FUTURE ASYMPTOTIC EXPANSIONS OF BIANCHI VIII VACUUM METRICS

HANS RINGSTRÖM

Abstract. Bianchi VIII vacuum solutions to Einstein's equations are causally geodesically complete to the future, given an appropriate time orientation, and the objective of this article is to analyze the asymptotic behaviour of solutions in this time direction. For the Bianchi class A spacetimes, there is a formulation of the field equations that was presented in an article by Wainwright and Hsu, and in a previous article we analyzed the asymptotic behaviour of solutions in these variables. One objective of this paper is to give an asymptotic expansion for the metric. Furthermore, we relate this expansion to the topology of the compactified spatial hypersurfaces of homogeneity. The compactified spatial hypersurfaces have the topology of Seifert fibred spaces and we prove that in the case of NUT Bianchi VIII spacetimes, the length of a circle fibre converges to a positive constant but that in the case of general Bianchi VIII solutions, the length tends to infinity at a rate we determine. Finally, we give asymptotic expansions for general Bianchi VII_0 metrics.

1. Introduction

In a previous article [12], we considered the Bianchi class A vacuum cosmologies in the expanding direction. In it, we analyzed the asymptotics in terms of the variables of Wainwright and Hsu (note also that an analysis of Bianchi VIII in the case where matter is present has been carried out in [6]). One may then ask what purpose the present article serves. The point is that the spacetime metric can be written

$$\bar{g} = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\xi^i \otimes \xi^i \tag{1}$$

on $I \times G$, where G is a three dimensional Lie group and the ξ^i are dual to a left invariant basis of the tangent bundle of G. The a_i can be computed using the variables of Wainwright and Hsu, but the computation involves an integration. In particular, in the case of Bianchi VII_0 and VIII, the relevant integrand tends to zero for some of the a_i, and thus a further analysis is necessary. A second question of interest which was left open in [12] is the question of how the geometry of the spatial hypersurfaces of homogeneity behaves with time, and how this connects with topology. Recently, general conjectures on how the future asymptotics of solutions to Einstein's equations should be have emerged, see e.g. [5] and [1]. We do not wish to describe these ideas in general here, but would like to state their implications for Bianchi VIII. The compactified spatial hypersurfaces have the structure of Seifert fibred spaces, and the conjecture is that by rescaling the induced metric on the spatial hypersurfaces by dividing by proper time squared for example, the circle

TABLE 1. Bianchi class A.

Type	n_1	n_2	n_3
I	0	0	0
II	+	0	0
VI_0	0	+	−
VII_0	0	+	+
VIII	−	+	+
IX	+	+	+

fibres in the Seifert fibration should collapse. Here we prove more than this, in fact we are able to give a rate of collapse and to distinguish between general Bianchi VIII spacetimes and the subclass of NUT spacetimes by considering how the length of a circle fibre evolves with time. Finally, note that the results presented here are of interest when trying to redo the analysis in [3] for non-trivial circle bundles over a higher genus surface.

In order to explain the results it is necessary to introduce some terminology. Let G be a 3-dimensional Lie group and e_i, $i = 1, 2, 3$ be a basis of the Lie algebra with structure constants determined by $[e_i, e_j] = \gamma_{ij}^k e_k$. If $\gamma_{ik}^k = 0$, then the Lie algebra and Lie group are said to be of *class A* and

$$\gamma_{ij}^k = \epsilon_{ijm} n^{km} \tag{2}$$

where the symmetric matrix n^{ij} is given by

$$n^{ij} = \frac{1}{2}\gamma_{kl}^{(i}\epsilon^{j)kl}. \tag{3}$$

Definition 1. *Class A vacuum initial data* for Einstein's equations consist of the following. A Lie group G of class A, a left invariant metric g on G and a left invariant symmetric covariant two-tensor k on G satisfying

$$R_g - k_{ij}k^{ij} + (\mathrm{tr}_g k)^2 = 0 \tag{4}$$

and

$$\nabla_i \mathrm{tr}_g k - \nabla^j k_{ij} = 0 \tag{5}$$

where ∇ is the Levi-Civita connection of g and R_g is the corresponding scalar curvature, indices are raised and lowered by g.

Definition 2. Consider class A vacuum initial data (G, g, k). A left invariant basis of the tangent bundle $\{e_i\}$ is called a *canonical basis* with respect to the initial data if g is orthonormal and k is diagonal with respect to it, and if the structure constants $\gamma_{jk}^i = \epsilon_{jkl}n^{li}$ associated with the basis have the property that n is diagonal and the diagonal elements n_i fall into one of the classes given in Table 1.

On pp. 3798–99 in [12] we prove that there are canonical bases and that the Bianchi class is well defined.

Definition 3. Consider class A vacuum initial data (G, g, k) of type VII_0 or VIII and let $\{e_i\}$ be a canonical basis. If $k_2 = k_3$ and $n_2 = n_3$, then the initial data are said to be of *Taub* type, and the corresponding basis is called a basis of *NUT* type in the Bianchi VIII case.

In Lemma 19 we sort out the relation between different canonical bases for Bianchi VIII initial data. The important point to keep in mind here is that e_1 is well defined up to a sign. Note also that if the initial data allow a basis of NUT type, it only allows such bases.

Given class A initial data (G, g, k) and a canonical basis $\{e_i\}$, one can construct a globally hyperbolic Lorenz metric $\bar{g}$ on $I \times G$ for some open interval I such that, $\text{Ric}[\bar{g}] = 0$, the metric induced on $\Sigma = \{0\} \times G$ is g and the second fundamental form of Σ is k, assuming one identifies Σ with G in the obvious way. The metric has the form (1), where the ξ^i are the duals of the canonical basis $\{e_i\}$. By Lemma 21 the development is independent of the canonical basis chosen if the initial data is of Bianchi type VIII. Consider Bianchi VII_0 and VIII initial data that are not of Taub type. Then the corresponding developments are inextendible and future causally geodesically complete, assuming one has chosen a suitable time orientation. These statements are proven in [11]. The developments constructed in [11] will be called *class A developments*. The construction can also be found on pp. 3798–99 of [12].

Theorem 1. *Consider Bianchi VII_0 initial data which are not of Taub type. Then the class A development can be written in the form (1), where the ξ^i are the duals of a canonical basis. Furthermore, there are positive constants α_i, $i = 1, 2, 3$, C and T such that*

$$a_i(t) = \alpha_i (\ln t)^{1/2} + O(1) \tag{6}$$

for $i = 2, 3$ and $t \geq T$. Also, there are sequences $t_k, s_k \to \infty$ and a constant $c > 0$ such that

$$|a_i(t_k) - \alpha_i(\ln t_k)^{1/2}| \geq c \quad \text{and} \quad |a_i(s_k) - \alpha_i(\ln s_k)^{1/2}| \leq \frac{C \ln \ln s_k}{(\ln s_k)^{1/2}} \tag{7}$$

for $i = 2, 3$. Finally

$$a_1(t) = \alpha_1 t[1 + O(\frac{\ln \ln t}{\ln t})]$$

for all $t \geq T$.

Remark. The inequalities (7) are intended to emphasize the optimality of (6). Observe that they show that any lower order corrections to (6) have to oscillate.

The proof is to be found at the end of Section 4. Concerning Bianchi VIII, it is of interest to contrast general solutions with those arising from Taub initial data. We will refer to Bianchi VIII Taub initial data as NUT initial data.

Theorem 2. *Consider NUT initial data. Then the class A development can be written in the form (1), where the ξ^i are the duals of a canonical basis. Furthermore, there are positive constants α_1, α_2 and T such that*

$$a_1(t) = \alpha_1 + O(t^{-1}) \quad \text{and} \quad a_2(t) = a_3(t) = \alpha_2 t[1 + O(\frac{\ln t}{t})]$$

for $i = 2, 3$ and $t \geq T$.

The proof is to be found at the end of Section 5.

Theorem 3. *Consider Bianchi VIII initial data that are not of NUT type. Then the class A development can be written in the form (1), where the ξ^i are the duals* *of a canonical basis. Furthermore, there are positive constants α_i, $i = 1, 2, 3$, c_0 and a T such that*

$$a_1(t) = \alpha_1 (\ln t)^{1/2}[1 + O(\frac{\ln \ln t}{\ln t})], \quad a_i(t) = \alpha_i t[1 + O(\frac{\ln \ln t}{\ln t})]$$

for $i = 2, 3$ and $t \geq T$, and

$$\frac{a_2(t)}{a_3(t)} = c_0 + O(t^{-1})$$

for $t \geq T$.

Remark. By the results obtained in this article, it is possible to obtain further terms in the expansion. However, the computations involved are long and have for this reason not been carried out.

The proof is to be found at the end of Section 7.

Let us describe the connection between Theorem 2, 3 and the topology of the compactified spatial hypersurfaces, starting by clarifying what we mean by compactifications. Consider class A vacuum initial data (G, g, k). We will call a diffeomorphism ϕ of G an *isometry of the initial data* if $\phi^* g = g$ and $\phi^* k = k$. Let Γ be a free and properly discontinuous group of isometries of the initial data. We get a solution $(\tilde{g}, \tilde{k})$ of Einstein's constraint equations on G/Γ, and if this manifold is compact, we say that $(G/\Gamma, \tilde{g}, \tilde{k})$ is a *compactification* of (G, g, k).

Assume now that G is simply connected and let $(I \times G, \bar{g})$ be a class A development. It follows from Lemma 20 that if ϕ is an isometry of the initial data, then the diffeomorphism $(1, \phi)$ from $I \times G$ to itself defined by $(1, \phi)(t, h) = [t, \phi(h)]$ is an isometry of the development. Consequently if Γ is a free and properly discontinuous group of isometries of the initial data, $(1, \Gamma)$ is a free and properly discontinuous group of isometries of the class A development. Thus we get a vacuum Lorentz metric on $I \times G/\Gamma$ consistent with the compactified initial data. We will refer to the corresponding Lorentz manifold as a *spatially compactified class A development.*

A 3-manifold is said to be a *Seifert fibred space* if it satisfies the following two conditions:

(1) It can be written as a disjoint union of circles.
(2) Each circle fibre has an open neighbourhood U satisfying:
   (a) U can be written as a disjoint union of circle fibres.
   (b) U is isomorphic either to a solid torus or a cylinder where the ends have been identified after a rotation by a rational angle.

When we say that U is isomorphic to a solid torus, we mean that U is diffeomorphic to a solid torus and that the circle fibres of U are mapped to the natural circle fibres of the solid torus. Note that there are different definitions of Seifert fibred spaces in the literature. In particular, our definition coincides with the original definition by Seifert but not with that of Scott [13].

Theorem 4. *Consider Bianchi VIII initial data (G, g, k) where G is simply connected and assume that Γ is a free and properly discontinuous group of isometries of the initial data such that G/Γ is compact. Then G/Γ is a Seifert fibred space, and if e_1 is the first element in a canonical basis for the initial data, the integral curves of e_1 map to the circle fibres. As noted above, the action of Γ extends to the* *class A development, so that given a circle fibre we can speak of its length $l(t)$ at time t. By Theorem 2 we conclude that if the initial data are of NUT type, there is an $l_0 > 0$ such that*

$$l(t) = l_0 + O(t^{-1})$$

and by Theorem 3 we conclude that if the initial data are not of NUT type, there is a $c_0 > 0$ such that

$$l(t) = c_0(\ln t)^{1/2}[1 + O(\frac{\ln\ln t}{\ln t})].$$

Remark. For any initial data there are Γ such that G/Γ is compact. In fact, let $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ denote the universal covering group of $\mathrm{Sl}(2,\mathbb{R})$, which is a simply connected Lie group of Bianchi type VIII. Then, for any higher genus surface Σ_p with hyperbolic metric h, there is a subgroup Ξ_p of $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ such that Ξ_p acting on the left is a free and properly discontinuous group of diffeomorphims with $\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Xi_p$ diffeomorphic to the unit tangent bundle of Σ_p with respect to the metric h. This is of course well known, but we include a proof in Lemma 16 since there seems to be some confusion in the GR literature.

The proof is to be found at the very end of the article.

The associated question of what the possible topologies are for compactifications of initial data is not addressed here. If the initial data are of NUT type, the question has been answered, see [13] and references therein, since the isometry group for NUT initial data is the same as the isometry group for the Thurston geometry on $\tilde{\mathrm{Sl}}(2,\mathbb{R})$. Furthermore, if Γ consists only of left translations, what the possible topologies are has been sorted out in [10]. However, the question of greatest interest when discussing Bianchi VIII is what the possible topologies are when Γ is a subgroup of the isometry group of general initial data. This case falls between the cases that have been handled, cf. Lemma 20.

Let us try to describe what makes the analysis possible in the expanding direction of Bianchi VIII vacuum spacetimes. For a definition of the variables of Wainwright and Hsu, see Section 2. It turns out that for general, i.e. non NUT, Bianchi VIII solutions,

$$h = \Sigma_-^2 + \frac{3}{4}(N_2 - N_3)^2 = \frac{1}{4\tau} + O(\frac{\ln\tau}{\tau^2}),$$

see (70), where the first equality is a definition, but that

$$(\Sigma_-')^2 + \frac{3}{4}(N_2' - N_3')^2 = c\tau^{-5/2}e^{3\tau}[1 + O(\frac{\ln\tau}{\tau})],$$

where $c > 0$ is a constant, which follows from (40), (41), (82) and (70). This is a quantitative illustration of the oscillatory behaviour of Σ_- and $N_2 - N_3$, which we give more detailed description of in Section 6. On the other hand, Σ_+ and $N_1(N_2 + N_3)$ have bounded derivatives. Interestingly enough, the fact that some variables have bounded derivatives and some have derivatives that tend to infinity exponentially is what makes the problem manageable analytically. By averaging over the variables with bounded derivatives during a period of the oscillation in the rapidly varying variables, one obtains an iteration. Since the frequency of the oscillations tends to infinity exponentially with time, this approximation is very good. In fact, one can obtain series expansions of the form

$$\Sigma_+(\tau) = \frac{1}{2} - \frac{1}{4\tau} + \frac{\ln\tau}{8\tau^2} + \frac{c_u}{\tau^2} - \frac{\ln^2\tau}{16\tau^3} + O(\frac{\ln\tau}{\tau^3}),$$

see (75), where $u = \Sigma_+ - 1/2$ and c_u is some constant. One can also obtain similar expansions for the other slowly varying variables, and there seems to be no reason to believe that one could not obtain further terms in the expansions. Note that heuristic arguments supporting expansions of this form are provided in [6]. This article furthermore deals with the case when matter is present.

The article is structured as follows. In Section 2 we describe the variables of Wainwright and Hsu and in Section 3 we give the necessary background from [12]. In Sections 4 and 5 we then analyze Bianchi VII_0 and Bianchi VIII NUT. The heart of the paper then consists of Sections 6 and 7, which contain the analysis of general Bianchi VIII solutions. The remaining sections contain the material necessary to connect the asymptotic expansions of the metric with the Seifert fibred structure of the compactified spatial hypersurfaces. The main point is of course to identify the e_1 appearing in a canonical basis with the fibre direction in the Seifert fibration. We do this in a very explicit way, deriving the isometry group for the relevant initial data, and proving that the cocompact subgroups of the isometry group have to have a particular structure. While doing this, we always keep track of e_1 in order to achieve the desired objective. The argument to prove that the compactifications are Seifert fibred is taken from [17]. In fact most of the arguments presented are available in the literature. However, they are spread out and are not always so accessible to outsiders. For this reason the last two chapters also have a pedagogical purpose. We wish to carry out the necessary proofs without assuming more of the reader than basic differential geometry, Lie group theory and knowledge of covering spaces.

2. The equations of Wainwright and Hsu

Here we formulate the equations we will use and state some properties. The equations were obtained in [8] and are based on a formulation given in [4]. A brief derivation can be found in [12]. For Bianchi class A spacetimes, Einstein's vacuum equations take the following form in the formulation due to Wainwright and Hsu

$$\begin{aligned} N_1' &= (q - 4\Sigma_+)N_1 \\ N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \\ N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \\ \Sigma_+' &= -(2-q)\Sigma_+ - 3S_+ \\ \Sigma_-' &= -(2-q)\Sigma_- - 3S_- \end{aligned} \tag{8}$$

where the prime denotes derivative with respect to τ and

$$\begin{aligned} q &= 2(\Sigma_+^2 + \Sigma_-^2) \\ S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)] \\ S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3). \end{aligned} \tag{9}$$

The vacuum Hamiltonian constraint is

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_1N_3)] = 1. \tag{10}$$

The above equations have certain symmetries described in [8]. By permuting N_1, N_2, N_3 arbitrarily we get new solutions if we at the same time carry out appropriate combinations of rotations by integer multiples of $2\pi/3$ and reflections in the (Σ_+, Σ_-)-plane. Changing the sign of all the N_i at the same time does not change the equations. Since the sets $N_i > 0$, $N_i < 0$ and $N_i = 0$ are invariant under the flow of the equation we may classify solutions to (8)-(10) accordingly. Taking the symmetries into account, we get Table 1. When we speak of Bianchi VIII solutions we will assume that $N_2,\ N_3 > 0$ and that $N_1 < 0$. We only consider solutions to (8)-(10) which are not of Bianchi IX type. By the constraint (10) we conclude that $q \leq 2$ for such solutions. As a consequence the N_i cannot grow faster than exponentially due to (8). The solutions we consider can thus not blow up in a finite time so that we have existence intervals of the form $(-\infty, \infty)$. The set $\Sigma_- = 0$, $N_2 = N_3$ is invariant under the flow of (8)-(10). Applying the symmetries to this set we get new invariant sets. Solutions which are contained in this invariant set are said to be of *Taub* type.

3. Background

This article is a continuation of the analysis presented in [12], and for the benefit of the reader, we wish to restate the terminology and main conclusions of that paper. Let us first note that the trace of the second fundamental form of the spatial hypersurfaces of homogeneity θ satisfies

$$\theta' = -(1+q)\theta. \tag{11}$$

This is equation (17) of [12]. Observe that it is decoupled from (8)-(10). The prime is with respect to the same τ-time as in (8)-(10). This time is related to the t-time in (1) by

$$\frac{dt}{d\tau} = \frac{3}{\theta}, \tag{12}$$

which is equation (16) of [12]. Finally, the a_i appearing in (1) satisfy

$$a_i(\tau) = \exp[\int_0^\tau (3\Sigma_i + 1)ds], \tag{13}$$

where

$$\Sigma_1 = -\frac{2}{3}\Sigma_+,\ \Sigma_2 = \frac{1}{3}\Sigma_+ + \frac{1}{\sqrt{3}}\Sigma_-,\ \Sigma_3 = \frac{1}{3}\Sigma_+ - \frac{1}{\sqrt{3}}\Sigma_-. \tag{14}$$

These are equations (25) and (26) of [12]. We will quite consistently abuse notation and write $a_i(\tau)$ when we mean $a_i[t(\tau)]$ etc. However, when we talk of N_i, Σ_+ and Σ_-, we will always stick to the time associated with the equations (8)-(10). Let us now state what we need to know concerning Bianchi VII_0. The main theorem concerning Bianchi VII_0 solutions is the following.

Theorem 5. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10) with $N_1 = 0$ and $N_2,\ N_3 > 0$. Then there is an $n_0 > 0$ such that*

$$\lim_{\tau\to\infty}(\Sigma_+, \Sigma_-, N_2, N_3) = (-1, 0, n_0, n_0).$$

This is Theorem 1.1 of [12]. Note the problem that arises when one wants to use this information to estimate the behaviour of a_2 and a_3; the relevant integrand tends to zero. When considering Bianchi VII_0 solutions, we will always assume that $N_2,\ N_3 > 0$ and $N_1 = 0$. We will also only be interested in non-Taub type solutions. Thus, $\Sigma_-^2 + (N_2 - N_3)^2 > 0$ for all times. This implies that $-1 < \Sigma_+ < 1$ for all times. Thus we can define the smooth functions

$$x = \frac{\Sigma_-}{(1-\Sigma_+^2)^{1/2}} \tag{15}$$

$$y = \frac{\sqrt{3}}{2}\frac{N_2 - N_3}{(1-\Sigma_+^2)^{1/2}}. \tag{16}$$

Let

$$\tilde{g} = 3(N_2 + N_3) + 2(1+\Sigma_+)xy. \tag{17}$$

Then $x' = -\tilde{g}y$ and $y' = \tilde{g}x$. By the constraint, $x^2 + y^2 = 1$, so that we can choose a ϕ_0 such that $[x(\tau_0), y(\tau_0)] = [\cos(\phi_0), \sin(\phi_0)]$. Define

$$\xi(\tau) = \int_{\tau_0}^{\tau} \tilde{g}(s)ds + \phi_0. \tag{18}$$

Then $x(\tau) = \cos[\xi(\tau)]$ and $y(\tau) = \sin[\xi(\tau)]$. Observe that if one combines Theorem 5 with (17), one gets the conclusion that $\tilde{g} \to 6n_0$. This means that x and y will oscillate forever and that they in the limit will behave more and more like $\cos(6n_0\tau)$ and $\sin(6n_0\tau)$. For future reference, let us just note that there is a τ_0 such that

$$\tilde{g}(\tau) \geq 5n_0 \tag{19}$$

for all $\tau \geq \tau_0$. We will need Lemma 7.2 and 7.3 of [12], which we now state.

Lemma 1. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10) with $N_2, N_3 > 0$, and let τ_0 be such that (19) is fulfilled for $\tau \geq \tau_0$. Let $\tau_0 \leq \tau_a < \tau_b$, and assume*

$$\xi(\tau_b) - \xi(\tau_a) = \pi.$$

Then there is a T such that

$$|1 - \frac{(1+\Sigma_+)(\tau_1)}{(1+\Sigma_+)(\tau_2)}| \leq C\frac{(1+\Sigma_+)(\tau_b)}{(N_2+N_3)(\tau_{\max})} \tag{20}$$

and

$$|1 - \frac{(N_2+N_3)(\tau_1)}{(N_2+N_3)(\tau_2)}| \leq C\frac{(1+\Sigma_+)(\tau_b)}{(N_2+N_3)(\tau_{\max})} \tag{21}$$

if $\tau_a \geq T$, where $\tau_{\max}$ yields the maximum value of $N_2 + N_3$ in $[\tau_a, \tau_b]$ and τ_1, τ_2 are arbitrary elements of $[\tau_a, \tau_b]$. The constant C only depends on the constant $5n_0$ appearing in (19).

Lemma 2. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10) with $N_2, N_3 > 0$, and let τ_0 be such that (19) is fulfilled for $\tau \geq \tau_0$. Let $\tau_0 \leq \tau_a < \tau_b$, and assume*

$$\xi(\tau_b) - \xi(\tau_a) = \pi.$$

Then there is a T such that

$$|\int_{\tau_a}^{\tau_b}[\Sigma_-^2 - (1+\Sigma_+)]ds| \leq C\int_{\tau_a}^{\tau_b}(1+\Sigma_+)^2 ds \tag{22}$$

if $\tau_a \geq T$, for some constant C depending only on the constant $5n_0$ appearing in (19).

Note that it is far from obvious that a statement such as (22) should hold. The integrand is of the order of magnitude $1+\Sigma_+$, but the average over a period is of the order of magnitude $(1+\Sigma_+)^2$. Finally, note that by the proof of Lemma 7.5 in [12], there is for every $0<\alpha<1$ a $0<c_\alpha<\infty$ and a T_α such that

$$|1+\Sigma_+(\tau)| \leq \frac{c_\alpha}{\tau^\alpha} \tag{23}$$

for all $\tau \geq T_\alpha$.

Concerning Bianchi VIII, we will only need to know the contents of Theorem 1.2 of [12].

Theorem 6. *Let $(\Sigma_+,\Sigma_-,N_1,N_2,N_3)$ be a Bianchi VIII solution to (8)-(10) with $N_1<0$ and $N_2,N_3>0$. Then*

$$\lim_{\tau\to\infty} N_1 = 0,\ \lim_{\tau\to\infty} N_2 = \infty,\ \lim_{\tau\to\infty} N_3 = \infty.$$

Furthermore

$$\lim_{\tau\to\infty} \Sigma_+ = \frac{1}{2},\ \lim_{\tau\to\infty} \Sigma_- = 0$$

and

$$\lim_{\tau\to\infty} N_1(N_2+N_3) = -\frac{1}{2},\ \lim_{\tau\to\infty}(N_2-N_3) = 0.$$

4. Bianchi VII_0

We start where [12] ended.

Proposition 1. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10). Then there is a T and a C such that*

$$|1+\Sigma_+(\tau)-\frac{1}{2\tau}| \leq \frac{C\ln\tau}{\tau^2} \tag{24}$$

for all $\tau \geq T$.

Proof. We have

$$(1+\Sigma_+)' = -(2-q)(1+\Sigma_+)$$

whence

$$\frac{(1+\Sigma_+)'}{(1+\Sigma_+)^2} = -2 - \frac{(2-q)-2(1+\Sigma_+)}{1+\Sigma_+}. \tag{25}$$

Let ξ be defined by (18) and assume $[\tau_a,\tau_b]$ is an interval such that

$$\xi(\tau_b)-\xi(\tau_a) = \pi.$$

By (20) and the fact that N_2+N_3 is bounded from below by a positive constant, we deduce the existence of a constant C such that if τ_a is big enough, then

$$|\frac{1}{1+\Sigma_+(\tau_1)} - \frac{1}{1+\Sigma_+(\tau_2)}| \leq C$$

for all $\tau_1,\tau_2 \in [\tau_a,\tau_b]$. Thus

$$\int_{\tau_a}^{\tau_b} \frac{(2-q)-2(1+\Sigma_+)}{1+\Sigma_+} ds = \frac{1}{1+\Sigma_+(\tau_a)} \int_{\tau_a}^{\tau_b} [(2-q)-2(1+\Sigma_+)] ds + \epsilon_1$$

where

$$|\epsilon_1| \leq C \int_{\tau_a}^{\tau_b} [1+\Sigma_+(\tau)] d\tau.$$

Observe now that by Lemma 2,

$$|\int_{\tau_a}^{\tau_b}[\Sigma_-^2-(1+\Sigma_+)]d\tau|\leq C\int_{\tau_a}^{\tau_b}[1+\Sigma_+(\tau)]^2d\tau.$$

Since

$$(2-q)-2(1+\Sigma_+)=2(1+\Sigma_+)-2\Sigma_-^2-2(1+\Sigma_+)^2, \tag{26}$$

we conclude that

$$|\int_{\tau_a}^{\tau_b}[(2-q)-2(1+\Sigma_+)]d\tau|\leq C\int_{\tau_a}^{\tau_b}[1+\Sigma_+(\tau)]^2d\tau\leq$$

$$\leq C[1+\Sigma_+(\tau_a)]\int_{\tau_a}^{\tau_b}[1+\Sigma_+(\tau)]d\tau.$$

Thus

$$|\int_{\tau_a}^{\tau_b}\frac{(2-q)-2(1+\Sigma_+)}{1+\Sigma_+}ds|\leq C\int_{\tau_a}^{\tau_b}[1+\Sigma_+(\tau)]d\tau. \tag{27}$$

Since

$$|\int_{\tau_1}^{\tau_2}\frac{(2-q)-2(1+\Sigma_+)}{1+\Sigma_+}ds|\leq C$$

if $[\tau_1,\tau_2]$ corresponds to less than one multiple of π, we conclude that

$$|\frac{1}{1+\Sigma_+(0)}-\frac{1}{1+\Sigma_+(\tau)}+2\tau|\leq C+C\int_0^{\tau}[1+\Sigma_+(s)]ds \tag{28}$$

for $\tau\geq 0$ where we have also used (25) and (27). Observe that the estimate (27) may only be applicable for $\tau\geq T$, but as this is a bounded time, the constant takes care of the discrepancy. By (23) we conclude that for every $\alpha>0$, there is a constant C_α such that

$$|-\frac{1}{1+\Sigma_+(\tau)}+2\tau|\leq C_\alpha\tau^\alpha$$

for all $\tau\geq 1$. Inserting this information into (28) we get the conclusion that

$$|-\frac{1}{1+\Sigma_+(\tau)}+2\tau|\leq C\ln\tau$$

for all $\tau\geq 2$. This proves the proposition. □

Corollary 1. *Consider a non-Taub Bianchi VII$_0$ solution to (8)-(10). There are constants c_{s+} and C such that*

$$|\int_1^{\tau}[1+\Sigma_+(s)]ds-\frac{1}{2}\ln\tau-c_{s+}|\leq\frac{C\ln\tau}{\tau}$$

for all $\tau\geq 2$.

Proof. Let

$$c_{s+}=\int_1^{\infty}[1+\Sigma_+(\tau)-\frac{1}{2\tau}]d\tau,$$

which is convergent due to the previous proposition. The estimate (24) yields the conclusion of the corollary. □

Corollary 2. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10). There are constants c_q and C such that*

$$|\int_1^\tau (2-q)d\tau - \ln\tau - c_q| \leq \frac{C\ln\tau}{\tau}$$

for all $\tau \geq 2$.

Proof. Observe that by the proof of Proposition 1,

$$\int_1^\tau [(2-q) - 2(1+\Sigma_+)]ds$$

converges as $\tau \to \infty$. Let α be the limit. Then Lemma 2 and (26) yield

$$|\int_1^\tau [(2-q) - 2(1+\Sigma_+)]ds - \alpha| \leq C\int_\tau^\infty [1+\Sigma_+(s)]^2 \leq \frac{C}{\tau},$$

if τ is big enough, where we have also used (24). The conclusion now follows from Corollary 1. □

Corollary 3. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10). There is a constant C such that*

$$|\frac{N_2}{N_3} - 1| \leq \frac{C}{\tau^{1/2}} \tag{29}$$

for all $\tau \geq 1$. Furthermore, there are sequences $\tau_k, \hat{\tau}_k \to \infty$ and a constant $c > 0$ such that

$$|\frac{N_2(\tau_k)}{N_3(\tau_k)} - 1| \geq \frac{c}{\tau_k^{1/2}} \quad \text{and} \quad |\frac{N_2(\hat{\tau}_k)}{N_3(\hat{\tau}_k)} - 1| = 0. \tag{30}$$

Remark. The equation (30) is merely intended to emphasize the optimality of (29).

Proof. By the constraint,

$$\frac{3}{2}(N_2 - N_3)^2 = 2 - q \leq 2(1-\Sigma_+^2) = 2(1-\Sigma_+)(1+\Sigma_+),$$

which, when combined with (24) and the fact that N_3 is bounded from below by a positive constant yields (29). Due to the oscillatory behaviour of the solutions noted in the paragraph preceeding (19), we conclude the existence of time sequences $\tau_k, \hat{\tau}_k \to \infty$ such that $\Sigma_-(\tau_k) = 0$ and $(N_2 - N_3)(\hat{\tau}_k) = 0$. Since N_3 is bounded from below by a positive constant, we conclude that the second part of (30) holds. Finally, the first part of (30) follows from (24), the constraint, $\Sigma_-(\tau_k) = 0$ and the fact that N_3 is bounded from below by a positive constant. □

Corollary 4. *Consider a non-Taub Bianchi VII_0 solution to (8)-(10). Then there is a constant C and a T such that*

$$|\psi(\tau)| \leq \frac{C}{\tau^{1/2}}$$

for all $\tau \geq T$, where

$$\psi(\tau) = \ln\frac{N_2(0)}{N_3(0)} + \int_0^\tau 4\sqrt{3}\Sigma_- ds.$$

Furthermore, there are sequences $\tau_k, \hat{\tau}_k \to \infty$ and a constant $c > 0$ such that

$$|\psi(\tau_k)| \geq \frac{c}{\tau_k^{1/2}} \quad \text{and} \quad \psi(\hat{\tau}_k) = 0.$$

Proof. Observe that

$$\frac{N_2(\tau)}{N_3(\tau)} = e^{\psi(\tau)}.$$

The statements thus follow from the previous corollary. □

Proposition 2. *Consider a non-Taub Bianchi VII$_0$ solution to (8)-(10). There are positive constants α_i, $i = 1, 2, 3$, C and a T such that*

$$|a_i(\tau) - \alpha_i \tau^{1/2}| \leq C \tag{31}$$

for $i = 2, 3$ and all $\tau \geq T$. Furthermore, there are time sequences $\tau_k,\ \hat{\tau}_k \to \infty$ and a $c > 0$ such that

$$|a_i(\tau_k) - \alpha_i \tau_k^{1/2}| \geq c \quad \text{and} \quad |a_i(\hat{\tau}_k) - \alpha_i \hat{\tau}_k^{1/2}| \leq \frac{C \ln \hat{\tau}_k}{\hat{\tau}_k^{1/2}}$$

for $i = 2, 3$. Finally

$$|\tau a_1(\tau) e^{-3\tau} - \alpha_1| \leq \frac{C \ln \tau}{\tau} \tag{32}$$

for all $\tau \geq T$.

Proof. By Corollary 1 and 4, there is a constant $c_{s,2}$ such that

$$\kappa(\tau) = \int_0^\tau (1 + \Sigma_+ + \sqrt{3}\Sigma_-) ds - \frac{1}{2} \ln \tau - c_{s,2} = O(\tau^{-1/2})$$

for τ big enough, where the first equality defines κ. Furthermore, we conclude the existence of a time sequence $\tau_k \to \infty$ and a $c > 0$ such that

$$|\kappa(\tau_k)| \geq \frac{c}{\tau_k^{1/2}}.$$

There is also a time sequence $\hat{\tau}_k \to \infty$ such that

$$|\kappa(\hat{\tau}_k)| \leq \frac{C \ln \hat{\tau}_k}{\hat{\tau}_k}.$$

Since

$$|a_2(\tau) - e^{c_{s,2}} \tau^{1/2}| = e^{c_{s,2}} \tau^{1/2} |e^{\kappa(\tau)} - 1|,$$

the statements concerning a_2 follow. The proof for a_3 is similar, and one can see that the time sequences τ_k and $\hat{\tau}_k$ can be taken to be the same for a_2 and a_3. Since

$$a_1(\tau) = \exp[-\int_0^\tau (2\Sigma_+ - 1)] = \exp[3\tau - 2\int_0^\tau (1 + \Sigma_+) ds],$$

the statement concerning a_1 follows from Corollary 1. □

Lemma 3. *Consider a non-Taub Bianchi VII$_0$ solution to (8)-(10). Then there are constants C and $c_t > 0$ and a T such that*

$$|\tau \cdot t(\tau) e^{-3\tau} - c_t| \leq \frac{C \ln \tau}{\tau} \tag{33}$$

for all $\tau \geq T$.

Proof. Since (11) and Corollary 2 hold, we conclude that there is a constant $c_\theta > 0$ such that

$$|\frac{\tau e^{-3\tau}}{\theta(\tau)} - c_\theta| \leq \frac{C \ln \tau}{\tau}$$

for τ big enough. Since we have (12), it is of interest to observe that

$$|\tau e^{-3\tau} \int_1^\tau \frac{3}{s} e^{3s} ds - 1| \leq \frac{C}{\tau}$$

and that

$$|\tau e^{-3\tau} \int_1^\tau \frac{\ln s}{s^2} e^{3s} ds| \leq \frac{C \ln \tau}{\tau}.$$

These observations together yield the statement of the lemma. □

Proof of Theorem 1. Equation (33) implies that

$$\ln t = \ln c_t - \ln \tau + 3\tau + \ln[1 + \zeta(\tau)]$$

where

$$\zeta(\tau) = \frac{1}{c_t}[\tau \cdot t(\tau) e^{-3\tau} - c_t].$$

We thus get

$$3\tau = \ln t + \phi(\tau),$$

where $|\phi(\tau)| \leq C \ln \tau$. In consequence,

$$\tau = \frac{1}{3} \ln t + \rho(t),$$

where $|\rho(t)| \leq C \ln \ln t$. Thus

$$\tau^{1/2} = (\frac{1}{3} \ln t + \rho(t))^{1/2} = \frac{1}{\sqrt{3}} (\ln t)^{1/2} + \frac{1}{\sqrt{3}} (\ln t)^{1/2} [(1 + \frac{3\rho(t)}{\ln t})^{1/2} - 1].$$

Consequently,

$$|\tau^{1/2} - \frac{1}{\sqrt{3}} (\ln t)^{1/2}| \leq \frac{C \ln \ln t}{(\ln t)^{1/2}}.$$

Combining these observations with Proposition 2, we get the conclusions of the theorem as far as a_2 and a_3 are concerned, assuming we rename α_2 and α_3. Estimate

$$|\frac{a_1(t)}{t} - \frac{\alpha_1}{c_t}| = \frac{1}{\tau \cdot t(\tau)} e^{3\tau} |\tau a_1(\tau) e^{-3\tau} - \alpha_1 + \alpha_1 - \frac{\alpha_1}{c_t} \tau \cdot t(\tau) e^{-3\tau}| \leq \frac{C \ln \ln t}{\ln t}$$

where we have used (32) and (33). If we rename α_1 / c_t to α_1, the theorem follows. □

5. Bianchi VIII NUT

Let us consider Bianchi VIII solutions to (8)-(10) such that $N_2 = N_3$ and $\Sigma_- = 0$.

Proposition 3. *Consider a Bianchi VIII NUT solution to (8)-(10). Then there are constants C and $c_N > 0$ and a T such that*

$$|\Sigma_+(\tau) - \frac{1}{2}| \leq C e^{-3\tau/2}, \tag{34}$$

$$|(N_1 N_2)(\tau) + \frac{1}{4}| \leq C e^{-3\tau/2} \tag{35}$$

and

$$|N_2 - c_N e^{3\tau/2}| \leq C \tag{36}$$

for all $\tau \geq T$.

Proof. Let us reformulate the expression for Σ_+'. Using the constraint (10), we can express $N_1 N_2$ in terms of the other variables in order to obtain

$$\Sigma_+' = -2(1-\Sigma_+^2)\Sigma_+ - 2(1-\Sigma_+^2)(-\frac{1}{2}) + \frac{9}{4}N_1^2 = -2(1-\Sigma_+^2)(\Sigma_+ - \frac{1}{2}) + \frac{9}{4}N_1^2.$$

Letting $\hat{g} = \Sigma_+ - \frac{1}{2}$, $f = 2(1-\Sigma_+^2)$ and $h = 9N_1^2/4$, we get the conclusion that

$$\hat{g}' = -f\hat{g} + h. \tag{37}$$

By Theorem 6 we know that $\Sigma_+ \to \frac{1}{2}$. By (8) we have

$$|h(\tau)| = \frac{9}{4}N_1^2(\tau) \leq Ce^{-5\tau/2}$$

for $\tau \geq 0$. Since $f \to \frac{3}{2}$, we conclude that

$$|h(\tau)\exp[\int_0^\tau f(s)ds]| \leq Ce^{-\tau/2}$$

for all $\tau \geq 0$. By (37),

$$\left(\exp[\int_0^\tau f(s)ds]\hat{g}\right)' = h\exp[\int_0^\tau f(s)ds],$$

so that

$$|\hat{g}(\tau)| \leq C\exp[-\int_0^\tau f(s)ds]. \tag{38}$$

As a preliminary result, this yields

$$|\Sigma_+(\tau) - \frac{1}{2}| \leq Ce^{-\tau},$$

so that

$$|\int_0^\tau f(s)ds - 3\tau/2| = |\int_0^\tau 2(\frac{1}{4} - \Sigma_+^2)ds| \leq C\int_0^\tau e^{-s}ds \leq C,$$

which, together with (38) proves (34). By (8), we have

$$(N_1N_2)(\tau) = (N_1N_2)(0)\exp[\int_0^\tau 4\Sigma_+(\Sigma_+ - \frac{1}{2})ds] =$$

$$= -\frac{1}{4}\exp[-\int_\tau^\infty 4\Sigma_+(\Sigma_+ - \frac{1}{2})ds],$$

where we have used the fact that $N_1N_2 \to -1/4$, cf. Theorem 6. Thus

$$|(N_1N_2)(\tau) + \frac{1}{4}| \leq C\int_\tau^\infty |\Sigma_+ - \frac{1}{2}|ds$$

proving (35). Since

$$e^{-3\tau/2}N_2(\tau) = N_2(0)\exp[\int_0^\tau 2(\Sigma_+ + \frac{3}{2})(\Sigma_+ - \frac{1}{2})ds]$$

we conclude the existence of a constant c_N such that

$$|e^{-3\tau/2}N_2(\tau) - c_N| \to 0.$$

In consequence,

$$|e^{-3\tau/2}N_2(\tau) - c_N| = c_N|\exp[-\int_\tau^\infty 2(\Sigma_+ + \frac{3}{2})(\Sigma_+ - \frac{1}{2})ds] - 1|$$

and (36) follows. □

Corollary 5. *Consider a Bianchi VIII NUT solution to (8)-(10). Then there are positive constants α_1, α_2 and C such that*

$$|a_1(\tau) - \alpha_1| \leq Ce^{-3\tau/2} \quad \text{and} \quad |a_i(\tau) - \alpha_2 e^{3\tau/2}| \leq C$$

for $i = 2, 3$ and all $\tau \geq 0$.

Proof. Due to (13) and the fact that $\Sigma_- = 0$, the statements follow by arguments similar to those given in the proof of Proposition 3. □

Proof of Theorem 2. Due to (11) there is a constant $c_\theta > 0$ such that

$$|\frac{1}{\theta} - c_\theta e^{3\tau/2}| \leq C.$$

By (12) we conclude that

$$t(\tau) = 2c_\theta e^{3\tau/2} + \delta(\tau)$$

where

$$|\delta(\tau)| \leq C\tau$$

for $\tau \geq 1$. The theorem follows from these observations and the previous corollary. □

6. Approximations for general Bianchi VIII

In this section we exclusively consider non-NUT Bianchi VIII solutions. The behaviour of a such solutions as $\tau \to \infty$ is in some sense oscillatory. The quantities that oscillate are Σ_- and $N_2 - N_3$, and it will turn out that the frequency of the oscillations goes to infinity exponentially. Note that these expressions go to zero as $\tau \to \infty$ by Theorem 6. On the other hand, Σ_+ and $N_1(N_2 + N_3)$ do not converge to zero and their derivatives are bounded. In order to analyze this situation, we will proceed as follows. First we will approximate the oscillations and try to write the quantities that oscillate as multiples of sine and cosine. Then we use these approximations in the expressions for the derivatives of the slowly varying quantities. Integrating over a period, we get an iteration. In order to be able to carry this out, we will however need to know something concerning the variation of different objects within a time interval corresponding to a period, and we will have to spend some time writing down such estimates. Finally, we reformulate the iterations in the way most suitable to our purposes.

Let

$$\tilde{x} = \Sigma_-, \ \tilde{y} = \frac{\sqrt{3}}{2}(N_2 - N_3). \tag{39}$$

We have

$$\tilde{x}' = -3(N_2 + N_3)\tilde{y} + \epsilon_x, \ \epsilon_x = -(2 - q)\Sigma_- + 3N_1\tilde{y}. \tag{40}$$

Furthermore,

$$\tilde{y}' = 3(N_2 + N_3)\tilde{x} + \epsilon_y, \ \epsilon_y = (q + 2\Sigma_+)\tilde{y}. \tag{41}$$

Observe that by the constraint (10) $|\tilde{x}|, |\tilde{y}|, |\epsilon_x|$ and $|\epsilon_y|$ are bounded by numerical constants whereas $N_2 + N_3 \to \infty$ by Theorem 6. Let $g = 3(N_2 + N_3)$,

$$A = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}$$

$\tilde{\mathbf{x}} = {}^t(\tilde{x}, \tilde{y})$ and $\epsilon = {}^t(\epsilon_x, \epsilon_y)$ so that $\tilde{\mathbf{x}}' = A\tilde{\mathbf{x}} + \epsilon$. Let

$$\xi(\tau) = \int_{\tau_0}^{\tau} g(s)ds + \phi_0 \tag{42}$$

for some ϕ_0 and $x_1(\tau) = \cos[\xi(\tau)]$, $y_1(\tau) = \sin[\xi(\tau)]$. Then, if $\mathbf{x}_1 = {}^t(x_1, y_1)$, $\mathbf{x}_1' = A\mathbf{x}_1$. Define

$$\Phi = \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}.$$

Then $\Phi' = -A\Phi$ and $[A, \Phi] = 0$. Let

$$r(\tau) = [\tilde{x}^2(\tau) + \tilde{y}^2(\tau)]^{1/2} \tag{43}$$

and

$$\mathbf{x}(\tau) = {}^t[x(\tau), y(\tau)] = {}^t(r(\tau_0)\cos[\xi(\tau)], r(\tau_0)\sin[\xi(\tau)]) \tag{44}$$

where ϕ_0 has been chosen so that $\mathbf{x}(\tau_0) = \tilde{\mathbf{x}}(\tau_0)$. Since $[\Phi(\mathbf{x} - \tilde{\mathbf{x}})]' = -\Phi\epsilon$ and $\Phi(\tau) \in SO(2)\ \forall\tau$ we have

$$\|\tilde{\mathbf{x}}(\tau) - \mathbf{x}(\tau)\| \leq |\int_{\tau_0}^{\tau} \|\epsilon(s)\| ds|. \tag{45}$$

We are mainly interested in intervals $[\tau_1, \tau_2]$ such that

$$\xi(\tau_2) - \xi(\tau_1) = 2\pi. \tag{46}$$

Note that if $\tau_{\min} \in [\tau_1, \tau_2]$ corresponds to the minimum of $N_2 + N_3$ in this interval, then

$$\tau_2 - \tau_1 \leq \frac{2\pi}{3(N_2 + N_3)(\tau_{\min})}.$$

Furthermore, since $\Sigma_+ \to 1/2$ and $\Sigma_- \to 0$, due to Theorem 6, there is for every $\epsilon > 0$ a T_ϵ such that

$$\exp[(3/2 - \epsilon)\tau] \leq (N_2 + N_3)(\tau) \leq \exp[(3/2 + \epsilon)\tau]$$

for all $\tau \geq T_\epsilon$. The corresponding bounds on $\tau_2 - \tau_1$ will be used without further notice in the following. Let us begin by considering the variation of r in a time interval $[\tau_1, \tau_2]$ satisfying (46).

Lemma 4. *Let $[\tau_1, \tau_2]$ be a time interval such that (46) is fulfilled. If τ_1 is big enough, depending on the initial data, then if $\tau_{\min}$ corresponds to the minimum of r in this interval,*

$$|r(\tau_a) - r(\tau_b)| \leq 2r(\tau_{\min})|\tau_2 - \tau_1| \tag{47}$$

for any $\tau_a, \tau_b \in [\tau_1, \tau_2]$. Furthermore,

$$\|\tilde{\mathbf{x}}(\tau) - \mathbf{x}(\tau)\| \leq 2r(\tau_{\min})|\tau_2 - \tau_1|, \tag{48}$$

for $\tau \in [\tau_1, \tau_2]$ where the $\tau_0 \in [\tau_1, \tau_2]$ needed to define $\mathbf{x}$ is arbitrary.

Proof. Let $\tau_{\max}$ correspond to a maximum of r in the interval. Since $\Sigma_+ \to 1/2$, $\Sigma_- \to 0$ and $N_1 \to 0$, we get

$$\|\epsilon(s)\|^2 \leq \frac{10}{4} r^2(s) \leq \frac{10}{4} r^2(\tau_{\max}) \tag{49}$$

for $s \in [\tau_1, \tau_2]$, assuming τ_1 big enough. Thus, by (45), we have

$$|r(\tau) - r(\tau_{\max})| \leq \left(\frac{10}{4}\right)^{1/2} r(\tau_0)|\tau - \tau_{\max}|$$

assuming we have chosen $\tau_0 = \tau_{\max}$ when defining of $\mathbf{x}$. Since $|\tau_2 - \tau_1|$ can be assumed to be arbitrarily small if τ_1 is big enough, (49) implies

$$\|\epsilon(s)\| \leq 2r(\tau_{\min}).$$

The estimates (47) and (48) follow. □

The next lemma collects some technical estimates. We present them here in order not to interrupt the flow of later proofs.

Lemma 5. *Consider a Bianchi VIII solution of (8)-(10) and let C be a constant. Then there is a T depending on C and the initial values such that if $[\tau_1, \tau_2]$ is a time interval with $\tau_1 \geq T$ and*

$$|\tau_2 - \tau_1| \leq \frac{C}{(N_2 + N_3)(\tau_3)}$$

for some $\tau_3 \in [\tau_1, \tau_2]$, then

$$|\Sigma_+(t_1) - \Sigma_+(t_2)| \leq \frac{C_1}{(N_2 + N_3)(t_3)} \tag{50}$$

$$|[N_1(N_2 + N_3)](t_1) - [N_1(N_2 + N_3)](t_2)| \leq \frac{C_2}{(N_2 + N_3)(t_3)} \tag{51}$$

$$|1 - \frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)}| \leq \frac{C_3}{(N_2 + N_3)(t_3)} \tag{52}$$

for arbitrary $t_1, t_2, t_3 \in [\tau_1, \tau_2]$ where C_1, C_2 and C_3 are constants only depending on C.

Proof. We can assume T to be great enough that $\Delta\tau = \tau_2 - \tau_1 \leq 1$. The inequality

$$|\frac{(N_2 + N_3)'}{N_2 + N_3}| \leq 8$$

follows from (8)-(10). Thus

$$|\frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)}| \leq e^{8(\tau_2 - \tau_1)}$$

so that if $(N_2 + N_3)(t_1) \geq (N_2 + N_3)(t_2)$,

$$|1 - \frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)}| \leq |e^{8\Delta\tau} - 1| \leq e^8 \Delta\tau \tag{53}$$

since $\Delta\tau \leq 1$. Multiplying this inequality by $(N_2 + N_3)(t_2)/(N_2 + N_3)(t_1) \leq 1$ we conclude that the assumption $(N_2 + N_3)(t_1) \geq (N_2 + N_3)(t_2)$ is not essential. There is thus a constant $c < \infty$ such that

$$\frac{(N_2 + N_3)(t_1)}{(N_2 + N_3)(t_2)} \leq c$$

for any $t_1, t_2 \in [\tau_1, \tau_2]$. Consequently

$$\Delta\tau \leq \frac{C}{(N_2+N_3)(\tau_3)} \leq \frac{cC}{(N_2+N_3)(t_3)}. \tag{54}$$

By (8)-(10) $|\Sigma_+'|$ and $|[N_1(N_2+N_3)]'|$ are bounded by constants independent of Bianchi VIII solution. Equations (50) and (51) follow from (54). Equation (52) follows from (53) and (54). □

Lemma 6. *Let $[\tau_1, \tau_2]$ be a time interval such that (46) holds. Then, if τ_1 is big enough, there is a constant $C > 0$ such that*

$$\int_{\tau_1}^{\tau_2} \sin^2 \xi(\tau) d\tau = \frac{1}{2}(\tau_2 - \tau_1) + \eta_1 \tag{55}$$

and

$$\int_{\tau_1}^{\tau_2} \sin^4 \xi(\tau) d\tau = \frac{3}{8}(\tau_2 - \tau_1) + \eta_2 \tag{56}$$

where $|\eta_i| \leq C/(N_2+N_3)^2(\tau_3)$ for any $\tau_3 \in [\tau_1, \tau_2]$.

Proof. We have

$$\int_{\tau_1}^{\tau_2} \sin^2 \xi(\tau) d\tau = \int_{\xi(\tau_1)}^{\xi(\tau_2)} \sin^2 \xi \frac{d\xi}{g} = \frac{\pi}{g(\tau_1)} + \delta_1$$

where δ_1 satisfies an estimate of the same form as η_1 by (52). However

$$\tau_2 - \tau_1 = \int_{\xi(\tau_1)}^{\xi(\tau_2)} \frac{d\xi}{g} = \frac{2\pi}{g(\tau_1)} + \delta_2,$$

where δ_2 is of the same type as δ_1. Equation (55) follows, and the argument to prove (56) is similar. □

Consider

$$h = r^2 = \tilde{x}^2 + \tilde{y}^2.$$

Lemma 7. *Consider a Bianchi VIII solution to (8)-(10). There is a constant $C > 0$, such that if τ_1 is big enough and $[\tau_1, \tau_2]$ is an interval such that (46) is fulfilled, then*

$$h_2 - h_1 = [4(z_1 - \frac{1}{2})(1+z_1) + 2h_1 + \beta]h_1 \cdot (\tau_2 - \tau_1)$$

where $z_1 = \Sigma_+(\tau_1)$, $h_i = h(\tau_i)$, and $|\beta| \leq C/(N_2+N_3)(\tau_1)$.

Proof. We have

$$\frac{dh}{d\tau} = 2\tilde{x}\tilde{x}' + 2\tilde{y}\tilde{y}' = 2\tilde{x}[-3(N_2+N_3)\tilde{y} + \epsilon_x] + 2\tilde{y}[3(N_2+N_3)\tilde{x} + \epsilon_y] =$$
$$= 2\tilde{x}\epsilon_x + 2\tilde{y}\epsilon_y = -2(2-q)\tilde{x}^2 + 6N_1\tilde{x}\tilde{y} + 2(q+2\Sigma_+)\tilde{y}^2 =$$
$$= -4(1-\Sigma_+^2)\tilde{x}^2 + 4\tilde{x}^4 + 6N_1\tilde{x}\tilde{y} + 4\Sigma_+(1+\Sigma_+)\tilde{y}^2 + 4\tilde{x}^2\tilde{y}^2.$$

Consider an interval $[\tau_1, \tau_2]$ such that (46) is fulfilled and let $z_1 = \Sigma_+(\tau_1)$. In such an interval, we have, using Lemma 4, 5 and the fact that $|N_1| \leq C/(N_2+N_3)$,

$$\frac{dh}{d\tau} = -4(1-z_1^2)x^2 + 4x^4 + 4z_1(1+z_1)y^2 + 4x^2y^2 + \delta$$

where we have chosen $\tau_0 = \tau_1$ in the definition of $\mathbf{x}$ and δ satisfies

$$|\delta| \leq C \frac{r^2(\tau_1)}{(N_2+N_3)(\tau_1)}$$

where C is a numerical constant for τ_1 big enough. Applying Lemma 6, we conclude that

$$h_2 - h_1 = [-2(1-z_1^2) + \frac{3}{2}h_1 + 2z_1(1+z_1) + \frac{1}{2}h_1]h_1(\tau_2-\tau_1) + \eta$$

where $|\eta| \leq Ch_1 \cdot (\tau_2-\tau_1)/(N_2+N_3)(\tau_1)$ and $h_i = h(\tau_i)$. The lemma follows. □

It will be convenient to reformulate this expression. Observe that, by the constraint,

$$h + \frac{3}{4}N_1^2 + \frac{3}{2}(w-\frac{1}{2}) = (\frac{1}{2}-z)(\frac{1}{2}+z) = -(z-\frac{1}{2}) - (z-\frac{1}{2})^2$$

or

$$h + \frac{3}{4}N_1^2 + \frac{3}{2}v = -u - u^2, \tag{57}$$

where

$$z = \Sigma_+, \quad u = z - \frac{1}{2}, \quad w = -N_1(N_2+N_3) \quad \text{and} \quad v = w - \frac{1}{2}.$$

Lemma 8. *Consider a Bianchi VIII solution to (8)-(10). There is a constant $C > 0$, such that if τ_1 is big enough and $[\tau_1, \tau_2]$ is an interval such that (46) is fulfilled, then*

$$h_2 - h_1 = [-4h_1 - 9v_1 - 2u_1^2 + \delta]h_1(\tau_2 - \tau_1) \tag{58}$$

where $u_1 = u(\tau_1)$, $v_1 = v(\tau_1)$, $h_i = h(\tau_i)$, and $|\delta| \leq C/(N_2+N_3)(\tau_1)$.

Proof. By (57) we conclude that

$$4(z-\frac{1}{2})(1+z) + 2h = 4u^2 + 6u + 2h = 4u^2 - 6h - \frac{9}{2}N_1^2 - 9v - 6u^2 + 2h =$$

$$= -4h - 9v - 2u^2 - \frac{9}{2}N_1^2.$$

The lemma follows by applying Lemma 7. □

Observe that, since u converges to zero, (57) implies

$$u^2 \leq C_1 h^2 + C_2 v^2 + C_3 N_1^4, \tag{59}$$

so that the first two terms in the first factor on the right hand side of (58) are the important ones.

Lemma 9. *Consider a Bianchi VIII solution to (8)-(10). There is a constant $C > 0$, such that if τ_1 is big enough and $[\tau_1, \tau_2]$ is an interval such that (46) is fulfilled, then*

$$v_2 - v_1 = [h_1^2 - \frac{3}{2}v_1 + \psi(h_1, u_1, v_1) + \delta](\tau_2 - \tau_1) \tag{60}$$

where $|\delta| \leq C/(N_2+N_3)(\tau_1)$. Here

$$\psi(h,u,v) = [2h^2 - 3v + \phi(h,u,v)]v + \frac{1}{2}\phi(h,u,v), \tag{61}$$

$$\phi(h,u,v) = 2[\frac{9}{4}v^2 + u^4 + 2hu^2 + 3hv + 3vu^2], \tag{62}$$

and we use the standard notation, i.e. $v_i = v(\tau_i)$ etc.

Proof. Consider

$$w' = -[N_1(N_2+N_3)]' = (2q - 2\Sigma_+)w - 2\sqrt{3}\Sigma_- N_1(N_2 - N_3).$$

With the standard notations, we have

$$w' = 4z_1(z_1 - \frac{1}{2})w_1 + 4x^2 w_1 + \delta$$

in $[\tau_1, \tau_2]$, where $|\delta| \leq C/(N_2+N_3)(\tau_1)$. Consequently

$$w_2 - w_1 = [4z_1(z_1 - \frac{1}{2}) + 2h_1]w_1(\tau_2 - \tau_1) + \delta_1$$

where $|\delta_1| \leq C(\tau_2-\tau_1)/(N_2+N_3)(\tau_1)$. Let us reformulate, using (57),

$$4z(z-\frac{1}{2}) + 2h = 2u + 4u^2 + 2h = -2u^2 - 2h - \frac{3}{2}N_1^2 - 3v + 4u^2 + 2h =$$

$$= 2u^2 - 3v - \frac{3}{2}N_1^2 = 2h^2 - 3v + \phi(h,u,v) + \delta_2,$$

where $\phi(h,u,v)$ is defined by (62), so that in particular,

$$|\phi(h,u,v)| \leq C_1 v^2 + C_2 u^4 + C_3 h u^2 + C_4 h|v|$$

Furthermore $|\delta_2| \leq C(\tau_2-\tau_1)/(N_2+N_3)(\tau_1)$. Thus

$$v_2 - v_1 = [2h_1^2 - 3v_1 + \phi(h_1,u_1,v_1) + \delta_2]w_1(\tau_2-\tau_1) + \delta_1.$$

Letting ψ be defined by (61), we get the conclusion of the lemma. □

It is nice to have an iteration, but it is nicer to have integral expressions.

Lemma 10. *Consider a Bianchi VIII solution to (8)-(10). There is a constant $C > 0$ such that if τ_a is big enough and $[\tau_a, \tau_b]$ is an interval such that $\xi(\tau_b) - \xi(\tau_a)$ is an integer multiple of 2π, then*

$$\frac{h(\tau_b)}{h(\tau_a)} = \exp[-\int_{\tau_a}^{\tau_b}(4h + 9v + 2u^2)d\tau + \delta_5], \tag{63}$$

where

$$|\delta_5| \leq C\int_{\tau_a}^{\tau_b} e^{-s}ds. \tag{64}$$

Proof. Consider (58). Observe that

$$|h_1 \cdot (\tau_2 - \tau_1) - \int_{\tau_1}^{\tau_2} h(s)ds| \leq \frac{C(\tau_2-\tau_1)}{(N_2+N_3)(\tau_1)}$$

by (47), and similarly for the other expressions by (50) and (51), so that

$$\frac{h_2}{h_1} - 1 = -\int_{\tau_1}^{\tau_2}(4h + 9v + 2u^2)d\tau + \delta_3$$

where $|\delta_3| \leq C(\tau_2-\tau_1)/(N_2+N_3)(\tau_1)$. Thus

$$\frac{h_2}{h_1} = \exp[-\int_{\tau_1}^{\tau_2}(4h + 9v + 2u^2)d\tau + \delta_4]$$

where δ_4 satisfies an estimate similar to that of δ_3. The reason is the following. Let

$$a = -\int_{\tau_1}^{\tau_2}(4h + 9v + 2u^2)d\tau + \delta_3.$$

We have

$$\frac{h_2}{h_1} = \exp[\ln(1+a)],$$

so that we need to estimate $\ln(1+a) - a$. However,

$$|\ln(1+a) - a| \leq a^2$$

if $|a| \leq 1/2$. This criterion is fulfilled for τ_1 big enough, and the desired conclusion follows, since

$$a^2 \leq \frac{C(\tau_2 - \tau_1)}{(N_2+N_3)(\tau_1)}.$$

Observe furthermore that for τ_1 big enough

$$|\delta_4| \leq C\int_{\tau_1}^{\tau_2} e^{-s}ds,$$

since $\Sigma_+ \to 1/2$ and $\Sigma_- \to 0$ due to Theorem 6. The lemma follows. □

Observe that (63) implies that v and h cannot both be $L^1([\tau_0,\infty))$, using the bound (59). Similarly, we get

$$v(\tau_b) - v(\tau_a) = \int_{\tau_a}^{\tau_b} [h^2 - \frac{3}{2}v + \psi(h,u,v)]ds + \delta_6 \tag{65}$$

where

$$|\delta_6| \leq C\int_{\tau_a}^{\tau_b} e^{-s}ds. \tag{66}$$

7. Asymptotics of general Bianchi VIII

In the following we wish to use the tools we developed in the previous section in order to analyze the details of the asymptotics of h,u,v. Note that we already know that (h,u,v) converges to zero by Theorem 6. Let us start by bounding v from below.

Lemma 11. *Consider a Bianchi VIII solution to (8)-(10). Let* $0<\alpha<1$. *Then there is a* T_α *such that*

$$v(\tau) \geq -e^{-\alpha\tau} \tag{67}$$

for all $\tau \geq T_\alpha$.

Proof. Consider (65). If v is negative and τ is big enough, then

$$h^2 - \frac{3}{2}v + \psi(h,u,v) \geq -v.$$

Consider now the situation where $[\tau_a,\tau_b]$ corresponds to a single multiple of 2π and τ_a is big enough. Let $0<\alpha<\beta<1$, and assume that

$$|v(\tau_a)| \geq e^{-\beta\tau_a}.$$

Observe that under these assumptions, v cannot change sign in $[\tau_a,\tau_b]$ if τ_a is big enough, since the maximum variation of v in a time interval corresponding to 2π is bounded by $C/(N_2+N_3)(\tau_a)$. Since

$$|\frac{v(\tau)-v(\tau_a)}{v(\tau_a)}| \leq Ce^{-(1-\beta)\tau_a}$$

under these circumstances, assuming τ_a big enough, we have

$$\frac{v(\tau_b)}{v(\tau_a)} \leq 1 - \int_{\tau_a}^{\tau_b} \frac{v(\tau)}{v(\tau_a)} d\tau + \frac{\delta_6}{v(\tau_a)} \leq 1 - (\tau_b - \tau_a) + \delta_7$$

where

$$|\delta_7| \leq C \int_{\tau_a}^{\tau_b} e^{-(1-\beta)\tau} d\tau.$$

Thus

$$\frac{v(\tau_b)}{v(\tau_a)} \leq \exp[-(\tau_b - \tau_a) + \delta_8]$$

where δ_8 satisfies an estimate similar to that of δ_7. Observe that we at this point can drop the restriction that $[\tau_a, \tau_b]$ correspond to 2π. It can be any interval corresponding to an integer multiple of 2π, as long as $v(\tau) \leq -e^{-\beta\tau}$ in the interval. If we assume that $v \leq -\exp(-\beta\tau)$ for all $\tau \geq T$, we get a contradiction. At late enough times, once v has satisfied the inequality $v \geq -\exp(-\beta\tau)$, it cannot at a later point in time fulfill $v \leq -\exp(-\alpha\tau)$. The lemma follows. □

It will be convenient to have the following rough estimate.

Lemma 12. *Consider a Bianchi VIII solution to (8)-(10). For every $\alpha > 0$, there is a T such that $\tau_1, \tau_2, \tau \geq T$ implies*

$$h(\tau) \geq e^{-\alpha\tau}$$

and, assuming $\tau_2 \geq \tau_1$ corresponds to an integral multiple of 2π,

$$h(\tau_2) \geq h(\tau_1) \exp[-\alpha(\tau_2 - \tau_1)].$$

Proof. The lemma follows from the fact that the integrand appearing in (63) tends to zero. □

Lemma 13. *Consider a Bianchi VIII solution to (8)-(10). Then there is a T such that*

$$|v(\tau)| \leq 2h^2(\tau)$$

for all $\tau \geq T$.

Proof. Let us first prove that for every T, there is $\tau \geq T$ such that $v(\tau) \leq \frac{10}{9} h^2(\tau)$. Assume that there is a T such that

$$v(\tau) \geq \frac{10}{9} h^2(\tau) \tag{68}$$

for all $\tau \geq T$. Consider (65). If assumption (68) is satisfied and τ_a is big enough, we conclude that,

$$v(\tau_b) - v(\tau_a) \leq -\frac{1}{2} \int_{\tau_a}^{\tau_b} v ds$$

where $[\tau_a, \tau_b]$ corresponds to an integral multiple of 2π, and we have used (65) and Lemma 12. Consider now the situation where $[\tau_a, \tau_b]$ corresponds to a single multiple of 2π. Observe that

$$|v(\tau)| \geq e^{-\tau/2}$$

if τ big enough, using (68) and Lemma 12. Since

$$|\frac{v(\tau) - v(\tau_a)}{v(\tau_a)}| \leq Ce^{-\tau_a/2}$$

for $\tau \in [\tau_a, \tau_b]$ under these circumstances, assuming τ_a big enough, we have

$$\frac{v(\tau_b)}{v(\tau_a)} \leq 1 - \frac{1}{2}(\tau_b - \tau_a) + \delta_{11}$$

where

$$|\delta_{11}| \leq C \int_{\tau_a}^{\tau_b} e^{-\tau/2} d\tau.$$

Thus

$$\frac{v(\tau_b)}{v(\tau_a)} \leq \exp[-\frac{1}{2}(\tau_b - \tau_a) + \delta_{12}]$$

where δ_{12} satisfies an estimate similar to that of δ_{11}. Observe that we at this point can drop the restriction that $[\tau_a, \tau_b]$ correspond to 2π. It can correspond to any integer multiple of 2π. Combining (68) with this estimate, we get a result incompatible with Lemma 12. For every T, there must thus be a $\tau \geq T$ such that

$$v(\tau) \leq \frac{10}{9} h^2(\tau).$$

Assume now that for every T, there is a $\tau \geq T$ such that

$$v(\tau) \geq 2h^2(\tau).$$

By the above, we can then find intervals $[\tau_1, \tau_2]$ with τ_1 arbitrarily great,

$$\frac{10}{9} h^2(\tau) \leq v(\tau) \leq 2h^2(\tau)$$

for $\tau \in [\tau_1, \tau_2]$ and

$$v(\tau_1) = \frac{10}{9} h^2(\tau_1) \quad \text{and} \quad v(\tau_2) = 2h^2(\tau_2).$$

Observe that the interval $[\tau_1, \tau_2]$ will contain a subinterval corresponding to an arbitrarily large number of multiples of 2π by choosing τ_1 big enough. This is due to Lemma 12 and the fact that v does not vary more than roughly speaking $\exp[-3\tau/2]$ in a time interval corresponding to 2π. Let τ_{2-} be such that $[\tau_1, \tau_{2-}]$ corresponds to an integer multiple of 2π, but $[\tau_{2-}, \tau_2]$ corresponds to less than an integer multiple of 2π. For any $\alpha > 0$, we can choose an interval $[\tau_1, \tau_2]$ as above such that Lemma 12 yields

$$h^2(\tau_{2-}) \geq h^2(\tau_1) \exp[-\alpha(\tau_{2-} - \tau_1)].$$

The above arguments yield

$$v(\tau_{2-}) \leq \frac{10}{9} h^2(\tau_1) \exp[-\frac{1}{2}(\tau_{2-} - \tau_1) + \delta_{12}]$$

with δ_{12} satisfying an estimate as above. Consequently

$$v(\tau_{2-}) \leq \frac{10}{9} h^2(\tau_{2-})$$

if τ_1 is big enough. Since the interval $[\tau_{2-}, \tau_2]$ is to short to remedy this, the lemma follows, at least without the absolute value sign. If v is negative, we use (67) and Lemma 12. □

Lemma 14. *Consider a Bianchi VIII solution to (8)-(10) and let* $0 < \alpha_1 < 1 < \alpha_2$ *be arbitrary numbers such that* $\alpha_1\alpha_2 = 1$*. Then, if* τ_a *is big enough,*

$$\frac{\alpha_1 h(\tau_a)}{[1+4h(\tau_a)(\tau-\tau_a)]^{\alpha_2/\alpha_1}} \leq h(\tau) \leq \frac{\alpha_2 h(\tau_a)}{[1+4h(\tau_a)(\tau-\tau_a)]^{\alpha_1/\alpha_2}}$$

for all $\tau \geq \tau_a$*.*

Proof. Given $\alpha_1 < 1 < \alpha_2$ such that $\alpha_1\alpha_2 = 1$, there is a T such that if $\tau_a \geq T$ and $\tau \in (\tau_a - \epsilon_a, \infty)$, where $\epsilon_a > 0$ depends on τ_a, then

$$\alpha_1 \exp[-4\alpha_2 \int_{\tau_a}^{\tau} h(s)ds] \leq \frac{h(\tau)}{h(\tau_a)} \leq \alpha_2 \exp[-4\alpha_1 \int_{\tau_a}^{\tau} h(s)ds]. \tag{69}$$

Note first of all that it holds if $\tau \geq \tau_a$ is such that $[\tau_a, \tau]$ corresponds to an integer multiple of 2π, due to (63), Lemma 13, (59) and Lemma 12. However, moving around in a time interval that corresponds to less than a multiple of 2π doesn't change any of the constituents of the equation all that much. Let

$$g(\tau) = 4\int_{\tau_a}^{\tau} h(s)ds.$$

By (69), we have

$$\alpha_1 \exp[-\alpha_2 g(\tau)] \leq \frac{\dot{g}(\tau)}{\dot{g}(\tau_a)} \leq \alpha_2 \exp[-\alpha_1 g(\tau)].$$

Integrating the left inequality, we conclude that

$$1 + \dot{g}(\tau_a)(\tau - \tau_a) \leq \exp[\alpha_2 g(\tau)],$$

and by integrating the right, we get

$$\exp[\alpha_1 g(\tau)] \leq 1 + \dot{g}(\tau_a)(\tau - \tau_a).$$

Inserting these inequalities in (69) yields the conclusion of the lemma. □

Corollary 6. *Consider a Bianchi VIII solution to (8)-(10) and let* $0 < \alpha_1 < 1 < \alpha_2$ *be arbitrary numbers. Then, if* τ *is great enough,*

$$\tau^{-\alpha_2} \leq h(\tau) \leq \tau^{-\alpha_1}.$$

Proposition 4. *Consider a Bianchi VIII solution to (8)-(10). There is a* T *and a constant* $C > 0$*, such that for all* $\tau \geq T$*,*

$$|h(\tau) - \frac{1}{4\tau}| \leq \frac{C \ln \tau}{\tau^2}.$$

Proof. By the proof of Lemma 7,

$$\frac{dh}{d\tau} = -4(1-\Sigma_+^2)\tilde{x}^2 + 4\tilde{x}^4 + 6N_1\tilde{x}\tilde{y} + 4\Sigma_+(1+\Sigma_+)\tilde{y}^2 + 4\tilde{x}^2\tilde{y}^2,$$

so that

$$\frac{1}{h^2}\frac{dh}{d\tau} = -4(1-\Sigma_+^2)\frac{\tilde{x}^2}{h^2} + 4\frac{\tilde{x}^4}{h^2} + 6N_1\frac{\tilde{x}\tilde{y}}{h^2} + 4\Sigma_+(1+\Sigma_+)\frac{\tilde{y}^2}{h^2} + 4\frac{\tilde{x}^2\tilde{y}^2}{h^2}.$$

Consider a time interval $[\tau_1, \tau_2]$ such that (46) is satisfied. Observe that

$$|\frac{1}{h^2(\tau)} - \frac{1}{h^2(\tau_1)}| = \frac{[h(\tau)+h(\tau_1)]|h(\tau)-h(\tau_1)|}{h^2(\tau)h^2(\tau_1)}.$$

Since

$$|h(\tau) - h(\tau_1)| \leq Ch(\tau_1)|\tau_2 - \tau_1|$$

by (47), and Corollary 6 holds, we conclude that

$$|\frac{1}{h^2(\tau)} - \frac{1}{h^2(\tau_1)}| \leq Ce^{-\tau_1}$$

for $\tau \in [\tau_1, \tau_2]$, if τ_1 is big enough. Due to Corollary 6, Lemma 5 and (48), we also conclude that

$$\frac{1}{h^2}\frac{dh}{d\tau} = -4(1-z_1^2)\frac{x^2}{h_1^2} + 4\frac{x^4}{h_1^2} + 4z_1(1+z_1)\frac{y^2}{h_1^2} + 4\frac{x^2y^2}{h_1^2} + \delta_{13},$$

where $|\delta_{13}| \leq C\exp(-\tau_1)$. By Lemma 6 we conclude that

$$-\frac{1}{h(\tau_2)} + \frac{1}{h(\tau_1)} = [-2(1-z_1^2)\frac{1}{h_1} + \frac{3}{2} + 2z_1(1+z_1)\frac{1}{h_1} + \frac{1}{2}](\tau_2 - \tau_1) + \delta_{14}$$

where

$$|\delta_{14}| \leq C\int_{\tau_1}^{\tau_2} e^{-\tau}d\tau.$$

Let us now reformulate

$$-2(1-z_1^2) + 2z_1(1+z_1) = 4(z_1 - \frac{1}{2})(z_1+1) = 6u_1 + 4u_1^2.$$

Using (57), we get

$$6u_1 + 4u_1^2 = -6u_1^2 - 6h_1 - \frac{9}{2}N_1^2(\tau_1) - 9v_1 + 4u_1^2 = [-6 + \delta_{15}]h_1$$

where $|\delta_{15}| \leq Ch_1$, due to Lemma 13, Corollary 6 and (59). We conclude that

$$-\frac{1}{h(\tau_2)} + \frac{1}{h(\tau_1)} = -4(\tau_2 - \tau_1) + \delta_{16}$$

where

$$|\delta_{16}| \leq C\int_{\tau_1}^{\tau_2} h(s)ds.$$

At this point, we can drop the restriction that $[\tau_1, \tau_2]$ correspond to one multiple of 2π. The statement holds as long as it corresponds to an integer multiple of 2π. Fix τ_1 big enough so that the above analysis holds, and define

$$\zeta(\tau) = \frac{1}{h(\tau)} - 4\tau.$$

If $[\tau_1, \tau_2]$ corresponds to an integral multiple of 2π, then

$$\zeta(\tau_2) = \frac{1}{h(\tau_1)} - 4\tau_1 - \delta_{16}.$$

Fix $\alpha > 0$. Using Corollary 6, and assuming τ to be big enough, we conclude that

$$|\zeta(\tau)| \leq C_\alpha \tau^\alpha.$$

At this point, we see that the same inequality holds without any restriction on τ except that it be big enough. Going through the same argument using this improved knowledge concerning the asymptotic behaviour of h, we conclude that for τ great enough, $|\zeta(\tau)| \leq C\ln\tau$. The proposition follows. $\square$

Proposition 5. *Consider a Bianchi VIII solution to (8)-(10). There is a T and a constant $C > 0$ such that*

$$|v(\tau) - \frac{1}{24\tau^2}| \leq \frac{C \ln \tau}{\tau^3}$$

for all $\tau \geq T$.

Proof. Let

$$\chi = h^2 - \frac{3}{2} v$$

and consider an interval $[\tau_1, \tau_2]$ such that (46) is satisfied. Use (58) and (47) in order to deduce

$$h_2^2 - h_1^2 = (h_2 + h_1)(h_2 - h_1) = (h_2 + h_1)[-4h_1 - 9v_1 - 2u_1^2 + \delta] h_1 (\tau_2 - \tau_1) =$$

$$= 2h_1^2[-4h_1 - 9v_1 - 2u_1^2 + \delta_{17}](\tau_2 - \tau_1),$$

where $|\delta_{17}| \leq C/(N_2 + N_3)(\tau_1)$. Combining this with (60), we get the conclusion

$$\chi_2 - \chi_1 = [-\frac{3}{2}\chi_1 - \frac{3}{2}\psi(h_1, u_1, v_1) - 8h_1^3 - 18 v_1 h_1^2 - 4u_1^2 h_1^2 + \delta_{18}](\tau_2 - \tau_1),$$

where δ_{18} satisfies a bound similar to that of δ_{17}. Similarly to earlier arguments, we conclude that if $[\tau_a, \tau_b]$ corresponds to an integer multiple of 2π and τ_a is big enough, then

$$\chi(\tau_b) - \chi(\tau_a) = \int_{\tau_a}^{\tau_b} [-\frac{3}{2}\chi - \frac{3}{2}\psi(h, u, v) - 8h^3 - 18vh^2 - 4u^2h^2] d\tau + \delta_{19},$$

where

$$|\delta_{19}| \leq C \int_{\tau_a}^{\tau_b} e^{-\tau} d\tau.$$

Using Lemma 13, (59), Proposition 4 and the definition of ψ, one deduces

$$\chi(\tau_b) - \chi(\tau_a) = -\frac{3}{2} \int_{\tau_a}^{\tau_b} \chi d\tau + \delta_{20},$$

Where

$$|\delta_{20}| \leq C_1 \int_{\tau_a}^{\tau_b} \tau^{-3} d\tau.$$

Assuming that there is a T such that $\tau \geq T$ implies

$$|\chi(\tau)| \geq 2C_1 \tau^{-3}$$

yields the conclusion

$$\chi(\tau_b) - \chi(\tau_a) \leq - \int_{\tau_a}^{\tau_b} \chi d\tau,$$

if χ is positive, and

$$\chi(\tau_b) - \chi(\tau_a) \geq - \int_{\tau_a}^{\tau_b} \chi d\tau,$$

if χ is negative, assuming τ_a is big enough. Arguments similar to ones given in the proof of Lemma 13 yield the conclusion that

$$|\chi(\tau)| \leq C\tau^{-3}$$

if τ is big enough. The proposition follows. □

Proposition 6. *Consider a Bianchi VIII solution to (8)-(10). There are constants* $C > 0$ *and* $c_{h,1}$ *and a* T *such that*

$$|h(\tau) - \frac{1}{4\tau} + \frac{\ln\tau}{8\tau^2} - \frac{c_{h,1}}{\tau^2}| \leq \frac{C\ln^2\tau}{\tau^3} \tag{70}$$

for all $\tau \geq T$.

Proof. Consider

$$\tilde{h}(\tau) = g(\tau)(\tau h(\tau) - \frac{1}{4})$$

where

$$g(\tau) = \exp(\int_1^\tau 4h(s)ds - c_1)$$

and

$$c_1 = \int_1^\infty (4h(s) - \frac{1}{s})ds.$$

That the integral defining c_1 converges follows from Proposition 4. As a consequence, we have

$$|\int_1^\tau (4h(s) - \frac{1}{s})ds - c_1| \leq C\int_\tau^\infty \frac{\ln s}{s^2}ds \leq C\frac{\ln\tau}{\tau}.$$

Thus

$$|g(\tau) - \tau| = \tau|\exp[\int_1^\tau (4h(s) - \frac{1}{s})ds - c_1] - 1| \leq C\ln\tau$$

for τ great enough. Let $[\tau_1, \tau_2]$ be an interval such that (46) is satisfied. With the usual notation, we have

$$\tilde{h}_2 - \tilde{h}_1 = (g_2 - g_1)(\tau_2 h_2 - \frac{1}{4}) + g_1(\tau_2 h_2 - \tau_1 h_1) = \tag{71}$$

$$= (g_2 - g_1)(\tau_2 h_2 - \tau_1 h_1) + (g_2 - g_1)(\tau_1 h_1 - \frac{1}{4}) + g_1(\tau_2 h_2 - \tau_1 h_1).$$

Compute

$$g_2 - g_1 = g_1[\exp(\int_{\tau_1}^{\tau_2} 4h(s)ds) - 1] = 4g_1 h_1(\tau_2 - \tau_1) + \delta_{20},$$

where δ_{20} is of the order of magnitude $(\tau_2 - \tau_1)^2$. Consider

$$\tau_2 h_2 - \tau_1 h_1 = (\tau_2 - \tau_1)h_2 + \tau_1(h_2 - h_1) =$$

$$= (\tau_2 - \tau_1)(h_2 - h_1) + (\tau_2 - \tau_1)h_1 + \tau_1(h_2 - h_1).$$

Observe that, even if we multiply the first term with g_1, it is still of the order of magnitude $(\tau_2 - \tau_1)^2$, using (58), and will for this reason be of no importance. By (58), we conclude

$$(\tau_2 - \tau_1)h_1 + \tau_1(h_2 - h_1) =$$

$$= [h_1 - 4\tau_1 h_1^2 - 9\tau_1 v_1 h_1 - 2\tau_1 u_1^2 h_1 + \tau_1 \delta h_1](\tau_2 - \tau_1)$$

Consider the first term on the far right hand side of (71). By the above it will be of the order of magnitude $(\tau_2 - \tau_1)^2$. We conclude that

$$\tilde{h}_2 - \tilde{h}_1 =$$

$$= [4g_1 h_1(\tau_1 h_1 - \frac{1}{4}) + g_1 h_1 - 4\tau_1 g_1 h_1^2 - 9\tau_1 g_1 v_1 h_1 - 2\tau_1 g_1 u_1^2 h_1](\tau_2 - \tau_1) + \delta_{21}$$

where

$$|\delta_{21}| \leq C \int_{\tau_1}^{\tau_2} e^{-s} ds.$$

In other words,

$$\tilde{h}_2 - \tilde{h}_1 = [-9\tau_1 g_1 v_1 h_1 - 2\tau_1 g_1 u_1^2 h_1](\tau_2 - \tau_1) + \delta_{21}.$$

As in previous arguments, we conclude that

$$\tilde{h}(\tau_b) - \tilde{h}(\tau_a) = \int_{\tau_a}^{\tau_b} [-9\tau g v h - 2\tau g u^2 h] ds + \delta_{22}$$

where

$$|\delta_{22}| \leq C \int_{\tau_a}^{\tau_b} e^{-s} ds$$

and $[\tau_a, \tau_b]$ corresponds to an integer multiple of 2π. Observe that

$$-9\tau g v h - 2\tau g u^2 h = -\frac{9\tau^2}{24\tau^2 \cdot 4\tau} - \frac{2\tau^2}{64\tau^3} + O(\frac{\ln\tau}{\tau^2}) = -\frac{1}{8\tau} + O(\frac{\ln\tau}{\tau^2})$$

In other words, we see that

$$\tilde{h}(\tau) + \frac{1}{8}\ln\tau$$

converges. Let $c_{h,1}$ denote the value to which it converges. Then by the above estimates, we have

$$|\tilde{h}(\tau) + \frac{1}{8}\ln\tau - c_{h,1}| \leq C\frac{\ln\tau}{\tau}.$$

In other words,

$$|h(\tau) - \frac{1}{4\tau} + \frac{1}{8}\frac{\ln\tau}{\tau g(\tau)} - \frac{c_{h,1}}{\tau g(\tau)}| \leq C\frac{\ln\tau}{\tau^3}. \tag{72}$$

However, the above estimates imply

$$|\frac{\ln\tau}{\tau g(\tau)} - \frac{\ln\tau}{\tau^2}| \leq C\frac{\ln^2\tau}{\tau^3}$$

and

$$|\frac{1}{\tau g(\tau)} - \frac{1}{\tau^2}| \leq C\frac{\ln\tau}{\tau^3}$$

so that the proposition follows. □

Observe that the estimate (70) can be improved in the following way. Using (70), we conclude that

$$\int_1^{\tau} (4h(s) - \frac{1}{s}) ds - c_1 = -\int_{\tau}^{\infty} (4h(s) - \frac{1}{s}) ds = \int_{\tau}^{\infty} [\frac{\ln s}{2s^2} + O(\frac{1}{s^2})] ds.$$

Consequently

$$|\int_1^{\tau} (4h(s) - \frac{1}{s}) ds - c_1 - \frac{\ln\tau}{2\tau}| \leq \frac{C}{\tau}.$$

Thus

$$|g(\tau) - \tau - \frac{1}{2}\ln\tau| = \tau|\exp[\int_1^{\tau} (4h(s) - \frac{1}{s}) ds - c_1 - \frac{\ln\tau}{2\tau}]\exp(\frac{\ln\tau}{2\tau}) - 1 - \frac{\ln\tau}{2\tau}|.$$

By the above

$$|\exp[\int_1^{\tau} (4h(s) - \frac{1}{s}) ds - c_1 - \frac{\ln\tau}{2\tau}] - 1| \leq \frac{C}{\tau}.$$

We also have

$$|\exp(\frac{\ln\tau}{2\tau})-1-\frac{\ln\tau}{2\tau}|\leq C\frac{\ln^2\tau}{\tau^2}.$$

Adding up, we get

$$|g(\tau)-\tau-\frac{1}{2}\ln\tau|\leq C.$$

Thus

$$|\frac{1}{g(\tau)}-\frac{1}{\tau}+\frac{\ln\tau}{2\tau^2}|=|\frac{\tau-g(\tau)}{\tau g(\tau)}+\frac{\ln\tau}{2\tau^2}|=$$

$$=|-\frac{\ln\tau}{2\tau g(\tau)}+\frac{\tau+\ln\tau/2-g(\tau)}{\tau g(\tau)}+\frac{\ln\tau}{2\tau^2}|\leq\frac{C}{\tau^2}$$

Inserting this estimate in (72), we get

$$|h(\tau)-\frac{1}{4\tau}+\frac{1}{8}\frac{\ln\tau}{\tau^2}-\frac{c_{h,1}}{\tau^2}-\frac{1}{16}\frac{\ln^2\tau}{\tau^3}|\leq C\frac{\ln\tau}{\tau^3} \tag{73}$$

Corollary 7. *Consider a Bianchi VIII solution to (8)-(10). There is a constant C and a T such that*

$$|v(\tau)-\frac{1}{24\tau^2}+\frac{\ln\tau}{24\tau^3}|\leq\frac{C}{\tau^3} \tag{74}$$

for all $\tau\geq T$.

Proof. By the proof of Proposition 5, we have

$$|h^2(\tau)-\frac{3}{2}v|\leq\frac{C}{\tau^3}.$$

Combining this with (70) yields the conclusion of the corollary. □

It seems reasonable to think that one should be able to obtain more terms in the expansions, but we will be satisfied at this point.

Corollary 8. *Consider a Bianchi VIII solution to (8)-(10). There are constants c_u and C and a T such that*

$$|u(\tau)+\frac{1}{4\tau}-\frac{\ln\tau}{8\tau^2}-\frac{c_u}{\tau^2}+\frac{\ln^2\tau}{16\tau^3}|\leq\frac{C\ln\tau}{\tau^3}, \tag{75}$$

for all $\tau\geq T$.

Proof. The result follows by combining (73), (74) and (57). □

In what follows we will not try to obtain as detailed expansions as possible, since it is only a matter of work to do so. The interested reader is encouraged to calculate the expansions to higher orders.

Proposition 7. *Consider a non-NUT Bianchi VIII solution to (8)-(10). Then there are constants α_i, $i=1,2,3$, C and a T such that*

$$a_1(\tau)=\alpha_1\tau^{1/2}[1+O(\frac{\ln\tau}{\tau})] \tag{76}$$

$$a_i(\tau)=\frac{\alpha_i}{\tau^{1/4}}\exp(3\tau/2)[1+O(\frac{\ln\tau}{\tau})] \tag{77}$$

for $i=2,3$ and all $\tau\geq T$.

Proof. Using (75), we conclude the existence of a constant c_{s+} such that

$$\int_0^\tau [\frac{1}{2} - \Sigma_+(s)]ds = \frac{1}{4}\ln\tau + c_{s+} + O(\frac{\ln\tau}{\tau}). \tag{78}$$

Thus, if $\alpha_1 = \exp[2c_{s+}]$,

$$a_1(\tau) = \exp[\int_0^\tau 2[\frac{1}{2} - \Sigma_+(s)]ds] = \alpha_1 \tau^{1/2}[1 + O(\frac{\ln\tau}{\tau})].$$

Consider the integral

$$\int_0^\tau \Sigma_-(s)ds. \tag{79}$$

Observe that Σ_- is not $L^1([0,\infty))$, but the above mentioned expression will turn out to converge quite rapidly all the same. By (8) we conclude that

$$\frac{N_2(\tau)}{N_3(\tau)} = \exp\psi(\tau),$$

where

$$\psi(\tau) = \ln\frac{N_2(0)}{N_3(0)} + \int_0^\tau 4\sqrt{3}\Sigma_-(s)ds.$$

Since

$$\left(\frac{N_2(\tau)}{N_3(\tau)} - 1\right)^2 \leq \frac{4}{3}\frac{h(\tau)}{N_3^2(\tau)},$$

we conclude that

$$|\psi(\tau)| \leq \frac{C\cdot h^{1/2}(\tau)}{N_3(\tau)}.$$

Thus the integral expression (79) converges exponentially. We have

$$a_2(\tau) = \exp[\int_0^\tau [1 + \Sigma_+(s) + \sqrt{3}\Sigma_-(s)]ds]$$

$$= \exp(3\tau/2)\exp[\int_0^\tau [\Sigma_+(s) - \frac{1}{2} + \sqrt{3}\Sigma_-(s)]ds].$$

Thus

$$a_2(\tau) = \frac{\alpha_2}{\tau^{1/4}}\exp(3\tau/2)[1 + O(\frac{\ln\tau}{\tau})].$$

The argument concerning a_3 is the same. □

Proof of Theorem 3. In order to relate Wainwright Hsu time to proper time, we need to consider

$$\int_0^\tau [1 + q(s)]ds = \frac{3}{2}\tau + 2\int_0^\tau \Sigma_-^2(s)ds + 2\int_0^\tau [\Sigma_+(s) - \frac{1}{2}][\Sigma_+(s) + \frac{1}{2}]ds =$$

$$= \frac{3}{2}\tau + 2\int_0^\tau \Sigma_-^2(s)ds + 2\int_0^\tau [\Sigma_+(s) - \frac{1}{2}]ds + 2\int_0^\tau [\Sigma_+(s) - \frac{1}{2}]^2 ds.$$

Let us start by considering the integral involving Σ_-. Let $[\tau_1, \tau_2]$ be an interval such that (46) is fulfilled. Since we have the approximation (48), we get

$$\int_{\tau_1}^{\tau_2} \Sigma_-^2(s)ds = h(\tau_1)\int_{\tau_1}^{\tau_2} \cos^2\xi(\tau)d\tau + h(\tau_1)\epsilon_1 = \frac{1}{2}h(\tau_1)(\tau_2 - \tau_1) + h(\tau_1)\epsilon_2 =$$

$$= \frac{1}{2}\int_{\tau_1}^{\tau_2} h(s)ds + h(\tau_1)\epsilon_3$$

where $|\epsilon_i| \leq C/(N_2+N_3)^2$, $i=1,2,3$. In order to obtain this expression, we have also used (55) and (47). Consequently, there is a constant c_{s-} such that

$$\phi(\tau)=\int_0^\tau \Sigma_-^2(s)ds-\frac{1}{2}\int_0^\tau h(s)ds\to c_{s-}.$$

Furthermore,

$$|\phi(\tau)-c_{s-}|\leq Ce^{-\tau}.$$

Combining this estimate with (73), we get the conclusion

$$2\int_0^\tau \Sigma_-^2(s)ds=\frac{1}{4}\ln\tau+c_{s-,1}+O(\frac{\ln\tau}{\tau}).$$

Using (75), we also have

$$2\int_0^\tau [\Sigma_+(s)-\frac{1}{2}]ds+2\int_0^\tau [\Sigma_+(s)-\frac{1}{2}]^2ds=$$

$$=-\frac{1}{2}\ln\tau+c_1+O(\frac{\ln\tau}{\tau}).$$

In consequence,

$$\int_0^\tau [1+q(s)]ds=\frac{3}{2}\tau-\frac{1}{4}\ln\tau+c_{q,1}+O(\frac{\ln\tau}{\tau}). \tag{80}$$

By (11), we conclude that

$$\frac{1}{\theta(\tau)}=\frac{\alpha_\theta}{\tau^{1/4}}e^{3\tau/2}[1+O(\frac{\ln\tau}{\tau})].$$

Using (12), we conclude that

$$t(\tau)=\frac{2\alpha_\theta}{\tau^{1/4}}e^{3\tau/2}[1+O(\frac{\ln\tau}{\tau})]. \tag{81}$$

This implies that $\tau=2\ln t/3+O(\ln\ln t)$ so that

$$\tau^{1/2}=(\frac{2}{3}\ln t)^{1/2}[1+O(\frac{\ln\ln t}{\ln t})]^{1/2}=(\frac{2}{3}\ln t)^{1/2}[1+O(\frac{\ln\ln t}{\ln t})].$$

Combining this with Proposition 7, we conclude that

$$a_1(t)=\alpha_1(\frac{2}{3}\ln t)^{1/2}[1+O(\frac{\ln\ln t}{\ln t})]$$

and

$$a_i(t)=\frac{\alpha_i}{2\alpha_\theta}t[1+O(\frac{\ln\ln t}{\ln t})]$$

for $i=2,3$. By renaming the α_i we get the conclusion of the theorem except for the last statement. Note that

$$N_2=(N_2N_3)^{1/2}+N_2^{1/2}\frac{N_2-N_3}{N_2^{1/2}+N_3^{1/2}}.$$

The second term tends to zero and will for this reason not be of interest to us. Combining (78) and (80) we get

$$\frac{(N_2N_3)^{1/2}(\tau)}{(N_2N_3)^{1/2}(0)}=\exp[\int_0^\tau (q+2\Sigma_+)ds]=e^c\tau^{-3/4}e^{3\tau/2}[1+O(\frac{\ln\tau}{\tau})].$$

Consequently, there is a positive constant c_0 such that

$$N_2(\tau)=c_0\tau^{-3/4}e^{3\tau/2}[1+O(\frac{\ln\tau}{\tau})]. \tag{82}$$

Since

$$\frac{a_2}{a_3} = \left(\frac{N_3(0)}{N_2(0)}\right)^{1/2}\left(\frac{N_2}{N_3}\right)^{1/2}$$

by (13) and (8), it is of interest to note that

$$|\left(\frac{N_2}{N_3}\right)^{1/2} - 1| \leq |\frac{N_2}{N_3} - 1| \leq \frac{2}{\sqrt{3}}\frac{h^{1/2}}{N_3} \leq C\frac{\tau^{-1/2}}{\tau^{-3/4}e^{3\tau/2}} \leq \frac{C}{t},$$

where we have used (82), (73) and (81). The theorem follows. □

8. The isometry group of Bianchi VIII initial data

In the disc model of Hyperbolic space, the underlying manifold is the open unit disc D, and the metric is given by

$$g_D = \frac{4}{(1-x^2-y^2)^2}g_0$$

where g_0 is the standard Euclidean metric on $\mathbb{R}^2$.

Lemma 15. *Consider an orientation preserving isometry ϕ of (D, g_D), i.e. an orientation preserving diffeomorphism of D such that*

$$\phi^* g_D = g_D.$$

Then there is an $\alpha \in \mathbb{R}$ and a $z_0 \in D$ such that

$$\phi(z) = \frac{e^{i\alpha}z + z_0}{1 + \bar{z}_0 e^{i\alpha}z}. \tag{83}$$

Proof. The form of the metric implies that ϕ preserves angles in the Euclidean sense of the word, at least up to a sign. The condition that ϕ be orientation preserving ensures that angles are preserved. As a consequence, ϕ must be a biholomorphic map from D to itself, i.e. a holomorphic map with a holomorphic inverse. Let

$$\psi(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$$

where $z_0 = \phi(0)$. Then $f = \psi \circ \phi$ is biholomorphic and $f(0) = 0$. By the chain rule

$$f'(0)(f^{-1})'(0) = 1$$

and by the Schwarz lemma,

$$|f'(0)| \leq 1, \quad \text{and} \quad |(f^{-1})'(0)| \leq 1.$$

In consequence $|f'(0)| = 1$, and the Schwarz lemma yields the conclusion

$$f(z) = e^{i\alpha}z$$

for some real number α. The conclusion follows. □

Let $H^2 = \{(x,y) \in \mathbb{R}^2 |\ y > 0\}$ be the upper half plane. The map

$$\phi_{HD}(z) = \frac{z - i}{z + i} \tag{84}$$

defines a biholomorphic map from H^2 to D. If we give H^2 the Riemannian metric

$$g_H = \frac{1}{y^2}g_0,$$

then ϕ_{HD} is an isometry. We can of course identify the group of isometries of (H^2, g_H) with the group of isometries of (D, g_D) using ϕ_{HD}. Using this observation together with Lemma 15, we conclude that the orientation preserving isometries of (H^2, g_H) are the maps of the form

$$\phi(z) = \frac{az+b}{cz+d} \tag{85}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - cb = 1$. Obviously, we have a map from $\mathrm{Sl}(2,\mathbb{R})$ onto the orientation preserving isometry group. This map is a homomorphism, and we have a smooth Lie group action of $\mathrm{Sl}(2,\mathbb{R})$ on the left of H^2. Since the kernel is generated by $-\mathrm{Id}$, we get a smooth action of $\mathrm{PSl}(2,\mathbb{R})$ on H^2. The action of this group is effective. We can thus identify the group of orientation preserving isometries of hyperbolic space with $\mathrm{PSl}(2,\mathbb{R})$. The action of $\mathrm{PSl}(2,\mathbb{R})$ on H^2 can be extended to an action on UH^2, the unit tangent bundle of hyperbolic space, by for each group element g considering the push forward of the corresponding map (below, the push forward of the action of a Lie group element g on hyperbolic space will be denoted by g_*). In this way we get a smooth Lie group action on UH^2 on the left which is free and transitive, and if we fix an element $e \in UH^2$, the map obtained by letting the Lie group act on this element has non-degenerate derivative at each Lie group element. Consequently, after having chosen an identity element $e \in UH^2$, we have a diffeomorphism from $\mathrm{PSl}(2,\mathbb{R})$ to UH^2 given by $\phi(g) = g_* e$. It will be convenient to use different representations of hyperbolic space at different times. Conclusions that have been obtained using one representation can trivially be carried over to the others and we will do so without comment. For instance, we will consider the map defined by (83) to be an element of $\mathrm{PSl}(2,\mathbb{R})$ and consider UH^2 and UD to be the same. One can define a map $\rho : \mathbb{R} \times D \to UD$ by

$$\rho(\alpha, z) = \cos\alpha e_{1,z} + \sin\alpha e_{2,z}$$

where e_i is the unit vector in the direction of ∂_i. This defines a covering map of UD, and by going to the quotient, one obtains a diffeomorphism from $S^1 \times D$ to UD. Locally, ρ defines what we shall refer to as αxy-coordinates. Observe that the above means that $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ is $\mathbb{R}^3$ topologically.

We have a diffeomorphism from $\mathrm{PSl}(2,\mathbb{R})$ to UD. It can be used to give UD a group structure such that the diffeomorphism becomes a Lie group isomorphism. Let $e = e_{1,0} \in UD$ be the identity element. If ϕ_i are orientation preserving isometries of (D, g_D) for $i = 1, 2$, we have

$$(\phi_{1*}e)(\phi_{2*}e) = (\phi_1\phi_2)_* e. \tag{86}$$

Since ρ is a covering map, there is a unique way of making $\mathbb{R} \times D$ a Lie group such that ρ becomes a homomorphism, assuming one has chosen an identity element, see e.g. [7]. Let $(0, 0) \in \mathbb{R} \times D$ be the identity. Then the fact that ρ is a covering map and a homomorphism can be used to conclude that

$$(2\pi t, 0) \cdot (\alpha, z) = (\alpha + 2\pi t, e^{i2\pi t} z), \quad (\alpha, z) \cdot (2\pi t, 0) = (\alpha + 2\pi t, z). \tag{87}$$

In particular, $(2\pi n, 0)$ commutes with all elements of $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ and corresponds to a translation by $2\pi n$. These translations constitute the kernel of ρ.

We are interested in finding compactifications of Bianchi VIII initial data, and in that context, the following observation is relevant. Note that this contradicts a statement in [15]. However, as was noted in [16], the statement found in [15] is incorrect.

Lemma 16. *For every $p \in \mathbb{N}$ with $p > 1$, there is a subgroup Ξ_p of $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ such that Ξ_p, considered as a group of diffeomorphisms acting on the left, is a free and properly discontinuous group of diffeomorphisms such that*

$$\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Xi_p \cong U\Sigma_p,$$

where Σ_p is the compact orientable 2-manifold of genus p, and $U\Sigma_p$ is the unit tangent bundle of this surface with respect to a suitable hyperbolic metric. Here $\cong$ symbolizes the existence of a diffeomorphism.

Remark. The above statement is of course trivial. Something which is far from trivial is to classify all the subgroups Γ of $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ that are free and properly discontinuous when acting on the left and yield compact quotients. This is done in [10], where the corresponding topological spaces are also described.

Proof. Let $\phi : \mathrm{PSl}(2,\mathbb{R}) \to UH$ be defined by $\phi(g) = g_* e$, where e is a fixed element of UH. Then $hg = \phi^{-1}[h_*\phi(g)]$, so that a subgroup Γ of $\mathrm{PSl}(2,\mathbb{R})$ acting on the left is a free and properly discontinuous group of diffeomorphisms of $\mathrm{PSl}(2,\mathbb{R})$ if and only if the group $\Gamma_* = \{g_* | g \in \Gamma\}$ is a free and properly discontinuous group of diffeomorphisms of UH^2. In that case, ϕ defined above yields a diffeomorphism

$$\hat{\phi} : \mathrm{PSl}(2,\mathbb{R})/\Gamma \to UH^2/\Gamma_*.$$

For every $p > 1$, there is a subgroup Γ_p of $\mathrm{PSl}(2,\mathbb{R})$, such that Γ_p acting on H^2 is a free and properly discontinuous group of orientation preserving isometries of H^2 such that H^2/Γ_p is diffeomorphic to Σ_p, the compact orientable 2-manifold with genus p. The push forward of the projection map $\pi_p : H^2 \to \Sigma_p$ defines a smooth map $\pi_{p*} : UH^2 \to U\Sigma_p$, where $U\Sigma_p$ is the unit tangent bundle of Σ_p with respect to the natural hyperbolic metric induced by taking the quotient. The map π_{p*} identifies v_1 and v_2 if and only if $v_2 = h_* v_1$ for $h_* \in \Gamma_{p*} = \{h_* | h \in \Gamma_p\}$. Since Γ_{p*} is a free and properly discontinuous group of diffeomorphisms on UH^2, the map π_{p*} defines an isometry

$$\psi : UH^2/\Gamma_{p*} \to U\Sigma_p.$$

By the above correspondence, we get a covering projection

$$\pi_{\Sigma_p} : \mathrm{PSl}(2,\mathbb{R}) \to U\Sigma_p$$

where the covering transformations are the left translations by elements of Γ_p. Furthermore, we have the covering projection

$$\pi_{\mathrm{PSl}} : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to \mathrm{PSl}(2,\mathbb{R}).$$

Observe that $\Xi_p = \pi_{\mathrm{PSl}}^{-1}\Gamma_p$ is a subgroup of $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ since the projection is a homomorphism. Composing two covering projections, one does not necessarily get a covering projection, but in our situation, this is the case, see for instance Spanier [14]. One can also check that the group of covering transformations consists of the subgroup Ξ_p acting on the left on $\tilde{\mathrm{Sl}}(2,\mathbb{R})$. The lemma follows. □

The discussion below, until and including the proof of Lemma 17, is essentially taken from [13]. It will be necessary in order to prove Theorem 4. Note that one form of isometry is a reflection in a geodesic. By this we mean a map which takes the geodesic to itself and maps every vector v perpendicular to the geodesic to $-v$. Explicitly, we have the reflections

$$\phi(x,y) = (2x_0 - x, y), \quad \phi(z) = \frac{r^2}{\bar{z} - a} + a,$$

where $r, a, x_0 \in \mathbb{R}$ and $r > 0$, in the upper half plane. Given the representation (85) of the orientation preserving isometries, one can check that any orientation preserving isometry can be represented as a product of two reflections in geodesics. In fact, two reflections in straight geodesics gives the translations, two reflections in circular geodesics yield the maps of the form $az + b$ for $a, b \in \mathbb{R}$ and $a > 0$, $a \neq 1$. Finally, a combination of a reflection in a straight geodesic and a reflection in a circular geodesic yields the remaining isometries. Note that if α is a reflection in a geodesic γ and ϕ is an isometry, then $\phi\alpha\phi^{-1}$ is a reflection in the geodesic $\phi\gamma$. Furthermore, there are three distinct combinations of geodesics. i) The geodesics intersect in an interior point of hyperbolic space. ii) The geodesics intersect on the boundary when viewed in the disc model. iii) The geodesics do not intersect in the interior or on the boundary. The respective names for the corresponding isometries are *rotation*, *parabolic* isometry and *hyperbolic* isometry. Let us describe the possibilities in somewhat greater detail. Consider a rotation ψ. Let ϕ be an isometry which takes the intersection point of the two geodesics to the origin in the disc model. The geodesics then become straight lines through the origin and $\phi\psi\phi^{-1}$ is a rotation in the Euclidean sense of the word. The angle of rotation is twice the angle between the two geodesics. Thus non-trivial rotations leave one interior point of hyperbolic space fixed but no other points. Two geodesics which intersect at infinity become, after applying an isometry, two straight lines in the upper half plane. In the upper half plane, the resulting isometry is translation in the x-direction twice the Euclidean distance between the lines. It leaves exactly one point on the boundary of hyperbolic space fixed, but no point in the interior. Let ψ be a hyperbolic isometry. The two geodesics can be assumed to be a straight line and a circle with center on the real line in the upper half plane. One sees that there must be a unique geodesic γ intersecting each of the two geodesics at straight angles. Observe that both the reflections map $\gamma(\mathbb{R})$ to itself, so that the composition maps the geodesic to itself and preserves the orientation. Let ϕ be an isometry taking γ to the real line intersected with the disc. Then $\phi\psi\phi^{-1}$ is an isometry of the disc model taking the real line intersected with the disc into itself. Since every isometry of the disc can be written in the form

$$\chi(z) = \frac{e^{i\alpha}z + z_0}{1 + \bar{z}_0 e^{i\alpha} z},$$

we get the conclusion that z_0 has to be real and $e^{i\alpha} = \pm 1$. Since χ also preserves the orientation of the real line, $e^{i\alpha} = 1$. Thus

$$\chi(z) = \frac{z + t}{1 + tz}, \tag{88}$$

where $t \in \mathbb{R}$. For $t \neq 0$, such a map fixes two points on the boundary but no other points. Consider an orientation reversing isometry ψ of the unit disc model. Observe that the isometry can be extended to a holomorphic map from a neighbourhood of the closed disc into the complex plane. Furthermore, the map will have a non-degenerate derivative and will still map orthogonal vectors, with respect to the Euclidean metric, to orthogonal vectors. Since ψ maps the boundary to itself, a vector normal to the boundary will be mapped to a non-zero multiple of itself. Since the map also maps the interior of the disc to itself, it has to be a positive multiple. Since ψ is orientation reversing, the map from the boundary to itself thus has to be orientation reversing. Thus ψ has to have two fixed points on the boundary.

It will be of interest to find, given a non-trivial orientation preserving isometry ϕ, the group $C(\phi)$ of isometries commuting with ϕ.

Lemma 17. *Let ϕ be a non-trivial orientation preserving isometry of hyperbolic space.*

1. *If ϕ is a rotation, then there is an orientation preserving isometry χ such that $\chi C(\phi)\chi^{-1}$ is the group of rotations around the origin in the disc model.*
2. *If ϕ is a hyperbolic isometry, then there is an orientation preserving isometry χ such that $\chi C(\phi)\chi^{-1}$ is generated by maps of the form (88) and a reflection in the real axis.*
3. *If ϕ is parabolic, then there is an orientation preserving isometry χ such that $\chi C(\phi)\chi^{-1}$ is the group of translations by real numbers in the upper half plane.*

Proof. Note that if ϕ and ψ are two commuting maps, then $\psi = \phi\psi\phi^{-1}$ fixes $\phi[\text{fix}(\psi)]$, where $\text{fix}(\psi)$ is the set of fixed points of ψ. Thus ϕ leaves $\text{fix}(\psi)$ invariant. If ϕ is a rotation, then it fixes a point z of the interior, but no other point. Thus any $\psi \in C(\phi)$ must be a rotation around z. If χ takes z to the origin of the disc model, we get the conclusion that $\chi C(\phi)\chi^{-1}$ coincides with the Euclidean rotations around the origin. If ϕ is hyperbolic, let χ be such that $\chi\phi\chi^{-1}$ is of the form (88). Since $\chi C(\phi)\chi^{-1} = C(\chi\phi\chi^{-1})$, we can assume ϕ to be of the form (88). Note that $C(\phi)$ contains all isometries of this form plus the reflection in the real axis. We wish to prove that this is all there is. If $\psi \in C(\phi)$ and ψ is orientation preserving, then ψ fixes the points ± 1 or interchanges them. If it fixes ± 1, it must be of the form (88). If it interchanges ± 1, we can compose ψ with a rotation to get an orientation preserving isometry fixing ± 1. Thus the composition must be of the form (88) whence the non-trivial rotation must belong to $C(\phi)$ due to the group properties of this set. This is impossible. Given an orientation reversing isometry $\psi \in C(\phi)$, we compose it with a reflection in the real line. By the above, the composition is of the form (88). Finally, consider a non-trivial parabolic isometry ϕ. We can assume that ϕ is a non-trivial translation by a real number in the upper half plane. Since ϕ only leaves one point on the boundary fixed it cannot commute with an orientation reversing isometry. By the representation (85) one can check that the orientation preserving isometries that commute with a non-trivial translation by a real number are translations by a real number. □

Let us construct a basis for the Lie algebra of $\text{PSl}(2,\mathbb{R})$. By the above, we can identify $\text{PSl}(2,\mathbb{R})$ with UD. Let the unit vectorfields e_1 and e_2 be the normalizations of ∂_x and ∂_y, and let us identify the identity element of the group with the vector e_1 at zero. Consider the three curves in UD defined by $\gamma_1(t) = e_{1,0}\cos(t) + e_{2,0}\sin(t)$ $\gamma_2(t) = e_{1,t/2}$ and $\gamma_3(t) = e_{1,\mathrm{i}t/2}$. The derivatives of these curves for $t = 0$ define three tangent vectors at the identity which we will call $E_{i,0}$. In terms of αxy-coordinates close to $(0,0,0)$, we have

$$\gamma_1(t) = (t,0,0),\ \ \gamma_2(t) = (0,t/2,0) \text{ and } \gamma_3(t) = (0,0,t/2),$$

so that the tangent vectors to the different curves form a basis at the identity. We denote the corresponding left invariant vector fields $E_{i,p}$ for $p \in UD$. After some computations, one finds

$$E_1 = \partial_\alpha, \quad E_2 = \cos\alpha e_1 + \sin\alpha e_2 + f\partial_\alpha, \;\; E_3 = -\sin\alpha e_1 + \cos\alpha e_2 + g\partial_\alpha, \tag{89}$$

where

$$f = y\cos\alpha - x\sin\alpha, \quad g = -(x\cos\alpha + y\sin\alpha).$$

Note that the E_i also constitute a left invariant basis on $\mathbb{R} \times D \cong \tilde{\mathrm{Sl}}(2,\mathbb{R})$. The structure constants γ_{ij}^k, determined by

$$[E_i, E_j] = \gamma_{ij}^k E_k,$$

can be computed to be $\gamma_{ij}^k = \epsilon_{ijl} n^{lk}$ where $n = \mathrm{diag}(-1,1,1)$, which is the defining characteristic of Bianchi VIII. Later, it will be important to know that if we change the basis according to

$$\hat{E}_i = (A^{-1})_i{}^k E_k,$$

we get, for the structure constants $\hat{\gamma}_{ij}^k = \epsilon_{ijl}\hat{n}^{lk}$ corresponding to $\hat{E}_i$,

$$\hat{n} = \frac{1}{\det A} {}^t A n A. \tag{90}$$

Before we go on, let us note the following. If G_i are two simply connected Lie groups and ϕ is a Lie algebra isomorphism between their Lie algebras, there is an analytic group isomorphism

$$\psi : G_1 \to G_2$$

such that

$$\exp\circ\phi = \psi \circ \exp,$$

see e.g. [7].

Let g be a left invariant metric and k a left invariant symmetric and covariant 2-tensor. We assume that they are diagonal with respect to the basis E_i, but g need not be orthonormal with respect to this basis. If we rescale the E_i by positive numbers, we get an orthonormal basis $\tilde{E}_i$. With respect to this basis, n is still diagonal with $n_1 < 0$ and $n_2, n_3 < 0$. We will denote the diagonal elements by $k_1 = k(\tilde{E}_1, \tilde{E}_1)$ etc. We are interested in the subgroup $\mathcal{D}(g,k)$ of the group of diffeomorphisms $\phi : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to \tilde{\mathrm{Sl}}(2,\mathbb{R})$ such that

$$\phi^* k = k \quad \text{and} \quad \phi^* g = g. \tag{91}$$

First of all, left translations are in $\mathcal{D}(g,k)$. However, $\mathcal{D}(g,k)$ always contains three more elements which we now define. Consider a matrix A which is diagonal with two minus signs and one plus sign. Due to (90), A defines a Lie Algebra isomorphism. As noted above, this yields a Lie group isomorphism which also satisfies (91). Explicitly, these isometries are defined by

$$\phi_1(\alpha, z) = (\alpha, -z), \quad \phi_2(\alpha, x, y) = (-\alpha, -x, y), \quad \phi_3(\alpha, x, y) = (-\alpha, x, -y). \tag{92}$$

We will denote the group of diffeomorphisms generated by left translations and these additional isometries by $\mathcal{D}_s$. If $k_2 = k_3$ and $n_2 = n_3$, the matrix A leaving $\tilde{E}_1$ invariant and rotating $\tilde{E}_2, \tilde{E}_3$ by some angle θ is also a Lie Algebra isomorphism. It thus defines a Lie Group isomorphism satisfying (91). Adding these diffeomorphisms to $\mathcal{D}_s$, we get a group of diffeomorphisms we will refer to as $\mathcal{D}_e$ (s for standard and e for exceptional).

Let us make a few observations concerning $\mathcal{D}_s$ and $\mathcal{D}_e$.

Lemma 18. *For any $\theta \in \mathbb{R}$, the map T_θ defined by $T_\theta(\alpha, z) = (\alpha + \theta, z)$ is in $\mathcal{D}_e$ and the map defined by $\theta \mapsto T_\theta$ defines an injective homomorphism from $\mathbb{R}$ to $\mathcal{D}_e$. There is a map from $\mathcal{D}_e$ to maps from D to itself defined by*

$$p[\phi](z) = \pi_2[\phi(0, z)], \tag{93}$$

where π_2 is the projection to the second factor. Then p is a surjective homomorphism from $\mathcal{D}_e$ to $\mathrm{Isom}(H^2)$, and

$$p[\phi](z) = \pi_2[\phi(\alpha, z)]. \tag{94}$$

Furthermore, we have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{D}_e \xrightarrow{p} \mathrm{Isom}(H^2) \longrightarrow \{e\}. \tag{95}$$

Translations by $n\pi$ are in $\mathcal{D}_s$ and the map defined by $n \mapsto T_{n\pi}$ defines an injective homomorphism from $\mathbb{Z}$ to $\mathcal{D}_s$ and the following sequence is exact

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{D}_s \xrightarrow{p} \mathrm{Isom}(H^2) \longrightarrow \{e\}.$$

Finally $\mathcal{D}_e$ can be given the structure of a Lie group with exactly two connected components such that the connected component of the identity $\mathcal{D}_{e,o}$ is isomorphic with $\mathbb{R} \times \tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})/K$, where K is the subgroup generated by $[2\pi, (-2\pi, 0)]$. If we give $\mathrm{Isom}(H^2)$ a Lie group structure by identifying the orientation preserving part with $\mathrm{PSl}(2, \mathbb{R})$, then p is a smooth map taking open sets to open sets. Finally, translations commute with elements of $\mathcal{D}_{e,o}$.

Remark. Note that if we would have $\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$ instead of $\mathcal{D}_s$ we would have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R}) \xrightarrow{p} \mathrm{PSl}(2, \mathbb{R}) \longrightarrow \{e\}.$$

However, the $\mathbb{Z}$ would correspond to translations by multiples of 2π and not of π. For $t \in \mathbb{R}$ and $h \in \tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$, we will sometimes write th instead of $T_t(h)$.

Proof. Consider the diffeomorphism T_θ of $\mathbb{R} \times D$ to itself defined by $T_\theta(\alpha, z) = (\alpha + \theta, z)$. We have

$$T_{\theta *} E_{1,(\alpha,z)} = E_{1,(\alpha+\theta,z)}, \quad T_{\theta *} E_{2,(\alpha,z)} = \cos\theta E_{2,(\alpha+\theta,z)} - \sin\theta E_{3,(\alpha+\theta,z)} \tag{96}$$

and

$$T_{\theta *} E_{3,(\alpha,z)} = \sin\theta E_{2,(\alpha+\theta,z)} + \cos\theta E_{3,(\alpha+\theta,z)}. \tag{97}$$

If we put a metric $\tilde{g}$ on $\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$ such that E_i is an orthonormal basis, we get the conclusion that T_θ is an isometry of $(\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R}), \tilde{g})$. Furthermore, if we compose T_θ with a left translation corresponding to the element $(-\theta, 0)$ of $\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$ we get a map $L_{(-\theta,0)} T_\theta$ from $\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$ to itself taking the identity to itself and which is a rotation by an angle θ of (E_2, E_3) in the counter clockwise direction on the tangent space at the identity, cf. (96), (97) and (87). As noted above, there is however a Lie group isomorphism ψ_θ with exactly this effect on the tangent space at the identity. Thus $\psi_\theta^{-1} L_{(-\theta,0)} T_\theta$ is an isometry of $\tilde{g}$ taking the identity to the identity and which is the identity on the tangent space at the identity. Since $\tilde{\mathrm{S}}\mathrm{l}(2, \mathbb{R})$ is connected, $T_\theta = L_{(\theta,0)} \psi_\theta$. In particular all translations are in $\mathcal{D}_e$. Translations commute with left translations for the following reason. By (87), (96) and (97), $L_{(\alpha,z)} T_\theta$ and $T_\theta L_{(\alpha,z)}$ map the identity to the same point and have the same effect on the tangent space at the identity. Let us note the following. The map ϕ_1 in (92) can be written as a combination of a translation by π and a left translation. Thus translations by π are in $\mathcal{D}_s$. Left translations do not commute with ϕ_2, ϕ_3, but $\phi_i L_g = L_{\phi_i(g)}\phi_i$ for $i = 2, 3$. Similarly, for translations, $T_\theta\phi_i = \phi_i T_{-\theta}$. Finally $\phi_2\phi_3 = \phi_1$. Consequently, any element of $\psi \in \mathcal{D}_e$ can be written in the form

$$\psi = \phi T_\theta L_g, \tag{98}$$

where ϕ is either ϕ_2 or the identity, T_θ is a translation and L_g is a left translation.

Let p be defined by (93). For some ϕ, we also have (94). In fact, if ϕ is a translation, one of (92) or a left translation, this is true. If $\phi = \chi_1\chi_2$ where χ_1 is such a map, we get

$$p[\phi](z) = \pi_2[\chi_1\chi_2(0, z)] = p[\chi_1](\pi_2[\chi_2(0, z)]) = p[\chi_1]p[\chi_2](z).$$

Applying this observation repeatedly using (98), we get the conclusion that p is a homomorphism and that (94) holds for all ϕ. Let us compute p if ϕ is a left translation. Note that π_2 can be factored through ρ, the covering map from $\mathbb{R} \times D$ to UD. If $\pi_D : UD \to D$ is the projection to the base, we have $\pi_2 = \pi_D \circ \rho$. We get

$$p[L_{(\alpha,z)}](\zeta) = \pi_2[(\alpha, z) \cdot (0, \zeta)] = \pi_D[\rho(\alpha, z) \cdot \rho(0, \zeta)] = \pi_D[\phi_*\rho(0, \zeta)] = \phi(\zeta),$$

where ϕ is the orientation preserving isometry of (D, g_D) such that $\phi_* e_{1,0} = \rho(\alpha, z)$, cf. (86). Thus $p[L_{(\alpha,z)}]$ is an isometry of hyperbolic space. If ϕ is one of (92) it is not so difficult to compute $p[\phi]$, and if ϕ is a translation, one obtains the identity. Thus p is a homomorphism from $\mathcal{D}_e$ to the isometry group of hyperbolic space. Since the image of the left translations is the orientation preserving isometries and $p[\phi_2]$ is orientation reversing, p is surjective. By the representation (98) we get the conclusion that the kernel of p is the group of translations. Let us give $\mathrm{Isom}(H^2)$ the structure of a Lie group by identifying the orientation preserving part with $\mathrm{PSl}(2, \mathbb{R})$. By our previous discussions, we can identify $\mathrm{PSl}(2, \mathbb{R})$ with UD by equating $\phi \in \mathrm{PSl}(2, \mathbb{R})$ with $\phi_* e_{1,0}$, the push forward of the unit vector in the x-direction at the origin. With these identifications, p restricted to $\tilde{\mathrm{Sl}}(2, \mathbb{R})$ is simply the covering projection ρ from $\mathbb{R} \times D$ to UD.

Let $\mathcal{D}_{e,o}$ be the subgroup of $\mathcal{D}_e$ generated by translations and left translations. Note that $\phi_2\mathcal{D}_{e,o}$ is disjoint from $\mathcal{D}_{e,o}$ since $p(\mathcal{D}_{e,o})$ consists of orientation preserving isometries and $p(\phi_2\mathcal{D}_{e,o})$ of orientation reversing isometries. We have a surjective homomorphism from $\mathbb{R} \times \tilde{\mathrm{Sl}}(2, \mathbb{R})$ to $\mathcal{D}_{e,o}$, and the kernel K is generated by $[2\pi, (-2\pi, 0)]$. Thus $\mathcal{D}_{e,o}$ can be given the structure of a connected Lie group as stated in the lemma. Using ϕ_2, we can give $\mathcal{D}_e$ the structure of a Lie group with exactly two connected components. We can define a map $p' : \mathbb{R} \times \tilde{\mathrm{Sl}}(2, \mathbb{R}) \to \mathrm{Isom}(H^2)$ by $p'(t, h) = p[L_h]$. By the observations in the previous paragraph, p' can be identified with the map from $\mathbb{R} \times \mathbb{R} \times D$ to UD sending (t, α, z) to $\rho(\alpha, z)$. Consequently, p' descends to a smooth map from the quotient $\mathbb{R} \times \tilde{\mathrm{Sl}}(2, \mathbb{R})/K$ to $\mathrm{Isom}(H^2)$ which can be identified as p restricted to the identity component of $\mathcal{D}_e$. Thus we see that p is a smooth map with respect to the above mentioned topologies and that it takes open sets to open sets. □

Proposition 8. *Let (g, k) be a left invariant metric and symmetric covariant 2-tensor satisfying Einstein's constraint equations. Assume further that they are diagonal with respect to the basis E_i. Let $\tilde{E}_i$ be an orthonormal basis obtained by* *rescaling E_i by positive numbers and let the diagonal components of n and k with respect to $\tilde{E}_i$ be denoted n_i and k_i. Then if $n_2 = n_3$ and $k_2 = k_3$, $\mathcal{D}(g,k) = \mathcal{D}_e$. Otherwise $\mathcal{D}(g,k) = \mathcal{D}_s$.*

Proof. Let $\phi \in \mathcal{D}(g,k)$. Given a vectorfield X, we can define a vectorfield by

$$(\phi_{*d}X)_p = \phi_* X_{\phi^{-1}(p)}.$$

We will use the notation

$$E_i' = \phi_{*d}\tilde{E}_i \quad \text{and} \quad A_{ik} = < E_i', \tilde{E}_k > .$$

Note that A is an orthogonal matrix, but that it is not yet clear that it is constant. The strategy of determining $\mathcal{D}(g,k)$ will be to find a ψ in $\mathcal{D}_e(g,k)$ (if $n_2 = n_3$ and $k_2 = k_3$) or in $\mathcal{D}_s(g,k)$ such that $\psi[\phi(e)] = e$ and $\psi_*\phi_*$ is the identity on when restricted to the tangent space at the identity element. Since $\psi \circ \phi$ is an isometry and $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ is connected, this would imply $\psi \circ \phi = \mathrm{Id}$. Note that it is enough to prove that there is a ψ in the relevant group such that $\psi_{*d}\phi_{*d}\tilde{E}_i = \tilde{E}_i$ for all i, since a left translation takes $\psi[\phi(e)]$ to the identity element and leaves the E_i unchanged. Observe finally that if $\tilde{E}_i$ is a basis of NUT type, then the scale factor relating E_2 and $\tilde{E}_2$ is the same as the scale factor relating E_3 and $\tilde{E}_3$ so that a rotation of E_2 and E_3 corresponds to a rotation of $\tilde{E}_2$ and $\tilde{E}_3$ in this case.

Since ϕ is an isometry, it preserves the Levi-Civita connection so that

$$\nabla_{\phi_{*d}X}\phi_{*d}Y = \phi_{*d}(\nabla_X Y).$$

Note that

$$\mathrm{Ric}(E_i', E_j') = \mathrm{Ric}(\tilde{E}_i, \tilde{E}_j) \quad \text{and} \quad k(E_i', E_j') = k(\tilde{E}_i, \tilde{E}_j) \tag{99}$$

since ϕ satisfies (91). In principle, the right and the left hand sides should be evaluated at different points, but since the right hand sides are constant, it is not so important to keep this in mind. Let us define

$$\tilde{R}(X) = \sum_m \mathrm{Ric}(X, \tilde{E}_m)\tilde{E}_m \quad \text{and} \quad \tilde{k}(X) = \sum_m k(X, \tilde{E}_m)\tilde{E}_m$$

Observe that these operators are independent of the orthonormal basis used to define them. One can compute that

$$\mathrm{Ric}(\tilde{E}_i, \tilde{E}_j) = 2n_i{}^k n_{kj} - \mathrm{tr}(n)n_{ij} - n^{kl}n_{kl}\delta_{ij} + \frac{1}{2}[\mathrm{tr}(n)]^2\delta_{ij}.$$

Here, indices are raised and lowered with δ_{ij}. Thus $\tilde{E}_i$ and E_i' are eigenvalues of $\tilde{R}$ with the same eigenvalues, and similarly for $\tilde{k}$, due to (99). Note that since k and Ric are symmetric, eigenvectors of $\tilde{R}$ and $\tilde{k}$ with different eigenvalues are orthogonal. By the above it is clear that R_{ij} is diagonal, and we will refer to the diagonal components as R_i. Observe that

$$R_i = 2n_i^2 - \mathrm{tr}(n)n_i - n^j n_j + \frac{1}{2}[\mathrm{tr}(n)]^2. \tag{100}$$

The assumption $R_2 = R_3$ leads to the conclusion that

$$n_2^2 - (n_1 + n_3)n_2 = n_3^2 - (n_1 + n_2)n_3$$

whence

$$n_2(n_2 - n_1) = n_3(n_3 - n_1).$$

Observe that since $n_1 < 0$ and n_2 and n_3 are positive, this implies that $n_2 = n_3$. Thus $R_2 = R_3$ is equivalent to $n_2 = n_3$. Assuming $R_1 = R_2$ leads to the conclusion

$$n_1(n_1 - n_3) = n_2(n_2 - n_3).$$

Observe that the left hand side is positive, so that $n_2 > n_3$ is a consequence. In particular, all three eigenvalues of Ricci cannot be equal. Consider now the following cases of initial data.

i) If all eigenvalues of the Ricci tensor are different, we conclude that A is diagonal with entries ± 1. Since ϕ is an isometry, the structure constants corresponding to E_i' must coincide with those corresponding to $\tilde{E}_i$. Considering (90) we get the conclusion that A must be the identity or have two minus signs and one plus sign on the diagonal. In other words, $\phi \in \mathcal{D}_s$.

ii) If $n_2 = n_3$, then $A_{i1} = A_{1i} = 0$ for all $i \neq 1$, and $A_{11} = \pm 1$. If $k_2 \neq k_3$, then A has to be diagonal with diagonal entries ± 1 and we have the same situation as in the previous case. If $k_2 = k_3$, then there are extra isometries. By composing with an element of $\mathcal{D}_s$, we can assume that A_{ij} for $i, j = 2, 3$ is a rotation, but the angle of rotation may still depend on the point. By composing with an element of $\mathcal{D}_e$, we can assume that $\phi(e) = e$ and $A_{ij}(e)$ is diagonal with $A_{22}(e) = A_{33}(e) = 1$. If $A_{11}(e) = 1$ we are done, so the problem is to exclude the possibility $A_{11}(e) = -1$. This will be done below.

iii) Assume now that $R_1 = R_2$ or $R_1 = R_3$. If we are lucky, the relevant k_i:s are different and we are done, but we cannot assume that. Since $R_2 \neq R_3$, we do however get the conclusion that ϕ leaves invariant, or at worst changes the sign, of one of $\tilde{E}_2$ and $\tilde{E}_3$. This case will be pursued further below.

Since ϕ preserves the Levi-Civita connection, we get the conclusion that

$$\nabla_{E_i'} E_j' = \phi_{*d} \nabla_{\tilde{E}_i} \tilde{E}_j = < \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_m > A_{mn} \tilde{E}_n.$$

If we express E_i' and E_j' in the left hand side in terms of the basis $\tilde{E}_k$, take the scalar product of the result with $\tilde{E}_l$ and finally multiply the result with A in a suitable fashion, we obtain

$$\tilde{E}_o(A_{jl}) + A_{jm} < \nabla_{\tilde{E}_o} \tilde{E}_m, \tilde{E}_l > = < \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_m > A_{io} A_{ml}. \tag{101}$$

Consider the case ii). Then A_{1l} is constant since $R_1 \neq R_2 = R_3$. Letting $j = 1, o = 2$ and $l = 3$ in (101) we get

$$A_{11}(e) < \nabla_{\tilde{E}_2} \tilde{E}_1, \tilde{E}_3 > = < \nabla_{\tilde{E}_2} \tilde{E}_1, \tilde{E}_3 > A_{22}(e) A_{33}(e).$$

Since $A_{22}(e) = A_{33}(e) = 1$ and $< \nabla_{\tilde{E}_2} \tilde{E}_1, \tilde{E}_3 > = -n_1/2$, we get the conclusion that $A_{11}(e) = 1$.

Assume that $R_1 = R_2$. Let $(l, o, j) = (2, 2, 3)$ in (101). Since $A_{32} = 0$ by the orthogonality of eigenvectors of $\tilde{R}$ with different eigenvalues and since $< \nabla_{\tilde{E}_i} \tilde{E}_j, \tilde{E}_m > = 0$ unless i, j, m are all different, we get

$$< \nabla_{\tilde{E}_i} \tilde{E}_3, \tilde{E}_m > A_{i2} A_{m2} = 0.$$

Thus $(n_1 - n_2) A_{12} A_{22} = 0$. Since n_1 is negative and n_2 is positive, we conclude that $A_{12} A_{22} = 0$. Since A is an orthogonal matrix we have $A_{12} = \pm A_{21}$ and $A_{11} = \pm A_{22}$.

In any case, the matrix A_{ij} is a constant matrix, and since the commutator can be expressed in terms of the Levi-Civita connection, (90) implies

$$\frac{1}{\det A}{}^{t}AnA = n. \tag{102}$$

Since $A_{3j} = 0$ if $j \neq 3$ and $A \in O(3)$, we get the conclusion that $A \in SO(3)$. Combining this observation with (102), we exclude the possibility $A_{12} \neq 0$. Thus the only possibility is the situation when A is diagonal with diagonal elements ± 1 and determinant one. The argument concerning the case when $R_1 = R_3 \neq R_2$ is similar. □

Let us sort out the relation between two canonical bases for a given set of Bianchi VIII initial data. Let $\mathcal{S}$ be the subgroup of $SO(3)$ generated by the three diagonal matrices that have two minus signs and one plus sign on the diagonal, and the matrix with $a_{11} = -1$, $a_{23} = 1$, $a_{32} = 1$ and the remaining components zero. We say that $\{e_i\} \sim \{e_i'\}$ if there is a matrix $A \in \mathcal{S}$ (with components a_{ij}) such that $e_i = a_{ij}e_j'$. Let $\mathcal{S}_N$ be the subgroup generated by $\mathcal{S}$ and the matrices A with components $a_{11} = 1$, $a_{1i} = a_{i1} = 0$ if $i \neq 1$ and a_{ij}, $i, j = 2, 3$, a rotation matrix. We say that $\{e_i\} \sim_N \{e_i'\}$ if there is a matrix $A \in \mathcal{S}_N$ such that $e_i = a_{ij}e_j'$.

Lemma 19. *Consider two canonical bases $\{e_i\}$ and $\{e_i'\}$ with respect to Bianchi VIII initial data (G, g, k). Then they have the same orientation. If one of the bases is of NUT type, then $\{e_i\} \sim_N \{e_i'\}$. In particular, the other basis is of NUT type and $k_i = k_i'$ and $n_i = n_i'$, where k_i and k_i' are the diagonal components of k with respect to the different bases and similarly for n_i and n_i'. If one of the bases is not of NUT type, then $\{e_i\} \sim \{e_i'\}$.*

Proof. Assume that $\{e_i\}$ and $\{e_i'\}$ are two canonical bases corresponding to the same initial data. Then there is an orthogonal matrix A with components a_{ij} such that

$$e_i = a_{ij}e_j'.$$

If we take the determinant of (90), bearing in mind that n and $n'(= \hat{n})$ both have negative determinant, we get the conclusion that $\det A > 0$, whence $A \in SO(3)$. In other words, two canonical bases have the same orientation. Thus

$$n' = {}^{t}AnA. \tag{103}$$

Assume for the sake of argument that we know that $e_1 = \pm e_3'$. Then (103) implies

$$n_3' = \sum_k n_k a_{k3}^2 = n_1,$$

contradicting the definition of a canonical basis. Similarly, the assumption that $e_1 = \pm e_2'$ leads to the conclusion that $n_2' = n_1$, and thus to a contradiction.

Assume that e_i is a NUT basis. By (100), we know that $R_1 \neq R_2 = R_3$, with notation as in the proof of Proposition 8. Let us use the notation $R_i' = \mathrm{Ric}(e_i', e_i')$, and note that the e_i and the e_i' are eigenvectors of $\tilde{R}$ and $\tilde{k}$, defined in the proof of Proposition 8. If $R_1' \neq R_1$, then either $R_2' = R_1$ or $R_3' = R_1$, i.e. either $e_1 = \pm e_2'$ or $e_1 = \pm e_3'$. By the above argument, this cannot be. Thus $R_1' = R_1$ and thus $R_2' = R_3' = R_2$ and $e_1 = \pm e_1'$. We conclude that $A \in \mathcal{S}_N$. Note that this implies $k_i = k_i'$ and $n_i = n_i'$ due to (103) and the fact that $k_2 = k_3$ and $n_2 = n_3$.

Let $\{e_i\}$ and $\{e_i'\}$ be canonical bases corresponding initial data which is not of NUT type. Assume that the R_i are all different or that the k_i are all different. Then the e_i are permutations of the e_i', possibly with some sign. If e_1 does not coincide with $\pm e_1'$, we get a contradiction as before. Since the matrix relating the different bases has to be in $SO(3)$, we get the conclusion that $\{e_i\} \sim \{e_i'\}$. In what follows, we can thus assume that two of the eigenvalues of $\tilde{R}$ coincide and that two of the eigenvalues of $\tilde{k}$ coincide.

1) Assume $R_1 \neq R_2 = R_3$. Then $k_2 \neq k_3$. Assuming $R_1' \neq R_1$ implies $e_1 = \pm e_2'$ or $e_1 = \pm e_3'$ and we get a contradiction as before. Thus $e_1 = \pm e_1'$ and $k_1 = k_1'$. We may by the above assume that $k_1 = k_2$ or $k_1 = k_3$. By applying an element of $\mathcal{S}$ to $\{e_i'\}$ and one to $\{e_i\}$, we can assume that $k_3' \neq k_2' = k_1'$ and $k_1 = k_2 \neq k_3$. Consequently, $e_3 = \pm e_3'$. Thus $\{e_i\} \sim \{e_i'\}$.

2) Assume $R_1 = R_2 \neq R_3$ or $R_1 = R_3 \neq R_2$. The second case can be transformed into the first by applying a matrix in $\mathcal{S}$ to the basis. If $R_1' = R_3$, we get $e_1' = \pm e_3$, which leads to a contradiction similarly to before. Thus $R_1' = R_1$, and by applying a matrix in $\mathcal{S}$ to $\{e_i'\}$ we can assume $R_1' = R_2' \neq R_3'$ and that $e_3 = e_3'$. Consequently, e_1', e_2' is a rotation of e_1, e_2. Combining this observation with (103), one can conclude that $\{e_i\} \sim \{e_i'\}$.

Since the R_i cannot all coincide, the lemma follows. □

Lemma 20. *Consider Bianchi VIII initial data (G, g, k) where G is simply connected. Let $\{e_i\}$ be a canonical basis. If the basis is not of NUT type, the isometry group of is generated by the left translations and the three Lie group isomorphisms which on the Lie algebra level change the sign of two of the e_i and leave the remaining one unchanged. If the basis is of NUT type, the isometry group consists of the above mentioned diffeomorphims plus the Lie group isomorphims which on the Lie algebra level are rotations of e_2, e_3.*

Proof. Multiply the e_i by positive constants such that the resulting frame, say $\{e_i'\}$, has $n = \text{diag}(-1, 1, 1)$. Let (g', k') be data on $\tilde{\text{Sl}}(2, \mathbb{R})$ with $g(e_i', e_j') = g'(E_i, E_j)$ and $k(e_i', e_j') = k'(E_i, E_j)$. Then the map ϕ from the Lie algebra of $\tilde{\text{Sl}}(2, \mathbb{R})$ to the Lie algebra of G defined by $\phi(E_i) = e_i'$ yields a Lie algebra isomorphism. Since $\tilde{\text{Sl}}(2, \mathbb{R})$ is a simply connected Lie group, there is an analytic group isomorphism

$$\psi : \tilde{\text{Sl}}(2, \mathbb{R}) \to G$$

such that

$$\exp \circ \phi = \psi \circ \exp,$$

see e.g. [7]. Observe that if $e \in \tilde{\text{Sl}}(2, \mathbb{R})$ represents the identity and h and arbitrary element, then

$$e'_{i,\psi(h)} = \frac{d}{dt} \left(\psi(h) \exp[te'_{i,e}] \right)(0) = \frac{d}{dt} \psi(h \exp[tE_{i,e}])(0) = \psi_* E_{i,h}.$$

Thus

$$\psi^* g(E_{i,h}, E_{j,h}) = g(e'_{i,\psi(h)}, e'_{j,\psi(h)}) = g'(E_{i,h}, E_{j,h})$$

and similarly for k. Thus $\psi^* g = g'$ and $\psi^* k = k'$. The lemma now follows from Proposition 8. □

Lemma 21. *Consider Bianchi VIII initial data (G, g, k) and let $\{e_i\}$ and $\{e_i'\}$ be two canonical bases. Then the class A developments associated with these canonical bases coincide. Assume now that G is simply connected. Then there are initial data $(\tilde{\mathrm{Sl}}(2,\mathbb{R}), g', k')$ such that g' and k' are diagonal with respect to E_i and a Lie group isomorphim $\psi : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to G$ such that $\psi^* g = g'$ and $\psi^* k = k'$. Furthermore the intervals of existence appearing in the corresponding class A developments are the same and $(1, \psi)$ is an isometry of these developments.*

Proof. Let us first consider two canonical bases $\{e_i\}$ and $\{e_i'\}$ corresponding to the same initial data (G, g, k), where G is of Bianchi type VIII but is not necessarily simply connected. We wish to prove that the corresponding class A developments coincide. The construction of the class A development, given a canonical basis $\{e_i\}$ proceeds as follows (see pp. 3798–99 of [12]). We define $\theta(0) = -\mathrm{tr}_g k$ and $\sigma_i(0) = -k_i + \theta/3$ and call the diagonal components of the n matrix of the basis $n_i(0)$. We call the collection $[n_i(0), \sigma_i(0), \theta(0)]$ Ellis and MacCallum initial data corresponding to the basis $\{e_i\}$. Then we solve (11), (13) and (14) of [12] to obtain an existence interval I on which n_i, σ_i and θ are defined. The a_i in (1) are given by

$$a_i(t) = \exp[\int_0^t (\sigma_i + \frac{1}{3}\theta) ds] \tag{104}$$

and the ξ^i in (1) are the duals of the e_i. Let $A \in SO(3)$, with components a_{ij} be such that $e_i = a_{ij} e_j'$. If A has two minus signs on the diagonal and one plus sign, then the Ellis and MacCallum initial data corresponding to the different bases coincide. Thus the existence intervals I and I' coincide and the θ and σ_i coincide. By (104), we see that the a_i coincide. Thus the corresponding developments coincide. Assume that A has $a_{11} = -1$, $a_{23} = 1$, $a_{32} = 1$ and that the remaining components are zero. On the level of the Ellis and MacCallum initial data, this corresponds to interchanging $n_2(0)$ with $n_3(0)$ and similarly for the $\sigma_i(0)$. Considering (11), (13) and (14) of [12] one can see that this operation can be extended to the entire solution. That is, interchanging n_2 n_3 and σ_2 σ_3 at the same time maps solutions to solutions. Consequently $I = I'$, and since we get $a_2 = a_3'$ and $a_3 = a_2'$, we conclude that the developments coincide (note that $e_2 = e_3'$ and that $e_3 = e_2'$). Finally, assume that one of the bases is a NUT basis and that $a_{11} = 1$, $a_{i1} = a_{1i} = 0$ if $i \neq 1$ and that a_{ij}, $i, j = 2, 3$ is a rotation matrix. Then the Ellis and MacCallum initial data have to coincide, so that $I = I'$. Considering (11), (13) and (14) of [12], one concludes that $\sigma_2 = \sigma_3$ so that $a_2 = a_3 = a_2' = a_3'$ for all time. It is then easy to see that the developments coincide.

Assuming that G is simply connected, the proof that one can find a Lie group isomorphism $\psi : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to G$ and initial data $(\tilde{\mathrm{Sl}}(2,\mathbb{R}), g', k')$ as in the statement of the lemma is similar to the arguments presented in the proof of Lemma 20. The map ψ will on the Lie algebra level map one canonical basis to another. Therefore, the corresponding Ellis and MacCallum initial data will coincide. Consequently the different intervals of existence and a_i coincide. Thus, $(1, \psi)$ defines an isometry of developments. □

9. Compactifications

The essence of the arguments in this section are taken from [17], Proposition 4.7.2 and Corollary 4.7.3 and from [2], Theorem 12.8.

Lemma 22. *Assume that* Γ *is a free and properly discontinuous subgroup of* $\mathcal{D}_e$ *such that* $\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Gamma$ *is compact. Then* $p(\Gamma)$ *is a discrete subgroup of the isometry group of hyperbolic space.*

Remark. We here assume that $\mathrm{Isom}(H^2)$ has been given a Lie group structure by identifying the orientation preserving part with $\mathrm{PSl}(2,\mathbb{R})$. When we say that $p(\Gamma)$ is discrete, we mean that it is a discrete subset of the topological space $\mathrm{Isom}(H^2)$.

Proof. Note that we have the short exact sequence (95). Let U be an open connected neighbourhood of the identity of $\mathcal{D}_e$ such that $[U,U]\subseteq V$, where $V\cap\Gamma=\{e\}$. Here

$$[U,U]=\{ghg^{-1}h^{-1}:g,h\in U\}.$$

Two elements of $g,h\in\Gamma$ which project to $p(U)$ commute. The reason is that there are elements $t,s\in\mathbb{R}$ such that $tg,sh\in U$. Since the elements of $\mathbb{R}$ are central in G, we have

$$(tg)(sh)(tg)^{-1}(sh)^{-1}=ghg^{-1}h^{-1}\in[U,U]\cap\Gamma\subseteq\{e\}.$$

The subgroup H of $\mathrm{Isom}(H^2)$ generated by $p(\Gamma)\cap p(U)$ is thus abelian. Furthermore, if $\gamma\in\Gamma$, choose an open connected neighbourhood $U_\gamma\subseteq U$ of the identity such that $[\gamma,U_\gamma]\subseteq V$. Then the group H_γ generated by $p(\Gamma)\cap p(U_\gamma)$ satisfies $H_\gamma\subseteq H$, so that it is abelian, and every element of H_γ commutes with $p(\gamma)$. Note that since U is a connected neighbourhood of the identity, H must consist of orientation preserving isometries of hyperbolic space. In other words $H_\gamma\subseteq H\subseteq\mathrm{PSl}(2,\mathbb{R})$. We now wish to prove that if $p(\Gamma)$ is not discrete, then $\bar{H}$ and $\bar{H}_\gamma$, where the closure is taken in $\mathrm{PSl}(2,\mathbb{R})$, are equal and constitute a one parameter subgroup of $\mathrm{PSl}(2,\mathbb{R})$. Consider $\mathrm{Sl}(2,\mathbb{R})$. We have the exponential map from the vector space of tracefree 2×2 matrices T_2 to $\mathrm{Sl}(2,\mathbb{R})$. Locally around the origin, this is a diffeomorphism. The covering homomorphism $\pi:\mathrm{Sl}(2,\mathbb{R})\to\mathrm{PSl}(2,\mathbb{R})$ is also locally a diffeomorphism. Composing, we get the local diffeomorphism $\pi\circ\exp$ from a neighbourhood of the origin of T_2 to a neighbourhood of the identity of $\mathrm{PSl}(2,\mathbb{R})$. Note that the inverse image of $\bar{H}$ under $\pi\circ\exp$ is a closed subset S of T_2 with the property that if $v\in S$, then $nv\in S$ for all $n\in\mathbb{Z}$. Since $p(U)$ is an open neighbourhood of the identity, the assumption that $p(\Gamma)$ is not discrete leads to the conclusion that the origin in T_2 is not isolated in S. Note that we can give T_2 a norm $\|\cdot\|$ through the isomorphism of T_2 with $\mathbb{R}^3$. For $n\in\mathbb{N}$, let $x_n\in B_{1/n}(0)\cap S-\{0\}$ and let $k_n\geq n$ be an integer such that $1\leq\|k_nx_n\|\leq 2$. We can assume that $y_n=k_nx_n$ converges to an element y. Let $t\neq 0$ be a real number. There is a sequence of integers r_n such that $|r_n/k_n-t|\leq 1/n$. Since $r_nx_n\to ty$, we get the conclusion that $ty\in S$. Thus S contains the line $\mathbb{R}y$, and $\bar{H}$ contains a one parameter subgroup of $\mathrm{PSl}(2,\mathbb{R})$. The same argument yields the conclusion that $\bar{H}_\gamma$ contains a one parameter subgroup. Since $\bar{H}$ is abelian, orientation preserving and contains a non-trivial orientation preserving isometry we can apply Lemma 17. The only possibilities for $\bar{H}$ are, up to conjugation, rotations around the origin in the disc model, the group of isometries of the form (88) and the translations by real numbers in the upper half plane. Since $\bar{H}_\gamma\subseteq\bar{H}$ also contains a one parameter subgroup of isometries, the two groups must coincide. Since $p(\gamma)$ commutes with $\bar{H}_\gamma = \bar{H}$ for all $\gamma \in \Gamma$, we get the conclusion that $p(\Gamma)$ commutes with $\bar{H}$. Consequently, $p(\Gamma)$ is a subgroup of one of the groups mentioned in Lemma 17.

Assume that $\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Gamma$ is compact and let g be a metric on $\tilde{\mathrm{Sl}}(2,\mathbb{R})$ whose isometry group contains $\mathcal{D}_e$. Then there is an $r > 0$ such that for any $p \in \tilde{\mathrm{Sl}}(2,\mathbb{R})$ the projection $\pi : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to \tilde{\mathrm{Sl}}(2,\mathbb{R})/\Gamma$ is injective when restricted to $B_r(p)$, where the ball is defined using the topological metric induced by g. The reason is that if this were not the case, then there would be $p_n \in \tilde{\mathrm{Sl}}(2,\mathbb{R})$ and $0 < r_n \leq 1/n$ such that π would not be injective on $B_{r_n}(p_n)$. Since the quotient is compact, we can assume that $\pi(p_n)$ converges to a point q. Let $p \in \tilde{\mathrm{Sl}}(2,\mathbb{R})$ satisfy $\pi(p) = q$ and let $r > 0$ be such that π is injective on $B_r(p)$. This means that π is injective on $\gamma B_r(p) = B_r(\gamma p)$ for every $\gamma \in \Gamma$ and that $\gamma B_r(p) \cap B_r(p) \neq \emptyset$ implies $\gamma = e$. Since $\pi[B_r(p)]$ contains an open neighbourhood of q, we get the conclusion that $\pi[B_{r_n}(p_n)] \subseteq \pi[B_r(p)]$ for n large enough. Thus $B_{r_n}(p_n) \subseteq \gamma B_r(p)$ for some $\gamma \in \Gamma$. We have a contradiction.

Let us define a map $\tilde{p} : \tilde{\mathrm{Sl}}(2,\mathbb{R}) \to D$ by

$$\tilde{p}(h) = p[L_h](0). \tag{105}$$

If $\psi \in \mathcal{D}_e$, we wish to prove that

$$\tilde{p}[\psi(h)] = p[\psi]\tilde{p}(h). \tag{106}$$

If $\psi = L_g$, then (106) holds. If ψ is a Lie group isomorphism, then

$$\tilde{p}[\psi(h)] = p[L_{\psi(h)}](0) = p[\psi L_h \psi^{-1}](0) = p[\psi]p[L_h]p[\psi^{-1}](0) = p[\psi]\tilde{p}(h),$$

since $p[\psi^{-1}](0) = 0$ in this case. Since p is a homomorphism and the elements of $\mathcal{D}_e$ can be written as combinations of left translations and Lie group isomorphisms, we get the conclusion that (106) holds in general.

As mentioned above, there are three possibilities for $p(\Gamma)$. Assume there is an isometry χ such that $\chi\, p(\Gamma)\chi^{-1}$ is a subgroup of the rotations around the origin in the disc model. The presence of χ is only a technical nuisance, and for that reason we will assume it to be the identity below. Let $\gamma_t \in \mathrm{PSl}(2,\mathbb{R})$ be defined by (88) and let $\tilde{\gamma}_t \in \tilde{\mathrm{Sl}}(2,\mathbb{R})$ satisfy $p(L_{\tilde{\gamma}_t}) = \gamma_t$. Let $r > 0$ be such that π is injective on $B_r(h)$ $\forall h \in \tilde{\mathrm{Sl}}(2,\mathbb{R})$. Assume we have found $t_1, ..., t_n$ such that π is injective on $\cup_{k=1}^n \tilde{\gamma}_{t_k} B_r(e)$. We wish to find a t_{n+1} such that we can increase the union to $n+1$ and preserve the injectivity. Since π is injective on $\tilde{\gamma}_{t_{n+1}} B_r(e)$ by the choice of r, the only conceivable problem would be the existence of a $\gamma \in \Gamma$ and a $k \leq n$ such that $\tilde{\gamma}_{t_{n+1}} B_r(e) \cap \gamma\tilde{\gamma}_{t_k} B_r(e) \neq \emptyset$. However, this would imply

$$p[L_{\tilde{\gamma}_{t_{n+1}}}]\tilde{p}[B_r(e)] \cap p[\gamma]p[L_{\tilde{\gamma}_{t_k}}]\tilde{p}[B_r(e)] \neq \emptyset.$$

Note however that $\cup_{k=1}^n p[L_{\tilde{\gamma}_{t_k}}]\tilde{p}[B_r(e)]$ is inside a ball of radius strictly less than 1 in D and that $p[\gamma]$ leaves this ball invariant for $\gamma \in \Gamma$. Choosing t_{n+1} to be close enough to 1, one can show that $p[L_{\tilde{\gamma}_{t_{n+1}}}]\tilde{p}[B_r(e)]$ is outside of this ball. Thus for arbitrary $n \in \mathbb{N}$, one can choose a sequence $t_1, ..., t_n$ such that π is injective on the union of $\tilde{\gamma}_{t_k} B_r(e)$. Since these sets all have the same fixed volume, we get the conclusion that the quotient has infinite volume.

The arguments in the other two cases are similar. If $\chi\, p(\Gamma)\chi^{-1}$ is a subgroup of the group of translations by real numbers in the upper half plane, the isometries that should replace the γ_t:s in the above argument are the dilations of the upper half plane. Consider the case when $\chi\ p(\Gamma)\chi^{-1}$ is a subgroup of the isometries of the form (88). The γ_t should in this case be replaced with ψ_t defined by

$$\psi_t(z) = \frac{z+it}{1-itz}.$$

The reason is the following. Let U be contained in a compact subset of D. Then $\psi_t(U)$ tends to the point i as $t \to 1$. On the other hand, if ϕ_t is defined by (88), then the imaginary part of $\phi_t(U)$ can be bounded from above by a number strictly less than 1 independent of t. □

Lemma 23. *Assume that Γ is a free and properly discontinuous subgroup of $\mathcal{D}_e$ such that $\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Gamma$ is compact. As noted above, $p(\Gamma)$ is a discrete subgroup of the isometry group of hyperbolic space. Furthermore, $D/p(\Gamma)$ is compact and $\Gamma\cap\mathbb{R}$ is an infinite cyclic group. Here, $\mathbb{R}$ is to be interpreted as the subgroup of translations. Finally, $\tilde{\mathrm{Sl}}(2,\mathbb{R})/\Gamma$ is a Seifert fibred space with a compact hyperbolic orbifold as a base.*

Proof. Let us prove that if $z \in D$, then $p(\Gamma)z$ is a closed discrete set. It is enough to prove that for every $\zeta \in D$, there is an open set $U \subseteq D$ such that $\zeta \in U$ and U only contains a finite number of elements in $p(\Gamma)z$. Assume that $p(\gamma_n)z \to \zeta$ and that all the $p(\gamma_n)z$ are distinct. We can choose a subsequence so that $p(\gamma_n)$ are all orientation preserving or orientation reversing. If they are all orientation reversing, we can furthermore consider the sequence $p(\gamma_1\gamma_n)$ instead. This sequence will have the same properties, with ζ replaced by $p(\gamma_1)\zeta$, so we can assume that the original sequence consists of orientation preserving isometries. Note that we can identify $\mathrm{PSl}(2,\mathbb{R})$ with UD and that the above assumption implies that if one projects $p(\gamma_n)$ to D, one remains in the same compact set for all n. Thus, $p(\gamma_n)$ belongs to a compact subset of $\mathrm{PSl}(2,\mathbb{R})$. Thus we can assume that the sequence converges. Consequently, $p(\gamma_n\gamma_{n+1}^{-1})$ is a sequence of elements of $p(\Gamma)$ separate from the identity, converging to the identity. This is not possible since $p(\Gamma)$ is discrete.

In what follows it will be useful to know that if $p[\phi]$ has a fixed point and is orientation reversing, then ϕ has a fixed point. Assume to that end that $p[\phi](\zeta) = \zeta$ and that $p[\phi]$ is orientation reversing. Let χ be an element of the orientation preserving isometry group of hyperbolic space such that $\chi^{-1}(0) = \zeta$ and let $\hat{\chi}$ satisfy $p(\hat{\chi}) = \chi$. If we let $\hat{\phi} = \hat{\chi}\phi\hat{\chi}^{-1}$, then $p[\hat{\phi}](0) = 0$. Due to the representation (98) we have

$$\hat{\phi} = \phi_2 T_\theta L_{(\alpha,z)}$$

with notation as in the proof of Lemma 18. Thus $z = 0$ and we conclude that $(-(\alpha+\theta)/2, 0)$ is a fixed point of $\hat{\phi}$. Consequently, ϕ has a fixed point.

Consider the subgroup $p(\Gamma)_o$ of $p(\Gamma)$ fixing the origin $o \in D$. Since Γ is free and properly discontinuous, we conclude by the above that $p(\Gamma)_o$ consists of rotations around the origin. Since it is finite due to the discreteness of $p(\Gamma)$, it is generated by a rotation through an angle $2\pi q$ where $q \in \mathbb{Q}$. Let

$$l = \inf\{d(o, p(\gamma)o) | p(\gamma)o \neq o,\ \gamma \in \Gamma\}.$$

Since $p(\Gamma)o$ is a discrete set, $l > 0$. Let $0 < r < l/2$. Then if $p(\gamma)B_r(o) \cap B_r(o) \neq \emptyset$, $p(\gamma)$ has to belong to $p(\Gamma)_o$. Let $U \subseteq B_r(o)$ be an open and non-empty set contained in a sector of an angle less that $2\pi q$.

Consider (95). The subgroup K of $\mathbb{R}$ obtained by taking the inverse image of Γ under the injection from $\mathbb{R}$ to $\mathcal{D}_e$ has to be discrete due to the discreteness of Γ. Thus it is trivial or an infinite cyclic group. Assume that it is trivial. Then p restricted to Γ is an isomorphism. Let $\tilde{U} = \mathbb{R} \times U$. Assume $\phi\tilde{U} \cap \tilde{U} \neq \emptyset$ for $\phi \in \Gamma$. Then $\phi(\alpha_1, z_1) = (\alpha_2, z_2)$ for $z_i \in U$, $i = 1, 2$. However,

$$\tilde{p}(\alpha_2, z_2) = p[L_{(\alpha_2,z_2)}](0) = \pi_2[L_{(\alpha_2,z_2)}(0, 0)] = z_2,$$

where $\tilde{p}$ is defined in (105) and, similarly,

$$\tilde{p}[\phi(\alpha_1, z_1)] = p[\phi](z_1).$$

Thus $p[\phi]U \cap U \neq \emptyset$. Consequently, $p[\phi] = \mathrm{Id}$ and thus $\phi = \mathrm{Id}$, since p restricted to Γ is injective. Take any ball B in $\tilde{U}$. By translations by $\mathbb{R}$, one can then construct an infinite number of disjoint balls in $\tilde{U}$ all of which are isometric to B. This implies that the volume of $\tilde{\mathrm{Sl}}(2, \mathbb{R})/\Gamma$ is infinite. Consequently, K has to be an infinite cyclic group. Let $\pi_O : D \to D/p(\Gamma)$. Then $\pi_O\tilde{p}[\phi(g)] = \pi_O\tilde{p}(g)$ for $\phi \in \Gamma$. Thus we get a continuous map from $\tilde{\mathrm{Sl}}(2, \mathbb{R})/\Gamma$ onto $D/p(\Gamma)$. In other words, $D/p(\Gamma)$ is compact.

Let us prove that the quotient is Seifert fibred. Recall that $\tilde{\mathrm{Sl}}(2, \mathbb{R})$ is $\mathbb{R} \times D$ topologically and denote the generator of $\Gamma \cap \mathbb{R}$ by $\delta > 0$. Let $(\alpha, z) \in \mathbb{R} \times D$ and assume that only the identity element of $p(\Gamma)$ fixes z. Let $B_\epsilon(z) \subseteq D$ be an open ball such that $p(\gamma)B_\epsilon(z) \cap B_\epsilon(z) \neq \emptyset$ for $\gamma \in \Gamma$ implies $p(\gamma) = \mathrm{Id}$. Consider the set $U = [\alpha, \alpha + \delta] \times B_\epsilon(z)$. Assume that $\gamma p_1 = p_2$ for $p_i \in U$ and $\gamma \in \Gamma$. Then $p(\gamma)\tilde{p}(p_1) = \tilde{p}(p_2)$. Since $\tilde{p}(p_i)$ are both in $B_\epsilon(z)$ we get the conclusion that $p(\gamma) = \mathrm{Id}$. Consequently γ is a translation by δ. In other words, the image of U in $\tilde{\mathrm{Sl}}(2, \mathbb{R})/\Gamma$ is a cylinder. Thus there is a neighbourhood of the image of (α, z) homeomorphic to a cylinder. Assume that the subgroup $p(\Gamma)_z$ of $p(\Gamma)$ fixing z is non-trivial. Let χ be an orientation preserving isometry of hyperbolic space such that $\chi^{-1}(0) = z$ and let $\hat{\chi}$ be such that $p(\hat{\chi}) = \chi$. Assume that $p(\gamma) \in p(\Gamma)_z$. Then $\chi p(\gamma)\chi^{-1}(0) = 0$. Since $p(\gamma)$ is orientation preserving, we conclude that

$$\hat{\chi}\gamma\hat{\chi}^{-1} = T_\theta L_{(\beta,0)}$$

using (98). Let δ be defined by

$$\delta = \inf\{\theta + \beta : \hat{\chi}^{-1}T_\theta L_{(\beta,0)}\hat{\chi} \in \Gamma, \theta + \beta > 0\}.$$

Since Γ is free and properly discontinuous, $\delta > 0$ and there are θ and β such that $\delta = \theta + \beta$ and $\hat{\chi}^{-1}T_\theta L_{(\beta,0)}\hat{\chi} \in \Gamma$. Note that θ and β are unique, since if θ_1 and β_1 would have the same properties as θ and β but be different from them, then one could construct a map in Γ with a fixed point which would be different from the identity. Since $p[T_\theta L_{(\beta,0)}]$ is a rotation through an angle β and $p(\Gamma)$ is discrete, we get the conclusion that $\beta = 2\pi q$ where $q \in \mathbb{Q}$. Let $\epsilon > 0$ be such that if $p(\gamma)B_\epsilon(z) \cap B_\epsilon(z) \neq \emptyset$ then $p(\gamma)z = z$. Let $U = \hat{\chi}^{-1}\{[\alpha, \alpha + \delta] \times B_\eta(0)\}$, where $\eta > 0$ is such that $\tilde{p}(U) \subseteq B_\epsilon(z)$. Assume that $\gamma p_1 = p_2$ where $p_i = \hat{\chi}^{-1}(\alpha_i, z_i) \in U$ and $\gamma \in \Gamma$. Then $p(\gamma)\tilde{p}(p_1) = \tilde{p}(p_2)$ so that $p(\gamma)z = z$ since $\tilde{p}(p_i) \in B_\epsilon(z)$. Thus $\gamma = \hat{\chi}^{-1}T_{\theta'}L_{(\beta',0)}\hat{\chi}$, whence

$$(\alpha_1 + \theta' + \beta', e^{i\beta'}z_1) = (\alpha_2, z_2).$$

Since $(\alpha_i, z_i) \in [\alpha, \alpha + \delta] \times B_\eta(0)$, we get the conclusion that $(\theta', \beta') = \pm(\theta, \beta)$ or $(\theta', \beta') = (0, 0)$. Consequently, the image of U in $\tilde{\mathrm{Sl}}(2, \mathbb{R})/\Gamma$ is homeomorphic to a cylinder on which the ends have been identified after a rotation by an angle $2\pi q$. We conclude that $\tilde{\text{Sl}}(2,\mathbb{R})/\Gamma$ is a Seifert fibred space. □

Proof of Theorem 4. Let (G, g, k) be Bianchi VIII initial data with G simply connected and assume that Γ is a free and properly discontinuous subgroup of the isometry group of the initial data such that G/Γ is compact. Let g', k' and ψ be as in the statement of Lemma 21. Then $\Gamma' = \psi^{-1}\Gamma\psi$ is a free and properly discontinuous group of isometries of $(\tilde{\text{Sl}}(2,\mathbb{R}), g', k')$, so that $\Gamma' \subset \mathcal{D}_e$. As was noted in the introduction, $(1, \Gamma)$ is a free and properly discontinuous group of isometries of the class A development due to Lemma 20 and similarly for Γ'. The isometry $(1, \psi)$ between the Bianchi class A developments induces an isometry $(1, \psi')$ between the compactified class A developments $I \times \tilde{\text{Sl}}(2,\mathbb{R})/\Gamma'$ and $I \times G/\Gamma$. Since $\Gamma' \subseteq \mathcal{D}_e$, Lemma 23 implies that $\tilde{\text{Sl}}(2,\mathbb{R})/\Gamma'$ is Seifert fibred. Furthermore, when one unwraps the circle fibres, they correspond to E_1. Due to the existence of the isometry $(1, \psi')$ we get the conclusion that G/Γ is Seifert fibred. Since ψ takes E_1 to e_1, where e_1 is the first element of a canonical basis for (G, g, k), see the proofs of Lemma 20 and 21, we get the conclusion that the integral curves of e_1 become the circle fibres in the Seifert fibred space under the compactification. □

References

[1] Anderson M 2001 On long-time evolution in general relativity and geometrization of 3-manifolds *Commun. Math. Phy.* **222** 533–67

[2] Bredon G 1993 *Topology and Geometry* Springer-Verlag New York

[3] Choquet-Bruhat Y Moncrief V 2001 Future Global in Time Einsteinian Spacetimes with U(1) Isometry Group *Ann. Henri Poincaré* **2** 1001–64

[4] Ellis G and MacCallum M 1969 A class of homogenous cosmological models *Commun. Math. Phys.* **12** 108–41

[5] Fischer A and Moncrief V 2001 The reduced Einstein equations and the conformal volume collapse of 3-manifolds *Class. Quantum Grav.* **18** 4493–515

[6] Horwood J T *et al* Late-time asymptotic dynamics of Bianchi VIII cosmologies gr-qc/0210031

[7] Hilgert J and Neeb K-H 1991 *Lie-Gruppen und Lie-Algebren* Vieweg

[8] Wainwright J and Hsu L 1989 A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A *Class. Quantum Grav.* **6** 1409–31

[9] O'Neill B 1983 *Semi Riemannian Geometry With Applications to Relativity* Academic Press

[10] Raymond F and Vasquez T 1981 3-Manifolds whose universal coverings are Lie groups *Topology and its Application* **12** 161–79

[11] Ringström H 2001 The Bianchi IX attractor *Ann. Henri Poincaré* **2** 405–500

[12] Ringström H 2001 The future asymptotics of Bianchi VIII vacuum solutions *Class. Quantum Grav.* **18** 3791–823

[13] Scott P 1983 The geometries of 3-manifolds *Bull. London Math. Soc.* **15** 401–87

[14] Spanier E 1989 *Algebraic Topology* Springer

[15] Tanimoto M, Koike T and Hosoya A 1994 Compact homogeneous universes *J. Math. Phys.* **35** 4855–88

[16] Tanimoto M, Koike T and Hosoya A 1997 Hamiltonian structures for compact homogeneous universes *J. Math. Phys.* **38** 6560–77

[17] Thurston W 1997 *Three-Dimensional Geometry and Topology* vol 1, ed S Levy (Princeton, NJ: Princeton University Press)

Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, D-14476 Golm, Germany