximum of the corresponding C_x we obtain the required estimate.

Remark 4.3. More generally, an analogous estimate is obtained if $A(x)$ is the direct sum of a matrix $B(x)$ whose eigenvalues have positive real parts and the zero matrix. This is obvious since in that case $t^{A(x)}$ is the direct sum of $t^{B(x)}$ and the identity.

5. SETTING UP THE REDUCED EQUATIONS

In this section we introduce the adapted frame $\{e_a\}$ and the auxiliary exponents $\{q_a\}$ which will be used in the curvature estimate.

Let x_0 be given. Let ${}^0g_{ab}$, ${}^0k_b^a$, be solutions of the velocity dominated evolution and constraint equations (2.11) and (2.10). Let p_a be the eigenvalues of $K_b^a = -t^0k_b^a$. Assume that $\{p_a\}$ are such that $p_a(x_0) > 0$, $a = 1, 2, 3$, $\sum_a p_a = 1$ (Kasner condition) and p_a are ordered so that $p_a \leq p_b$, for $a \leq b$. Fix an initial time $t_0 \in (0, 1)$. We will in the following restrict our considerations to $t \in (0, 1)$.

If K has a double eigenvalue at x_0 , then in general the eigenvalues and eigenvectors of K are not analytic in a neighbourhood of x_0 , and therefore in general it is not possible to introduce an analytic frame diagonalizing K in a given neighbourhood. We will avoid this problem by using the well known fact that if $p_{a'}$ is a double eigenvalue of $K(x_0)$, the eigenspace of the pair of eigenvalues corresponding to $p_{a'}$ is analytic in a neighbourhood of x_0 . This means in particular that it is possible to choose an analytic frame which is adapted to the eigenspace of the pair of eigenvalues. This will play a central role in what follows.

Choose numbers $\alpha_0, \epsilon > 0$ so that $\epsilon = \alpha_0/4 < \min\{p_a(x_0)\}/40$.

1. **Cases I, II, III:** We will distinguish between the following cases:

- (I) (near Friedmann) $\max_{a,b} |p_a - p_b| < \epsilon/2, a = 1, 2, 3$.
- (II) (near double eigenvalue) $\max_{a,b} |p_a - p_b| > \epsilon/2$, and $|p_{a'} - p_{b'}| < \epsilon/2$ for some pair $a', b', a' \neq b'$. Denote by p_\perp the distinguished exponent not equal to $p_{a'}, p_{b'}$.
- (III) (diagonalizable) $\min_{\substack{a,b \\ a \neq b}} |p_a - p_b| > \epsilon/2$

By reducing ϵ if necessary we can make sure that condition I, II or III holds at x_0 if the maximum multiplicity of an eigenvalue at x_0 is three,

two or one respectively. The conditions I, II, III are open, and hence there is an open neighbourhood $U_0 \ni x_0$ such that, if condition I, II or III holds at x_0 , the condition holds in U_0 and further, for $x \in U_0$, $\min_a \{p_a(x)\} > 20\epsilon$.

2. **Auxiliary exponents** $\{q_a\}$: Let $U_0 \ni x_0$ be as in point 1. We will define analytic functions $q_a, a = 1, 2, 3$, called auxiliary exponents, in U_0 , with the properties
- (a) $q_a > 0$ (positivity)
 - (b) $q_a \leq q_b$ if $a \leq b$ (ordering)
 - (c) $\sum_a q_a = 1$ (Kasner)

The auxiliary exponents q_a will be defined in terms of the eigenvalues p_a of K depending on whether at x_0 we are in case I, II, or III.

Let $q_a(x_0) = p_a(x_0)$, $a = 1, 2, 3$, and choose q_a on U_0 satisfying the positivity, ordering and Kasner conditions such that in cases I, II, III, the following holds.

- (I) $q_a = 1/3, a = 1, 2, 3$
- (II) $q_\perp = p_\perp, q_{a'} = q_{b'} = \frac{1}{2}(1 - q_\perp)$
- (III) $q_a = p_a$

Then q_a are analytic on U_0 . Note that it follows from the definition of q_a that $q_1 \geq \min_i \{p_i\}$ and that $\max_a |q_a - p_a| < \epsilon/2$.

3. **The frame** $\{e_a\}$: In each case I, II, III define an analytic frame $\{e_a\}$ with dual frame $\{\theta^a\}$, by the following prescription: $\{e_a\}$ is an ON frame w.r.t. ${}^0g(t_0)$, and in case II, III the following additional conditions hold.
- (II) e_\perp is the eigenvector of K corresponding to q_\perp and $e_{a'}, e_{b'}$ span the eigenspace of K corresponding to the eigenvalues $p_{a'}, p_{b'}$.
 - (III) e_a are eigenvectors of K corresponding to the eigenvalues q_a .

We will call $\{q_a\}, \{e_a\}, \{\theta^a\}$, satisfying the above conditions **adapted**. In the following we will work in a neighbourhood U_0 defined as above and assume that we are given adapted $\{q_a\}, \{e_a\}, \{\theta^a\}$, on U_0 .

The role of α_b^a will be to shift the spectrum of the system matrix to be positive. We need to choose α_0 larger than ϵ to compensate for the fact that the q_a are not the exact eigenvalues of K .

In the following T_{ab} will denote frame components of the tensor T w.r.t. the frame $\{e_a\}$. Define the rescaled frame $\tilde{e}_a = t^{-q_a} e_a$ with dual frame $\tilde{\theta}^a = t^{q_a} \theta^a$ and denote by \tilde{T}_{ab} the \tilde{e}_a frame components of the tensor T . Then we have

$$(5.1) \quad \tilde{T}_{ab} = t^{-q_a - q_b} T_{ab}, \quad \tilde{T}_b^a = t^{q_a - q_b} T_b^a.$$

It follows from the definitions, that in case III,

$$(5.2) \quad K_b^a = \delta_b^a q_b, \quad {}^0k_b^a = -t^{-1} \delta_b^a q_b$$

while in case II, the tensors K , 0k and 0g are block diagonal in the frame $\{e_a\}$,

$$(5.3) \quad K_{\perp}^{a'} = 0, \quad {}^0k_{\perp}^{a'} = 0, \quad {}^0g_{a'\perp} = 0$$

For $s \in \mathbf{R}$, let $(s)_+ = \max(s, 0)$, for $a, b \in \{1, 2, 3\}$, let

$$(5.4) \quad \alpha_b^a = 2(q_b - q_a)_+ + \alpha_0,$$

and $\tilde{\alpha}_b^a = |q_b - q_a| + \alpha_0$. In view of the relation $2(s)_+ - s = |s|$ we have

$$(5.5) \quad \alpha_b^a + q_a - q_b = \tilde{\alpha}_b^a.$$

The following identities are an immediate consequence of equations (5.2) and (5.3), together with the fact that in case I, $q_a = 1/3$, $a = 1, 2, 3$.

$$(5.6a) \quad {}^0g_{ab}t^{\alpha^b c} = {}^0g_{ab}t^{\alpha^a c}$$

$$(5.6b) \quad {}^0g_{ab}t^{q_b} = {}^0g_{ab}t^{q_a}$$

Note that in (5.6) no summation over indices is implied.

Lemma 4.3 implies the following estimate for the rescaled frame components of ${}^0\tilde{g}$.

Lemma 5.1.

$$(5.7) \quad \|{}^0\tilde{g}_{ab}\|_{\infty} \leq Ct^{-\epsilon}, \quad \|{}^0\tilde{g}^{ab}\|_{\infty} \leq Ct^{-\epsilon}$$

Proof. Let $Q_b^a = \delta_b^a q_b$. A direct computation starting from the matrix form of the velocity dominated evolution equation gives

$$t\partial_t \tilde{G}_{ab} = 2\tilde{G}_{ac}(K_b^c - Q_b^c)$$

which has the solution

$$\tilde{G}_{ab}(t) = \tilde{G}_{ac}(t_0) \left(\frac{t}{t_0} \right)^{2(K_b^c - Q_b^c)}$$

From the definition of q_a and the frame e_a the spectrum of $K - Q$ is of the form $p_a - q_a$. Therefore since $|p_a - q_a| < \epsilon/2$, we find that the spectrum of $2(K - Q)$ is contained in the interval $(-\epsilon, \epsilon)$. Therefore it follows from Lemma 4.3 that $\|\tilde{G}\|_{\infty} \leq Ct^{-\epsilon}$.

Similarly, using the fact that \tilde{G}^{-1} satisfies the equation

$$t\partial_t \tilde{G}_{ab}^{-1} = 2(Q_a^c - K_a^c)\tilde{G}_{cb}^{-1}$$

an application of Lemma 4.3 yields the estimate $\|\tilde{G}^{-1}\|_{\infty} \leq Ct^{-\epsilon}$. \square

We are now ready to describe the ansatz which will be used to write the Einstein–scalar field and Einstein–stiff fluid systems in Fuchsian form. Assume that a solution of the velocity dominated constraint and evolution equations, ${}^0g_{ab}$, ${}^0k_{ab}$, with adapted frame, coframe, and auxiliary exponents $\{e_a\}$, $\{\theta^a\}$, $\{q_a\}$, is given. Let α_b^a be defined by (5.4).

Let $\zeta = \epsilon/200$. We will consider metrics and second fundamental forms g, k of the form

$$(5.8a) \quad g_{ab} = {}^0g_{ab} + {}^0g_{ac}t^{\alpha^c_b}\gamma^c_b \quad \gamma^c_b = o(1)$$

$$(5.8b) \quad g^{ab} = {}^0g^{ab} + t^{\alpha^a_b}\bar{\gamma}^a_c {}^0g^{cb} \quad \bar{\gamma}^a_c = o(1)$$

$$(5.8c) \quad e_c(\gamma^a_b) = t^{-\zeta}\lambda^a_{bc} \quad \lambda^a_{bc} = o(1)$$

$$(5.8d) \quad k_{ab} = g_{ac}({}^0k^c_b + t^{-1+\alpha^c_b}\kappa^c_b) \quad \kappa^c_b = o(1)$$

The form (5.8b) is a consequence of (5.8a). To see this, note that ${}^0g^{ac}g_{cb} = \delta^a_b + t^{\alpha^a_b}\gamma^a_b$ and that $g^{ac}g_{cb} = ({}^0g^{ac}g_{cb})^{-1}$. Thus the desired result follows from the matrix identity $(I+A)^{-1} = I - A + (I+A)^{-1}A^2$ and the fact that, using $2(x)_+ - x = |x|$, it can be concluded that

$$(5.9) \quad \alpha^a_e + \alpha^e_b - \alpha^a_b = |q_e - q_a| + |q_b - q_e| - |q_b - q_a| + \alpha_0 \geq \alpha_0.$$

The latter relation shows that each component of the square of γ^a_b vanishes faster than the corresponding component of γ^a_b itself.

Let $\beta = \epsilon/100$. In addition to (5.8) we will use the following ansatz for the scalar field

$$(5.10a) \quad \phi = {}^0\phi + t^\beta\psi \quad \psi = o(1)$$

$$(5.10b) \quad e_a(\psi) = t^{-\zeta}\omega_a \quad \omega_a = o(1)$$

$$(5.10c) \quad t\partial_t\psi + \beta\psi = \chi \quad \chi = o(1)$$

and for the stiff fluid case,

$$(5.11a) \quad \mu = {}^0\mu + t^{-2+\beta_1}\nu \quad \nu = o(1)$$

$$(5.11b) \quad u_a = {}^0u_a + t^{1+\beta_2}v_a \quad v_a = o(1)$$

Equations (5.8), (5.10) and (5.11) with the exception of (5.8c) and (5.10b) will be used to derive the first reduced form of the field equations, and equations (5.8c) and (5.10b) for the spatial derivatives of γ^a_b and ψ , will be used to derive the second reduced system.

Note that in view of (5.1), we have

$$(5.12a) \quad \tilde{g}_{ab} = {}^0\tilde{g}_{ab} + {}^0\tilde{g}_{ac}t^{\tilde{\alpha}^c_b}\gamma^c_b$$

$$(5.12b) \quad \tilde{g}^{ab} = {}^0\tilde{g}^{ab} + t^{\tilde{\alpha}^a_b}\bar{\gamma}^a_c {}^0\tilde{g}^{cb}$$

$$(5.12c) \quad \tilde{k}_{ab} = \tilde{g}_{ac}({}^0\tilde{k}^c_b + t^{-1+\tilde{\alpha}^c_b}\kappa^c_b)$$

We use the following **conventions** throughout:

- indices on velocity dominated fields ${}^0g_{ab}, {}^0k_{ab}, {}^0u_a$ are raised and lowered with ${}^0g_{ab}$, while indices on other tensors are raised and lowered with g_{ab} .
- the dynamic tensor fields γ^a_b, κ^a_b in g_{ab}, k_{ab} are always used in mixed form and only in $\{e_a\}$ frame components.
- the dynamic 1-form v_a in the velocity field u_a is always used with lower index.

5.1. The reduced Einstein–matter system. In this section, we describe the first reduced system for the Einstein–scalar field evolution equations, derived from (2.3) using the ansatz given by equations (5.8) and (5.10) for g_{ab}, k_{ab}, ϕ in terms of the velocity dominated solution ${}^0g_{ab}, {}^0k_{ab}, {}^0\phi$, the auxiliary exponents $\{q_a\}$ and the dynamical fields $\gamma_b^a, \kappa_b^a, \psi$. Similarly, we describe the first reduced system for the Einstein–stiff fluid evolution equations obtained using the equations (5.11). For convenience we use the term ‘Einstein–matter system’ to describe the Einstein–scalar field and Einstein–stiff fluid system collectively.

The tensor g_{ab} of the form (5.8a) is not a priori symmetric, but it will follow from Lemma 5.2 that the solution to the Fuchsian form of the Einstein–matter evolution equations will be symmetric. It is convenient to introduce the symmetrized tensor

$${}^Sg_{ab} = \frac{1}{2}(g_{ab} + g_{ba}).$$

Let ${}^SR_{ab}$ be the Ricci tensor computed w.r.t. the symmetrized metric ${}^Sg_{ab}$, see section 6 for details.

By substituting into the evolution equations (2.3) with R_{ab} replaced by ${}^SR_{ab}$, defined in terms of γ_b^a and λ_{bc}^a , we get the following system for $\gamma_b^a, \kappa_b^a, \lambda_{bc}^a$.

(5.13a)

$$t\partial_t\gamma_b^a + \alpha_b^a\gamma_b^a + 2\kappa_b^a + 2\gamma_e^a(t^0k_b^e) - 2(t^0k_e^a)\gamma_b^e = -2t^{\alpha_e + \alpha_b - \alpha_b}\gamma_e^a\kappa_b^e$$

(5.13b)

$$t\partial_t\lambda_{bc}^a = t^\zeta e_c(t\partial_t\gamma_b^a) + \zeta t^\zeta e_c(\gamma_b^a)$$

(5.13c)

$$t\partial_t\kappa_b^a + \alpha_b^a\kappa_b^a - (t^0k_b^a)(\text{tr}\kappa) = t^{\alpha_0}(\text{tr}\kappa)\kappa_b^a$$

(5.13d) $+ t^{2-\alpha_b}({}^SR_b^a - M_b^a)$

where M_{ab} is given by

$$(5.14a) \quad M_{ab} = 8\pi e_a(\phi)e_b(\phi) \quad \text{for the Einstein–scalar field system}$$

$$(5.14b) \quad M_{ab} = 16\pi\mu u_a u_b \quad \text{for the Einstein–stiff fluid system}$$

M_b^a will be estimated in section 6. Note that the power of t occurring on the right hand side of equation (5.13a) is positive due to (5.9).

The wave equation (2.6) becomes the following system of equations for ψ, ω_a, χ .

$$(5.15a) \quad t\partial_t\psi + \beta\psi - \chi = 0$$

$$(5.15b) \quad t\partial_t\omega_a = t^\zeta[e_a(\chi) + (\zeta - \beta)e_a(\psi)]$$

$$(5.15c) \quad t\partial_t\chi + \beta\chi = t^{\alpha_0 - \beta}\text{tr}\kappa(A + t^\beta\chi) + t^{2-\beta}\Delta^0\phi + t^{2-\zeta}\nabla^a\omega_a$$

Let $\mathcal{U} = (\gamma_b^a, \kappa_b^a, \psi, \chi, \lambda_{bc}^a, \omega_a)$. Then we can write the second reduced system in the Einstein–scalar field case, which consists of equations (5.13)

and (5.15) in the form

$$t\partial_t\mathcal{U} + \mathcal{A}\mathcal{U} = \mathcal{F}(t, x, \mathcal{U}, \mathcal{U}_x).$$

for a matrix \mathcal{A} and a function \mathcal{F} . We will prove that this system is in Fuchsian form.

The Einstein-stiff fluid equations will be treated in a similar way. However, due to the complexity of the equations in that case, they will not be written out more explicitly than is absolutely necessary to understand the essential features of their structure. The second reduced system in the stiff fluid case can be brought into the generalized Fuchsian form

$$(5.16) \quad t\partial_t\mathcal{U} + \mathcal{A}\mathcal{U} = \mathcal{F}(t, x, \mathcal{U}, \mathcal{U}_x) + \mathcal{G}(t, x, \mathcal{U})\partial_t\mathcal{U}$$

already introduced in section 4. In order to do this it is useful to introduce some abbreviations for certain terms in (2.9) so that the equations become

$$(5.17a) \quad \partial_t\mu - 2(\text{tr}k)\mu = -2|u|^2\partial_t\mu - 4\mu u^a\partial_t u_a + F_1$$

$$(5.17b) \quad \begin{aligned} \partial_t u_a + (\text{tr}k)u_a + (1/2)\mu^{-1}e_a(\mu) &= -\mu^{-1}[\partial_t\mu - 2(\text{tr}k)\mu]u_a \\ + (1 + |u|^2)^{-1}u^b u_a \partial_t u_b - [(1 + |u|^2)^{-1/2} - 1]\mu^{-1}e_a(\mu) &+ F_2 \end{aligned}$$

The expressions F_1 and F_2 contain only terms which can be incorporated into \mathcal{F} in (5.16). Next the ansatz (5.11) must be substituted into these equations. The result is:

$$(5.18a) \quad t\partial_t\nu + \beta_1\nu = -2t^{3-\beta_1}(1 + t\text{tr}k)^0\mu + 2(1 + t\text{tr}k)\nu + t^{3-\beta_1}[\partial_t\mu - 2(\text{tr}k)\mu]$$

$$(5.18b) \quad t\partial_t v_a + \beta_2 v_a = -(1 + t\text{tr}k)v_a + t^{-\beta_2}[\partial_t u_a + (\text{tr}k)u_a + (1/2)^0\mu^{-1}e_a(^0\mu)]$$

The expressions on the left hand side of the above form of the Euler equations written in terms of the basic variables μ and u_a occur on the right hand sides of the above evolution equations for ν and v_a . In order to get a fully explicit form it would be necessary to substitute for these expressions and then express the final result in terms of ν and v_a . This is, however, neither necessary nor even helpful for the analysis to be done here.

Next we will consider the matrix \mathcal{A} and prove that \mathcal{A} is a direct sum of a matrix with spectrum bounded from below by a positive number, with a zero matrix. (The arguments in the scalar field and stiff fluid cases are very similar.) It is therefore of a form such that the theory presented in section 4 applies. In addition we must show that $\mathcal{F}(t, x, \mathcal{U}, \mathcal{U}_x) = O(t^\delta)$ for some $\delta > 0$ and, in the stiff fluid case, that $\mathcal{G}(t, x, \mathcal{U})$ satisfies a similar estimate. This will be done in the next section.

The matrix \mathcal{A} is block diagonal and therefore it is enough to consider each block separately. The rows and columns of \mathcal{A} corresponding to λ_{bc}^a, ω_a are zero, and therefore this \mathcal{A} is the direct sum of a matrix corresponding to $\gamma, \kappa, \psi, \chi$, with a zero matrix in the scalar field case and the direct sum of a matrix corresponding to γ, κ, ν, v_a with a zero matrix in the stiff fluid case.

We now consider the spectrum of this matrix. The submatrix corresponding to γ, κ is upper block triangular. The γ, γ block is given by

$$\gamma_b^a \mapsto \alpha_b^a \gamma_b^a + 2[\gamma, t^0 k]_b^a$$

To estimate the spectrum of this, it is necessary to consider the cases I, II, III separately. Working in a frame which diagonalizes ${}^0 k$, $t^0 k_b^a = -\delta_b^a p_b$, and hence in this case

$$2[\gamma, t^0 k]_b^a = -2(p_b - p_a) \gamma_b^a$$

Therefore, in case III, we get using the definition of α_b^a ,

$$\alpha_b^a \gamma_b^a + 2[\gamma, t^0 k]_b^a = 2((p_b - p_a)_+ - (p_b - p_a) + \alpha_0) \gamma_b^a$$

and hence using $(x)_+ - x \geq 0$ for all $x \in \mathbf{R}$, the spectrum of the γ, γ block is bounded from below by α_0 in case III.

Next consider case I. In this case, $\alpha_b^a = \alpha_0 = 4\epsilon$ and the spectrum of $\gamma_b^a \mapsto 2[\gamma, t^0 k]_b^a$ is bounded from below by ϵ , which shows that the spectrum of the γ, γ block is bounded from below by 3ϵ in case II.

Finally, in case II, ${}^0 k_b^a$ is block diagonal in the adapted frame. The spectrum of $\gamma_b^a \mapsto 2[\gamma, t^0 k]_b^a$ consists of $2(p_{a'} - p_{b'})$, $2(p_{a'} - p_\perp)$, and 0. Now using the definition of α_b^a for case II and arguing as above, we get the lower bound 3ϵ for the spectrum of the γ, γ block in case II. Therefore the spectrum of the γ, γ of \mathcal{A} is bounded from below by 3ϵ .

Next we consider the κ, κ block. This is of the form

$$\kappa_b^a \mapsto \alpha_b^a \kappa_b^a - (t^0 k_b^a) \text{tr} \kappa$$

First consider the action on the trace-free part of κ_b^a . Then the spectrum is given by $\alpha_b^a > \alpha_0$. On the other hand, restricting to the trace part of κ_b^a , which is diagonal, we see that the spectrum is $\alpha_0 + 1$. Therefore the spectrum of the κ, κ block is bounded from below by α_0 .

The ψ, χ block is of the form

$$\begin{pmatrix} \beta & -1 \\ 0 & \beta \end{pmatrix}$$

which has spectrum $\beta > 0$. The ν, ν_a block is diagonal with eigenvalues β_1 and β_2 .

Therefore, in view of the facts that $3\epsilon > \beta > 0$, $\beta_1 > 0$ and $\beta_2 > 0$, the desired properties of the spectrum of \mathcal{A} have been verified.

Given a solution $\mathcal{U} = (\gamma_b^a, \kappa_b^a, \psi, \chi, \lambda_{bc}^a, \omega_a)$ of the reduced system for the Einstein–scalar field equations, define g_{ab} , k_{ab} and ϕ by (5.8) and (5.10). Similarly, given a solution $\mathcal{U} = (\gamma_b^a, \kappa_b^a, \nu, \nu_a \lambda_{bc}^a)$ of the reduced system for the Einstein–stiff fluid equations, define $(g_{ab}, k_{ab}, \mu, u_a)$ by (5.8) and (5.11).

If it can be shown that g_{ab} and k_{ab} are symmetric then a solution of the Einstein–scalar field equations is obtained. The next lemma gives sufficient conditions for this to be true.

Lemma 5.2. *Let a solution of the velocity dominated Einstein–matter system be given on $S \times (0, \infty)$ with all eigenvalues of $-t^0 k_b^a$ positive and*

$t\text{tr}^0k = -1$. Let \mathcal{U} be a solution of the reduced system for the Einstein–scalar field or Einstein–stiff fluid system corresponding to the given velocity dominated solution, with $\mathcal{U} = o(1)$. Define (g_{ab}, k_{ab}, ϕ) by (5.8) and (5.10) in the scalar field case, and define $(g_{ab}, k_{ab}, \mu, u_a)$ by (5.8) and (5.11) in the stiff fluid case. Then g_{ab} and k_{ab} are symmetric.

Proof. From the evolution equation for γ^a_b and the definitions of g_{ab} and k_{ab} it follows that $\partial_t g_{ab} = -2k_{ab}$. Similarly an equation close to the usual evolution equation for k^a_b can be recovered from (5.13d). It differs from the usual one only in the fact that R^a_b is replaced by ${}^S R^a_b$. From these equations we can derive the equations:

$$(5.19a) \quad \partial_t(g_{ab} - g_{ba}) = -2(k_{ab} - k_{ba})$$

$$(5.19b) \quad \partial_t(k_{ab} - k_{ba}) = (\text{tr}k)(k_{ab} - k_{ba})$$

It follows from the assumptions on γ^a_b and κ^a_b together with the definition of k_{ab} , that the components of $k_{ab} - k_{ba}$ are $o(t^{-1+\eta})$ for some $\eta > 0$. Hence the quantity $\Omega_{ab} = t^{1-\eta}(k_{ab} - k_{ba})$ tends to zero as $t \rightarrow 0$. It satisfies the equation:

$$(5.20) \quad t\partial_t\Omega_{ab} + \eta\Omega_{ab} = (t\text{tr}k + 1)\Omega_{ab}$$

From Theorem 4.2 we conclude that $\Omega_{ab} = 0$. Thus k_{ab} is symmetric. It then follows immediately from (5.19a) and the fact that $g_{ab} = o(1)$ that g_{ab} is also symmetric. \square

6. CURVATURE ESTIMATES

Let ${}^S R_{ab}$ be the Ricci tensor computed w.r.t. the symmetrized metric ${}^S g_{ab} = \frac{1}{2}(g_{ab} + g_{ba})$. In order to get a Fuchsian form for the Einstein–matter evolution equations, we need the following estimate for the frame components of ${}^S R^a_b$,

$$(6.1) \quad t^{2-\alpha^a_b} {}^S R^a_b = O(t^\delta), \quad \text{for some } \delta \in (0, \epsilon).$$

In doing the estimates we will use the notion of comparing the size of functions introduced in Definition 4.1. In proving that the second reduced system

$$t\partial_t u + A(x)u = f(t, x, u, u_x)$$

is in Fuchsian form, one essential step is to prove an estimate of the form

$$f \preceq t^\delta$$

for some $\delta > 0$. In the present section, we accomplish this task for the expression $t^{2-\alpha^a_b} {}^S R^a_b$, which is now considered as a function $r(t, x, v(x, t), u, u_x)$, where $v(x, t)$ is defined in terms of the solution ${}^0 g_{ab}, {}^0 k_{ab}$ to the velocity dominated system and the data $\{e_a\}, \{\theta^a\}, \{q_a\}$, etc. defined in section 5, and u consists of the variables $\gamma^a_b, \lambda^a_{bc}$. In terms of the relation \preceq , the goal is to prove

$$(6.2) \quad t^{2-\alpha^a_b} {}^S R^a_b \preceq t^\delta, \quad \text{for some } \delta \in (0, \epsilon).$$

By assumption, $\alpha_0 = 4\epsilon$, so $\alpha_b^a - 2\epsilon = 2(q_b - q_a)_+ + \alpha_0/2$. Therefore, the arguments that apply to α_b^a also apply to $\alpha_b^a - 2\epsilon$.

The symmetrized metric tensor satisfies

$$(6.3a) \quad {}^S\tilde{g}_{ab} = {}^0\tilde{g}_{ab} + t^{\tilde{\alpha}_b^a - 2\epsilon} {}^0\tilde{g}_{ac} {}^S\gamma_b^c, \quad {}^S\gamma_b^c = o(1),$$

$$(6.3b) \quad {}^S\tilde{g}^{ab} = {}^0\tilde{g}^{ab} + t^{\tilde{\alpha}_b^a - 2\epsilon} {}^S\tilde{\gamma}_c^a {}^0\tilde{g}^{cb}, \quad {}^S\tilde{\gamma}_c^a = o(1)$$

To see this, note the identity

$${}^S\gamma_b^c = \frac{1}{2}t^{2\epsilon} \left(\gamma_b^c + {}^0\tilde{g}^{cd} {}^0\tilde{g}_{bf} \gamma_d^f \right),$$

which in view of Lemma 5.1 shows that ${}^S\gamma_b^c = o(1)$. The argument that ${}^S\tilde{\gamma}_c^a = o(1)$ is the same as for $\tilde{\gamma}_c^a = o(1)$.

It is convenient to estimate the rescaled frame components. Note $R_b^a = t^{-q_a + q_b} \tilde{R}_b^a$. Hence in view of (5.5) we need to consider

$$t^{2 - \tilde{\alpha}_b^a} \tilde{R}_b^a$$

Using Lemma 5.1 and (5.8) gives

$$\|t^{2 - \tilde{\alpha}_b^a} \tilde{R}_b^a\|_\infty \leq Ct^{-\epsilon} \|t^{2 - \tilde{\alpha}_b^a} \tilde{R}_{ab}\|_\infty$$

To see this, we compute using (5.6a) and (5.12b)

$$\begin{aligned} \|t^{2 - \tilde{\alpha}_b^a} \tilde{R}_b^a\|_\infty &= \|t^{2 - \tilde{\alpha}_b^a} \tilde{g}^{ac} \tilde{R}_{cb}\|_\infty \\ &\leq \|t^{2 - \tilde{\alpha}_b^a} {}^0\tilde{g}^{ac} \tilde{R}_{cb}\|_\infty + \|t^{2 - \tilde{\alpha}_b^a + \tilde{\alpha}_c^a} \tilde{\gamma}_d^a {}^0\tilde{g}^{dc} \tilde{R}_{cb}\|_\infty \\ &\leq C \|t^{2 - \epsilon - \tilde{\alpha}_b^c} \tilde{R}_{cb}\|_\infty \end{aligned}$$

where we used the triangle inequality in the form $-\tilde{\alpha}_b^a + \tilde{\alpha}_c^a \geq -\tilde{\alpha}_b^c$.

Let $\gamma_{ab}^c = \theta^c([e_a, e_b])$ be the structure coefficients of the frame $\{e_a\}$. The structure coefficients $\tilde{\gamma}_{ab}^c = \tilde{\theta}^c([\tilde{e}_a, \tilde{e}_b])$ of the frame \tilde{e}_a are given by

$$(6.4) \quad \tilde{\gamma}_{ab}^c = t^{q_c - q_a - q_b} \gamma_{ab}^c - \log(t) (t^{-q_a} e_a(q_b) \delta_b^c - t^{-q_b} e_b(q_a) \delta_a^c)$$

It is convenient to define $\tilde{\Gamma}_{abc} = \langle \nabla_{\tilde{e}_a} \tilde{e}_b, \tilde{e}_c \rangle$. Then $\tilde{\Gamma}_{abc}$ is given in terms of \tilde{g}_{ab} by

$$(6.5) \quad \begin{aligned} 2\tilde{\Gamma}_{abc} &= \tilde{e}_a(\tilde{g}_{bc}) + \tilde{e}_b(\tilde{g}_{ac}) - \tilde{e}_c(\tilde{g}_{ab}) \\ &\quad + \tilde{\gamma}_{ab}^d \tilde{g}_{cd} - \tilde{\gamma}_{ac}^d \tilde{g}_{bd} - \tilde{\gamma}_{bc}^d \tilde{g}_{ad} \end{aligned}$$

and \tilde{R}_{dcab} is given in terms of $\tilde{\Gamma}_{abc}$ and $\tilde{\gamma}_{bc}^a$ by

$$(6.6) \quad \begin{aligned} \tilde{R}_{dcab} &= \tilde{e}_a \tilde{\Gamma}_{bcd} - \tilde{e}_b \tilde{\Gamma}_{acd} - \tilde{\gamma}_{ab}^f \tilde{\Gamma}_{fcd} \\ &\quad - \tilde{g}^{fg} \tilde{\Gamma}_{bcf} \tilde{\Gamma}_{adg} + \tilde{g}^{fg} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{bdg} \\ &= (R1)_{dcab} - (R2)_{dcab} - (R3)_{dcab} - (R4)_{dcab} + (R5)_{dcab} \end{aligned}$$

Let

$$z_{ab} = \begin{cases} 0, & \text{if } a = b \\ 1, & \text{if } a \neq b \end{cases}$$

Define

$$(6.7) \quad \begin{aligned} Z_{abc}(t) &= t^{-q_a} + t^{-q_b} + t^{-q_c} \\ &+ t^{q_a - q_b - q_c} z_{bc} + t^{q_b - q_c - q_a} z_{ca} + t^{q_c - q_a - q_b} z_{ab} \end{aligned}$$

Note that $(R1)_{dcab}, \dots, (R5)_{dcab}, z_{ab}, Z_{abc}$ are not tensors. In the rest of this section, the frame $\{e_a\}$ is fixed and the estimates will be done for tensor components in this frame.

We will use the following lemma as the starting point of the estimates in this section.

Lemma 6.1.

$$(6.8a) \quad \tilde{g}_{ab} \preceq t^{-\epsilon}$$

$$(6.8b) \quad \tilde{g}^{ab} \preceq t^{-\epsilon}$$

$$(6.8c) \quad \tilde{e}_a \tilde{g}_{bc} \preceq t^{-q_a - 2\epsilon}$$

$$(6.8d) \quad \tilde{e}_a \tilde{e}_b \tilde{g}_{cd} \preceq t^{-q_a - q_b - 3\epsilon}$$

$$(6.8e) \quad \tilde{g}^{ab} t^{-q_b} \preceq t^{-\epsilon - q_a}$$

$$(6.8f) \quad \tilde{e}_c (t^{q_a} \tilde{g}_{ab}) \preceq t^{-q_c + q_b - 2\epsilon}$$

$$(6.8g) \quad \tilde{g}^{ab} Z_{abc} \preceq t^{-\epsilon} (t^{-q_h} + t^{-q_c} + t^{q_c - 2q_h}) \quad h = \min(a, b)$$

$$(6.8h) \quad \tilde{\gamma}_{ab}^c \tilde{g}^{bd} \preceq t^{-2\epsilon} (t^{q_c - q_a - q_d} + t^{-q_a} + t^{-q_d})$$

Proof. The inequalities (6.8a) and (6.8b) are immediate from Lemma 5.1, (5.12) and definition 4.1. Recalling that the variables u, u_x occurring in the second reduced system contain $\gamma_b^a, \lambda_{bc}^a$ and its first order derivatives gives (6.8c) and (6.8d). (Here the inequality $\zeta < \epsilon$ has been used.)

The estimate (6.8e) follows from Lemma 5.1, (5.6) and (5.12) together with the triangle inequality in the form $-q_b \leq -q_a + |q_a - q_b|$. The estimate (6.8f) follows from (5.6) and (5.12) together with the observation that since ${}^0g_{ab}$ is block diagonal for $x \in U_0$, $e_c {}^0g_{ab}$ is also block diagonal. The estimates (6.8g) and (6.8h) follow in a similar way starting from (6.7) and (6.4). In (6.8h) a $\log(t)$ term is dominated by $t^{-\epsilon}$. \square

The following lemma gives estimates of $\tilde{\Gamma}_{abc}$ in terms of Z_{abc}

Lemma 6.2.

$$(6.9a) \quad \tilde{\Gamma}_{abc} \preceq t^{-2\epsilon} Z_{abc}$$

$$(6.9b) \quad \tilde{\Gamma}_{abb} \preceq t^{-2\epsilon - q_a}$$

$$(6.9c) \quad \tilde{\Gamma}_{aab} \preceq t^{-2\epsilon} (t^{-q_a} + t^{-q_b})$$

$$(6.9d) \quad \tilde{\Gamma}_{aba} \preceq t^{-2\epsilon} (t^{-q_a} + t^{-q_b})$$

$$(6.9e) \quad \tilde{e}_a \tilde{\Gamma}_{bcd} \preceq t^{-3\epsilon - q_a} Z_{bcd}$$

Proof. First observe that $\gamma_{bc}^f \preceq z_{bc}$. From (6.4) and (6.8) we have

$$\tilde{\gamma}_{bc}^f \tilde{g}_{fa} \preceq t^{-2\epsilon} (t^{q_a - q_b - q_c} z_{bc} + t^{-q_b} + t^{-q_c})$$

Now noting $\tilde{e}_a(\tilde{g}_{bc}) \preceq t^{-q_a-2\epsilon}$, (6.9a) follows.

To estimate $\tilde{\Gamma}_{abb}$ we compute

$$\tilde{\Gamma}_{abb} = \frac{1}{2}t^{-q_a}e_a\tilde{g}_{bb} \preceq t^{-2\epsilon-q_a}$$

The estimates for $\tilde{\Gamma}_{aab}, \tilde{\Gamma}_{aba}$ follow directly from (6.9a) and the definition of Z_{abc} .

Finally we consider (6.9e). Expanding out $\tilde{e}_a\tilde{\Gamma}_{bcd}$, we see that it contains (up to permutations of the indices) terms of the form $\tilde{e}_a\tilde{e}_b(\tilde{g}_{cd})$, $(\tilde{e}_a\tilde{\gamma}_{cd}^f)\tilde{g}_{fb}$ and $\tilde{\gamma}_{cd}^f\tilde{e}_a\tilde{g}_{fb}$. The first and second type of terms are estimated using (6.8d) and (6.8e), using the form of $\tilde{\gamma}_{cd}^f$. Finally, the third type of term is estimated using (6.8f). \square

An important consequence of the Kasner relation $\sum_a q_a = 1$, is

$$(6.10) \quad 2 + 2(q_1 - q_2 - q_3) = 4q_1$$

which implies

$$(6.11) \quad t^{2+2(q_j-q_k-q_l)} \preceq t^{4q_1} \quad \text{if at least one of } k, l \text{ is different from } 3,$$

The strategy will be to eliminate as much as possible the occurrence of repeated negative exponents, in order to be able to use this relation.

We make note of the following useful relations.

$$(6.12a) \quad Z_{abc} \preceq t^{q_1-q_2-q_3}$$

$$(6.12b) \quad Z_{hhc} \preceq t^{-q_h} + t^{-q_c}$$

$$(6.12c) \quad e_a Z_{bcd} \preceq t^{-\epsilon} Z_{bcd}$$

The estimates (6.12a) and (6.12b) are immediate from (6.7).

For the rest of this section, we will assume that g_{ab} is symmetric and is of the form given by (5.8a). Let R_{ab} be the Ricci tensor defined with respect to g_{ab} . Under these assumptions we will estimate R_b^a . The estimate then applies after a small modification to ${}^S R_b^a$.

We now proceed to estimate the rescaled components of the Ricci tensor $\tilde{R}_{ad} = \tilde{g}^{bc}\tilde{R}_{dcab}$. Corresponding to the terms (R1), ..., (R5) we have

$$\tilde{R}_{ad} = (\text{Ric1})_{ad} - (\text{Ric2})_{ad} - (\text{Ric3})_{ad} - (\text{Ric4})_{ad} + (\text{Ric5})_{ad}$$

We make the following simplifying observations.

- By the symmetry $\tilde{R}_{ad} = \tilde{R}_{da}$ we can assume without loss of generality that $a \leq d$.
- \tilde{R}_{dcab} is skew symmetric in the first and second pair of indices, therefore we can assume without loss of generality that $c \neq d, b \neq a$.

Therefore, in the following, we can without loss of generality use the following **convention**: The indices a, b, c, d satisfy the relations

$$(6.13) \quad a \leq d, \quad c \neq d, \quad a \neq b$$

We will now estimate \tilde{R}_{ad} by considering each term $(\text{Ric1})_{ad}, \dots, (\text{Ric5})_{ad}$ in turn. The estimate we will actually prove is of the form

$$t^{2+q_a-q_d}\tilde{R}_{ad} \lesssim t^{4q_1-6\epsilon}$$

which will imply the needed estimate for ${}^S R^a_d$.

6.1. (Ric1).

$$(\text{Ric1})_{ad} = \tilde{g}^{bc}\tilde{e}_a\tilde{\Gamma}_{bcd}$$

where we are summing over repeated indices. Let $F = t^{2+q_a-q_d}(\text{Ric1})_{ad}$.

Using Lemma 6.2 and (6.8g), we have

$$\begin{aligned} F &\lesssim t^{2-4\epsilon}t^{q_a-q_d}t^{-q_a}(t^{-q_c} + t^{-q_d} + t^{q_d-2q_c}) \\ &\lesssim t^{2-4\epsilon}(t^{-q_d-q_c} + t^{-2q_d} + t^{-2q_c}) \\ &\lesssim t^{4q_1-4\epsilon} \lesssim t^{4q_1-6\epsilon} \end{aligned}$$

6.2. (Ric2).

$$(\text{Ric2})_{ad} = \tilde{g}^{bc}\tilde{e}_b\tilde{\Gamma}_{acd}$$

Let $F = t^{2+q_a-q_d}(\text{Ric2})_{ad}$. By Lemma 6.2 and (6.8e),

$$F \lesssim t^{2-4\epsilon}t^{q_a-q_d-q_c}Z_{acd}$$

By (6.13), $d \neq c$ and using (6.12a) and (6.11) this gives

$$F \lesssim t^{4q_1-4\epsilon} \lesssim t^{4q_1-6\epsilon}$$

which is the required estimate.

6.3. (Ric3).

$$(\text{Ric3})_{ad} = \tilde{g}^{bc}\tilde{\gamma}_{ab}^f\tilde{\Gamma}_{fcd}$$

Let $F = t^{2+q_a-q_d}(\text{Ric3})_{ad}$. We estimate using (6.8h) and Lemma 6.2,

$$\begin{aligned} F &\lesssim t^{2-4\epsilon}t^{q_a-q_d}(t^{q_f-q_a-q_c} + t^{-q_a} + t^{-q_c})Z_{fcd} \\ &\lesssim t^{2-4\epsilon}(t^{q_f-q_d-q_c} + t^{-q_d} + t^{q_a-q_d-q_c})Z_{fcd} \end{aligned}$$

use $c \neq d$ by (6.13), (6.12a) and (6.11)

$$\lesssim t^{4q_1-4\epsilon} \lesssim t^{4q_1-6\epsilon}$$

6.4. (Ric4).

$$(\text{Ric4})_{ad} = \tilde{g}^{bc}\tilde{g}^{fg}\tilde{\Gamma}_{bcf}\tilde{\Gamma}_{adg}$$

Let $F = t^{2+q_a-q_d}(\text{Ric4})_{ad}$. We have by Lemma 6.2,

$$F \lesssim t^{2+q_a-q_d-4\epsilon}\tilde{g}^{bc}\tilde{g}^{fg}Z_{bcf}Z_{adg}$$

In case $a = d$ this gives, with $h = \min(b, c)$, $m \in \{f, g\}$, using (6.12) and (6.8),

$$\begin{aligned} t^2(\text{Ric4})_{aa} &\lesssim t^{2-6\epsilon}(t^{-q_h} + t^{-q_m} + t^{q_m-2q_h})(t^{-q_a} + t^{-q_m}) \\ &\lesssim t^{4q_1-6\epsilon} \end{aligned}$$

Next we consider the case $a < d$. In case $g = d$, Lemma 6.2 together with (6.8e) and (6.8g) gives

$$\begin{aligned} F &\preceq t^{2+q_a-q_d} \tilde{g}^{bc} \tilde{g}^{fd} \tilde{\Gamma}_{bcf} t^{-2\epsilon-q_a} \\ &\preceq t^{2-6\epsilon} (t^{-q_f-q_h} + t^{-2q_f} + t^{-2q_h}) \quad h = \min(b, c) \\ &\preceq t^{4q_1-6\epsilon} \end{aligned}$$

In case $a = g$ we have arguing as above

$$\begin{aligned} F &\preceq t^{2+q_a-q_d} \tilde{g}^{bc} \tilde{g}^{fa} \tilde{\Gamma}_{bcf} t^{-2\epsilon} (t^{-q_a} + t^{-q_d}) \\ &\preceq t^{2-5\epsilon} \tilde{g}^{fa} (t^{-q_h} + t^{-q_f} + t^{q_f-2q_h}) (t^{-q_d} + t^{q_a-2q_d}), \quad h = \min(b, c) \\ &\preceq t^{2-6\epsilon} (t^{-q_h} + t^{-q_m} + t^{q_m-2q_h}) (t^{-q_d} + t^{q_m-2q_d}), \quad m = \min(a, f) \end{aligned}$$

From $h = \min(b, c)$ and $c \neq d$ which holds by (6.13), we find that either $h < 3$ or $d < 3$ must hold. Using this it follows using (6.11) that in case $a = g < d$, $F \preceq t^{4q_1-6\epsilon}$.

It remains to consider the case when $a < d$ and a, d, g are distinct. In this case, the estimates used above give

$$\begin{aligned} F &\preceq t^{2+q_a-q_d-6\epsilon} (t^{-q_h} + t^{-q_g} + t^{q_g-2q_h}) Z_{adg}, \quad h = \min(b, c) \\ &\preceq t^{2-6\epsilon} (t^{-q_h} + t^{-q_g} + t^{q_g-2q_h}) \\ &\quad (t^{-q_d} + t^{q_a-2q_d} + t^{q_a-q_d-q_g} + t^{2q_a-2q_d-q_g} + t^{-q_g} + t^{q_g-2q_d}) \end{aligned}$$

By construction, $h < 3$ or $d < 3$ must hold, which in conjunction with the fact that in the present case, $g \neq d$ gives using (6.11),

$$F \preceq t^{4q_1-6\epsilon}$$

The above proves that the required estimate $t^{2+q_a-q_d}(\text{Ric}4)_{ad} \preceq t^{4q_1-6\epsilon}$ holds.

6.5. (Ric5).

$$(\text{Ric}5)_{ad} = \tilde{g}^{bc} \tilde{g}^{fg} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{bdg}$$

The estimate for $(\text{Ric}5)_{ad}$ is the most complicated, and will be done in several steps. We review the steps which will be used here. In each step the conditions on the indices a, c, f, b, d, g which leads to the required estimate may be excluded from our considerations. Recall that $a \leq d$ may be assumed and also note by (6.13) we may assume without loss of generality that $a \neq b$, $c \neq d$.

The steps we will use are:

1. $a = d$ can be excluded, so $a < d$ may be assumed.
2. $g = d$ can be excluded, so $g \neq d$ may be assumed.
3. $g = a$ can be excluded, so $g \neq a$ may be assumed.
4. $b = d$ can be excluded, so $b \neq d$ may be assumed.
5. $a = c$ can be excluded, so $a \neq c$ may be assumed.

When all the above claims are verified, we may restrict our considerations to the indices satisfying the conditions

$$(6.14) \quad a < d, \quad a \neq b, \quad c \neq d, \quad g \neq d, \quad g \neq a, \quad b \neq d, \quad a \neq c$$

These conditions imply that the indices $\{a, d, g\}$, $\{a, d, c\}$ and $\{a, d, b\}$ are distinct, so (6.14) implies $g = b = c$, as all indices take values in $\{1, 2, 3\}$. Therefore the required estimate for $(\text{Ric5})_{ad}$ will hold if we can verify that it holds under (6.14) in conjunction with the condition $g = b = c$, which is the final step.

Let

$$F = t^{2+q_a-q_d}(\text{Ric5})_{ad}.$$

Case $a = d$: In case $a = d$, Lemma 6.2 and (6.12a) give

$$\begin{aligned} F &\preceq t^{2-6\epsilon} t^{2(q_1-q_2-q_3)} \\ &\preceq t^{4q_1-6\epsilon} \end{aligned}$$

Therefore we may assume $a < d$ in the following. Further by (6.13), $a \neq b$ and $c \neq d$.

Case $g = d$: Next consider the case $g = d$. Then using Lemma 6.2, (6.7) and (6.8e) we have

$$\begin{aligned} F &= t^{2+q_a-q_d} \tilde{g}^{bc} \tilde{g}^{fd} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{bdd} \\ &\preceq t^{2+q_a-q_d} \tilde{g}^{bc} \tilde{g}^{fd} \tilde{\Gamma}_{acf} t^{-2\epsilon-q_b} \\ &\preceq t^{2-6\epsilon+q_a-q_d-q_c} (t^{-q_a} + t^{-q_c} + t^{-q_d} + t^{q_a-q_c-q_d} + t^{q_c-q_d-q_a} + t^{q_d-q_a-q_c}) \\ &\preceq t^{2-6\epsilon} (t^{-q_d-q_c} + t^{q_a-q_d-2q_c} \\ &\quad + t^{q_a-2q_d-q_c} + t^{2(q_a-q_d-q_c)} + t^{-2q_d} + t^{-2q_c}). \end{aligned}$$

By (6.13), $d \neq c$, this gives $F \preceq t^{4q_1-6\epsilon}$ in the case $g = d$.

Case $g = a$: Next consider the case $g = a$. Then we have

$$F = t^{2+q_a-q_d} \tilde{g}^{bc} \tilde{g}^{fa} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{bda}$$

use Lemma 6.2 and (6.8g)

$$\preceq t^{2-5\epsilon} t^{q_a-q_d} \tilde{g}^{bc} (t^{-q_a} + t^{-q_c} + t^{q_c-2q_a}) Z_{bda}$$

use Lemma 5.1

$$\preceq t^{2-6\epsilon} (t^{-q_d} + t^{q_a-q_d-q_c} + t^{q_c-q_a-q_d}) Z_{bda}$$

use (6.12a) and $a < d$, $d \neq c$

$$\preceq t^{4q_1-6\epsilon}$$

At this stage we may assume

$$(6.15) \quad a < d, \quad a \neq b, \quad c \neq d, \quad g \neq d, \quad g \neq a$$

Case $b = d$: Next consider the case $b = d$. In this case

$$\begin{aligned} F &= t^{2+q_a-q_d} \tilde{g}^{dc} \tilde{g}^{fg} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{ddg} \\ &\preceq t^{2+q_a-q_d} \tilde{g}^{dc} \tilde{g}^{fg} \tilde{\Gamma}_{acf} t^{-2\epsilon} (t^{-q_d} + t^{-q_g}) \\ &\preceq t^{2-6\epsilon} (t^{q_a-q_d-q_c} + t^{q_a-q_d-q_g}) Z_{acf} \end{aligned}$$

use $d \neq c$ and $d \neq g$ from (6.15)

$$\preceq t^{4q_1-6\epsilon}$$

Case $a = c$: Next consider the case $a = c$. In this case we have using Lemma 6.2

$$F \preceq t^{2+q_a-q_d} \tilde{g}^{ba} \tilde{g}^{fg} t^{-2\epsilon} (t^{-q_a} + t^{-q_f}) t^{-2\epsilon} Z_{bdg}$$

use (6.8e)

$$\preceq t^{2-6\epsilon} (t^{-q_d} + t^{q_a-q_d-q_g}) Z_{bdg}$$

use $g \neq d$ from (6.15) and (6.12a)

$$\preceq t^{4q_1-6\epsilon}$$

At this stage we may assume

$$(6.16) \quad a < d, \quad a \neq b, \quad c \neq d, \quad g \neq d, \quad g \neq a, \quad b \neq d, \quad a \neq c$$

As discussed above, if we can prove that the required estimate holds under the condition $g = c = b$ we are done.

Case $g = b = c$: Next consider the case $g = b = c$. In this case, after making the substitutions $g = c$ and $b = c$,

$$F = t^{2+q_a-q_d} \tilde{g}^{cc} \tilde{g}^{fc} \tilde{\Gamma}_{acf} \tilde{\Gamma}_{cdc}$$

use Lemma 6.2

$$(6.17) \quad \preceq t^{2-3\epsilon} t^{q_a-q_d} \tilde{g}^{fc} \tilde{\Gamma}_{acf} (t^{-q_d} + t^{-q_c})$$

To estimate F we must now consider the cases $f = d$, $f = c$, $f = a$ separately.

Case $g = b = c$ and $f = d$: In case $g = b = c$ and $f = d$, we have from (6.17)

$$F \preceq t^{2-3\epsilon} t^{q_a-q_d} \tilde{g}^{dc} \tilde{\Gamma}_{acd} (t^{-q_d} + t^{-q_c})$$

use (6.8e) and Lemma 6.2

$$\lesssim t^{2-6\epsilon} t^{q_a - q_d - q_c} Z_{acd}$$

use $c \neq d$ from (6.16) and (6.12a)

$$\lesssim t^{4q_1 - 6\epsilon}$$

Therefore we may exclude condition (6.16) in conjunction with $f = d$ from our considerations.

Case $g = b = c$ and $f = c$: In case $g = b = c$ and $f = c$ we have from (6.17)

$$F \lesssim t^{2-3\epsilon} t^{q_a - q_d} \tilde{g}^{cc} \tilde{\Gamma}_{acc} (t^{-q_d} + t^{-q_c})$$

use Lemma 6.2

$$\begin{aligned} &\lesssim t^{2-4\epsilon} t^{q_a - q_d} t^{-2\epsilon - q_a} (t^{-q_d} + t^{-q_c}) \\ &= t^{2-6\epsilon} (t^{-2q_d} + t^{-q_d - q_c}) \\ &\lesssim t^{4q_1 - 6\epsilon} \end{aligned}$$

Case $g = b = c$ and $f = a$: The only remaining case is $g = b = c$ and $f = a$. In this case we get from (6.17) using Lemma 6.2

$$F \lesssim t^{2-5\epsilon} t^{q_a - q_d} \tilde{g}^{ac} Z_{aca} (t^{-q_d} + t^{-q_c})$$

use (6.12b) and (6.8g)

$$\begin{aligned} &\lesssim t^{2-6\epsilon} (t^{q_a - 2q_d} + t^{q_a - q_d - q_c}) t^{-q_a} \\ &\lesssim t^{2-6\epsilon} (t^{-2q_d} + t^{-q_d - q_c}) \\ &\lesssim t^{4q_1 - 6\epsilon} \end{aligned}$$

Therefore it now follows that under (6.16), the required estimate

$$F \lesssim t^{4q_1 - 6\epsilon}$$

holds and hence by the above argument it follows that this estimate holds under (6.13).

This proves for a symmetric metric satisfying (5.8) the estimate

$$(6.18) \quad t^{2-\alpha^a_b} R^a_b \lesssim t^{4q_1 - 6\epsilon}$$

We wish to apply this to the symmetrized metric ${}^S g_{ab}$, which has the property that the rescaled symmetrized metric ${}^S \tilde{g}_{ab}$ satisfies (6.3). The estimate (6.18) translates to an estimate for a metric satisfying (6.3) after replacing α_0 by $\alpha_0 - 2\epsilon$, which by the definition of α_0 satisfies $\alpha_0 - 2\epsilon > \epsilon > 0$. Therefore we get in view of the discussion at the beginning of this section

$$t^{2-\tilde{\alpha}^a_d} {}^S \tilde{R}^a_d \lesssim t^{4p_1 - 9\epsilon - \alpha_0}$$

or

$$t^{2-\alpha_a} S R_d^a \lesssim t^{4p_1-9\epsilon-\alpha_0}$$

Now recall $\epsilon = \alpha_0/4 = \min_a \{p_a(x_0)\}/40$ and $q_1 > 20\epsilon$ by construction. This gives

$$\begin{aligned} t^{2-\alpha_a} S R_d^a &\lesssim t^{4q_1-13\epsilon} \\ &\lesssim t^{3q_1} \\ &\lesssim t^{3p_1} \end{aligned}$$

where we used that fact that $q_1 \geq p_1$ by construction.

This finishes the proof of

Lemma 6.3 (Curvature estimate).

$$t^{2-\alpha_a} S R_b^a \lesssim t^{3p_1}$$

□

Now some estimates will be obtained for matter variables. These will be used to check that the right hand side of the second reduced system has the properties required for a Fuchsian system. Let w_a be a one-form with the property that $w_a \lesssim t^{q_a}$. We have

$$\begin{aligned} t^2 \tilde{g}^{ab} \nabla_a w_b &= t^2 \tilde{g}^{ab} \tilde{e}_a w_a - t^2 \tilde{g}^{ab} \tilde{g}^f g \tilde{\Gamma}_{abf} w_g \\ &\lesssim t^{2-\epsilon-2q_a} + t^{2-4\epsilon} t^{q_1+q_2-q_3} t^{-q_g} \\ &\lesssim t^{4q_1-4\epsilon} \lesssim t^{4p_1-4\epsilon} \end{aligned}$$

Here it has been assumed that w_a behaves in a suitable way upon taking derivatives. In the context of the matter variables this will be the case for the relevant choices of w_a , namely $e_a(\phi)$ and $t^{-1}u_a$. Note that this estimate requires no use of cancellations, since it only uses the relation (6.12a) and not (6.8g). In this situation the estimate for a given quantity is never more difficult than that for the corresponding velocity dominated part, since the difference between the two is always of higher order. This gives the estimates required for the matter equations in the scalar field case. For the stiff fluid some more work is needed.

The aim now is to estimate the terms on the right hand side of equations (5.18) by a positive power of t . For most of these terms no cancellations are required to get the desired estimate. There are only two exceptions to this and they will be discussed explicitly now. The first is the following combination which arises if the evolution equation for v_a is written out explicitly:

$$(6.19) \quad {}^0\mu^{-1} e_a({}^0\mu) - \mu^{-1} e_a(\mu)$$

This expression is equal to

$$(6.20) \quad t^{\beta_1} [-A^{-1}\nu(1 + A^{-1}\nu t^{\beta_1})^{-1} A^{-1}(\nabla_a A + \nabla_a \nu) + A^{-1} t^{\beta_1} \nabla_a \nu]$$

which is $O(t^{\beta_1})$. The contribution of this expression to the right hand side of the evolution equation for v_a is as a consequence $O(t^{\beta_1-\beta_2})$ which shows that it is necessary to choose $\beta_2 < \beta_1$. The second expression where a cancellation is necessary is $1 + t\text{tr}k$. Now $\text{tr}k = -t^{-1} + \text{tr}\kappa t^{-1+\alpha_0}$ and hence $1 + t\text{tr}k = \text{tr}\kappa t^{\alpha_0}$. It follows that this expression is $O(t^{\alpha_0})$.

The analysis of the other terms is rather straightforward, although lengthy, and will not be carried out explicitly here. However some comments may be useful. The terms which are a priori most difficult to estimate are those involving covariant derivatives of u_a . For those it is convenient to use the components in the rescaled frame \tilde{e}_a . For all other terms the original frame e_a can be used straightforwardly. In order that all terms can be estimated by a positive power of t it suffices to choose β_1 and β_2 small enough. One possible choice is $\beta_2 < \beta_1 < q_1 - 5\epsilon$.

In order to show that the reduced systems for the Einstein–scalar field system and the Einstein–stiff fluid systems, are in Fuchsian form, we need to show that $t^{2-\alpha^a_b}M^a_b = o(t^\delta)$ for some $\delta > 0$.

We consider first the Einstein–scalar field case. In this case,

$$M^a_b = g^{ac}e_c(\phi)e_b(\phi)$$

Arguing as above for the estimate of $t^{2-\alpha^a_b}{}^S R^a_b$, we have

$$t^{2-\alpha^a_b}M^a_b \lesssim t^{2-\tilde{\alpha}^a_b-\epsilon}\tilde{M}_{ab}$$

Therefore it is enough to show

$$t^{2-\tilde{\alpha}^a_b}\tilde{M}_{ab} \lesssim t^{2\epsilon}$$

By definition $\tilde{M}_{ab} = t^{-q_a-q_b}e_a(\phi)e_b(\phi)$ and hence

$$\tilde{M}_{ab} \lesssim t^{-q_a-q_b}$$

This give using $\tilde{\alpha}^a_b = |q_a - q_b|$,

$$t^{2-\tilde{\alpha}^a_b}M^a_b \lesssim t^{3q_1}$$

For the scalar field case this gives, together with the above,

$$t^{2-\alpha^a_b}({}^S R^a_b - M^a_b) \lesssim t^\delta, \quad \text{for some } \delta > 0$$

which is the estimate required for proving that the second reduced system for the Einstein–scalar field system is in Fuchsian form. The argument for the Einstein–stiff fluid system is very similar.

7. THE CONSTRAINTS

The main aim of this section is to show that if a solution of the evolution equations is given which corresponds to a solution of the velocity dominated equations as in Theorem 2.1 or 2.2 then it satisfies the full constraints. It will also be shown how the existence of a large class of solutions of the velocity dominated constraints can be demonstrated.

The first result on the propagation of the constraints relies on rough computations which prove the result in the case where all p_a are close to $1/3$. An analytic continuation argument then gives the general case.

Lemma 7.1. *Let $({}^0g_{ab}, {}^0k_{ab}, {}^0\phi)$ be a solution of the velocity dominated system as in the hypotheses of Theorem 2.1 with $|p_a - 1/3| < \alpha_0/10$. Let $(\gamma^a_b, \kappa^a_b, \psi)$ be a solution of (5.13) and (5.15) modelled on this velocity dominated solution and define g_{ab} , k_{ab} and ϕ by (3.1). Suppose that this solution satisfies the properties 1.-6. of the conclusions of the Theorem 2.1. Then the Einstein constraints are also satisfied.*

Proof Define:

$$(7.1) \quad C = -k_{ab}k^{ab} + (\text{tr}k)^2 - R - 16\pi\rho$$

$$(7.2) \quad C_b = \nabla_a k^a_b - \nabla_b(\text{tr}k) - 8\pi j_b$$

These quantities satisfy the evolution equations

$$(7.3) \quad \partial_t C - 2(\text{tr}k)C = \nabla^a C_a$$

$$(7.4) \quad \partial_t C_a - (\text{tr}k)C_a = \frac{1}{2}\nabla_a C$$

Define rescaled quantities by $\bar{C} = t^{2-\eta_1}C$ and $\bar{C}_a = t^{1-\eta_2}C_a$ for some positive real numbers η_1 and η_2 . Then the above equations can be written in the form:

$$(7.5) \quad t\partial_t\bar{C} + \eta_1\bar{C} = 2[1 + t\text{tr}k]\bar{C} - t^{2-\eta_1+\eta_2}\nabla^a\bar{C}_a$$

$$(7.6) \quad t\partial_t\bar{C}_a + \eta_2\bar{C}_a = [1 + t\text{tr}k]\bar{C}_a - (1/2)t^{\eta_1-\eta_2}\nabla_a\bar{C}$$

Choose η_1 and η_2 so that $\eta_1 - \eta_2 > 0$. The aim is to apply Theorem 4.2 to show that \bar{C} and \bar{C}_a vanish. In order to do this we should show that these two quantities vanish as fast as a positive power of t as $t \rightarrow 0$, that $1 + t\text{tr}k$ vanishes like a positive power of t and that the term $\nabla^a\bar{C}_a$ is not too singular. In obtaining these estimates it is necessary to use the behaviour of the derivatives of the solution mentioned in the remark following Theorem 4.2. Note that since the velocity dominated constraints are satisfied by assumption, it is enough to estimate the differences of the constraint quantities corresponding to the velocity dominated and full solutions, since these are in fact equal to C and C_a . By property 2. of the conclusions of Theorem 2.1 it follows that $1 + t\text{tr}k = O(t^{\alpha_0})$ which gives one of the desired statements. Similarly it follows that

$$(7.7) \quad -k^{ab}k_{ab} + (\text{tr}k)^2 = -({}^0k^{ab})({}^0k_{ab}) + (\text{tr}{}^0k)^2 + O(t^{-2+\alpha_0})$$

It follows from the curvature estimates done in section 6 that the scalar curvature is also $O(t^{-2+\alpha_0})$. Now consider the expression $\rho - {}^0\rho$. The components of the inverse metric can be estimated by $t^{-2+40\alpha_0}$, so that the terms in ρ involving spatial derivatives can be estimated by $t^{-2+\alpha_0}$ as well. The difference of the time derivatives can be estimated by $t^{-2+\beta}$. These are the estimates for the Hamiltonian constraint that will be needed.

The estimates just carried out were independent of the restriction on the p_a in the hypotheses of the lemma. The following estimates for the momentum constraint are of a cruder type and do use the restriction. First note that the metric and its inverse can be estimated by the powers of t equal to $2p_1$ and $-2p_3$ respectively. It follows from the assumption on the p_a in the hypotheses of the theorem that $2p_1 > 2/3 - \alpha_0/5$ and $-2p_3 > -2/3 - \alpha_0/5$. Using (3.4) then shows that the connection coefficients can be estimated in terms of the power $-2\alpha_0/5$. The effect on the order of a term of taking a divergence can be estimated by the powers $-2\alpha_0/5$ and $-2/3 - 3\alpha_0/5$ for upper and lower indices respectively.

The gradient of the mean curvature is $O(t^{-1+\alpha_0} \log t)$ while the difference of j_a is $O(t^{-1+\beta} \log t)$. The difference of the divergence of the second fundamental form produces the power $-1 + 3\alpha_0/5$. Evidently the last power and that containing β are the limiting ones and determine the estimate for C_a . Similarly the divergence of C_a can be estimated by the powers $-5/3$ and $-5/3 + \beta - 3\alpha_0/5$. Note that it follows from the definition of α_0 that $\alpha_0 < 1/30$. Thus given the hypothesis of the lemma it can be concluded that η_1 and η_2 can be chosen in such a way that all terms on the right hand side of the propagation equations for the constraint quantities vanish like positive powers of t . Thus these equations are Fuchsian and the conclusion follows from Theorem 4.2.

The analogue of this lemma with the scalar field replaced by a stiff fluid is also true and can be proved in the same way. Next the restriction on the exponents p_a will be lifted. Consider a solution $({}^0g_{ab}, {}^0k_{ab}, {}^0\mu, {}^0v_a)$ of the velocity dominated constraints for the Einstein-stiff fluid system. Let ${}^0\hat{k}_{ab}$ be the trace-free part of ${}^0k_{ab}$. The velocity dominated constraints become:

$$(7.8) \quad -{}^0\hat{k}^{ab}{}^0\hat{k}_{ab} + (2/3)(\text{tr}{}^0k)^2 = 16\pi\mu$$

$$(7.9) \quad \nabla_a {}^0\hat{k}^a_b = 8\pi\mu u_b$$

Now let ${}^\lambda k_{ab} = (1 - \lambda){}^0k_{ab} + (1/3)(\text{tr}{}^0k){}^0g_{ab}$ and

$$(7.10) \quad {}^\lambda\mu = (1/16\pi)[-(1 - \lambda)^2({}^0\hat{k}^{ab})({}^0\hat{k}_{ab}) + (2/3)(\text{tr}{}^0k)^2]$$

$$(7.11) \quad {}^\lambda\mu_b = 2(1 - \lambda)\nabla^{a0}\hat{k}_{ab}[-(1 - \lambda)^2({}^0\hat{k}^{ab})({}^0\hat{k}_{ab}) + (2/3)(\text{tr}{}^0k)^2]^{-1}$$

Then $({}^0g_{ab}, {}^\lambda k_{ab}, {}^\lambda\mu, {}^\lambda u_a)$ is a one parameter family of solutions of the velocity dominated constraints which depends analytically on the parameter λ . There exists a corresponding family of solutions of the velocity dominated evolution equations which also depends analytically on λ . Next, Theorem 4.2 provides a corresponding analytic family of solutions of the full evolution equations. (Cf. the second remark following that theorem.) These define constraint quantities depending analytically on λ . For λ close to one Lemma 7.1 shows that these quantities are zero. Hence by analyticity they are zero for all values of λ , including $\lambda = 0$. This means that the conclusion of Lemma 7.1 holds for all positive p_a and a stiff fluid. Since any solution of

the second reduced system for a scalar field defines a solution of the second reduced system for a stiff fluid, this extension also holds for the scalar field.

A variant of the conformal method for solving the full Einstein constraints can be used to analyse the velocity dominated constraints. Consider the following set of free data: a Riemannian metric \bar{g}_{ab} , a symmetric trace-free tensor σ_{ab} on S and two scalar functions $\bar{\phi}$ and $\bar{\phi}_t$ on S . Next consider the following ansatz:

$$(7.12) \quad g_{ab} = \omega^4 \bar{g}_{ab}$$

$$(7.13) \quad k_{ab} = -(1/3)t_0^{-1}g_{ab} + \omega^{-2}l_{ab}$$

$$(7.14) \quad \phi = \omega^{-2}\bar{\phi}$$

$$(7.15) \quad \phi_t = \omega^{-4}\bar{\phi}_t$$

where

$$(7.16) \quad l_{ab} = \sigma_{ab} + \nabla_a W_b + \nabla_b W_a - (2/3)\bar{g}_{ab}\bar{g}^{cd}\nabla_c W_d$$

Putting this into (2.12a) and defining $\bar{\rho} = \frac{1}{2}(\bar{\phi}_t)^2$ gives $\rho = \omega^{-8}\bar{\rho}$, a relation well known from the usual conformal method. As a result of the Hamiltonian constraint the function ϕ satisfies the following algebraic analogue of the Lichnerowicz equation:

$$(7.17) \quad -\omega^{-12}l_{ab}l_{cd}\bar{g}^{ac}\bar{g}^{bd} + \frac{2}{3}t_0^{-2} - 16\pi\omega^{-8}\bar{\rho} = 0$$

Solving this comes down to looking for positive roots of the equation $a\zeta^3 + b\zeta^2 - c = 0$ where a and b are non-negative and c is positive. The derivative of the function on the left hand side of this equation is $\zeta(3a\zeta + 2b)$. Thus unless a and b are both zero the derivative has no positive roots. Moreover the function tends to plus infinity for large ζ and is negative at $\zeta = 0$. Hence the equation has a unique solution for each a and b not both zero and if a and b depend analytically on some parameter then the solution does so too. If a and b are both zero then of course there is no positive solution. In the case of interest here a and b are both positive. The function ω is given by $\omega = \Omega(l_{ab}l_{cd}\bar{g}^{ac}\bar{g}^{bd}, t_0, \bar{\rho})$ where the analytic function Ω is defined as the solution of the algebraic Lichnerowicz equation. The momentum constraint implies the elliptic equation

$$(7.18) \quad \bar{g}^{as}\nabla_a[\nabla_s W_b + \nabla_b W_s - (2/3)\bar{g}_{sb}\bar{g}^{cd}\nabla_c W_d] = 8\pi[\bar{j}_b - 2\bar{\phi}_t\bar{\phi}\omega\nabla_b\omega] - \nabla_a\sigma^a_b$$

for W_a . Here $\bar{j}_a = \bar{\phi}_t\nabla_a\bar{\phi}$. Note that, when ω is expressed in terms of the function Ω of the basic variables, it depends on the first derivatives of W_a . Thus the expression $\nabla_a\omega$ involves second derivatives of W_a and is not simply a lower order term.

Consider now the linearization of (7.18), with respect to W_a , where ω has been reexpressed using Ω . In particular, consider the linearization in the particular case where \bar{g}_{ab} is the metric of constant negative curvature on a compact hyperbolic manifold, the tensor σ_{ab} is zero, $\bar{\phi}$ and $\bar{\phi}_t$ are

constant and the background value of W_a is zero. Because ω is a function of an expression quadratic in W_a , the right hand side of (7.18) makes no contribution to the linearization. Since \bar{g}_{ab} has no non-trivial conformal Killing vectors it follows from the standard theory of the York operator that the operator obtained by linearization of the equation (7.18) is invertible as a map between appropriate Sobolev spaces. Then an application of the implicit function theorem gives solutions of (7.18) for arbitrary choices of the free data sufficiently close (with respect to a Sobolev norm) to the particular free data at which the linearization was carried out. This shows the existence of solutions of the velocity dominated constraints which are as general as the solutions of the full Einstein constraints (at least in the crude sense of function counting).

The conformal method can be applied in a similar way in the stiff fluid case and it turns out to be easier than in the scalar field case. This might seem paradoxical, since the scalar field problem can be identified with a subcase of the stiff fluid problem. The explanation is that it is difficult to identify which free data in the procedure for constructing stiff fluid data which will be presented correspond to data for a scalar field. The ansatz used is $\mu = \omega^{-8}\bar{\mu}$ and $u_a = \omega^2\bar{u}_a$. This gives the scaling $\rho = \omega^{-8}\bar{\rho}$ and $j_a = \omega^{-6}\bar{j}_a$ which is often used in the conformal method. The quantities describing the geometry are scaled as in the case of the scalar field. The equations for ω and W_a are very similar in both cases, with the notable difference that in the stiff fluid case the term involving the derivative of ω is missing from the equation for W_a . This means that the equation for W_a is independent of ω and can be solved by standard theory, as long as the metric \bar{g}_{ab} has no conformal Killing vectors. Once this has been done the algebraic equation for ω can be solved straightforwardly.

8. DISCUSSION

We have shown the existence of a family of solutions of the Einstein equations coupled to a scalar field or a stiff fluid whose singularity structure we can analyse. No symmetry assumptions are made and the solutions are general in the sense that they depend on the same number of free functions as general initial data for the same system on a regular Cauchy surface. These solutions agree with the picture of general spacetime singularities proposed by Belinskii, Khalatnikov and Lifshitz in two important ways. Firstly, the evolution at different spatial points decouples, in the sense that the solutions of the full equations are approximated near the singularity by a solution of a system of ordinary differential equations. Secondly there exists a Gaussian coordinate system which covers a neighbourhood of the singularity in which the singularity is situated at $t = 0$. It is easily seen that the curvature invariant $R_{\alpha\beta}R^{\alpha\beta} = 64\pi^2(\nabla_\alpha\phi\nabla^\alpha\phi)^2$ blows up uniformly for $t \rightarrow 0$. In fact the leading term is proportional to $A^4(x)t^{-4}$ and in the solutions we consider A can never vanish, as a consequence of the Hamiltonian constraint. Thus these singularities are all consistent with the strong cosmic censorship

hypothesis. The mean curvature of the hypersurfaces of constant Gaussian time tends uniformly to infinity as $t \rightarrow 0$ so that the singularity in crushing in the sense of [10]. It then follows from well-known results that a neighbourhood of the singularity can be covered by a foliation consisting of constant mean curvature hypersurfaces. This is the most general class of spacetimes in which all these suggested properties of general spacetimes have been demonstrated. A subclass of these spacetimes is covered by the results of Anguige and Tod[1]. The connection between their results and those of the present paper deserves to be examined more closely but intuitively their spacetimes should correspond to the case where, in our notation, the p_i are everywhere equal to $1/3$.

The spacetimes constructed have been shown to be general in the sense of function counting. It would, however, be desirable to prove that the assumption of analyticity of the data can be replaced by smoothness and that, this having been done, the spacetimes constructed include all those arising from a non-empty open set of initial data on a regular Cauchy surface which, in particular, contains the initial data for a Friedmann model. This would be a statement on the stability of the Friedmann singularity. A model for this kind of generalization is provided by the work of Kichenassamy[12] on nonlinear wave equations.

It was indicated in the introduction that the results on the Einstein-scalar equations can be interpreted in more than one way. The interpretation which has been emphasized here is that where the metric occurring in this system is considered to be the physical metric. In the interpretation in terms of string cosmology the physical metric is (up to a multiplicative constant) $e^\phi g_{\mu\nu}$. This means that for $A(x)$ sufficiently negative the limit $t \rightarrow 0$ does not correspond to a singularity at all, but rather to a phase which lasts for an infinite proper time. It is the time reverse of this situation which plays a role in the pre-big bang model[7]. Another interpretation is in terms of the vacuum field equations in Brans-Dicke theory. This is very similar to the string cosmology case, with the difference that the conformal factor e^ϕ is replaced by $e^{C\phi}$ where C is a constant which depends on the Brans-Dicke coupling constant.

All the results in this paper have concerned the case of three space dimensions. There are reasons to believe that if the space dimension is at least ten then the vacuum Einstein equations allow stable quiescent singularities, similar in some ways to those of the Einstein-scalar field equations in three space dimensions[8]. The techniques developed in this paper might allow this to be proved rigorously. It would also be interesting to know what happens to the picture when further matter fields are added. There are several possibilities here. One is to add some other field, not directly coupled to the scalar field, to the Einstein-scalar field system. A second is to reinstate some of the extra fields (axion, moduli) which have been discarded in passing from the low energy limit of string theory to the Einstein-dilaton theory. A third is to add extra matter fields to the Brans-Dicke theory.

Another direction in which the results on the Einstein-scalar field and Einstein-stiff fluid equations could be generalized is to start with situations where the solution has one Kasner exponent negative and investigate whether it moves (in the direction towards the singularity) towards the region where all Kasner exponents are non-negative. If this were true, then the singularities in generic solutions of these equations could be quiescent. The set of initial data concerned would be not just open, but also dense. This question is sufficiently difficult that it would seem advisable to first try and investigate it rigorously in the spatially homogeneous case.

Acknowledgements This research was supported in part by the Swedish Natural Sciences Research Council (SNSRC), contract no. F-FU 4873-307, and the US National Science Foundation under Grant No. PHY94-07194. Part of the work was done while the authors were enjoying the hospitality of the Institute for Theoretical Physics, Santa Barbara. We gratefully acknowledge stimulating discussions with V. Moncrief which had an important influence on the development of the strategy used in this work.

REFERENCES

- [1] Anguige, K., Tod, K. P. (1999) Isotropic cosmological singularities 1: Polytropic perfect fluid spacetimes. *Ann. Phys. (NY)* 276, 257-293.
- [2] Baouendi, M. S., Goulaouic, C. (1977) Remarks on the abstract form of nonlinear Cauchy-Kowalewski theorems. *Comm. P. D. E.* 2, 1151-1162.
- [3] Barrow, J. D. (1978) Quiescent cosmology. *Nature* 272, 211-215.
- [4] Belinskii, V. A., Khalatnikov, I. M. and Lifshitz, E. M. (1970) Oscillatory approach to a singular point in the relativistic cosmology. *Adv. Phys.* 19, 525-573.
- [5] Belinskii, V. A. and Khalatnikov, I. M. (1973) Effect of scalar and vector fields on the nature of the cosmological singularity. *Sov. Phys. JETP* 36, 591-597.
- [6] Belinskii, V. A., Khalatnikov, I. M. and Lifshitz, E. M. (1982) A general solution of the Einstein equations with a time singularity. *Adv. Phys.* 31, 639-667.
- [7] Buonanno, A., Damour, T. and Veneziano, G. (1999) Pre-big bang bubbles from the gravitational instability of generic string vacua. *Nucl. Phys. B* 543, 275-320.
- [8] Demaret, J., Henneaux, M. and Spindel, P. (1985) Non-oscillatory behaviour in vacuum Kaluza-Klein cosmologies. *Phys. Lett. B* 164, 27-30.
- [9] Eardley, D., Liang, E. and Sachs, R. (1972) Velocity-dominated singularities in irrotational dust cosmologies. *J. Math. Phys.* 13, 99-106.
- [10] Eardley, D., Smarr, L. (1979) Time functions in numerical relativity: marginally bound dust collapse. *Phys. Rev. D* 19, 2239-2259.
- [11] Isenberg, J. and Moncrief, V. (1990) Asymptotic behaviour of the gravitational field and the nature of singularities in Gowdy spacetimes. *Ann. Phys. (NY)* 199, 84-122.
- [12] Kichenassamy, S. (1996) The blow-up problem for exponential nonlinearities. *Comm. P. D. E.* 21, 125-162.
- [13] Kichenassamy, S. and Rendall, A. D. (1998) Analytic description of singularities in Gowdy spacetimes. *Class. Quantum Grav.* 15, 1339-1355.
- [14] Lifshitz, E. M. and Khalatnikov, I. M. (1963) Investigations in relativistic cosmology. *Adv. Phys.* 12, 185-249.
- [15] Rendall, A. D. (1990) Convergent and divergent perturbation series and the post-Minkowskian approximation scheme. *Class. Quantum Grav.* 7:803-812.

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN,

E-mail address: `larsa@math.kth.se`

MAX-PLANCK-INSTITUT FÜR GRAVITATIONSPHYSIK, AM MÜHLENBERG 1, 14424 GOLM, GERMANY

E-mail address: `rendall@aei-potsdam.mpg.de`