

Bogomol'nyi Limit For Magnetic Vortices In Rotating Superconductor

Brandon Carter, David Langlois, Reinhard Prix
*Département d'Astrophysique Relativiste et de Cosmologie,
Centre National de la Recherche Scientifique,
Observatoire de Paris, 92195 Meudon, France.
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Abstract. This work is the sequel of a previous investigation of stationary and cylindrically symmetric vortex configurations for simple models representing an incompressible non-relativistic superconductor in a rigidly rotating background. In the present paper, we carry out our analysis with a generalized Ginzburg-Landau description of the superconductor, which provides a prescription for the radial profile of the normal density within the vortex. Within this framework, it is shown that the Bogomol'nyi limit condition marking the boundary between type I and type II behavior is unaffected by the rotation of the background.

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I. INTRODUCTION

The present work is the sequel of a previous investigation¹ on the energy of vortices in a rotating superconductor, where it was shown in particular that the magnetic and kinetic contributions to the energy density that are proportional to the background angular velocity remarkably cancel. The motivation which brought us to study vortices in rotating superconductors is the study of the interior of neutron stars, more specifically their inner core which is believed to contain a proton superconductor. Although we worked out the macroscopic description of an array of magnetic vortices and superfluid vortices in a general relativistic framework² necessary for refined analysis of neutron stars, thus generalizing the earlier work of Lindblom and Mendell³ in a Newtonian approach, we have here restricted our analysis to a Newtonian framework for simplicity.

In our previous work,¹ we left undetermined the explicit structure of the vortex core. One of the purposes of the present work is to provide a specification for the profile of the condensate particle density, based on a Ginzburg-Landau type approach. Our treatment will however not be restricted just to the widely used standard Ginzburg-Landau description, but will also be valid for a generalized version (more readily justifiable by heuristic considerations^{4,5}), leaving arbitrary the coupling constant g_c that enters the gradient energy density.

The second purpose of this work is to reexamine the question of the Bogomol'nyi limit⁶ in the context of a superconductor in a rotating background. The conclusion will be that the usual boundary between type I and type II superconductors remains unaffected by the rotation of the background.

Before entering the details of this work, let us recall the essential features of the model and define the relevant quantities. The superconducting matter will consist of a charged superfluid component, represented by a locally variable number density n_s of bosonic particles characterized by an effective mass m , a charge q , and of an ordinary component of opposite charge which locally compensates the charge of the first component. The essential property that distinguishes the superfluid constituent from ordinary matter is that its momentum is directly related to the phase variable φ (a scalar with period 2π) of the boson condensate, according to the expression

$$m\vec{v} + q\vec{A} = \hbar\vec{\nabla}\varphi. \quad (1)$$

In this formula, \hbar is the Dirac-Planck constant, \vec{A} is the magnetic vector potential and \vec{v} the velocity of the Bose condensate. The whole system will be described, in addition to the relation (1), by the Maxwell equations,

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{j}, \quad (2)$$

where \vec{B} is the magnetic field, related to the magnetic potential vector by the usual relation $\vec{B} = \vec{\nabla} \times \vec{A}$, and \vec{j} is the electric current, which consists of the sum of the currents due to the condensate component and to the ordinary component. The ordinary component current will be supposed to be that of a rigidly rotating fluid. To be able to solve the coupled system of equations, one needs a prescription concerning the spatial evolution of the condensate particle number density. It will be given by an energy minimization principle, using the energy functional corresponding to a generalized Ginzburg-Landau approach.

The plan of the paper will be the following. In section II, we shall use the cylindrical symmetry and introduce new variables to simplify the system of coupled equations. Section III will be devoted to the energy minimization principle, which will give a prescription for the determination of n_s . And finally, section IV will deal with the Bogomol'nyi limit condition.

II. SYSTEM OF COUPLED EQUATIONS

The scenarios we shall consider will be of the usual kind, in which each individual vortex is treated as a stationary, cylindrically symmetric configuration consisting of a rigidly rotating background medium with uniform angular velocity Ω_∞ , say, together with a charged superfluid constituent in a state of differential rotation with a velocity v , which tends at large distance towards the rigid rotation value given by $\Omega_\infty r$, where r is the cylindrical radial distance from the axis. It will be supposed that the superfluid particle number density n_s is a monotonically increasing function of the cylindrical radius variable r , tending asymptotically to a constant value, n_∞ , say, at large distances from the axis. It will be supposed that the local charge density is canceled by the background so that there is no electric field, but that there is a magnetic induction field with magnitude B and direction parallel to the axis, whose source is the axially oriented electromagnetic current whose magnitude j will be given by

$$j = qn_s(v - \Omega_\infty r). \quad (3)$$

The relevant Maxwellian source equation for the magnetic field (2) will have the familiar form

$$\frac{dB}{dr} = -4\pi j. \quad (4)$$

The other relevant Maxwellian equation is the one governing the axial component A (which in an appropriate gauge will be the only component) of the electromagnetic potential covector, which will be related to the magnetic induction by

$$\frac{d(rA)}{dr} = rB. \quad (5)$$

The essential property distinguishing the superconducting case from its “normal” analogue is the London flux quantization condition (1), which in the present context (where all physically relevant quantities depend only on the cylindrical radius r) will be expressible in the well known form⁷

$$mv + qA = \frac{N\hbar}{r}, \quad (6)$$

where N is the phase winding number, which must be an integer.

Before proceeding, it will be useful to take advantage of the possibility of transforming the preceding system of equations to a form that is not just linear but also homogeneous, by replacing the variables v , B , A by corresponding variables \mathcal{V} , \mathcal{B} , \mathcal{A} that are defined by

$$\mathcal{V} = v - \Omega_\infty r, \quad (7)$$

$$\mathcal{B} = B - B_\infty, \quad (8)$$

$$\mathcal{A} = A - \frac{1}{2}rB_\infty, \quad (9)$$

where B_∞ is the uniform background magnetic field value that would be generated by a rigidly rotating superconductor, which is given by the London formula

$$B_\infty = -\frac{2m}{q}\Omega_\infty, \quad (10)$$

obtained by combining (6) and (5) in the special case of rigid corotation, i.e., with $v = \Omega_\infty r$.

In terms of these new variables the equation (5) will be transformed into the form

$$\frac{d(r\mathcal{A})}{dr} = r\mathcal{B}, \quad (11)$$

while the other differential equation (4) will be transformed into the form

$$\frac{d\mathcal{B}}{dr} = -4\pi j, \quad (12)$$

in which, rewriting equation (3), we shall have

$$j = qn_s\mathcal{V}. \quad (13)$$

Finally, the flux quantization condition (6) will be converted into the form

$$m\mathcal{V} + q\mathcal{A} = \frac{N\hbar}{r}, \quad (14)$$

which can be used to transform (11) into

$$\frac{m}{qr} \frac{d(\mathcal{V}r)}{dr} = -\mathcal{B}. \quad (15)$$

The advantage of this reformulation is that unlike v , B , and A , the new variables \mathcal{V} , \mathcal{B} and \mathcal{A} are subject just to homogeneous boundary conditions, which are simply that they all tend to zero as $r \rightarrow \infty$.

III. ENERGY MINIMIZATION PRINCIPLE

The equations of the previous section are not sufficient by themselves to fully determine the system. In order to specify the radial distribution of the condensate particle number density n_s we will use an energy minimization principle based on a model in which the condensate energy density is postulated to be given as the sum of a gradient contribution and a potential energy contribution by an expression of the form

$$\mathcal{E}_{\text{con}} = \mathcal{E}_{\text{grad}} + V, \quad (16)$$

where the contribution $\mathcal{E}_{\text{grad}}$ is proportional to the square of the gradient of n_s with a coefficient that, like the potential energy contribution V , is given as an algebraic function of n_s by some appropriate ansatz. The use of such a model as a fairly plausible approximation is justifiable by heuristic considerations⁴ that motivate the use of an ansatz of what we shall refer to as the Ginzburg type, according to which the energy contribution is postulated to have the form

$$\mathcal{E}_{\text{grad}} = \frac{g_c^2 \hbar^2}{8m n_s} \left(\frac{dn_s}{dr} \right)^2, \quad (17)$$

where g_c is a dimensionless coupling constant, while the potential energy density V is given in terms of some constant proportionality factor \mathcal{E}_c , say, by the formula

$$V = \mathcal{E}_c \left(1 - \frac{n_s}{n_\infty} \right)^2, \quad (18)$$

which provides a particularly convenient ansatz for interpolation in the theoretically intractable intermediate region between the comparatively well understood end points of the allowed range $0 \leq n_s \leq n_\infty$. The constant \mathcal{E}_c is interpretable as representing the maximum condensation energy density. Its value is commonly expressed in terms of the corresponding critical value H_c , say, representing the strength of the maximum magnetic field that can be expelled from the superconductor by the Meissner effect, to which it is evidently related by the formula

$$\mathcal{E}_c = \frac{H_c^2}{8\pi}. \quad (19)$$

The total energy density associated with a vortex will be of the general form

$$\mathcal{E} = \mathcal{E}_{\text{mag}} + \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{con}}, \quad (20)$$

where \mathcal{E}_{mag} , \mathcal{E}_{kin} and \mathcal{E}_{con} are respectively the magnetic, kinetic and condensate energy contributions. More precisely, \mathcal{E}_{mag} is the extra magnetic energy density arising from a non-zero value of the phase winding number N , i.e., the local deviation from the magnetic energy density due just to the uniform field B_∞ (associated with of the state of rigid corotation characterized by the given background angular velocity Ω_∞), namely,

$$\mathcal{E}_{\text{mag}} = \frac{B^2}{8\pi} - \frac{B_\infty^2}{8\pi}, \quad (21)$$

while \mathcal{E}_{kin} is the corresponding deviation of the kinetic energy from that of the state of rigid corotation characterized by the given background angular velocity Ω_∞ , namely,

$$\mathcal{E}_{\text{kin}} = \frac{m}{2} n_s (v^2 - \Omega_\infty^2 r^2). \quad (22)$$

It is convenient for many purposes to express such a model in terms of a dimensionless amplitude ψ that varies in the range $0 \leq \psi \leq 1$ according to the conventional specification

$$n_s = \psi^2 n_\infty. \quad (23)$$

Within the general category of Ginzburg type models as thus described, the special case of the standard kind of Ginzburg-Landau model is characterized more specifically by the postulate that the gradient coupling constant g_c should be exactly equal to unity. This ansatz has the attractive feature of allowing the theory to be neatly reformulated in terms of a complex variable $\Psi \equiv \psi e^{i\varphi}$ where φ is the phase that appears in (1) in a manner that is evocative of the Schroedinger model for a single particle. Indeed, it is easy to verify that in the case of $g_c = 1$, the gradient term (17) and the kinetic term in (22) can be rewritten, using (1) and (23), to give the usual Ginzburg-Landau type gradient term, i.e.,

$$\frac{\hbar^2}{8mn_s} (\vec{\nabla} n_s)^2 + \frac{m}{2} n_s \vec{v}^2 = \frac{\hbar^2 n_\infty}{2m} |\vec{\mathcal{D}}\Psi|^2, \quad (24)$$

where the covariant derivative is defined as

$$\vec{\mathcal{D}} \equiv \vec{\nabla} - \frac{iq}{\hbar} \vec{A}. \quad (25)$$

However, although there are physical reasons⁴ for expecting that g_c should be comparable with unity, the seductive supposition that it should exactly satisfy the Landau condition $g_c = 1$ is more dubious.⁵ This more specialized ansatz will not be needed for the work that follows, which applies to the generalized Ginzburg category with no restriction on the parameter g_c .

Using the expression

$$\mathcal{E}_{\text{mag}} + \mathcal{E}_{\text{kin}} = \frac{\mathcal{B}^2}{8\pi} + \frac{m}{2q} j\mathcal{V} + \frac{B_\infty}{8\pi r} \frac{d}{dr} (r^2 \mathcal{B}) \quad (26)$$

for the first two terms in the combination (20), it can be seen that the total energy density arising from the presence of the vortex will be given by

$$\mathcal{E} = \frac{\mathcal{B}^2}{8\pi} + \frac{H_c^2}{8\pi} (1 - \psi^2)^2 + \frac{n_\infty}{2m} \left[\left(g_c \hbar \frac{d\psi}{dr} \right)^2 + (m\mathcal{V}\psi)^2 \right] + \frac{B_\infty}{8\pi r} \frac{d(r^2 \mathcal{B})}{dr}. \quad (27)$$

The equation governing the distribution of the condensate particle number density n_s is obtained by requiring that the integral of the energy density (27) be stationary with respect to variation of n_s or equivalently of ψ , which gives the field equation for the latter in the form

$$\frac{g_c^2 \hbar^2}{mr} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) = m\mathcal{V}^2 \psi - \frac{4}{n_\infty} \mathcal{E}_c \psi (1 - \psi^2). \quad (28)$$

When this equation is satisfied, it can be seen that the energy density (27) will reduce to a value given by

$$\mathcal{E} = \frac{\mathcal{B}^2}{8\pi} + \frac{H_c^2}{8\pi} (1 - \psi^4) + \frac{1}{r} \frac{d}{dr} \left(B_\infty \frac{r^2 \mathcal{B}}{8\pi} + r \frac{g_c^2 \hbar^2 n_\infty}{4m} \frac{d\psi^2}{dr} \right), \quad (29)$$

in which the last term is a divergence that goes out when integrated, so that for the total energy per unit length one is left simply with

$$U = \frac{1}{8\pi} \int (\mathcal{B}^2 + H_c^2 (1 - \psi^4)) dS. \quad (30)$$

IV. BOGOMOL'NYI INEQUALITY

We shall now try to rewrite the energy density associated with the vortex in a different form. Let us begin by writing the relation

$$\left(g_c \hbar \frac{d\psi}{dr}\right)^2 + (m\mathcal{V}\psi)^2 = \left(g_c \hbar \frac{d\psi}{dr} \mp m\mathcal{V}\psi\right)^2 \pm g_c \hbar q \psi^2 \mathcal{B} \pm \frac{g_c \hbar m}{qn_\infty} \frac{d(rj)}{rdr}, \quad (31)$$

which can be obtained by rewriting $\psi(d\psi/dr)$ in terms of dj/dr and $d\mathcal{V}/dr$, the latter term being transformed by use of the Maxwell equation (15). In the first term on the right hand side of (31), one takes the minus sign if \mathcal{V} is positive, the plus sign otherwise. We are thus able to obtain a Bogomol'nyi type reformulation of (27) that is given by

$$\begin{aligned} \mathcal{E} = & \frac{g_c \hbar q n_\infty}{2m} |\mathcal{B}| + \frac{1}{2\pi} \left(\frac{|\mathcal{B}|}{2} + \frac{\pi g_c \hbar q n_\infty}{m} (\psi^2 - 1) \right)^2 + \frac{n_\infty}{2m} \left(g_c \hbar \frac{d\psi}{dr} - m|\mathcal{V}\psi \right)^2 \\ & + \left(\frac{H_c^2}{8\pi} - \frac{\pi g_c^2 \hbar^2 q^2 n_\infty^2}{2m^2} \right) (1 - \psi^2)^2 + \frac{1}{r} \frac{d}{dr} \left(\frac{B_\infty r^2 \mathcal{B}}{8\pi} + \frac{g_c \hbar r |j|}{2q} \right), \end{aligned} \quad (32)$$

which generalizes the original version⁶ by inclusion of the term with the coefficient B_∞ , which allows for the effect of the background rotation velocity Ω_∞ . Since this extra term is just the divergence of a quantity that vanishes both on the axis and in the large distance limit it gives no contribution to the corresponding integral expression, which therefore has the same form as the usual Bogomol'nyi relation for the non rotating case.⁸ More generally, performing the Bogomol'nyi trick (31) just for a fraction f of the combined kinetic and gradient contribution, one can see that for a vortex with (relative) magnetic flux

$$\Phi = \int \mathcal{B} dS, \quad (33)$$

the total vortex energy per unit length and per unit of flux will be expressible in the form

$$\begin{aligned} U = & \frac{fH_c}{4\pi\kappa} |\Phi| + \frac{H_c^2}{8\pi} \left(1 - \frac{f^2}{\kappa^2}\right) \int (1 - \psi^2)^2 dS + \frac{H_c^2}{8\pi} \int \left(\frac{|\mathcal{B}|}{H_c} - \frac{f(1 - \psi^2)}{\kappa} \right)^2 dS \\ & + \frac{fn_\infty}{2m} \int \left(g_c \hbar \frac{d\psi}{dr} - m|\mathcal{V}\psi \right)^2 dS + \frac{(1-f)n_\infty}{2m} \int \left(\left(g_c \hbar \frac{d\psi}{dr} \right)^2 + (m\mathcal{V}\psi)^2 \right) dS, \end{aligned} \quad (34)$$

in which κ is a dimensionless constant given by the definition

$$\kappa = \frac{mH_c}{2g_c \pi \hbar q n_\infty}. \quad (35)$$

It can be convenient to introduce the so-called London penetration length λ , defined by the expression

$$\lambda^2 = \frac{m}{4\pi q^2 n_\infty}, \quad (36)$$

and the usual flux quantum

$$\tilde{\Phi} = \frac{2\pi\hbar}{q}. \quad (37)$$

It is to be observed that all the terms in this expression will be non-negative provided the quantity f is chosen not only so as to lie in the range $0 \leq f \leq 1$ but also so as to satisfy $f \leq \kappa$, a requirement that will be more restrictive if $\kappa \leq 1$. In the latter case we can maximize the first term on the right of (34) by choosing $f = \kappa$, thereby incidentally eliminating the second term, so that we obtain the lower limit

$$\kappa \leq 1 \quad \Rightarrow \quad \frac{U}{|\Phi|} \geq \frac{H_c}{4\pi}. \quad (38)$$

If $\kappa \geq 1$ we shall be able to choose $f = 1$, thereby eliminating the final term in (34), so that we obtain the inequality

$$\kappa \geq 1 \quad \Rightarrow \quad \frac{U}{|\Phi|} \geq \frac{H_c}{4\pi\kappa}. \quad (39)$$

More particularly it can be seen that choosing $f = 1$ will eliminate both the second term and the last term on the right of (34) in the special Bogomol'nyi limit case characterized the condition

$$\kappa = 1. \quad (40)$$

(Readers should be warned that much of the relevant literature^{9,8} follows a rather awkward tradition in which the symbol κ is used for what in the present notation scheme would be denoted by $\sqrt{2}\kappa$, which instead of (40) makes the critical condition come out to be $\kappa = 1/\sqrt{2}$.) In this Bogomol'nyi limit, the energy per unit flux per unit length (34) will be minimized by imposing the conditions

$$g_c \hbar \frac{d\psi}{dr} = m|\mathcal{V}|\psi, \quad (41)$$

(with the sign adjusted so as to make the right hand side positive), and

$$|\mathcal{B}| = \frac{g_c \tilde{\Phi}}{4\pi\lambda^2} (1 - \psi^2). \quad (42)$$

which (as when the background is non-rotating¹⁰) will automatically guarantee the solution of the field equations in this case, annihilating the last two terms in (34) so that one is left simply with

$$\frac{U}{|\Phi|} = \frac{H_c}{4\pi}. \quad (43)$$

The qualitative distinction between what are known¹¹ as Pippard type or type I superconductors on one hand and as London type or type II superconductors on the other hand is based on the criterion of whether or not, for a given total flux, the energy will be minimized by gathering the flux in a small number of vortices, each with large winding number N or by separating the flux in a large number of vortices, each just with unit winding number. Within the framework of our analysis, a model may be characterized as type I if the derivative with respect to $|\Phi|$ of $U/|\Phi|$ is always negative, i.e., if $dU/d|\Phi| < U/|\Phi|$, in which case the vortices will effectively be mutually attractive, and as being of type II if the derivative with respect to $|\Phi|$ of $U/|\Phi|$ is always positive, i.e., if $dU/d|\Phi| > U/|\Phi|$, in which case the vortices will effectively be mutually repulsive. One can of course envisage the possibility of models that are intermediate in the sense of having a sign for the derivative of $U/|\Phi|$ that depends on N , so that the minimum is obtained for some large but finite value of the winding number.

What can be seen directly from (43) is that within the category of models characterized by the Ginzburg Landau ansatz, the special Bogomol'nyi limit case lies precisely on the boundary between type I and type II, since it evidently satisfies the exact equality

$$4\pi\lambda^2 H_c = g_c \tilde{\Phi} \quad \Rightarrow \quad \frac{dU}{d|\Phi|} = \frac{U}{|\Phi|}, \quad (44)$$

for all values of $|\Phi|$. The implication of our work is that the well known conclusion^{9,8} that there is neither repulsion nor attraction between Ginzburg model vortices in the Bogomol'nyi limit case will remain valid even in the presence of a rotating background. (It has also been shown to be generalizable to cases where self gravitation is allowed for in a general relativistic framework^{12,13}).

Since the change of variables performed in section II has transformed the system to a representation that is formally identical to that of the case with a non-rotating background, it can also be concluded that the conclusions of the pioneering analysis of Kramer⁹ will remain valid, i.e., that in the sense of the preceding paragraph the system will be of the Pippard kind (type I), if $\kappa < 1$, i.e.,

$$4\pi\lambda^2 H_c < g_c \tilde{\Phi} \quad \Rightarrow \quad \frac{dU}{d|\Phi|} < \frac{U}{|\Phi|}, \quad (45)$$

and that it will be of the London kind (type II), if $\kappa > 1$, i.e., if

$$4\pi\lambda^2 H_c > g_c \tilde{\Phi} \quad \Rightarrow \quad \frac{dU}{d|\Phi|} > \frac{U}{|\Phi|}, \quad (46)$$

at least so long as the winding number N and the parameters κ and g_c are not too far from the neighborhood of unity, in which the numerical investigations have been carried out. It does not seem that the possibility of an intermediate scenario, with $U/|\Phi|$ minimized by a winding number N that is finite but larger than one, can occur within the framework of Ginzburg type models except perhaps for parameter values that are too extreme to be of likely physical relevance.

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