Energy of Magnetic Vortices in Rotating Superconductor

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Abstract. We carry out a systematic analytic investigation of stationary and cylindrically symmetric vortex configurations for simple models representing an incompressible non-relativistic superconductor in a background, which is rigidly rotating with the angular velocity Ω_∞. It is shown that although the magnetic and kinetic contributions to the energy per unit length of such a vortex are separately modified by the background angular velocity, its effect on the total energy per unit length cancels out. For a type II superconductor threaded by a parallel array of such vortices, this result implies that the relevant macroscopic magnetic field strength H will not be equal to the large scale average $\langle B\rangle$ of the local magnetic induction B (as has previously been suggested) but instead that H will simply be equal to the external London field value $B_\infty = -(2m/q)\Omega_\infty$ (where m and q are the mass and charge of the condensate particles) that characterizes the value of B outside the vortices.

I. INTRODUCTION

Because of the property of irrotationality of superfluidity, it is well known that the angular momentum of a zero temperature superfluid is "carried" only by the vortices it contains. Similarly, when a magnetic field is applied to a type II superconductor, there exist quantized flux tubes. When one considers a rotating superconductor, the two effects, velocity and electromagnetism, are combined. It is the purpose of this work to carefully analyze the energy of such a system.

The physical motivation behind this work is the need to clear up some confusion that has arisen in the context of neutron star matter[1] about the relation between the large scale magnetic field strength H and the average value $\langle B\rangle$ of the local magnetic induction B in a rotating type II superconductor threaded by a parallel array of vortices. In order to clarify the issue we shall proceed on the basis of the same kind of simplification that was postulated as the basis of the earlier discussion,[1] working in terms of a broad category of non-relativistic incompressible superfluid models that includes, but is not restricted to, the special case characterized by the standard Ginzburg-Landau ansatz. The main conclusion of our work, as will be shown in Section V, is that the conventionally defined macroscopic field strength H will simply be given by the value (9) of the external (London) limit B_∞ of the local induction B, and not by the average value $\langle B\rangle$ as was previously suggested.[1]

The essential feature of the models to be dealt with is just the usual postulate that the relevant charged superfluid constituent is represented by a locally variable number density n_s of bosonic particles that are characterized by an effective mass m, charge q and a momentum covector having space components

$$mv_i + qA_i = \hbar\nabla_i\varphi\,, \tag{1}$$

where $\hbar$ is the Dirac-Planck constant, A_i is the magnetic vector potential and φ is a scalar with period 2π representing the phase variable of the boson condensate. In ordinary laboratory applications the particles would represent Cooper type electron pairs, characterized in terms of the charge and mass of the electron by the exact relation $q = -2e$, and to a good approximation by $m \simeq 2m_e$, whereas in the context of neutron star matter they would represent proton pairs, characterized by $q = 2e$ and an effective mass given roughly in terms of that of the proton by $m \approx 2m_p$.

The scenarios we shall consider will be of the usual kind, in which each individual vortex is treated as a stationary cylindrically symmetric configuration consisting of a rigidly rotating background medium with uniform angular velocity Ω_∞, say, together with a charged superfluid constituent in a state of differential rotation with a velocity v, which tends at large distance towards the rigid rotation value given by $\Omega_\infty r$, where r is the cylindrical radial distance from the axis . It will be supposed that the superfluid particle number density n_s vanishes on the axis and is a monotonically increasing function of the cylindrical radius variable r, tending rapidly to a constant value n_∞ at large distances from the axis: $n_s = n_\infty$ for $r \gtrsim \xi$, say, where ξ is a parameter interpretable as the core radius. It will further be supposed that the local charge density is canceled by the background so that there is no electric field, but that there is a magnetic induction field with magnitude B and direction parallel to the axis, whose source is the axially oriented electromagnetic current whose magnitude j will be given by

$$j = qn_s(v - \Omega_\infty r)\,. \tag{2}$$

The relevant Maxwellian source equation for the magnetic field will have the familiar form

$$\frac{dB}{dr} = -4\pi j\,. \tag{3}$$

The other relevant Maxwellian equation is the one governing the axial component A (which in an appropriate gauge will be the only one) of the electromagnetic potential covector, which will be related to the magnetic induction by

$$\frac{d(rA)}{dr} = rB\,. \tag{4}$$

The essential property distinguishing the "superconducting case" from its "normal" analogue is the London flux quantization condition, which in the present context (where all physically relevant quantities depend only on the cylindrical radius r) will be expressible in the well known form[3]

$$mv + qA = \frac{N\hbar}{r}\,, \tag{5}$$

where N is the relevant phase winding number, which must be an integer.

It is to be noted of course that by themselves the foregoing equations are not quite sufficient to fully characterize the model: in order to obtain a complete system it is also necessary to have some well defined prescription for the radial dependence of the number density n_s, which will be referred to below as the *structure function.* The available literature does not seem to provide any fully adequate general purpose ansatz for such a structure function, though various, more or less satisfactory, phenomenological prescriptions have been put forward in particular contexts. One of the simplest proposals is to postulate that n_s falls discontinuously from its asymptotic constant value n_∞ to zero. Such a simple ansatz is in fact perfectly adequate for many purposes, since, as will be seen below, much of the relevant physics turns out to be insensitive to the detailed structure of the core. However, no such specific prescription for the structure function will be needed to obtain the general result of section V.

The plan of this paper is the following. In section II, we transform our system of equations to a simpler form by considering the deviations of all quantities with respect to their asymptotic values corresponding to rigid rotation. Section III is devoted to the demonstration of the cancellation between the rotation–induced terms of the kinetic energy and the magnetic energy. In section IV, we show that there is a simple relation between the total energy per unit length of the vortex and the total flux independently of the details of the structure of the vortex. Section VI and VII are concerned respectively with the external and the internal solution representing the vortex. In section VIII we evaluate explicitly the energy contributions as functions of the core parameters and finally, in section V we apply our results for one vortex to the case of an array of aligned vortices and obtain our main result concerning the macroscopic field strength H. Section IX summarizes this work.

II. HOMOGENIZATION OF THE SYSTEM

For given values of the relevant physical constants m and q and the rotation rate Ω_∞, and subject to the provision that the structure function for n_s has been prescribed in advance, the foregoing equations will constitute a linear differential system relating the variable functions v, B, A to the integer valued parameter N. Before proceeding, it will be useful to take advantage of the possibility of transforming the preceding system of equations to a form that is not just linear but also homogeneous by replacing the variables v, B, A by corresponding variables $\mathcal{V}$, $\mathcal{B}$, $\mathcal{A}$ that are defined by

$$\mathcal{V} = v - \Omega_\infty r\,, \tag{6}$$

$$\mathcal{B} = B - B_\infty\,, \tag{7}$$

$$\mathcal{A} = A - \tfrac{1}{2} r B_\infty\,. \tag{8}$$

Here B_∞ is the uniform background magnetic field value that would be generated by a rigidly rotating superconductor and is given by the London formula,

$$B_\infty = -\frac{2m}{q}\Omega_\infty\,, \tag{9}$$

obtained by combining (5) and (4) in the specialized case of rigid corotation, which is $\mathcal{V} = 0$.

In terms of these new variables the equation (4) will be transformed to the form

$$\frac{d(r\mathcal{A})}{dr} = r\mathcal{B}\,, \tag{10}$$

while the other differential equation (3) will be transformed to the form

$$\frac{d\mathcal{B}}{dr} = -4\pi j\,, \tag{11}$$

in which we shall have

$$j = q n_s \mathcal{V}\,. \tag{12}$$

Finally the flux quantization condition (5) will be converted to the form

$$m\mathcal{V} + q\mathcal{A} = \frac{N\hbar}{r}\,, \tag{13}$$

which can be used to transform (10) to the form

$$\frac{m}{qr}\frac{d(\mathcal{V}r)}{dr} = -\mathcal{B}\,. \tag{14}$$

The advantage of this reformulation is that unlike v, B and A, the new variables $\mathcal{V}$, $\mathcal{B}$ and $\mathcal{A}$ are subject just to homogeneous boundary conditions: they must all tend to zero as $r \to \infty$, while at the inner boundary, as $r \to 0$, there is just the regularity requirement that $\mathcal{B}$ should be bounded, so that we have $\mathcal{B} \to \mathcal{B}_0$ for some finite limit value $\mathcal{B}_0$, which by (10) entails automatically that $\mathcal{A}$ should tend to zero. Since the number density n_s is postulated to vanish at the origin there is no corresponding restriction on $\mathcal{V}$. We have thus obtained a homogeneous linear system of equations relating the integer N to the set of three functions consisting of the excess (with respect to the background) magnetic induction variable $\mathcal{B}$, and the corresponding excess potential variable $\mathcal{A}$ together with the relative velocity variable $\mathcal{V}$, or equivalently the current magnitude j as given by (12). This means that they will be expressible in the form

$$\mathcal{B} = N\tilde{\mathcal{B}}\,, \qquad \mathcal{A} = N\tilde{\mathcal{A}}\,, \tag{15}$$

$$\mathcal{V} = N\tilde{V}\,, \qquad j = N\tilde{j}\,, \tag{16}$$

in terms of corresponding rescaled functions $\tilde{\mathcal{B}}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{V}}$ and $\tilde{j}$ that will be fully determined (independently not just of the rotation parameter $B_\infty = -2m\Omega_\infty/q$ but also of the winding number N) just by the physical constants m and q and the specification of the structure function giving the radial dependence of the number density n_s.

III. ROTATION ENERGY CANCELLATION LEMMA

One of the main purposes of the present work is to demonstrate, in the present section, a useful lemma concerning mutual cancellation – *independently of the radial dependence* of the relevant particle density n_s – between the background rotation dependent term in the magnetic energy per unit length

$$U_{\rm mag} = \int \mathcal{E}_{\rm mag}\, dS\,, \tag{17}$$

and the corresponding term in the kinetic energy per unit length

$$U_{\rm kin} = \int \mathcal{E}_{\rm kin}\, dS\,, \tag{18}$$

with

$$dS = 2\pi r\, dr\,. \tag{19}$$

In the above expressions, $\mathcal{E}_{\rm mag}$ is the extra magnetic energy density arising from a non-zero value of the phase winding number N, i.e., the local deviation from the magnetic energy density due just to the uniform field B_∞ associated with the state of rigid corotation at the angular velocity Ω_∞, namely

$$\mathcal{E}_{\rm mag} = \frac{B^2}{8\pi} - \frac{B_\infty^{\,2}}{8\pi}\,, \tag{20}$$

while $\mathcal{E}_{\rm kin}$ is the corresponding deviation of the kinetic energy from that of the state of rigid corotation at the angular velocity Ω_∞, namely

$$\mathcal{E}_{\rm kin} = \frac{m}{2} n_s \big(v^2 - \Omega_\infty^2 r^2\big)\,. \tag{21}$$

Note that in addition to $U_{\rm mag}$ and $U_{\rm kin}$ the total energy per unit length $U_{\rm tube}$ associated with the vortex will contain an extra potential energy term allowing for effect of the breakdown of superfluid condensation in the core, but this will not be relevant for the work of the present section. In the limiting case of an ordinary superfluid, as characterized by vanishing charge $q = 0$, the kinetic contribution would be the dominant one, but in the context of superconductivity, i.e., when q is non-zero, it is commonly[1] overlooked, perhaps because of the small value of the electron mass that is relevant in laboratory applications. The purpose of the present section is to show not only that the kinetic contribution will not in general be negligible compared with the magnetic contribution, but also that its inclusion brings about considerable simplification.

To start with, using the the decomposition (7) of the magnetic field, it will be possible to express the magnetic energy density contribution in the form

$$\mathcal{E}_{\rm mag} = \frac{\mathcal{B}^2}{8\pi} + \frac{B_\infty \mathcal{B}}{4\pi}\,, \tag{22}$$

while similarly, using the decomposition (6) of the velocity, it will be possible to express the corresponding kinetic energy density in the analogous form

$$\mathcal{E}_{\rm kin} = \frac{m}{2} n_s \big(\mathcal{V}^2 + 2\Omega_\infty \mathcal{V} r\big)\,, \tag{23}$$

which can usefully be rewritten in terms of the current magnitude j, using (12) and (9) as

$$\mathcal{E}_{\rm kin} = \frac{m}{2} n_s \mathcal{V}^2 - \frac{j}{2} B_\infty r\,. \tag{24}$$

Using the Maxwell source equation (11) this can be converted to the form

$$\mathcal{E}_{\rm kin} = \frac{m}{2} n_s \mathcal{V}^2 + \frac{B_\infty}{r}\frac{d}{dr}\Big(\frac{r^2\mathcal{B}}{8\pi}\Big) - \frac{B_\infty \mathcal{B}}{4\pi}\,, \tag{25}$$

in which the second term can be seen to be a pure divergence, while the last term can be seen to be equal in magnitude but opposite in sign to the last term in (22), so that there will be a cancellation between them when the magnetic and kinetic contributions are combined.

At an integrated level, in view of (22), it will be possible to express the magnetic energy in terms of quantities $\widehat{U}_{\rm mag}$ and Φ that are specified independently of B_∞, in the form

$$U_{\rm mag} = \widehat{U}_{\rm mag} + \frac{B_\infty}{4\pi}\Phi\,, \tag{26}$$

where the part that would still be present if the background were non rotating is given by

$$\widehat{U}_{\rm mag} = \int \frac{\mathcal{B}^2}{8\pi}\, dS\,, \qquad (27)$$

and where the coefficient Φ is a flux integral of the simple form

$$\Phi = \int \mathcal{B}\, dS = N\phi\,, \qquad (28)$$

where ϕ is the usual flux quantum, given by

$$\phi = \frac{2\pi\hbar}{q}\,. \qquad (29)$$

In a similar manner, it will be possible to express the kinetic contribution in terms of quantities $\widehat{U}_{\rm kin}$ and $\Phi_{\rm kin}$ that are also specified independently of Ω_∞, or equivalently of B_∞, in the form

$$U_{\rm kin} = \widehat{U}_{\rm kin} + \frac{B_\infty}{4\pi}\Phi_{\rm kin}\,, \qquad (30)$$

where the part that would still be present if the background were non rotating is given by

$$\widehat{U}_{\rm kin} = \frac{m}{2q}\int j\mathcal{V}\, dS\,, \qquad (31)$$

and where the coefficient $\Phi_{\rm kin}$ is given by

$$\Phi_{\rm kin} = \pi\int r^2\frac{d\mathcal{B}}{dr}\, dr = \pi\int d\big(r^2\mathcal{B}\big) - \int 2\pi r\mathcal{B}\, dr\,, \qquad (32)$$

which corresponds to the last two terms on the right hand side of (25). The first term in this expression clearly vanishes when the integration is taken over the whole range from the center, where $r = 0$, to the large radius limit where $r^2\mathcal{B} \to 0$, as can be seen from the explicit solution (66). We are thus left with the second term, the kinetic analogue of the magnetic flux contribution, to which it is evidently equal in magnitude but opposite in sign, i.e., we obtain

$$\Phi_{\rm kin} = -\Phi\,. \qquad (33)$$

It can thus be seen that there is a remarkable cancellation whereby the dependence on B_∞, or equivalently on Ω_∞, in the separate magnetic and kinetic energy contributions will cancel out when they are combined, so that we are left simply with a result of the form

$$U_{\rm mag} + U_{\rm kin} = \widehat{U}_{\rm mag} + \widehat{U}_{\rm kin} = \int\Big(\frac{\mathcal{B}^2}{8\pi} + \frac{m}{2}n_s\mathcal{V}^2\Big)\, dS\,. \qquad (34)$$

Since the terms in this expression are both quadratically dependent on fields, namely $\mathcal{V}$ and $\mathcal{B}$, that by (15) and (16) will just be proportional to the winding number N, we obtain the following conclusion.

Rotation energy cancellation lemma: Whereas the separate values of the the magnetic and kinetic contributions (as defined using the formulae (26) and (30) above) to the energy per unit length of the vortex will be affected by the rate of rotation of the background Ω_∞ (or equivalently the corresponding London field $B_\infty = -2m\Omega_\infty/q$), the combination of these two contributions *will not depend directly on* Ω_∞ and can be simply expressed in the form (as a result of equations (15) and (16))

$$U_{\rm mag} + U_{\rm kin} = \tilde{U}N^2\,, \qquad (35)$$

where we recall that N is the winding number and $\tilde{U}$ depends only on the physical constants m and q and on the form of the radial distribution of the number density n_s. A simple form for $\tilde{U}$ will be given in the next section.

IV. AXIS-FIELD ENERGY FORMULA

The preceding result, namely the cancellation of the contributions due to the background rotation, was obtained simply with the background London equation (9) without using the full, i.e., local London quantization condition (5). The ultimate cancellation of the B_∞ – dependent contribution is attributable at a local level to the fact that the B_∞ – dependent contribution to the combined energy density is a pure divergence:

$$8\pi\big(\mathcal{E}_{\rm mag} + \mathcal{E}_{\rm kin}\big) = \mathcal{B}^2 + 4\pi\frac{m}{q}j\mathcal{V} + \frac{B_\infty}{r}\frac{d}{dr}\big(r^2\mathcal{B}\big)\,. \qquad (36)$$

We can obtain a stronger result if we now invoke the more specialized relation (14) which is a consequence of the quantization condition (5) that specifically characterizes superconductivity. This condition can be seen to imply that the whole of the right hand side of (36) will be expressible as a divergence, since we shall have

$$\mathcal{B}^2 + 4\pi\frac{m}{q}j\mathcal{V} = -\frac{m}{qr}\frac{d}{dr}\big(r\mathcal{V}\mathcal{B}\big)\,. \qquad (37)$$

It can thereby be seen, using (5) again, that the combined energy density will be expressible as

$$\mathcal{E}_{\rm mag} + \mathcal{E}_{\rm kin} = \frac{1}{8\pi r}\frac{d}{dr}\left(\Big(B_\infty r^2 + rA - \frac{N\hbar}{q}\Big)\mathcal{B}\right)\,. \qquad (38)$$

In the outer limit, as $r \to \infty$, the rapid fall off of $\mathcal{B}$ will ensure that the quantity inside the divergence will tend to zero. In the inner limit, as $r \to 0$, the first term in the divergence obviously gives no contribution, and the consideration that A should be bounded ensures that the second term also gives no contribution, so we shall be left with the contribution just from the final term, which is proportional to the winding number N. The final outcome of the integration of (38) can be stated as follows.

Axis-field energy lemma: Subject to the London quantization (as given by (5) above) the combination of the magnetic and kinetic contributions (as defined using the formulae (26) and (30) above) to the energy per unit length for a vortex with given winding number N and corresponding total flux Φ as specified by (28) will be provided just by the the axis-field value $\mathcal{B}_0$ according to the proportionality law

$$U_{\rm mag} + U_{\rm kin} = \frac{\Phi\mathcal{B}_0}{8\pi}\,, \tag{39}$$

where $\mathcal{B}_0$ is the value on the axis of the *relative* magnetic field value $\mathcal{B}$ as given by (7), i.e. it is the difference

$$\mathcal{B}_0 = B_0 - B_\infty \tag{40}$$

between the central value, B_0, of the magnetic induction B and its asymptotic London value B_∞.

A corollary of this second lemma is that the combination of the kinetic and magnetic energy per unit length will remain the same whatever the internal structure, as long as the total flux and the axis magnetic field are the same. The simplest such configuration is given by a field B retaining the same uniform central value B_0 out to a cut–off where it drops discontinuously to its asymptotic value B_∞. This cut–off radius, $\tilde R$, say, is adjusted so as to give the same total flux as in the actual model, i.e. so as to satisfy the specification

$$\pi\tilde R^2 = \frac{\Phi}{\mathcal{B}_0} = \frac{\phi}{\tilde{\mathcal{B}}_0}\,. \tag{41}$$

Since the quantity $\tilde{\mathcal{B}}_0$, i.e. the value on the axis of the rescaled field defined by (15), depends only on the physical constants m and q and on the form of the structure function specifying the radial dependence of the number density n_s, it follows that the same applies to the effective radius $\tilde R$, which will thus be independent of N, as well as of the background rotation rate $\Omega_\infty = -qB_\infty/2m$. The conclusion that the effective magnetic radius depends only on the structure function specifying n_s is interpretable as a restatement of our first lemma, since it can be seen that the coefficient $\tilde U$ in (35) will be given just in terms of this effective radius $\tilde R$ by the formula

$$\tilde U = \frac{\hbar^2}{2q^2\tilde R^2}\,. \tag{42}$$

V. AVERAGE OVER AN ARRAY OF ALIGNED VORTICES

Let us now consider the typical situation in a type II superconductor, in which we have not just a single vortex but a parallel array of such vortices with sufficiently low mean number density per unit surface area, ν, say, for the separation distance between neighboring vortices to be large compared with the penetration length λ. Since, according to (7) and (28), each vortex carries an extra magnetic flux Φ in addition to the contribution from the uniform London field B_∞, the large scale average magnetic field will be given by

$$\langle B\rangle = B_\infty + \nu\,\Phi\,. \tag{43}$$

As compared with the average energy density of a configuration in rigid corotation with the given angular velocity Ω_∞, but with no magnetic field, the extra energy density averaged over a large number of vortices will be given by

$$\langle\mathcal{E}\rangle = \mathcal{E}_{\rm Lon} + \langle\mathcal{E}_{\rm tube}\rangle\,, \tag{44}$$

where $\mathcal{E}_{\rm Lon}$ is the uniform contribution from the London magnetic field, i.e.

$$\mathcal{E}_{\rm Lon} = \frac{B_\infty^2}{8\pi}\,, \tag{45}$$

and where $\langle\mathcal{E}_{\rm tube}\rangle$ is the large scale average of the contribution given locally for the separate vortices by

$$\mathcal{E}_{\rm tube} = \mathcal{E}_{\rm mag} + \mathcal{E}_{\rm kin} + \mathcal{E}_{\rm con}\,. \tag{46}$$

where $\mathcal{E}_{\rm mag}$ and $\mathcal{E}_{\rm kin}$ are the magnetic and kinetic energy contributions discussed in the preceding sections and $\mathcal{E}_{\rm con}$ is the condensation energy contribution depending just on the radial distribution of the condensate number density n_s (in a manner that is unimportant for our present purpose), which means that the corresponding additional contribution

$$U_{\rm con} = \int \mathcal{E}_{\rm con}\, dS \tag{47}$$

to the vortex energy per unit length can be treated just as a constant as far as the present section is concerned. It follows that we shall have

$$\langle\mathcal{E}_{\rm tube}\rangle = \nu U\,, \tag{48}$$

where U is the total energy per unit length of an individual vortex as given by the combination

$$U = U_{\rm mag} + U_{\rm kin} + U_{\rm con}\,, \tag{49}$$

in which the first two contributions will separately depend on the background rotation velocity Ω_∞ (or equivalently on the London field B_∞) but in which, by the cancellation lemma expressed by (34), the total (like final term) will not.

Since each vortex is associated with a momentum circulation of magnitude $2\pi\hbar N$ there will be a corresponding generalized vorticity, in the sense of momentum circulation per unit area, with large scale average given by

$$\langle w\rangle = 2\pi\hbar N\nu\,. \tag{50}$$

In terms of this quantity the large scale average (44) of the extra energy due to the deviation from a configuration of unmagnetized rigid corotation will be given by

$$\langle\mathcal{E}\rangle = \frac{B_\infty^2}{8\pi} + \frac{\langle w\rangle}{2\pi\hbar}\frac{U}{N}\,. \tag{51}$$

For a large scale variational description it is convenient to use $\langle w\rangle$ and $\langle B\rangle$ as the independent variables. In terms of these, the London field can be seen from (43) to be expressible as

$$B_\infty = \langle B\rangle - \frac{\langle w\rangle}{q}\,, \tag{52}$$

so (51) gives

$$\langle\mathcal{E}\rangle = \frac{\langle B\rangle^2}{8\pi} - \frac{\langle B\rangle\langle w\rangle}{4\pi q} + \frac{\langle w\rangle^2}{8\pi q^2} + \frac{\langle w\rangle}{2\pi\hbar}\frac{U}{N}\,. \tag{53}$$

Since, by (39) the ratio U/N will be given by the formula

$$\frac{U}{N} = \frac{\tilde{\Phi}\mathcal{B}_0}{8\pi} + \frac{U_{\rm con}}{N}\,, \tag{54}$$

in which all dependence on B_∞ and thus also on $\langle B\rangle$ has canceled out, it can immediately be seen that the conventional definition (which is the same as the definition adopted in Ref. 1)

$$H = 4\pi\frac{\partial\langle\mathcal{E}\rangle}{\partial\langle B\rangle} \tag{55}$$

for the effective magnetic field strength H will simply give

$$H = B_\infty\,, \tag{56}$$

i.e. H is directly identifiable with the London field. The corresponding magnetic polarization $\mathcal{M}$, as defined in the usual way by

$$\langle B\rangle = H + 4\pi\mathcal{M} \tag{57}$$

will be expressible as

$$\mathcal{M} = \frac{\langle w\rangle}{4\pi q} = \frac{\nu N\hbar}{2q}\,. \tag{58}$$

VI. EXTERNAL SOLUTION

In the previous sections we have been able to establish very useful properties concerning the energy density per unit length of a vortex, without needing the specification of a internal structure. This section and the next one will consider this question and show in particular how the unspecified parameter of the previous section, the axis value of the relative magnetic field, or equivalently the effective radius $\tilde{R}$, can be explicitly computed depending on the modelization of the internal structure. We shall focus here on the solution outside the core, which is always the same up to a normalization constant, that can be determined only with the knowledge of the internal structure. This will be the purpose of the next section.

In the region outside the core, i.e. in the range $r \geq \xi$, where the the number density n_s is uniform with value

$$n_s = n_\infty\,, \tag{59}$$

the equation obtained from (11) and (14) by eliminating $\mathcal{B}$ will have the form

$$r^2\frac{d^2\mathcal{V}}{dr^2} + r\frac{d\mathcal{V}}{dr} - \left(\frac{r^2}{\lambda^2}+1\right)\mathcal{V} = 0\,, \tag{60}$$

where λ is a fixed lengthscale given by

$$\lambda^2 = \frac{m}{4\pi q^2 n_\infty}\,, \tag{61}$$

and which is called the London penetration length.

The equation (60) is of the well known Bessel–type, whose most general asymptotically bounded solution is expressible in the form (see Ref. 4)

$$\mathcal{V} = CK_1\{x\}\,, \tag{62}$$

where the independent variable x is defined by

$$x = \frac{r}{\lambda}\,, \tag{63}$$

C is a normalization constant and K_1 is a modified Bessel function.

It follows immediately from the flux quantization condition (13) that the magnetic potential deviation defined by (8) will be given by

$$\mathcal{A} = -\frac{m}{q}CK_1\{x\} + \frac{\Phi}{2\pi r}\,, \tag{64}$$

where Φ is the magnetic flux integral given by (28).

Using the fact that K_1 is related to the Bessel function K_0 by

$$K_1 = -K_0'\,, \qquad K_0 = -K_1' - x^{-1}K_1\,, \tag{65}$$

where a prime stands for differentiating with respect to the argument x, it is straightforward to obtain the corresponding solution of (14) for the magnetic field deviation, which will be expressible in the simple form

$$\mathcal{B} = \frac{m}{q\lambda}CK_0\{x\}\,. \tag{66}$$

The external configuration for the magnetic vortex has thus far been determined up to the normalization constant C. It will be seen in the next section how, on the basis of a suitable ansatz for the radial dependence of n_s, the solution inside the core can be used to fix this constant C, and thus to determine completely the configuration of the vortex.

VII. INTERNAL SOLUTION

Instead of directly specifying the way in which the number density n_s varies from zero on the axis ($r = 0$) to its external value n_∞ at the core radius (where $r = \xi$, i.e. where $x = \check{x} \equiv \xi/\lambda$), it is more convenient to work with an ansatz based on an explicit prescription for the current magnitude j, which will have a qualitatively similar behavior in the core, ranging from zero on the axis to a value $\check{j}$ at the core radius, that according to (12) and (62) will be given by

$$\check{j} = q n_\infty C \check{K}_1 \,, \tag{67}$$

using the obvious abbreviation $\check{K}_1 = K_1\{\check{x}\}$. The current in the core, i.e. where $x \leq \check{x}$, will therefore be expressible in the form

$$j = \sigma \check{j} \,, \tag{68}$$

where σ is a dimensionless function of x that is required to vanish, $\sigma = 0$, for $x = 0$ and to increase to unity, $\sigma = 1$ where $x = \check{x}$ (σ plays here the role of the structure function mentioned in the introduction). For any suitably prescribed function σ with these properties, there will be corresponding functions, χ and ζ, say, that are defined by the requirement that they too should vanish on the axis, i.e., $\chi = \zeta = 0$ for $x = 0$ and by the requirement that they should be obtained in the region $0 \leq x \leq \check{x}$ as solutions of the differential equations

$$\check{x}\,\frac{d\chi}{dx} = 2\sigma \,, \qquad \check{x}\,\frac{d(x\zeta)}{dx} = 4x\chi \,. \tag{69}$$

In terms of such a set of functions, the relevant solution of equation (11) will evidently be given by

$$\mathcal{B} = \mathcal{B}_0 - 2\pi\xi\check{j}\,\chi \,, \tag{70}$$

and the corresponding solution of (10) will be given by

$$\mathcal{A} = \tfrac{1}{2}\lambda\mathcal{B}_0\, x - \tfrac{1}{2}\pi\xi^2\check{j}\,\zeta \,. \tag{71}$$

The requirement that the magnetic field should be continuous (so that the current density j remains finite) entails that the internal solution (70) should match the corresponding external solution (66) where $x = \check{x}$, so we obtain a boundary condition of the form

$$\mathcal{B}_0 - 2\pi\xi\check{j}\,\check{\chi} = \frac{m}{q\lambda} C\check{K}_0 \,, \tag{72}$$

while the corresponding continuity requirement for the potential gives a second boundary condition of the form

$$\mathcal{B}_0 - \pi\xi\check{j}\,\check{\zeta} = \frac{\Phi}{\pi\xi^2} - \frac{2m}{q\xi} C\check{K}_1 \,. \tag{73}$$

This pair of boundary equations can be solved to give the central magnetic field difference in the form

$$\mathcal{B}_0 = \frac{m}{q\lambda} C\big(\check{K}_0 + \frac{\check{\chi}}{2}\check{x}\check{K}_1\big) \,, \tag{74}$$

while the required normalization constant C is finally obtained in the form

$$C = \frac{N\hbar}{\lambda m \check{\mathcal{K}}} \,, \tag{75}$$

in terms of a dimensionless quantity that is given by

$$\check{\mathcal{K}} = \big(1 + \frac{\check{\epsilon}}{8}\check{x}^2\big)\check{x}\check{K}_1 + \tfrac{1}{2}\check{x}^2\check{K}_0 = \frac{\check{x}^3}{8}\left(\check{\epsilon}\,\check{K}_1 + \frac{4}{\check{x}}\,\check{K}_2\right) \,, \tag{76}$$

in which the only dependence on the internal structure is that embodied in the dimensionless number $\check{\epsilon}$ which can be seen to be given in terms of the boundary values $\check{\chi}$ and $\check{\zeta}$ of the functions χ and ζ by the simple formula

$$\check{\epsilon} = 2\check{\chi} - \check{\zeta} \,. \tag{77}$$

This quantity $\check{\epsilon}$ can be interpreted as the value at the core boundary $x = \check{x}$ of a function ϵ of x given by

$$\epsilon = 2\frac{\check{\chi}}{\check{x}} x - \zeta \,, \tag{78}$$

in terms of which the solution for the potential difference $\mathcal{A}$ will be given by

$$\mathcal{A} = \frac{mC}{2q}\big(\check{K}_0\, x + \frac{\check{x}^2}{4}\check{K}_1\,\epsilon\big) \,. \tag{79}$$

The corresponding expression for the magnetic field excess will have the form

$$\mathcal{B} = \frac{mC}{q\lambda}\big(\check{K}_0 + \frac{\check{x}}{2}\check{K}_1(\check{\chi} - \chi)\big) \,. \tag{80}$$

Using the solution (75) for C, the central value needed for the energy formula (39) can be seen to be obtainable as

$$\mathcal{B}_0 = \frac{N\hbar}{q\lambda^2\check{x}^2}\Big(2 - \frac{\check{x}\check{K}_1(8 - \check{\zeta}\check{x}^2)}{4\check{\mathcal{K}}}\Big) \,. \tag{81}$$

VIII. EXPLICIT ENERGY CONTRIBUTIONS

Using the axis–field energy formula (39) together with the solution (81) for the axis–field $\mathcal{B}_0$, we can immediately obtain the total "dynamical" energy $U_{\rm dyn} \equiv U_{\rm mag} + U_{\rm kin} = \widehat{U}_{\rm mag} + \widehat{U}_{\rm kin}$ (which is the total energy less the condensation energy):

$$U_{\rm dyn} = U_0\,\frac{2\check{K}_0 + \check{x}\check{\chi}\check{K}_1}{2\check{\mathcal{K}}}, \tag{82}$$

where

$$U_0 \equiv \left(\frac{N\phi}{4\pi\lambda}\right)^2 . \qquad (83)$$

Using the external solution for $\mathcal{B}$ and $\mathcal{V}$, one can obtain the external magnetic and kinetic energy contributions, defined in (27) and (31), in the form

$$\widehat{U}^{\rm ext}_{\rm mag} = \left(\frac{mC}{2q}\right)^2 \frac{\check{x}^2}{2} \left(\check{K}_1^2 - \check{K}_0^2\right) \qquad (84)$$

and

$$\widehat{U}^{\rm ext}_{\rm kin} = \left(\frac{mC}{2q}\right)^2 \frac{\check{x}^2}{2} \left[\frac{2}{\check{x}}\,\check{K}_0\,\check{K}_1 - \left(\check{K}_1^2 - \check{K}_0^2\right)\right], \qquad (85)$$

where the core structure dependence is contained exclusively in the constant C. This constant C, using the solution (75), can be related to the (core structure independent) constant U_0 defined just above by the simple relation

$$\left(\frac{mC}{2q}\right)^2 = \frac{U_0}{\check{\mathcal{K}}^2}, \qquad (86)$$

where, obviously, all the dependence on the core structure is contained in the dimensionless term $\check{\mathcal{K}}$.

As a consequence, the total external energy contribution can be rewritten in the simple form

$$\widehat{U}^{\rm ext} = \frac{\check{x}\,\check{K}_0\,\check{K}_1}{\check{\mathcal{K}}^2}\, U_0, \qquad (87)$$

which can be decomposed in two similar expressions for the magnetic and kinetic contributions. The sum of the magnetic and kinetic *internal* contributions can then of course be obtained by using the relation $\widehat{U}^{\rm int} = U^{\rm dyn} - \widehat{U}^{\rm ext}$.

IX. CONCLUSIONS

Let us summarize the results of the present work. We have first shown that the contributions linearly dependent on Ω_∞ in the magnetic and kinetic energies *cancel* each other. As a consequence, we find that the effective magnetic field strength H is simply the London field. It is to be observed that the extra energy density contribution arising from the second term in (26) would give an extra contribution of the form $B_\infty \langle w\rangle / 4\pi q$ in (53). By including this extra term – overlooking the fact that, according to (33), it will be canceled by the second term in the kinetic contribution (30), which was not taken into account – the analysis by Mendell[1] provided the erroneous conclusion that there would be no polarization, or in other words that H should be identified not with the London field but simply with the mean induction, meaning the replacement of (56) by $H = \langle B\rangle$.

The identification (56) of H, as given by the conventional definition (55), with the asymptotic London field B_∞ has been established here as a precise mathematical relation in the framework only of a particularly simple model. The problem of generalization to more sophisticated models, allowing for compressibility, relativistic effects and other relevant complications, remains to be dealt with in future work.

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