Chapter 14

Infinite-dimensional symmetries in gravity

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14.1 Einstein theory

14.1.1 Introduction

In these lectures we review the symmetry properties of Einstein's theory when it is reduced from four to two dimensions. We explain how, in this reduction, the theory acquires an infinite-dimensional symmetry group, the Geroch group, whose associated Lie algebra is the affine extension of $SL(2, \mathbb{R})$. The action of the Geroch group, which is nonlinear and non-local, can be linearized, thereby permitting the explicit construction of many solutions of Einstein's equations with two commuting Killing vectors ∂_2 and ∂_3. A non-trivial example of this method for a colliding plane wave metric is given.

The lectures review some well-known material at a pedagogical level. Therefore, rather than including references in the text, we have chosen to collect some basic references at the end, which readers are invited to use as a guide to the vast literature on the subject of exact solutions, on the integrability of Einstein's equations in the reduction to two dimensions, and finally on the generalization of these structures to other theories, including supergravity.

14.1.2 Mathematical conventions

Our main interest is in studying the structural properties of Einstein's theory and its generalizations. We will first formulate it in D dimensions, with coordinates

$x^M = (x^0, \ldots, x^{D-1})$. The metric can be expressed in terms of the vielbein as

$$g_{MN} = E^A_M E^B_N \eta_{AB} \tag{14.1}$$

with the flat metric $\eta_{AB} \equiv (+, -, \ldots, -)$. For the following it will be important that the vielbein can be viewed as an element of a coset space according to

$$E^A_M \in GL(D, \boldsymbol{R})/SO(1, D-1). \tag{14.2}$$

The metric must be covariantly conserved

$$D_N(\Gamma) g_{MP} = 0 \tag{14.3}$$

where Γ is the Christoffel symbol of the metric g_{MN}. We next introduce a spin connection one-form, with coefficients $\omega_{MA}{}^B$. The vielbein postulate, that is the covariant constancy of the vielbein, which agrees exactly with Cartan's structure equation for the torsion two-form, is

$$D_M(\omega, \Gamma) E_N{}^A = 0. \tag{14.4}$$

Writing out this equation, we have

$$\partial_{[M} E_{N]}{}^A + \omega_M{}^A{}_B E_N{}^B = \Gamma_{[MN]}{}^P E_P{}^A. \tag{14.5}$$

We assume there is no torsion, so the Christoffel symbols are symmetric in spacetime indices, hence

$$\partial_{[M} E_{N]}{}^A + \omega_{[M}{}^A{}_{N]} = 0. \tag{14.6}$$

The coefficients of the anholonomy are

$$\Omega_{AB}{}^C = 2 E_{[A}{}^M E_{B]}{}^N \partial_M E_N{}^C. \tag{14.7}$$

Using the torsion-free condition for the spin connection and permuting the indices of the coefficients of the anholonomy we obtain the following equations

$$\begin{aligned} \Omega_{ABC} + \omega_{ACB} - \omega_{BCA} &= 0 \\ -\Omega_{BCA} - \omega_{BAC} + \omega_{CAB} &= 0 \\ \Omega_{CAB} + \omega_{CBA} - \omega_{ABC} &= 0. \end{aligned} \tag{14.8}$$

Employing then the property of the spin-connection $\omega_{ABC} = -\omega_{ACB}$, due to the fact that the generators of the algebra of the D-Lorentz Group are totally antisymmetric matrices, we have the expression of the spin connection as a function of Ω_{ABC}

$$\omega_{ABC} = \tfrac{1}{2}(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}). \tag{14.9}$$

The Riemann tensor (the curvature two-form) can be defined by

$$[D_M(\omega), D_N(\omega)]V^A = R_{MN}{}^A{}_B V^B. \tag{14.10}$$

The explicit expression in terms of the spin connection is

$$R_{MNA}{}^B = 2\partial_{[M}\omega_{N]A}{}^B + 2\omega_{[MA}{}^C\omega_{N]C}{}^B. \tag{14.11}$$

From the Riemann tensor the Ricci tensor and the curvature scalar are obtained in the usual way: $R_{MN} := R_{MPN}{}^P$ and $R := g^{MN}R_{MN}$. The metric determinant is

$$E = \det E_M{}^A = \sqrt{-g}. \tag{14.12}$$

14.1.3 The Einstein–Hilbert action

Now we have all the elements to define Einstein theory. The Einstein–Hilbert action is

$$S = \int d^4x\, \mathcal{L} \tag{14.13}$$

and the Lagrangian $\mathcal{L}$ can be expressed in function of the spin connection

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{4}ER = -\tfrac{1}{4}EE_A{}^M E_B{}^N R_{MN}{}^{AB}\\ =& -\tfrac{1}{2}EE_A{}^M E_B{}^N \partial_M\omega_N{}^{AB} - \tfrac{1}{4}E\omega_A{}^{AC}\omega_{BC}{}^B + \tfrac{1}{4}\omega_{BAC}\omega^{ACB}.\end{aligned} \tag{14.14}$$

Substituting now the expression $\omega = \omega(\Omega)$, integrating by parts and dropping total derivatives, we arrive at

$$-\tfrac{1}{4}ER = \tfrac{1}{16}E(\Omega_{ABC}\Omega^{ABC} - 2\Omega^{ABC}\Omega_{BCA} - 4\Omega_{AC}{}^C\Omega^A{}_D{}^D). \tag{14.15}$$

This is the expression best suited for dimensional reduction of Einstein's theory. In the remainder, we will now set $D = 4$, i.e. work in four spacetime dimensions.

14.1.4 Dimensional reduction $D = 4 \to D = 3$

'Dimensional reduction' is equivalent to searching for solutions of Einstein's equations with one Killing vector, which we take to be $\xi^M\partial_M \equiv \partial_3$. For this purpose, we proceed from the 'Kaluza–Klein ansatz' for the vierbein.

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/2}e_m{}^a & \Delta^{1/2}B_m \\ 0 & \Delta^{1/2}\end{pmatrix}, \quad E_A{}^M = \begin{pmatrix} \Delta^{1/2}e_a{}^m & -e_a{}^n B_n\Delta^{1/2} \\ 0 & \Delta^{-1/2}\end{pmatrix}. \tag{14.16}$$

The matrix $e_m{}^a$ is the three-bein; B_m is called the Kaluza–Klein vector and Δ the Kaluza–Klein scalar. The ansatz fixes a part of the $SO(1,3)$ Lorentz symmetry. The residual symmetry group preserving the gauge condition $E_3{}^a = 0$ is the gauge group $SO(1,2)$.

After some algebra, we find

$$\Omega_{abc} = \Delta^{\frac{1}{2}}(\Omega^{(3)}_{abc} - e_{[a}{}^{m}\eta_{b]c}\Delta^{-1}\partial_m\Delta) \tag{14.17}$$
$$\Omega_{ab}{}^{3} = \Delta^{3/2}e_a{}^{m}e_b{}^{n}B_{mn} \tag{14.18}$$
$$\Omega_{3b}{}^{3} = -\tfrac{1}{2}e_b{}^{m}\Delta^{-1/2}\partial_m\Delta \tag{14.19}$$
$$\Omega_{3b}{}^{c} = 0. \tag{14.20}$$

Substituting the ansatz for the vierbein in the field action and making use of the above decomposition of the Einstein action, after some calculations we arrive at the following result

$$-\tfrac{1}{4}ER(E) = -\tfrac{1}{4}eR^{(3)}(e) - \tfrac{1}{16}e\Delta^2B^{mn}B_{mn} + \tfrac{1}{8}eg^{mn}\Delta^{-2}\partial_m\Delta\partial_n\Delta \tag{14.21}$$

where $B_{mn} = \partial_m B_n - \partial_n B_m$.

Duality transformation

The very special feature of three dimensions is that the Kaluza–Klein vector field can be dualized to a scalar. This is achieved by adding to the Einstein–Hilbert Lagrangian the expression

$$\mathcal{L}' = \tfrac{1}{8}e\tilde{\epsilon}^{mnp}B_{mn}\partial_pB \tag{14.22}$$

where B is a Lagrange multiplier and $\tilde{\epsilon}^{mnp}$ the Levi-Civita totally antisymmetric symbol. The dualization makes the Lagrangian depend only on B. So, adding $\mathcal{L}'$ to $\mathcal{L}$ and varying B_n leads to

$$e\Delta^2B^{mn} = \epsilon^{mnp}\partial_pB \tag{14.23}$$

modulo a numerical constant. Here we have set $\epsilon^{mnp} = e\tilde{\epsilon}^{mnp}$. When we substitute this expression in the three-dimensional reduced Einstein–Hilbert Lagrangian, we get a new one with two scalar fields

$$\mathcal{L} = -\tfrac{1}{4}eR^{(3)}(e) + \tfrac{1}{8}eg^{mn}\Delta^{-2}(\partial_m\Delta\partial_n\Delta + \partial_mB\partial_nB). \tag{14.24}$$

This is consistent with the equation of motion $\partial_m(e\Delta^2B^{mn}) = 0$. In fact, the term we add to the Lagrangian, which is now three dimensional, can be dropped by an integration by parts and the use of the three-dimensional Bianchi identities for the tensor B^{mn}.

14.1.5 Dimensional reduction $D = 3 \to D = 2$

Then we perform a dimensional reduction from three to two, i.e. we have two Killing commuting vectors (∂_3 and ∂_2) and there is no dependence on x^2 at all.

$$x^m = (x^\mu, x^2). \tag{14.25}$$

Repeating the same steps as before, the two-bein now takes the form

$$e_m{}^a = \begin{pmatrix} e_\mu{}^\alpha & \rho A_\mu \\ 0 & \rho \end{pmatrix}, \quad e_a{}^m = \begin{pmatrix} e_\alpha{}^\mu & -e_\alpha^\sigma A_\sigma \\ 0 & \rho^{-1} \end{pmatrix}. \tag{14.26}$$

Detailed calculation shows that

$$-\tfrac{1}{4}e^{(3)}R^{(3)} = -\tfrac{1}{4}\rho e R^{(2)} - \tfrac{1}{16}\rho^3 e A_{\mu\nu}A^{\mu\nu} \tag{14.27}$$

with

$$eR^{(2)} = -2\partial_\mu(ee^{\alpha\mu}\Omega_{\alpha\beta}{}^\beta) \tag{14.28}$$

where e is the determinant of the two-bein. At this point we can write the equations of motion for the theory. The equation of motion for the Kaluza–Klein vector is given by

$$\partial_\mu(\rho^3 e A^{\mu\nu}) = 0. \tag{14.29}$$

In two dimensions, a Maxwell field does not propagate, as there are no transverse degrees of freedom. Neglecting topological effects (i.e. non-vanishing holonomies) we can, therefore, set $A_\mu = 0$.

For the remaining equations of motion, we can fix the gauge, and then calculate them in a particular gauge, called the conformal gauge. The term $eR^{(2)}(e)$ is Weyl-invariant. To see why this is so, let us consider the term

$$-\tfrac{1}{4}\rho R^{(2)} = \tfrac{1}{2}\rho\partial_\nu(ee_\alpha{}^\nu\Omega^{\alpha\gamma}{}_\gamma). \tag{14.30}$$

An integration by parts gives

$$-\tfrac{1}{4}\rho R^{(2)} \doteq -\tfrac{1}{2}e\Omega^{\alpha\gamma}{}_\gamma e_\alpha{}^\mu\partial_\mu\rho. \tag{14.31}$$

Then, using the definition of the anholonomy, we get

$$= -\tfrac{1}{2}e(e_\alpha{}^\nu e_\gamma{}^\tau\partial_\nu e_\tau{}^\gamma - e_\gamma{}^\nu e_\alpha{}^\tau\partial_\nu e_\tau{}^\gamma)e^{\alpha\mu}\partial_\mu\rho \tag{14.32}$$

$$= -\tfrac{1}{2}eg^{\mu\nu}e_\gamma{}^\tau\partial_\nu e_\tau{}^\gamma\partial_\mu\rho - \tfrac{1}{2}e\partial_\nu e_\alpha{}^\nu e^{\alpha\mu}\partial_\mu\rho \tag{14.33}$$

where another integration by parts and the definition of the two-bein have been used.

Now we can set the gauge, i.e. the 2D diffeomorphisms, by a condition on the two-bein. So we write

$$e_\mu{}^\alpha = \lambda\tilde{e}_\mu{}^\alpha \tag{14.34}$$

with $\det\tilde{e}_\mu{}^\alpha = 1$ and $\lambda = \lambda(x)$; hence, we are not considering the whole group $GL(2, \mathbf{R})$ but only its restriction to unimodular matrices $SL(2, \mathbf{R})$.

As we said before, we can set a particular gauge, the conformal gauge, by imposing the following condition on the two-bein. It is given by

$$\tilde{e}_\mu{}^\alpha = \delta_\mu^\alpha. \tag{14.35}$$

Then, after an integration by parts, we get

$$-\tfrac{1}{4}\rho R^{(2)} \doteq -\tilde{g}^{\mu\nu}\lambda^{-1}\partial_\nu\lambda\partial_\mu\rho + \tfrac{1}{2}\tilde{e}_\alpha{}^\nu\partial_\nu(\tilde{e}^{\alpha\mu}\partial_\mu\rho). \tag{14.36}$$

In this gauge it is obviously

$$\tilde{g}^{\mu\nu} = \tilde{e}_\alpha{}^\mu\tilde{e}^{\nu\alpha} = \lambda^2 g^{\mu\nu}. \tag{14.37}$$

So we have three fields: the dilaton ρ, the λ and the unimodular two-bein $\tilde{e}_\mu{}^\alpha$. We can calculate the equations of motion varying the Lagrangian with respect to all these fields. Varying it with respect to λ we get

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu\rho) \equiv \Box\rho = 0 \tag{14.38}$$

because in conformal gauge $\tilde{g}^{\mu\nu} = \eta^{\mu\nu}$. The solution of this equation is

$$\rho(x) = \rho_+(x^+) + \rho_-(x^-) \tag{14.39}$$

with $x^\pm = x^0 \pm x^1$.

The dilaton can be dualized: in two dimensions, the dual of a scalar field is again a scalar field. We will refer to the dual of the dilaton field as the 'axion'; it is defined by

$$\partial_\mu\rho + \epsilon_{\mu\nu}\partial^\nu\tilde{\rho} = 0 \tag{14.40}$$

where $\tilde{\rho}$ is just the axion. In the conformal gauge this field is

$$\tilde{\rho}(x) = \rho_+(x^+) - \rho_-(x^-). \tag{14.41}$$

The equation obtained by varying ρ is

$$\partial_\mu(\tilde{g}^{\mu\nu}\lambda^{-1}\partial_\nu\lambda) = \text{matter contribution.} \tag{14.42}$$

Note that with matter contribution we refer to the fields Δ and B coming out from dimensional reduction. The terminology *matter part* will be clear in the following section, where we will be able to identify this fields with the fields of a bosonic nonlinear σ-model Lagrangian.

Before writing the complete Lagrangian of the two-dimensional reduced gravity we must still consider the equation that is obtained from (14.36) by variation with respect to the unimodular two-bein. The corresponding equations must be interpreted as constraint equations (in standard conformal field theory, they would just correspond to the Virasoro constraints). We have

$$-\delta\tilde{e}_\alpha{}^\mu\tilde{e}^{\alpha\nu}\lambda^{-1}\partial_{(\mu}\lambda\partial_{\nu)}\rho + \tfrac{1}{2}\delta\tilde{e}_\alpha{}^\mu\partial_\mu(\tilde{e}^{\alpha\nu}\partial_\nu\rho) - \tfrac{1}{2}\partial_\mu\tilde{e}_\alpha{}^\mu\delta\tilde{e}^{\alpha\nu}\partial_\nu\rho + \text{matter} = 0. \tag{14.43}$$

In conformal gauge, $\tilde{e}_\alpha{}^\mu = \delta_\alpha^\mu$, this expression becomes

$$-\tfrac{1}{2}\delta\tilde{g}^{\mu\nu}\lambda^{-1}\partial_\mu\lambda\partial_\nu\rho + \tfrac{1}{4}\delta\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\rho = \text{matter} \tag{14.44}$$

where $\delta\tilde{g}^{\mu\nu} = 2\delta\tilde{e}_\alpha{}^\mu\tilde{e}^{\alpha\nu}$ has been used. The metric is diagonal in this gauge, so the equations for the conformal factors become

$$\text{(traceless part of)}\{-\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial_\nu\rho + \tfrac{1}{4}\partial_\mu\partial_\nu\rho\} = \text{matter}. \tag{14.45}$$

We have not explicitly written the matter sector yet. This will be done in the next section.

The pure gravity action written in two dimensions in the conformal gauge reads

$$-\tfrac{1}{4}e^{(3)}R^{(3)} = -\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial^\mu\rho \tag{14.46}$$

where the equation $\Box\rho = 0$ has been used to make the second term of (14.36) vanish. The whole Lagrangian is then

$$\mathcal{L}_\mathrm{E} = -\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial^\mu\rho + \tfrac{1}{8}\rho\Delta^{-2}(\partial_\mu\Delta\partial^\mu\Delta + \partial_\mu B\partial^\mu B) \tag{14.47}$$

where the second term with the Kaluza–Klein scalar and the dual of the Kaluza–Klein vector has been obtained considering only the two-dimensional part of the action. The subscript E stands for Ehlers, who did this analysis for the first time in the 1950s.

Here we have treated one possible way of performing dimensional reduction: we have seen it consists of many steps. One first reduces from four to three; then, dualizes the vector field and performs the reduction to two dimensions. However, this is not the whole story: actually, it is possible also to get the two-dimensional Lagrangian directly from the three-dimensional one, without dualization.

The procedure for doing the calculation is as follows: first we express the Kaluza–Klein vector in the form

$$B_m = (B_\mu, B_2 \equiv \tilde{B}) \tag{14.48}$$

and then we perform directly the dimensional reduction in conformal gauge by using the previous choice of the three-bein in triangular form. Proceeding in this way, we meet two electromagnetic fields in two dimensions, A_μ and B_μ, which can be set to zero because they do not propagate (we have already used this argument before) and there is no cosmological constant.

Let us note now the various steps of the calculation. The new writing for the Kaluza–Klein vector and the considerations on two-dimensional electromagnetism imply

$$B_\mu = 0 \to B_{\mu\nu} = 0. \tag{14.49}$$

So the only non-vanishing terms of the three-dimensional Lagrangian before the reduction are

$$\mathcal{L} = -\tfrac{1}{4}e^{(3)}R^{(3)}(e) - \tfrac{1}{8}e^{(3)}\Delta^2 B_{\mu 2}B^{\mu 2} + \tfrac{1}{8}e^{(3)}g^{mn}\Delta^{-2}\partial_m\Delta\partial_n\Delta. \tag{14.50}$$

Then we reduce the dimensions as before (we set $A_\mu = 0$) and choose the conformal gauge. Keeping in mind that

$$g^{\mu\nu} = \lambda^2\eta^{\mu\nu}, \quad g^{22} = -\rho^2 \tag{14.51}$$

we get a new 2D Lagrangian

$$\mathcal{L}_{\mathrm{MM}} = -\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial^\mu\rho + \tfrac{1}{8}\rho\Delta^{-2}\partial_\mu\Delta\partial^\mu\Delta + \tfrac{1}{8}\rho^{-1}\Delta^2\partial_\mu\tilde{B}\partial^\mu\tilde{B} \tag{14.52}$$

for the fields $\tilde{B}$ and Δ.

The subscript MM stands for Matzner–Misner, who performed this analysis at the end of the 1960s. It is worth noting that the link between the fields B and $\tilde{B}$ is given by three-dimensional duality. To see why it is so, let us consider the duality relation

$$e\Delta^2 B^{mn} = \epsilon^{mnp}\partial_p B \tag{14.53}$$

and then reduce the dimensionality using the properties of the Kaluza–Klein vector B_m. The duality relation becomes then

$$\rho^{-1}\Delta^2\partial_\mu\tilde{B} = \epsilon_{\mu\nu}\partial^\nu B. \tag{14.54}$$

The deep relation between these two distinct reducted Lagrangians will be explained in the next section, treating nonlinear σ-models. In the language of the nonlinear sigma models, these two reducted actions correspond to two distinct $SL(2,\mathbf{R})/SO(2)$ models.

14.2 Nonlinear σ-models

In this section, we introduce nonlinear σ-models and discover that the reduced gravity is a certain nonlinear σ-model, connected to a certain symmetry group. The expression of the Lagrangian of the model depends on this symmetry group.

Let us start from a non-compact Lie group G and consider the maximal compact Lie subgroup H of G. The Lie algebra decomposition is

$$\boldsymbol{G} = \boldsymbol{H} \oplus \boldsymbol{K} \tag{14.55}$$

with the following commutation rules

$$[\boldsymbol{H},\boldsymbol{H}] \subset \boldsymbol{H}, \quad [\boldsymbol{K},\boldsymbol{K}] \subset \boldsymbol{H}, \quad [\boldsymbol{H},\boldsymbol{K}] \subset \boldsymbol{K}. \tag{14.56}$$

This decomposition is invariant under the symmetric space automorphism

$$\tau(\boldsymbol{H}) = \boldsymbol{H}, \quad \tau(\boldsymbol{K}) = -\boldsymbol{K} \tag{14.57}$$

which can alternatively be formulated in terms of Lie group elements g directly through

$$\tau(g) = \eta^{-1}(g^{\mathrm{T}})^{-1}\eta \tag{14.58}$$

where the matrix η depends on the group G (e.g. $\eta = 1$ for $G = SL(n,\mathbb{R})$).

Example: $G = SL(2, \boldsymbol{R})$, $H = SO(2)$

The generators of the group are

$$Y^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14.59}$$

We have

$$\mathrm{Tr}(Y^1)^2 = \mathrm{Tr}(Y^2)^2 = -\mathrm{Tr}(Y^3)^2 = 2 \tag{14.60}$$

so Y^1 and Y^2 are the non-compact generators, while Y^3 generates the $SO(2)$ subgroup.

The group can be decomposed on its generators, as

$$\boldsymbol{H} = \boldsymbol{R}Y^3, \quad \boldsymbol{K} = \boldsymbol{R}Y^1 \oplus \boldsymbol{R}Y^2. \tag{14.61}$$

Let us introduce now an element of the group $v(x) \in G$ with the property

$$v(x) \to v'(x) = g^{-1}v(x)h(x) \tag{14.62}$$

where g is a rigid G transformation and $h(x)$ a local H transformation. This type of transformation is needed for the necessity of preserving gauge choice. In fact, you can fix the gauge choosing a particular element of the group v. Then, when you act on v by an arbitrary g, that gauge choice will be lost. To restore the gauge you have to introduce the local transformation $h(x)$ so that the rotation g can be compensed. It follows that h does not depend only on the coordinates x, but also on the vector v and the rotation g.

Therefore, equation (14.62) is called a nonlinear realization of symmetries, because h depends nonlinearly on v.

This is important for the following calculation, because we can fix a gauge, called triangular gauge, such that

$$v(x) = \exp\varphi(x), \quad \varphi(x) \in \mathrm{K} \to v \in G/H. \tag{14.63}$$

The next step is the construction of a Lagrangian with the required symmetry. To this aim, let us consider the Lie algebra valued expression

$$v^{-1}\partial_m v = Q_m + P_m, \quad Q_m \in \boldsymbol{H}, \; P_m \in \boldsymbol{K} \tag{14.64}$$

or equivalently

$$v^{-1}D_m v = v^{-1}(\partial_m v - vQ_m) = P_m \tag{14.65}$$

which defines the H-covariant derivative D_m. It is straightforward to verify that Q_m transforms like a gauge field with respect to the local group H, namely $Q'_m = h^{-1}Q_m h + h^{-1}\partial_m h$ and that $P'_m = h^{-1}P_m h$. The formula (14.62) implies the integrability relations

$$\partial_m Q_m - \partial_n Q_m + [Q_m, Q_n] = -[P_m, P_n]$$
$$D_m P_n - D_n P_m = 0.$$

The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{4} e g^{mn} \operatorname{Tr} P_m P_n \tag{14.66}$$

and then the field equations for P_m read

$$D_m(\sqrt{g} g^{mn} P_n) = 0. \tag{14.67}$$

In the following section we will show how it is possible to reproduce the Lagrangians obtained by dimensional reduction from this general construction of nonlinear σ-models.

14.2.1 Ehlers Lagrangian as a nonlinear σ-model

To link these arguments to the previous discussion, let us consider the groups

$$G = SL(2, \boldsymbol{R}), \qquad H = SO(2). \tag{14.68}$$

The quotient space has only two degrees of freedom. We enforce the triangular gauge choosing for v the following expression

$$v = \begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \tag{14.69}$$

and then

$$\begin{aligned} v^{-1}\partial_m v &= \begin{pmatrix} \frac{1}{2}\Delta^{-1}\partial_m\Delta & \Delta^{-1}\partial_m B \\ 0 & -\frac{1}{2}\Delta^{-1}\partial_m\Delta \end{pmatrix} \\ &= P_m^1 Y^1 + P_m^2 Y^2 + Q_m Y^3 \end{aligned} \tag{14.70}$$

where the coefficients of the generators of the algebra are given by

$$P_m^1 = \tfrac{1}{2}\Delta^{-1}\partial_m\Delta, \qquad P_m^2 = Q_m = \tfrac{1}{2}\Delta^{-1}\partial_m B. \tag{14.71}$$

The evaluation of the Lagrangian is straightforward, and we get

$$\mathcal{L} = \tfrac{1}{4} e g^{mn} \operatorname{Tr} P_m P_n = \tfrac{1}{8} e g^{mn} \Delta^{-2}(\partial_m\Delta\partial_n\Delta + \partial_m B \partial_n B). \tag{14.72}$$

This result matches exactly with the matter part of the Einstein–Hilbert Lagrangian found in the previous section. We have found that this expression can be directly reduced to two dimensions, and then, coupled to gravity, it becomes simply the Ehlers Lagrangian $\mathcal{L}_{\mathrm{E}}$ seen before.

The Ehlers Lagrangian after dimensional reduction is

$$\begin{aligned} \mathcal{L}_{\mathrm{E}} &= \text{gravity} + \tfrac{1}{4}\rho e^{(2)} g^{\mu\nu} \operatorname{Tr}(P_\mu P_\nu) \\ &= -\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial^\mu\rho + \tfrac{1}{8}\rho e^{(2)}\Delta^{-2} g^{\mu\nu}(\partial_\mu\Delta\partial_\nu\Delta + \partial_\mu B\partial_\nu B). \end{aligned} \tag{14.73}$$

We saw in the previous section that, by another type of dimensional reduction, we got a different reduced Lagrangian, the Matzner–Misner one.

This one can be constructed as a nonlinear σ-model too: we need only a different gauge choice, as we will see in the next section; before this, let us look at the equations of motion derived from the Ehlers Lagrangian.

14.2.2 The Ernst equation

The equations of motion for the fields Δ and B from the Lagrangian $\mathcal{L}_E$ are

$$\Delta\partial_\mu(\rho\partial^\mu\Delta) = \rho(\partial_\mu\Delta\partial^\mu\Delta - \partial_\mu B\partial^\mu B) \tag{14.74}$$
$$\Delta\partial_\mu(\rho\partial^\mu B) = 2\rho\partial^\mu\Delta\partial_\mu B. \tag{14.75}$$

Defining a complex function $\mathcal{E} = \Delta + \mathrm{i}B$ called the Ernst potential, these equations can be combined into a single one, called the 'Ernst equation':

$$\Delta\partial_\mu(\rho\partial^\mu\mathcal{E}) = \rho\partial^\mu\mathcal{E}\partial_\mu\mathcal{E}. \tag{14.76}$$

This equation figures prominently in studies of exact solutions of Einstein's equations.

Here we have got the Ernst equation from the fields equations for Δ and B. Actually equation (14.67) *is* the Ernst equation, in the sense that it reduces to it choosing the Ehlers triangular form for v in the conformal gauge.

14.2.3 The Matzner–Misner Lagrangian as a nonlinear σ-model

Recalling the shape of the Matzner–Misner Lagrangian as written before in the conformal gauge

$$\mathcal{L}_{\mathrm{MM}} = -\tfrac{1}{2}\lambda^{-1}\partial_\mu\lambda\partial^\mu\rho + \tfrac{1}{8}\rho\Delta^{-2}\partial_\mu\Delta\partial^\mu\Delta + \tfrac{1}{8}\rho^{-1}\Delta^2\partial_\mu B_2\partial^\mu B_2. \tag{14.77}$$

This can be thought of as a nonlinear σ-model, too. We suppose our Lagrangian to be composed by a term of pure gravity, but reduced to two dimensions, and a term coming from a two-dimensional nonlinear σ-model. We are in conformal gauge, namely $e_\mu{}^\alpha = \lambda\delta_\mu{}^\alpha$ and $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{2}\tilde{\lambda}^{-1}\partial^\mu\tilde{\lambda}\partial_\mu\rho + \tfrac{1}{4}\rho\eta^{\mu\nu}\,\mathrm{Tr}\,\tilde{P}_\mu\tilde{P}_\nu. \tag{14.78}$$

We have to choose a proper gauge, namely an expression for v, such that the two Lagrangians match together.

We refer now to the generators of $SL(2,\boldsymbol{R})$ introduced at the beginning of this section. Let us choose for $\tilde{v}$ the following triangular form

$$\tilde{v} = \begin{pmatrix} (\rho/\Delta)^{1/2} & B_2(\Delta/\rho)^{1/2} \\ 0 & (\Delta/\rho)^{1/2} \end{pmatrix}, \quad \tilde{v}^{-1} = \begin{pmatrix} (\Delta/\rho)^{1/2} & -B_2(\Delta/\rho)^{1/2} \\ 0 & (\rho/\Delta)^{1/2} \end{pmatrix}.$$

Evaluating now the matrix product $\tilde{v}^{-1}\partial_\mu\tilde{v}$ and decomposing it on the algebra generators. Following the standard procedure seen before, the Lagrangian is built using only the non-compact elements of this decomposition. After calculation, we have

$$\begin{aligned}\tilde{v}^{-1}\partial_\mu\tilde{v} = \tilde{P}^1_\mu + \tilde{P}^2_\mu + \tilde{Q}_\mu &= \frac{1}{2}(\rho^{-1}\partial_\mu\rho - \Delta^{-1}\partial_\mu\Delta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &+ \frac{1}{2}\left(\frac{\Delta}{\rho}\right)\partial_\mu B_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\left(\frac{\Delta}{\rho}\right)\partial_\mu B_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned} \tag{14.79}$$

Then, the trace is

$$\tfrac{1}{4}\operatorname{Tr}\tilde{P}_\mu\tilde{P}^\mu = \tfrac{1}{4}(\tilde{P}^1_\mu\tilde{P}^{\mu 1} + \tilde{P}^2_\mu\tilde{P}^{\mu 2}) = \tfrac{1}{8}\rho^{-1}\partial_\mu\rho(\rho^{-1}\partial^\mu\rho - 2\Delta^{-1}\partial^\mu\Delta)$$
$$+\frac{1}{8}\Delta^{-2}\partial_\mu\Delta\partial^\mu\Delta + \frac{1}{8}\left(\frac{\Delta}{\rho}\right)^2\partial_\mu B_2\partial^\mu B_2. \tag{14.80}$$

Now, the two Lagrangians coincide if $\tilde{\lambda}$ satisfies the condition

$$-\tfrac{1}{2}\tilde{\lambda}^{-1}\partial^\mu\tilde{\lambda}\partial_\mu\rho + \tfrac{1}{8}\rho^{-2}\partial_\mu\rho(\rho^{-1}\partial_\mu\rho - 2\Delta^{-1}\partial^\mu\Delta) = -\tfrac{1}{2}\lambda^{-1}\partial^\mu\lambda\partial_\mu\rho \tag{14.81}$$

namely if

$$\tilde{\lambda} \equiv \lambda\rho^{1/4}\Delta^{-1/2}. \tag{14.82}$$

Therefore, the two-dimensional reduced gravity in conformal gauge is given by a part of pure two-dimensional gravity, characterized by the conformal factor λ and the dilaton ρ, and a matter part given by the bosonic fields Δ and B, or $\tilde{B}$: this one has the structure of a nonlinear G/H sigma model.

Following the first section of this paper, the complete Lagrangian reduced to two dimensions in conformal gauge, for any G/H σ-model is

$$\mathcal{L} = -\tfrac{1}{2}\lambda^{-1}\partial^\mu\lambda\partial_\mu\rho + \tfrac{1}{4}\rho\operatorname{Tr}(P_\mu P^\mu) \tag{14.83}$$

and we can recover, as before, the field equation for the conformal factor λ, this time with the general σ-model matter part. It is given by *the traceless part of*

$$\lambda^{-1}\partial_\mu\lambda\partial_\nu\rho = \tfrac{1}{2}\operatorname{Tr}(P_\mu P_\nu) + \tfrac{1}{2}\partial_\mu\partial_\nu\rho. \tag{14.84}$$

This will be useful in the foregoing sections when recovering the colliding plane wave solutions of Einstein's theory.

The Kramer–Neugebauer transformation

Note now that the two models, that of Ehlers and that of Matzner–Misner, are related by the Kramer–Neugebauer transformation, defined by

$$\Delta \leftrightarrow \frac{\rho}{\Delta}, \quad B \leftrightarrow B_2$$

It is worth remembering that the fields B and B_2 are related by duality too, namely

$$\epsilon_{\mu\nu}\partial^\nu B = \frac{\Delta^2}{\rho}\partial_\mu B_2. \tag{14.85}$$

To sum up: in this section we have seen that the dimensional reduction of Einstein theory from $D = 4$ to $D = 2$ can be done in two ways, leading to two different $SL(2,\boldsymbol{R})/SO(2)$ σ-models.

We discover two different isometry groups, that of Ehlers and that of Matzer–Misner

$$SL(2,\boldsymbol{R})_{\mathrm{E}}, \quad SL(2,\boldsymbol{R})_{\mathrm{MM}}. \tag{14.86}$$

Combining these two groups, one gets the (infinite-dimensional) Geroch group.

14.3 Symmetries of nonlinear σ-models

We have seen that for preserving the gauge choice, in particular the triangular gauge, the symmetry

$$v \rightarrow v' = g^{-1}v, \quad g \in G \tag{14.87}$$

must be realized in a nonlinear way, namely

$$v \rightarrow v'(x) = g^{-1}v(x)h(x), \quad g \in G, \ h \in H. \tag{14.88}$$

Now consider the infinitesimal form of (14.88). The infinitesimal variation of v is

$$\delta v(x) = -\delta g^{-1}v(x) + v(x)\delta h(x) \tag{14.89}$$

applying now this linearized transformation to the two σ-models seen before.

Considering in particular the Chevalley–Serre generators for the $SL(2, \mathbb{R})$ Lie algebra

$$e \equiv T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \equiv T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \equiv T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14.90}$$

endowed with the following commutation rules

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \tag{14.91}$$

one can check that this nonlinear transformation has been introduced to preserve the gauge. Let us now analyse the action of the Ehlers and Matzner–Misner groups in turn.

14.3.1 Nonlinear realization of $SL(2, R)_{\mathrm{E}}$

We use the Chevalley–Serre generators for the algebra. Considering the triangular gauge

$$v = \begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \tag{14.92}$$

we now linearize the transformation (14.87). The variation of v is only due to the algebra element a:

$$\delta v = v' - v = -av. \tag{14.93}$$

Then, given the triangular form of v, it follows also

$$\delta v = \begin{pmatrix} \frac{1}{2}\Delta^{-1/2}\delta\Delta & -\frac{1}{2}\Delta^{-3/2}B\delta\Delta + \Delta^{-1/2}\delta B \\ 0 & -\frac{1}{2}\Delta^{-3/2}\delta\Delta \end{pmatrix}. \tag{14.94}$$

In the following we will refer to the variation $\delta\Delta$ by, for example, the generator e with the compact notation $e(\Delta)$ or $e(B)$ for B. Now, we realize the transformation using the Chevalley–Serre algebra generators, e, h and f. For e we have

$$e_1 : -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta^{-1/2} \\ 0 & 0 \end{pmatrix} \tag{14.95}$$

where the subscript 1 refers to the Ehlers group. From (14.94), one deduces

$$e_1(\Delta) = 0, \quad e_1(B) = -1 \tag{14.96}$$

and the triangular gauge is preserved. The calculation is analogous for the generator h

$$h_1 : -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} = \begin{pmatrix} -\Delta^{1/2} & -B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} \tag{14.97}$$

with $h_1(\Delta) = -2\Delta$ and $h_1(B) = -2B$. The triangular gauge is still preserved. This is not so for the third generator, f. Repeating the above steps we find

$$f_1 : -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Delta^{1/2} & B\Delta^{-1/2} \\ 0 & \Delta^{-1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\Delta^{1/2} & -B\Delta^{-1/2} \end{pmatrix} \tag{14.98}$$

namely the triangular gauge is not preserved. Therefore, we have to introduce a compensating term, i.e. we need the transformation rule (14.89). We introduce a local H transformation parametrized by a function ω, which is determined in such a way as to preserve the gauge. Remember that the H generator is Y^3:

$$f_1 : -f_1 v + v(-\omega Y^3) = \begin{pmatrix} 0 & 0 \\ -\Delta^{1/2} & -B\Delta^{-1/2} \end{pmatrix} + \omega\begin{pmatrix} B\Delta^{-1/2} & -\Delta^{1/2} \\ \Delta^{-1/2} & 0 \end{pmatrix}. \tag{14.99}$$

The triangular gauge is defined by the condition

$$-\sqrt{\Delta} + \frac{\omega}{\sqrt{\Delta}} = 0 \rightarrow \omega = \Delta \tag{14.100}$$

and so the transformation reads

$$f_1 : \delta v = \begin{pmatrix} B\Delta^{1/2} & -\Delta^{3/2} \\ 0 & -B\Delta^{-1/2} \end{pmatrix}. \tag{14.101}$$

Hence the variations of the fields Δ and B are

$$f_1(\Delta) = 2\Delta B, \quad f_1(B) = B^2 - \Delta^2 \tag{14.102}$$

clearly not linear in the fields.

Note that the $SL(2, \boldsymbol{R})$ transformations leave the fields ρ and λ unchanged, i.e.

$$\delta\lambda = 0, \quad \delta\rho = 0.$$

14.3.2 Nonlinear realization of $\boldsymbol{SL(2, R)_{MM}}$

On the other side, identical calculations can be done to evaluate the action of $SL(2, \boldsymbol{R})_{MM}$ on the fields (Δ, B_2). Also in this case the symmetry is realized in a nonlinear way. We have (with the suffix 0 for Matzner–Misner)

$$e_0(\Delta) = 0, \quad e_0(B_2) = -1 \tag{14.103}$$

$$h_0(\Delta) = 2\Delta, \quad h_0(B_2) = -2B_2 \tag{14.104}$$

$$f_0(\Delta) = -2\Delta B_2, \quad f_0(B_2) = B_2^2 - \left(\frac{\rho}{\Delta}\right)^2. \tag{14.105}$$

Again, the generator f_0 acts nonlinearly.

14.4 The Geroch group

The aim of this section is to combine the two groups, $SL(2, \mathbf{R})_E$ with fields (Δ, B) and $SL(2, \mathbf{R})_{MM}$, with (Δ, B_2), into a unified group, the infinite-dimensional Geroch group. The associated Lie algebra is an affine Kac–Moody algebra.

We return first to duality relation

$$\rho^{-1}\Delta^2\partial_\mu B_2 = \epsilon_{\mu\nu}\partial^\nu B \tag{14.106}$$

which is invariant under the Kramer–Neugebauer transformation. We need this equation because we now have to evaluate the action of $SL(2)_E$ on B_2 and of $SL(2)_{MM}$ on B.

14.4.1 Action of $SL(2, R)_E$ on $\tilde{\lambda}$, B_2

Keeping in mind that $\delta\rho = 0$, we have

$$B \to B + \delta B \Rightarrow \epsilon_{\mu\nu}\partial^\nu(\delta B) = \delta(\Delta^2\rho^{-1}\partial_\mu B_2) \tag{14.107}$$

after the functional differentiation and the usage of duality

$$\partial_\mu(\delta B_2) = \rho\epsilon_{\mu\nu}(\partial^\nu\delta B - 2\Delta^{-3}\delta\Delta). \tag{14.108}$$

Consequently, from the change of B calculated before, we have the variation of B_2 due to the $SL(2)_E$ generators.

$$e_1 : 0 = \partial_\mu(\delta B_2) \Rightarrow e_1(B_2) = c_1 (= \text{constant}) \tag{14.109}$$

$$h_1 : \partial_\mu(\delta B_2) = 2\rho\Delta^{-2}\epsilon_{\mu\nu}\partial^\nu B \Rightarrow h_1(B_2) = 2B_2 \tag{14.110}$$

$$f_1 : \epsilon^{\mu\nu}\partial_\nu(\delta B_2) = 2\rho(\Delta^{-2}B\partial^\nu B + \Delta^{-1}\partial^\nu\Delta) \Rightarrow f_1(B_2) = 2\phi_1. \tag{14.111}$$

Here a dual potential ϕ_1 has been introduced, which is defined such that

$$\rho^{-1}\epsilon_{\mu\nu}\partial^\nu\phi_1 = \Delta^{-2}(B\partial_\mu B + \Delta\partial_\mu\Delta). \tag{14.112}$$

Careful inspection of these relations now shows the following. The contributions due to e_1 and h_1 are linear in the fields and local; the difference is in the transformation generated by f_1, which is clearly nonlinear and non-local, because one has to perform an integration to calculate explicitly the dual potential.

We then evaluate also the action on $\tilde{\lambda}$. From the definition of (14.82), and observing that $\delta\lambda = 0$, it follows that

$$\tilde{\lambda}^{-1}\delta\tilde{\lambda} = -\tfrac{1}{2}\Delta^{-1}\delta\Delta \tag{14.113}$$

14.4.2 Action of $SL(2, R)_{\text{MM}}$ on λ, B

Exactly the same analysis has to be done for the other group, with generators (e_0, h_0, f_0)

$$e_0 :\Rightarrow e_0(B) = c_0 \tag{14.114}$$

$$h_0 :\Rightarrow h_0(B) = 2B \tag{14.115}$$

$$f_0 :\Rightarrow f_0(B) = 2\phi_0 \tag{14.116}$$

with

$$\rho\epsilon_{\mu\nu}\partial^\nu\phi_0 = -\Delta^2 B_2\partial_\mu B_2 + \rho\Delta\partial_\mu\left(\frac{\rho}{\Delta}\right). \tag{14.117}$$

14.4.3 The affine Kac–Moody $SL(2, R)$ algebra

The transformations we have just derived are to be identified with an affine $SL(2, \boldsymbol{R})$ Kac–Moody algebra. The latter is characterized by the Cartan matrix

$$A_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{14.118}$$

and the standard Chevalley–Serre presentation defining the algebra which can be read off from the Cartan matrix:

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= A_{ij} e_j \\ [h_i, f_j] &= -A_{ij} f_j \\ [e_i, f_j] &= \delta_{ij} h_j \\ [e_i[e_i[e_i, e_j]]] &= 0 \\ [f_i[f_i[f_i, f_j]]] &= 0. \end{aligned}$$

Here $i = j = 0, 1$; note that there is no summation on repeated indices and that the first relation defines the Cartan subalgebra. To see the relation with the $SL(2, \boldsymbol{R})$ transformations dealt with before, we make the identifications

$$e_1 = T_0^+, \quad f_1 = T_0^-, \quad h_1 = T_0^3 \tag{14.119}$$

$$e_0 = T_1^-, \quad f_0 = T_-^+, \quad h_0 = c - T_0^3. \tag{14.120}$$

Using the known commutation rules of the Chevalley–Serre generators of $SL(2, \boldsymbol{R})$ it is possible to directly check the algebra.

For example, it is straightforward to see that

$$[h_1, e_0] = [T_0^3, T_1^-] = -2T_1^- = -2e_0 \tag{14.121}$$

or that

$$\begin{aligned}[e_1, e_0] &= [T_0^+, T_1^-] = 2T_1^3 \\ [e_1, [e_1, e_0]] &= -4T_1^+ \Rightarrow [e_1[e_1[e_1, e_0]]] = 0\end{aligned} \tag{14.122}$$

and so on for the other commutators.

The full current algebra is now built by taking multiple commutators in all possible ways. The Lie algebra element $c = h_0 + h_1$ is the central charge. It has a trivial action on the fields Δ, B, B_2, $\tilde{\lambda}$, λ.

14.5 The linear system

The aim of this section is to linearize and localize the action of the Geroch group seen in the previous section.

Let us start from the Lagrangian for arbitrary G/H in three dimensions and then reduce to two

$$\mathcal{L} = -\tfrac{1}{4}\rho e R(e) + \tfrac{1}{4}\rho e g^{mn}\,\mathrm{Tr}\, P_m P_n. \tag{14.123}$$

We pick now the conformal gauge for the three-bein, as before

$$e_m^a = \begin{pmatrix} \lambda\delta_\mu^\alpha & 0 \\ 0 & \rho \end{pmatrix} \tag{14.124}$$

where we have dropped the two-dimensional Kaluza–Klein vector because it carries no physical degrees of freedom any more. It is well known that the choice $e_\mu^\alpha = \lambda e_\mu^\alpha$ is preserved under conformal diffeomorphisms

$$\delta x^+ = \xi_-(x^+), \quad \delta x^- = \xi_+(x^-) \tag{14.125}$$

with the light cone coordinates $x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. This residual coordinate freedom can be gauged away, for example, by employing the dilaton and the axion fields. One can fix the residual conformal diffeomorphisms by identifying the field ρ or $\tilde{\rho}$ with one of the coordinates.

14.5.1 Solving Einstein's equations

Let us now focus our attention on the way of solving Einstein's equation. First note that by substituting the gauge (14.124) into the scalar equation (14.67), we arrive at

$$\rho^{-1} D^\mu(\rho P_\mu) = 0. \tag{14.126}$$

The dependence of this equation on ρ is all that remains of three-dimensional gravity. Equation (14.126) reduces to the Ernst equation for $G = SL(2, \mathbb{R})$, but we will return to this later. For the moment, note only that this equation works for the σ-model degrees of freedom, namely on Δ.

The remaining equations, which follow from higher dimensions, are the equations for the dilaton ρ and for the conformal factor λ. ρ is a free field in two dimensions which can be solved for in terms of two arbitrary functions (left-movers and right-movers)

$$\Box\rho = 0 \Rightarrow \rho(x) = \rho_+(x^+) + \rho_-(x^-). \tag{14.127}$$

The equations of motion for the conformal factor in light-cone coordinates

$$\rho^{-1}\partial_\pm\rho\lambda^{-1}\partial_\pm\lambda = \tfrac{1}{2}\,\mathrm{Tr}(P_\pm P_\pm) + \tfrac{1}{2}\rho^{-1}\partial_\pm^2\rho \tag{14.128}$$

can be written as

$$\partial_\pm\rho\partial_\pm\hat{\sigma} = \tfrac{1}{2}\rho\,\mathrm{Tr}\,P_\pm P_\pm \tag{14.129}$$

where the second term on the right-hand side of (14.128) has been reabsorbed into the Liouville scalar $\hat{\sigma} = \lambda(\partial_+\rho)^{-\frac{1}{2}}(\partial_-\rho)^{-\frac{1}{2}}$. Note that this equation determines λ only up to a constant factor. Observe also that this equation has no analogue in flat space theories, and this, together with the presence of ρ, makes a great difference. For instance, we cannot simply put ρ = constant, for this would imply the vanishing of the right-hand side of (14.129), which by the positivity of the Killing metric on the subalgebra $\mathbb{K}$ would imply $P_\pm = 0$ and leave us only with the trivial solution v = constant (modulo H gauge transformations).

Now specializing to general relativity, i.e. $G/H = SL(2, \mathbb{R})/SO(2)$ coset space. As anticipated before, we start from the equation of motion (14.126), employ the triangular gauge in the Ehlers form, so to have explicit expressions for P_μ and Q_ν. After a little algebra we have again the Ernst equation

$$\Delta\partial_\mu(\rho\partial^\mu\mathcal{E}) = \rho\partial_\mu\mathcal{E}\partial^\mu\mathcal{E} \tag{14.130}$$

in terms of the complex potential $\mathcal{E} = \Delta + \mathrm{i}B$. Solving Einstein's equations is now simply a matter of choosing the appropriate $\rho(x)$, finding a solution of the nonlinear partial differential equation (14.130) and finally determining the conformal factor λ by integration of (14.129). For the colliding plane wave solutions, that will be recovered in the next sections, one distinguishes waves with collinear polarization, where $B = 0$ and waves with non-collinear polarization. For collinearly polarized waves, the nonlinear Ernst equation can be reduced to a *linear* partial differential equation through the replacement $\Delta = \exp\psi$.

So, for collinearly polarized waves, with $B = 0$, the four-bein is

$$E_M{}^A = \begin{pmatrix} \lambda\Delta^{-1/2} & 0 & 0 & 0 \\ 0 & \lambda\Delta^{-1/2} & 0 & 0 \\ 0 & 0 & \rho\Delta^{-1/2} & 0 \\ 0 & 0 & 0 & \Delta^{1/2} \end{pmatrix} \tag{14.131}$$

and then the four dimensional line element is

$$ds^2 = 2\Delta^{-1}\lambda^2\,dx^+\,dx^- - \Delta^{-1}\rho^2(dx^2)^2 - \Delta(dx^3)^2 \tag{14.132}$$

where Δ, ρ and λ depend only on x^+ and x^-.

14.5.2 The linear system

The integrability of the nonlinear equation of motion (14.126) is reflected in the existence of a linear system. This means that there is a set of *linear* differential equations, whose compatibility conditions yield just the nonlinear equations that one tries to solve.

To formulate the linear system one must introduce a so-called spectral parameter t as an extra variable and replace $v(x)$ by a matrix $\hat{v}(x)$ which also depends on t.

$$v(x^0, x^1) \rightarrow \hat{v}(x^0, x^1; t). \tag{14.133}$$

We postulate

$$\hat{v}^{-1}\partial_\mu \hat{v} = Q_\mu + \frac{1+t^2}{1-t^2}P_\mu + \frac{2t}{1-t^2}\epsilon_{\mu\nu}P^\nu. \tag{14.134}$$

This is a generalization of $v^{-1}\partial_\mu v = Q_\mu + P_\mu$, which is obtained from (14.134) in the case $t = 0$. (14.134) is equivalent to

$$\hat{v}\partial_\pm \hat{v} = Q_\pm + \frac{1 \mp t}{1 \pm t}P_\pm. \tag{14.135}$$

Here we have an integrability condition, written as

$$\partial_+(\hat{v}^{-1}\partial_-\hat{v}) - \partial_-(\hat{v}^{-1}\partial_+\hat{v}) + [\hat{v}^{-1}\partial_+\hat{v}, \hat{v}^{-1}\partial_-\hat{v}] = 0 \tag{14.136}$$

which using (14.135) can be directly checked by calculation, making use of the integrability condition seen before and of the equation of motion for ρ. We define explicitly

$$\mathcal{A} = \partial_+(\hat{v}^{-1}\partial_-\hat{v}) - \partial_-(\hat{v}^{-1}\partial_+\hat{v}) \tag{14.137}$$

$$\mathcal{B} = [\hat{v}^{-1}\partial_+\hat{v}, \hat{v}^{-1}\partial_-\hat{v}]. \tag{14.138}$$

Employing (14.135) these relations become

$$\mathcal{A} = \partial_+ Q_- - \partial_- Q_+ + \frac{1+t}{1-t}\partial_+ P_- - \frac{1-t}{1+t}\partial_- P_+ + \partial_+\left(\frac{1+t}{1-t}\right)P_- - \partial_-\left(\frac{1-t}{1+t}\right)P_+ \tag{14.139}$$

$$\mathcal{B} = [Q_+, Q_-] + [P_+, P_-] + \frac{1+t}{1-t}[Q_+, P_-] - \frac{1-t}{1+t}[Q_-, P_+]. \tag{14.140}$$

The sum now reads

$$\mathcal{A}+\mathcal{B}=\frac{1+t}{1-t}D_+P_- - \frac{1-t}{1+t}D_-P_+ + \frac{2t}{(1-t)^2}t^{-1}\partial_+ t P_- + \frac{2t}{(1+t)^2}t^{-1}\partial_- t P_+ \tag{14.141}$$

where the integrability relation seen in section 14.2 has been used. Now let us postulate

$$t^{-1}\partial_\pm t = \frac{1\mp t}{1\pm t}\rho^{-1}\partial_\pm \rho \tag{14.142}$$

so it follows

$$\mathcal{A}+\mathcal{B}=\frac{1+t^2}{1-t^2}(D_+P_- - D_-P_+) + \frac{2t}{1-t^2}(D_+P_- + D_-P_+) + \frac{2t}{1-t^2}(\rho^{-1}\partial_+\rho P_- + \rho^{-1}\partial_-\rho P_+). \tag{14.143}$$

Now the first term is null for the integrability relation $D_+P_- = D_-P_+$ and the second for the equation (14.126). Therefore, the integrability condition is checked.

Let us now focus on equation (14.142): it is integrable once one has a solution of $\Box\rho = 0$. This can be explicitly verified, as it follows. First, let us multiply (14.142) by $(1-t^2)$; after a little algebra this equation reduces to

$$\partial_\pm\left[\rho\left(t+\frac{1}{t}\right)-2\tilde{\rho}\right]=0 \tag{14.144}$$

where the axion $\tilde{\rho}$ has been introduced. So one must have

$$\frac{1}{2}\rho\left(t+\frac{1}{t}\right)-\tilde{\rho}=w \tag{14.145}$$

where w is an integration constant. When we substitute in this relation the explicit expression of the dilaton and the axion as functions of incoming and outgoing fields, we get

$$t(x;w)=\frac{\sqrt{w+\rho_+(x^+)}-\sqrt{w-\rho_-(x^-)}}{\sqrt{w+\rho_+(x^+)}+\sqrt{w-\rho_-(x^-)}}. \tag{14.146}$$

For fixed x, the function $t(x;w)$ lives on a two-sheeted Riemann surface over the complex w-plane, with an x-dependent cut extending from $\rho_-(x^-)$ to $\rho_+(x^+)$. The integration constant w can be regarded as an alternative spectral parameter.

The inverse of the spectral parameter is also important

$$y\equiv\frac{1}{w}=\frac{2t}{\rho(1+t^2)-2t\tilde{\rho}}=\begin{cases}\dfrac{2t}{\rho}+\cdots, & t\sim 0\\ \dfrac{2}{\rho t}+\cdots, & t\sim\infty\end{cases} \tag{14.147}$$

where we consider the expansion around zero and infinity. What is the significance of the replacement (14.133)? A spectral parameter is required if one wants to enlarge the finite Lie group to its affine extension, and the appearance of t in (14.133) fits nicely with this expectation. There is now an infinite hierarchy of fields, as one can see by expanding $\hat{v}$ in t. For convenience let us pick a generalized triangular gauge, defined by the requirement that $\hat{v}$ should be regular at $t = 0$, or

$$\hat{v}(x;t) = \exp \sum_{n=0}^{\infty} t^n \varphi_n(x). \tag{14.148}$$

Another important feature of the linear system is the invariance under a generalization of the symmetric space automorphism. Let us define it for $\eta = 1$

$$\tau^{\infty} \hat{v}(t) = (\hat{v}^{\mathrm{T}})^{-1} \left(\frac{1}{t}\right). \tag{14.149}$$

In terms of the Lie algebra, the action of τ^{∞} reads

$$Q_\mu \to Q_\mu, \quad P_\mu \to -P_\mu. \tag{14.150}$$

It is straightforward to verify that

$$\tau^{\infty}(\hat{v}^{-1}\partial_\mu \hat{v}) = \hat{v}^{-1}\partial_\mu \hat{v}. \tag{14.151}$$

We can say that it is $\hat{v}\partial_\mu \hat{v} \in \boldsymbol{H}^{\infty}$, which is the subalgebra of the Geroch group G_t^{∞} which is t^{∞}-invariant, as happens for finite-dimensional symmetric spaces. It is worth noticing that this property does not hold for $v^{-1}\partial_\mu v$ if we replace τ^{∞} with the transformation τ defined in section two.

14.5.3 Derivation of the colliding plane metric by factorization

At this point we can convince ourselves that the results obtained so far can be used to construct exact solutions of Einstein's equations. Of central importance for this task is the monodromy matrix, which is defined as follows

$$\mathcal{M} = \hat{v}(x;t)\hat{v}^{\mathrm{T}}\left(x; \frac{1}{t}\right). \tag{14.152}$$

A short calculation reveals that

$$\partial_\mu \mathcal{M} = \hat{v}(\hat{v}^{-1}\partial_\mu \hat{v} - \tau^{\infty}(\hat{v}^{-1}\partial_\mu \hat{v}))\tau^{\infty}\hat{v}^{-1} = 0 \tag{14.153}$$

where the relation (14.151) was used. Consequently, $\mathcal{M}$ can only depend on w. The solutions generating procedure now consists in choosing a matrix $\mathcal{M}(w)$ and finding a factorization as in (14.152).

The simplest non-trivial example, that will be considered here, permits us to recover the Ferrari–Ibanez colliding plane wave metric. Let us consider for this aim the monodromy matrix

$$\mathcal{M}(w) = \begin{pmatrix} \frac{w_0 - w}{w_0 + w} & 0 \\ 0 & \frac{w_0 + w}{w_0 - w} \end{pmatrix} \in SL(2, \boldsymbol{C}) \tag{14.154}$$

and use

$$w - w_0 = -\frac{\rho}{4t_0}(t - t_0)\left(\frac{1}{t} - t_0\right) \tag{14.155}$$

with the special value $w_0 = \frac{1}{2}$ and $t_0 \equiv t(x; w_0)$.

We use light cone coordinates, with the following notation to facilitate the comparison with the standard literature

$$u \equiv x^+, \quad v \equiv x^- \tag{14.156}$$

then the remaining conformal invariance is entirely fixed by choosing the coordinates in such a way that

$$\rho_+(u) = \tfrac{1}{2}(1 - 2u^2), \quad \rho_-(v) = \tfrac{1}{2}(1 - 2v^2) \Rightarrow \rho(u, v) = 1 - u^2 - v^2 \tag{14.157}$$

where $\rho(u, v) > 0$ because the interaction region, where the waves collide, is $u^2 + v^2 < 1$. Substituting (14.155) into (14.154) and defining two particular solutions (14.146) in our gauge as

$$t_1(u, v) \equiv t\left(u, v; w = \frac{1}{2}\right) = \frac{\sqrt{1 - u^2} - v}{\sqrt{1 - u^2} + v} > 0 \tag{14.158}$$

$$t_1(u, v) \equiv t\left(u, v; w = -\frac{1}{2}\right) = -\frac{\sqrt{1 - v^2} + u}{\sqrt{1 - v^2} - u} < 0 \tag{14.159}$$

where the inequalities hold in the interaction region, we obtain in a straightforward way the desired factorization form for the monodromy matrix. Then it follows that

$$\hat{v}(u, v; t) = \begin{pmatrix} \sqrt{-\frac{t_2}{t_1}\frac{t - t_1}{t - t_2}} & 0 \\ 0 & \sqrt{-\frac{t_1}{t_2}\frac{t - t_2}{t - t_1}} \end{pmatrix}. \tag{14.160}$$

Putting $t = 0$ we recover $v(u, v)$ in the triangular gauge, and then read directly the result for Δ by virtue of (14.69). We get

$$\Delta = -\frac{t_1}{t_2} = \frac{1 - \xi}{1 + \xi}, \quad B = 0 \tag{14.161}$$

where the oblate spherical coordinates have been introduced

$$\xi \equiv u\sqrt{1 - v^2} + v\sqrt{1 - u^2} \tag{14.162}$$

$$\eta \equiv u\sqrt{1 - v^2} - v\sqrt{1 - v^2}. \tag{14.163}$$

From (14.161) we have $P^2_\pm = Q_\pm = 0$, with $P^1_\pm = \frac{1}{2}\Delta^{-1}\partial_\pm\Delta$. Putting equation (14.161) into this second relation, we gain

$$\tfrac{1}{2}\Delta^{-1}\partial_\pm\Delta = \tfrac{1}{2}t_1^{-1}\partial_\pm t_1 - \tfrac{1}{2}t_2^{-1}\partial_\pm t_2 \tag{14.164}$$

which using the formulae

$$t^{-1}\partial_+ t = \rho^{-1}\partial_+\rho\frac{1-t}{1+t}, \qquad t^{-1}\partial_- t = \rho^{-1}\partial_-\rho\frac{1+t}{1-t} \tag{14.165}$$

becomes

$$P^1_\pm = \frac{1}{2}\rho^{-1}\partial_\pm\rho\left(\frac{1\mp t_1}{1\pm t_1} - \frac{1\mp t_2}{1\pm t_2}\right). \tag{14.166}$$

Now we use the expression given here for P^1_+ to integrate the equation for the conformal factor. Some further calculations show that

$$\lambda^2 = 8uv\frac{(1-t_1t_2)^2}{(1-t_1^2)(1-t_2^2)} \tag{14.167}$$

where the undetermined overall factor has been chosen for convenience. Then, this result yields the four-dimensional metric

$$\mathrm{d}s^2 = (1+\xi)^2\left(\frac{\mathrm{d}\xi^2}{1-\xi^2} - \frac{\mathrm{d}\eta^2}{1-\eta^2}\right) - \rho^2\frac{1-\xi}{1+\xi}(\mathrm{d}x^2)^2 - \frac{1+\xi}{1-\xi}(\mathrm{d}x^3)^2. \tag{14.168}$$

This is (a special case of) the so-called Ferrari–Ibanez colliding plane wave solution.

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Chapter 15

Gyroscopes and gravitational waves

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The behaviour of a gyroscope in geodesic motion is studied in the field of a plane gravitational wave. We find that, with respect to a special set of frames, the compass of inertia undergoes a precession which, to first order in the dimensionless amplitude h of the wave, is dominated by the cross-polarization alone. This suggests that a gyro might act as a filter of the polarization state of the wave.

15.1 Introduction

The (direct) detection of gravitational waves is still an open question, although indirect evidence for their existence has been obtained from the observation of the binary pulsar system PSR 1913+16 [1]. Besides the well-known bar antennae, there is a growing interest in laser interferometry detectors, like LIGO and VIRGO, which are sensitive to the low frequency ($\sim$10 Hz) gravitational waves which are emitted by sources like coalescing binaries.

The purpose of this paper is to study the behaviour of a test gyroscope which is acted upon by a plane gravitational wave with the purpose to see whether this interaction leads to observable effects. It is well known that in the absence of significant coupling between the background curvature and the multipole moments of the energy–momentum tensor of an extended body, the spin vector is Fermi–Walker transported along the body's own trajectory (see [2] and references therein). The effects of a gravitational wave on a frame which is not Fermi–Walker transported, are best appreciated by studying the precession of a gyro at rest in that frame. Clearly it is essential to identify a class of frames which optimize the corresponding precession effect.

In section 15.2 we give a short review of the observer dependent spacetime splitting which enables one to describe in physical terms the motion of a test particle as well as that of a test gyroscope. In section 15.3 we discuss the spacetime of a plane gravitational wave and confine our attention to the family of static observers; we give an example of a tetrad frame adapted to these observers with respect to which the precession of a gyroscope is induced by the cross-polarization only. In section 15.4 we discuss the non-trivial problem of how to fix, in an operational and non-ambiguous way, a frame of reference which is not Fermi–Walker transported in the spacetime of a plane gravitational wave. Finally, in section 15.5 we calculate the precession of a gyroscope, in a general geodesic motion, with respect to the above frame.

In what follows, Greek indices run from 0–3, latin indices from 1–3.

15.2 Splitting formalism and test particle motion: a short review

A given family of test observers, namely a congruence of timelike lines with unit tangent vector field u (i.e. $u \cdot u = -1$) induces a splitting of the spacetime into space plus time through the orthogonal decomposition of the tangent space at each point into the local time direction along u and the local rest space LRS_u.

Projection of spacetime tensor fields onto LRS_u is accomplished using the projection operator

$$P_{(u)} = I + u \otimes u \tag{15.1}$$

and yields a family of spatial tensors (belonging to $\mathrm{LRS}_u \otimes \cdots \otimes \mathrm{LRS}_u$, i.e. for which any contraction with u vanishes). The collection of all the spatially projected tensor fields, associated to a given spacetime tensor field, will be referred to as the 'measure' of the spacetime tensor itself. For instance, the measure of the unit volume four-form $\eta_{\alpha\beta\mu\nu}$, gives only one non-trivial spatial field: $\epsilon_{(u)\alpha\beta\gamma} = u^\delta \eta_{\delta\alpha\beta\gamma}$ which can be used in turn to define the spatial cross product $\times_u$ in LRS_u.

One can also spatially project the various derivative operators so that the result of the derivative of any tensor field is itself a spatial tensor; examples are: the spatial Lie derivative, $\pounds_{(u)X} = P_{(u)}\pounds_X$ for any vector field X, the spatial covariant derivative $\nabla_{(u)\alpha} = P_{(u)} P_{(u)}{}^\beta_\alpha \nabla_\beta$, the Lie temporal derivative, $\nabla_{(\mathrm{lie},u)} = P_{(u)}\pounds_u$, the Fermi–Walker temporal derivative, $\nabla_{(\mathrm{fw},u)} = P_{(u)}\nabla_u$ and several other natural derivatives for which a detailed discussion can be found in [2].

The measure of the covariant derivative of the four-velocity of the observers, gives rise to the kinematical coefficients of the observer congruence, namely the acceleration, vorticity expansion

$$
\begin{aligned}
a^{\alpha}_{(u)} &= \nabla_{(\mathrm{fw},u)} u^{\alpha}, \\
\omega_{(u)\alpha\beta} &= P_{(u)}{}^{\gamma}{}_{\alpha} P_{(u)}{}^{\delta}{}_{\beta} \nabla_{[\gamma} u_{\delta]}, \\
\theta_{(u)\alpha\beta} &= P_{(u)}{}^{\gamma}{}_{\alpha} P_{(u)}{}^{\delta}{}_{\beta} \nabla_{(\gamma} u_{\delta)},
\end{aligned} \tag{15.2}
$$

and the spatial dual of the vorticity field

$$
\omega_{(u)}{}^{\alpha} = \tfrac{1}{2}\epsilon_{(u)}{}^{\alpha\beta\gamma}\omega_{(u)\beta\gamma}. \tag{15.3}
$$

When dealing with different families of test observers, say u and U, the mixed projection map $P_{(U,u)} = P_{(u)}P_{(U)}$ from LRS_U to LRS_u (and the analogous compositions of two or more projectors) will be useful. Let ℓ_U be the world line of a nonzero rest mass test particle with U as its unit timelike tangent vector. The orthogonal decomposition of U relative to the family of test observers u, identifies its relative velocity $\nu_{(U,u)} = \nu\hat{\nu}_{(U,u)}$ where $\nu = \|\nu_{(U,u)}\| = \|\nu_{(u,U)}\|$ and $\hat{\nu}_{(U,u)}$ is the unit spatial vector, so that

$$
U = \gamma[u + \nu\hat{\nu}_{(U,u)}]. \tag{15.4}
$$

Here $\gamma = (1-\nu^2)^{-1/2}$ is the local relative Lorentz factor. If the four acceleration of the particle

$$
a_{(U)} = \nabla_U U = \frac{D}{\mathrm{d}\tau_U} U
$$

is non-vanishing, then its projection onto LRS_u, leads to the acceleration-equals-force equation:

$$
P_{(U,u)} a_{(U)} \equiv \gamma F_{(U,u)}
$$

where $F_{(U,u)}$ is the spatial *force* acting on the particle as *seen* by the observer u. In a similar way, one defines a *spatial gravitoinertial force*

$$
\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,u)} &= -\gamma^{-1} P_{(u)} \frac{Du}{\mathrm{d}\tau_U} \\
&= -P_{(u)} \frac{Du}{\mathrm{d}\tau_{(U,u)}} \\
&= \gamma[g_{(u)} + \nu(\tfrac{1}{2}\hat{\nu}_{(U,u)} \times_u H_{(u)} - \theta_{(u)} \llcorner \hat{\nu}_{(U,u)})],
\end{aligned} \tag{15.5}
$$

where τ_U is a proper time parametrization for U and $\tau_{(U,u)} = \int_{\ell_U} \gamma\, \mathrm{d}\tau_U$ is the corresponding Cattaneo relative standard time parametrization; $g_{(u)} = -a_{(u)}$ and $H_{(u)} = 2\omega_{(u)}$ are, respectively, the electric- and magnetic-like components of the gravitoinertial force. This terminology is justified by the Lorentz form of the gravitoinertial force which appears in the last of equations (15.5).

If we define $p_{(U,u)} = \gamma\nu_{(U,u)}$, $E_{(U,u)} = \gamma$ and

$$
\frac{D_{(\mathrm{fw},U,u)}}{\mathrm{d}\tau_{(U,u)}} = \gamma^{-1} P_{(u)} \frac{D}{\mathrm{d}\tau_U} = \nabla_{(\mathrm{fw},u)} + \nu^{\alpha}_{(U,u)} \nabla(u)_{\alpha} \tag{15.6}
$$

the latter being the measure of the (rescaled) absolute derivative along U, the $(3+1)$ version of the equation of motion of the particle and of the energy theorem, acquires the Newtonian form

$$\frac{D_{(\mathrm{fw},U,u)}}{\mathrm{d}\tau_{(U,u)}} p_{(U,u)} = F_{(U,u)} + F^{(\mathrm{G})}_{(\mathrm{fw},U,u)},$$
$$\frac{\mathrm{d}E_{(U,u)}}{\mathrm{d}\tau_{(U,u)}} = [F_{(U,u)} + F^{(\mathrm{G})}_{(\mathrm{fw},U,u)}] \cdot_u \nu_{(U,u)}. \tag{15.7}$$

Let us now consider the motion of a test gyroscope. As it is well known, the spin vector $S_{(U)}$ of a gyroscope carried by an observer U, is Fermi–Walker transported along his worldline (i.e. $S_{(U)}$ does not precess with respect to spatial axes which are Fermi–Walker dragged along U), namely:

$$\frac{D_{(\mathrm{fw},U)}}{\mathrm{d}\tau_U} S_{(U)} = P_{(U)} \frac{D}{\mathrm{d}\tau_U} S_{(U)} = 0. \tag{15.8}$$

Suppose that we have chosen a spatial triad $\bar{e}(U)_{\hat{a}}$ which is adapted to the observer U and is *not* a Fermi–Walker frame. The observer U will then *see* the spin $S(U)$ of the gyroscope to precess with respect to these axes according to the law:

$$\left[\frac{\mathrm{d}S^{\hat{a}}_{(U)}}{\mathrm{d}\tau_U} - \epsilon^{\hat{a}}{}_{\hat{b}\hat{c}} \zeta^{\hat{b}}_{(\mathrm{fw},U,\bar{e}(U)_{\hat{a}})} S^{\hat{c}}_{(U)}\right] \bar{e}(U)_{\hat{a}} = 0 \tag{15.9}$$

where

$$\zeta^{\hat{a}}_{(\mathrm{fw},U,\bar{e}(U)_{\hat{a}})} \equiv \epsilon^{\hat{a}\hat{b}\hat{c}} \bar{e}(U)_{\hat{b}} \cdot \nabla_{(\mathrm{fw},U)} \bar{e}(U)_{\hat{c}} \tag{15.10}$$

is the precession rate vector.

However, we may want the gyroscope to be analysed by a different observer, u say, who is not comoving with the gyro's centre of mass. In this case we need a smooth family of these observers, each one intersecting the gyro's worldline at any of its spacetime points where he *measures* the instantaneous precession of the spin vector relative to a suitably defined frame, adapted to u. Of course, we require that the observer's u are synchronized so that their measurements can be compared. The results of these measurements are described by a smooth and at least once differentiable function of the proper-time of u.

Let $\{e(u)_{\hat{a}}\}$ be a field of spatial triads adapted to u; then the restriction of $\{e(u)_{\hat{a}}\}$ to the worldline ℓ_U of the gyroscope, allows one to define on ℓ_U a field of tetrad frames, adapted to U, given by $\{(U, \bar{e}(U)_{\hat{a}}\}$, where:

$$\bar{e}(U)_{\hat{a}} = B_{(\mathrm{lrs},u,U)} e(u)_{\hat{a}}, \tag{15.11}$$

$B_{(\mathrm{lrs},u,U)} = P_{(U)} B_{(u,U)} P_{(u)} : \mathrm{LRS}_u \to \mathrm{LRS}_U$ being the boost map between the rest spaces of the observers U and u; this map has been studied extensively in [2–4]. Since the boost is an isometry, the precession of $S_{(U)}$ with respect to the axes $\bar{e}(U)_{\hat{a}}$ is the same as the precession, with respect to the axes $e(u)_{\hat{a}}$, of the boosted spinvector $s_{(u)}$, which reads:

$$s_{(u)} = B_{(\mathrm{lrs},U,u)} S_{(U)} = \left[P_{(u)} - \frac{\gamma}{\gamma+1} \nu_{(U,u)} \otimes \nu_{(U,u)} \right] P_{(U,u)} S_{(U)}. \qquad (15.12)$$

Hence, from (15.9) and (15.11) and the acceleration-equals-force equation for U; we find:

$$\left[\frac{\mathrm{d}s^{\hat{a}}_{(u)}}{\mathrm{d}\tau_U} - \gamma \epsilon^{\hat{a}}{}_{\hat{b}\hat{c}} [\zeta_{(\mathrm{fw},U,u)} + \zeta_{(\mathrm{sc,fw},U,u)}]^{\hat{b}} s^{\hat{c}}_{(u)} \right] e(u)_{\hat{a}} = 0 \qquad (15.13)$$

where[1]

$$\zeta_{(\mathrm{fw},U,u)} = \frac{1}{\gamma+1} \nu_{(U,u)} \times_u [F^{(G)}_{(\mathrm{fw},U,u)} - \gamma F_{(U,u)}]$$
$$\zeta_{(\mathrm{sc,fw},U,u)} = \frac{1}{2} \delta^{\hat{a}\hat{b}} \frac{D_{(\mathrm{fw},U,u)}}{\mathrm{d}\tau_{(U,u)}} e(u)_{\hat{a}} \times_u e(u)_{\hat{b}} \qquad (15.14)$$

so that

$$\zeta_{(\mathrm{fw},U,\bar{e}(U)_{\hat{a}})} = \gamma B_{(\mathrm{lrs},u,U)} [\zeta_{(\mathrm{fw},U,u)} + \zeta_{(\mathrm{sc,fw},U,u)}]. \qquad (15.15)$$

Finally, by rescaling equation (15.13) to the proper-time of u, one has

$$\frac{\mathrm{d}s_{(u)}{}^{\hat{a}}}{\mathrm{d}\tau_{(U,u)}} - \epsilon^{\hat{a}}{}_{\hat{b}\hat{c}} [\zeta_{(\mathrm{fw},U,u)} + \zeta_{(\mathrm{sc,fw},U,u)}]^{\hat{b}} s_{(u)}{}^{\hat{c}} = 0 \qquad (15.16)$$

where

$$\zeta_{(\mathrm{fw},U,u)} + \zeta_{(\mathrm{sc,fw},U,u)} \equiv \tilde{\zeta}_{(\mathrm{fw},u,e(u)_{\hat{a}})} = \gamma^{-1} B_{(\mathrm{lrs},U,u)} \zeta_{(\mathrm{fw},U,\bar{e}(U)_{\hat{a}})} \qquad (15.17)$$

is the angular velocity precession of the gyroscope as *measured* by the observer u with respect to the axes $e(u)_{\hat{a}}$.

It is worth mentioning here that while the observer U, who is comoving with the gyro's centre of mass, measures the precession (15.10) along his own worldline, the observer's u can only compare the instantaneous measurements of $\tilde{\zeta}_{(\mathrm{fw},u,e(u)_{\hat{a}})}$ in (15.17), made by each of them along the gyro's worldline. Evidently either type of measurements requires the tetrad frames to be operationally well defined. This will be discussed in the following section.

15.3 The spacetime metric

The metric of a plane monochromatic gravitational wave, elliptically polarized and propagating along a direction which we fix as the coordinate x direction, can be written in the 'TT' gauge as [5]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + (1 - h_{22})\,\mathrm{d}y^2 + (1 + h_{22})\,\mathrm{d}z^2 - 2h_{23}\,\mathrm{d}y\,\mathrm{d}z \qquad (15.18)$$

[1] This notation for the Fermi–Walker relative angular velocity $\zeta_{(\mathrm{fw},U,u)}$ and the Fermi–Walker space-curvature relative angular velocity $\zeta_{(\mathrm{sc,fw},U,u)}$ has been introduced in [2].

with $h_{AB} = h_{AB}(t - x)$, $(A, B = 2, 3)$. Let u denote the tangent vector field of a family of geodesic, non-rotating and expanding observers defined by

$$u^\flat = -\mathrm{d}t, \quad u = \partial_t. \tag{15.19}$$

One can adapt to these observers an infinite number of spatial frames, by rotating any given one arbitrarily. For example, consider the following u-frame $\{e_{\hat\alpha}\} = \{e_{\hat 0} = u, e_{\hat a} = e(u)_{\hat a}\}$ with its dual $\{\omega^{\hat\alpha}\} = \{\omega^{\hat 0} = -u^\flat, \omega^{\hat a} = \omega(u)^{\hat a}\}$

$$\begin{aligned}
u &= \partial_t \\
e(u)_{\hat 1} &= \partial_x \\
e(u)_{\hat 2} &= (1 - h_{22})^{-1/2}\partial_y \simeq (1 + \tfrac{1}{2}h_{22})\partial_y \\
e(u)_{\hat 3} &= (1 - h_{22}^2 - h_{23}^2)^{-1/2}[(1 - h_{22})^{-1/2}h_{23}\partial_y + (1 - h_{22})^{1/2}\partial_z] \\
&\simeq h_{23}\partial_y + (1 - \tfrac{1}{2}h_{22})\partial_z \\
-u^\flat &= \mathrm{d}t \\
\omega(u)^{\hat 1} &= \mathrm{d}x \\
\omega(u)^{\hat 2} &= (1 - h_{22})^{1/2}\left[\mathrm{d}y - \frac{h_{23}}{1 - h_{22}}\,\mathrm{d}z\right] \simeq \left(1 - \frac{1}{2}h_{22}\right)\mathrm{d}y - h_{23}\,\mathrm{d}z \\
\omega(u)^{\hat 3} &= \left(\frac{1 - h_{22}^2 - h_{23}^2}{1 - h_{22}}\right)^{1/2}\mathrm{d}z \simeq \left(1 + \frac{1}{2}h_{22}\right)\mathrm{d}z,
\end{aligned} \tag{15.20}$$

where $\simeq$ denotes the corresponding weak-field limit. Any other spatial frame $\{\tilde e(u)_{\hat a}\}$, adapted to the observers (15.19), can be obtained from this one by a spatial rotation R

$$\tilde e(u)_{\hat a} = e(u)_{\hat b} R^{\hat b}{}_{\hat a}. \tag{15.21}$$

Among all the possible frames, there exists only one with respect to which the local compass of inertia experiences no precession. This frame is Fermi–Walker transported along u, namely it satisfies the condition

$$P_{(u)}\frac{D}{\mathrm{d}\tau_u}\tilde e(u)_{\hat a} \equiv \nabla_{(\mathrm{fw},u)}\tilde e(u)_{\hat a} = 0. \tag{15.22}$$

A Fermi–Walker frame is the most natural of the u-frames; its spatial directions, in fact, are fixed by three mutually orthogonal axes of small size comoving gyroscopes. However, if a metric perturbation causes a dragging of the local compass of inertia, the only way to detect and measure it, is to select a frame which is *not* Fermi–Walker transported. In this frame, in fact, a gyroscope would be seen to precess and indeed its precession contains all the informations about gravitational dragging. Nonetheless, it is quite non-trivial to identify, in an operational way, a frame which is *not* Fermi–Walker transported along the observer's worldline when it is acted upon by a gravitational wave.

Frame (15.20) is clearly Fermi–Walker transported in the absence of gravitational waves (h_{22} and h_{23} being time independent), but it is not so when they are present. The Fermi rotation of the frame, in this case, is described by the (antisymmetric) angular velocity spatial tensor [6]:

$$C_{(u)\hat{b}\hat{a}} = e(u)_{\hat{b}} \cdot_u \nabla_{(\mathrm{fw},u)} e(u)_{\hat{a}}, \qquad (15.23)$$

hence, a gyroscope carried by the observer u will precess with respect to frame (15.20) with an angular velocity tensor which has only one independent nonzero component, namely:

$$C_{(u)\hat{3}\hat{2}} = -\frac{[h_{23,t}(1-h_{22}) + h_{22,t}h_{23}]}{2(1-h_{22})\sqrt{1-h_{22}^2-h_{23}^2}} \simeq -\frac{1}{2}h_{23,t}. \qquad (15.24)$$

However, frame $e(u)_{\hat{a}}$ cannot be operationally defined, so result (15.24) is of little physical significance although it shows the existence of frames which respond to one state of polarization only, at least to first order in h_{AB}. We are, therefore, motivated to search for 'frames' that can be fixed from a viable experimental set up.

15.4 Searching for an operational frame

Let us consider the timelike geodesics of the metric (15.18). These are well known [7]; the four-velocity of a general such geodesic can be written as

$$\begin{aligned} U_g = &\frac{1}{2E}[(1+f+E^2)\partial_t + (1+f-E^2)\partial_x] \\ &+ \frac{1}{1-h_{22}^2-h_{23}^2}\{[\alpha(1+h_{22})+\beta h_{23}]\partial_y + [\beta(1-h_{22})+\alpha h_{23}]\partial_z\}, \end{aligned} \qquad (15.25)$$

where α, β and E are Killing constants and $f = g_{AB}U^AU^B$ is equal to

$$\begin{aligned} f &= \frac{1}{1-h_{22}^2-h_{23}^2}[\alpha^2(1+h_{22}) + \beta^2(1-h_{22}) + 2\alpha\beta h_{23}] \\ &\simeq \alpha^2(1+h_{22}) + \beta^2(1-h_{22}) + 2\alpha\beta h_{23}. \end{aligned} \qquad (15.26)$$

If $u = \partial_t$ is the family of observers who make the mesurements and $\{e(u)_{\hat{a}}\}$ is an adapted spatial frame, then the relative velocity $\nu_{(U_g,u)}{}^{\hat{a}}$ of U_g with respect to u is defined by the relation

$$U_g = \gamma_{(U_g,u)}[u + \nu_{(U_g,u)}{}^{\hat{a}} e(u)_{\hat{a}}]. \qquad (15.27)$$

where

$$\gamma_{(U_g,u)} = \frac{1+f+E^2}{2E}$$
$$\simeq \frac{1}{2E}[1+E^2+\alpha^2(1+h_{22})+\beta^2(1-h_{22})+2\alpha\beta h_{23}] \quad (15.28)$$

and the relative velocity components can be obtained by comparison of (15.28) and (15.25).

Hereafter, we restrict ourselves to the weak field approximation (first order in h_{AB}) and assume, without loss of generality that, in the absence of a gravitational wave, the spatial velocity $\nu_{(U_g,u)}$, was only in the y coordinate direction. This corresponds to the requirement that

$$\beta = 0, \quad E = \sqrt{1+\alpha^2}. \qquad (15.29)$$

In this case we find

$$\nu_{(U,u)}{}^{\hat{1}} = \frac{\alpha^2}{2(1+\alpha^2)}h_{22},$$
$$\nu_{(U,u)}{}^{\hat{2}} = \frac{\alpha}{\sqrt{1+\alpha^2}}\left[1+\frac{h_{22}}{2(1+\alpha^2)}\right], \qquad (15.30)$$
$$\nu_{(U,u)}{}^{\hat{3}} = \frac{\alpha}{\sqrt{1+\alpha^2}}h_{23}.$$

We now require that the four-velocity of a test gyroscope is $U = U_g$ as given by (15.25) with conditions (15.29). The assumption of geodesicity is justified if we consider the limit of zero size gyroscopes. In this case, in fact, not only can we neglect the multipole moments of the gyro's stress tensor higher than the dipole, but also ignore the tidal term which enters the Papapetrou–Dixon equations and arises from the coupling of the gyro's spin with the background curvature. This term is of the order of the ratio of the average size of the gyroscope and the gravitational-wave wavelength.

Let us consider the restriction of the vector field $u = \partial_t$ of stationary observers, to the gyroscope's worldline and require that these observers *monitor* the behaviour of the spin of the moving gyro, measuring the (instantaneous) precession vector $\tilde{\zeta}$ given by equation (15.17). As already mentioned, the spatial frame $e(u)_{\hat{a}}$ given by (15.20) is not operationally well defined, so we have to find one which is so.

To find a spatial frame which is suitable for actual experiments, note that the observer's u can unambiguously determine in their rest-frame, a spatial direction given by that of the relative velocity ν of the gyroscope. Suppose they fix, by guessing if necessary, a direction of propagation of the gravitational wave and term this as the x-axis with unit vector $e(u)_{\hat{1}}$. Then from these two directions, namely that of the relative velocity of the gyro and of the wave propagation, it is possible to construct a spatial triad as follows

$$\begin{aligned} \lambda(u)_{\hat 1} &= e(u)_{\hat 1} = \partial_x \\ \lambda(u)_{\hat 2} &= \hat\nu_{(U,u)} \times_u e(u)_{\hat 1} \\ \lambda(u)_{\hat 3} &= \lambda(u)_{\hat 1} \times_u \lambda(u)_{\hat 2}. \end{aligned} \tag{15.31}$$

This frame can be operationally constructed apart from guessing the direction of propagation of the wave. Indeed such a guess is also required to fit data from bar antenna detectors, for example. Obviously this frame is not unique: any other spatial triad obtained from it after a rotation which depends at most on the (known) modulus of the relative velocity (or is constant) is equally useful.

The spatial triads $e(u)_{\hat a}$ in (15.20) and $\lambda(u)_{\hat a}$ in (15.31), differ by a rotation

$$\lambda(u)_{\hat a} = \mathcal{R}^{\hat b}{}_{\hat a} e(u)_{\hat b}.$$

In the weak field limit, the only non-trivial components of $\mathcal{R}^{\hat b}{}_{\hat a}$, are

$$\mathcal{R}^{\hat 1}{}_{\hat 1} = -\mathcal{R}^{\hat 2}{}_{\hat 3} = \mathcal{R}^{\hat 3}{}_{\hat 2} = 1, \quad \mathcal{R}^{\hat 2}{}_{\hat 2} = \mathcal{R}^{\hat 3}{}_{\hat 3} = h_{23}$$

so that

$$\begin{aligned} \lambda(u)_{\hat 1} &= \partial_x \\ \lambda(u)_{\hat 2} &\simeq h_{23} e(u)_{\hat 2} - e(u)_{\hat 3} \simeq -(1 - \tfrac{1}{2}h_{22})\partial_z \\ \lambda(u)_{\hat 3} &\simeq e(u)_{\hat 2} + h_{23} e(u)_{\hat 3} \simeq (1 + \tfrac{1}{2}h_{22})\partial_y + h_{23}\partial_z. \end{aligned} \tag{15.32}$$

15.5 Precession of a gyroscope in geodesic motion

The precession of the gyro which is measured by the observer's u all along its worldline, is the image of the precession measured by the comoving observer U, under the boost $B(U, u)$ as shown in (15.17). In order to study the spin precession seen by the observer comoving with the gyro, we must first decide with respect to what axes (non-Fermi–Walker transported but operationally well defined) the precession will be measured, as explained in section 15.2.

Since the observer's u intersect the worldline of the observer U carrying the gyro, the u-frames $\{\lambda(u)_{\hat a}\}$ in (15.31) form a smooth field of frames on it, so the observer U can identify spatial directions in his rest space simply by boosting the directions $\lambda(u)_{\hat a}$.

At each event along his worldline in fact he will see the axes $\lambda(u)_{\hat a}$ defined in (15.31) to be in relative motion, therefore the boost of these axes, namely

$$\bar\lambda(U)_{\hat a} = B_{(\mathrm{lrs},u,U)}\lambda(u)_{\hat a} = \lambda(u)_{\hat a} + \frac{\gamma}{\gamma+1}[\nu_{(U,u)} \cdot \lambda(u)_{\hat a}](u + U), \tag{15.33}$$

from (15.11), will be *seen* by U as the corresponding axes with the same orientation which are 'momentarily at rest'. The orientation of the spin vector S with respect to the axes $\bar{\lambda}(U)_{\hat{a}}$ is also the orientation of S with respect to the moving axes $\lambda(u)_{\hat{a}}$.

The velocity of spin precession then corresponds to the spatial dual of the Fermi–Walker structure functions of $\bar{\lambda}(U)_{\hat{a}}$, namely $C_{(\mathrm{fw})}(U, \bar{\lambda}(U)_{\hat{a}})_{\hat{b}\hat{a}}$, according to relation (15.23).

Confining our attention to the weak-field approximation, the components of the precession velocity with respect to the triad $\bar{\lambda}(U)_{\hat{a}}$ are

$$\begin{aligned}
\zeta_{(\mathrm{fw},\mathrm{U},\bar{\lambda}(\mathrm{U})_{\hat{a}})}{}^{\hat{1}} &\simeq -\tfrac{1}{2}h_{23,t}\\
\zeta_{(\mathrm{fw},\mathrm{U},\bar{\lambda}(\mathrm{U})_{\hat{a}})}{}^{\hat{2}} &\simeq \alpha/2h_{22,t}\\
\zeta_{(\mathrm{fw},\mathrm{U},\bar{\lambda}(\mathrm{U})_{\hat{a}})}{}^{\hat{3}} &\simeq \alpha/2h_{23,t}.
\end{aligned} \tag{15.34}$$

We observe that in the limit of small linear momentum, $\alpha \ll 1$, the dominant precession is in the direction of wave propagation $e(u)_{\hat{1}}$ (to zeroth order, $\bar{\lambda}(U)_{\hat{1}} \simeq e(u)_{\hat{1}}$) and is induced by the cross-polarization only. (Note that the precession in the direction of propagation of the wave does not depend on α.) In this case we can conclude that the gyro can act as a polarization filter for gravitational waves.

In the opposite limit of large linear momentum, $\alpha \gg 1$, the precession vector lies mainly in the plane orthogonal to the propagation direction and is contributed likewise by both polarizations. Indeed the measurement of the precession induced by a plane gravitational wave, of a gyroscope set in relativistic motion, would enable one to identify the local direction of propagation of the wave. A similar situation will be encountered in the rest frame of u, where from (15.17), (15.28) and (15.29) we have:

$$\begin{aligned}
\tilde{\zeta}_{(\mathrm{fw},u,\lambda(u)_{\hat{a}})}^{\hat{1}} &= \gamma_{(U,u)}^{-1} B_{(\mathrm{lrs},U,u)} \zeta_{(\mathrm{fw},U,\bar{\lambda}(U)_{\hat{a}})}{}^{\hat{1}} \simeq -\frac{1}{2}\frac{1}{\sqrt{1+\alpha^2}} h_{23,t}\\
\tilde{\zeta}_{(\mathrm{fw},u,\lambda(u)_{\hat{a}})}^{\hat{2}} &= \gamma_{(U,u)}^{-1} B_{(\mathrm{lrs},U,u)} \zeta_{(\mathrm{fw},U,\bar{\lambda}(U)_{\hat{a}})}{}^{\hat{2}} \simeq \frac{1}{2}\frac{\alpha}{\sqrt{1+\alpha^2}} h_{22,t}\\
\tilde{\zeta}_{(\mathrm{fw},u,\lambda(u)_{\hat{a}})}^{\hat{3}} &= \gamma_{(U,u)}^{-1} B_{(\mathrm{lrs},U,u)} \zeta_{(\mathrm{fw},U,\bar{\lambda}(U)_{\hat{a}})}{}^{\hat{3}} \simeq \frac{1}{2}\frac{\alpha}{\sqrt{1+\alpha^2}} h_{23,t}.
\end{aligned} \tag{15.35}$$

Finally, let us note that results (15.35) are only slightly modified after rotating the frame (15.31) by a constant angle ϕ around the propagation direction of the wave. In fact, in this case, the new spatial u-frame becomes

$$\begin{aligned}
f(u)_{\hat{1}} &= \lambda(u)_{\hat{1}}\\
f(u)_{\hat{2}} &= \cos\phi\lambda(u)_{\hat{2}} + \sin\phi\lambda(u)_{\hat{3}}\\
f(u)_{\hat{3}} &= -\sin\phi\lambda(u)_{\hat{2}} + \cos\phi\lambda(u)_{\hat{3}};
\end{aligned} \tag{15.36}$$

when $\phi = 0$ it reduces to (15.31). Once the boosted frame $\bar{f}(U)_{\hat{a}} = B_{(\mathrm{lrs},U,u)} f(u)_{\hat{a}}$ in LRS_U is obtained, the components of the precession velocity turn out to be

$$\begin{aligned}
\zeta_{(\mathrm{fw},U,\bar{f}(U)_{\hat{a}})}{}^{\hat{1}} &\simeq -\tfrac{1}{2}h_{23,t} \\
\zeta_{(\mathrm{fw},U,\bar{f}(U)_{\hat{a}})}{}^{\hat{2}} &\simeq \alpha/2[\cos\phi h_{22,t} - \sin\phi h_{23,t}] \\
\zeta_{(\mathrm{fw},U,\bar{f}(U)_{\hat{a}})}{}^{\hat{3}} &\simeq \alpha/2[-\sin\phi h_{22,t} + \cos\phi h_{23,t}],
\end{aligned} \tag{15.37}$$

again showing that, in the limit of small linear momentum α, the precession mainly occurs about the direction of propagation of the wave.

15.6 Conclusions

We have operationally defined a tetrad frame adapted to a family of static observers in the background of a plane gravitational wave. Then we have used this family to study the precession angular velocity of a gyroscope moving along a spacetime geodesic. The results show that, to first order in h and in the case of non-relativistic motion ($\alpha \ll 1$), the observed gyroscopic precession is mainly induced by the cross-polarization only so a gyro appears to behave as a polarization filter.

Assuming the form $h_{23} = h_\times \sin(\frac{2\pi c}{\lambda_{\mathrm{GW}}}(t - x/c) + \psi)$ for the cross-polarization, with an obvious meaning for the symbols, the precession frequency (in conventional units) would be

$$\Omega_{(\mathrm{gyro})}(t) \simeq -\frac{\pi c h_\times}{\lambda_{\mathrm{GW}}} \cos\left(\frac{2\pi c}{\lambda_{\mathrm{GW}}}(t - x/c) + \psi\right). \tag{15.38}$$

The values of the precession frequency are very small, as expected. With a typical amplitude of 10^{-21} at the Earth and a frequency of 10^3 Hz, we could hope for a maximum precession of the order of 10^{-18} s^{-1}.

Clearly the precession effect is larger with high-frequency gravitational waves or when the gyroscope is close enough to the wave source to allow for a higher value of $h_\times$. This situation is encountered by a spinning neutron star, say, in a compact binary system.

The type of analysis we have considered, is most suitable to describe the interaction of a moving gyroscope with a *continuous* flow of plane gravitational waves with metric form as in (15.18). These are expected to be emitted by sources like compact binaries. If we have an impulsive source, like a supernova, then one expects a burst of gravitational radiation which is better described by a gravitational sandwich. This latter case is now under investigation.

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