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On Superpotentials for D-Branes in Gepner Models

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Abstract

A large class of D-branes in Calabi-Yau spaces can be constructed at the Gepner points using the techniques of boundary conformal field theory. In this note we develop methods that allow to compute open string amplitudes for such D-branes. In particular, we present explicit formulas for the products of open string vertex operators of untwisted A-type branes. As an application we show that the boundary theories of the quintic associated with the special Lagrangian submanifolds $\Im \omega_i z_i = 0$ where $\omega_i^5 = 1$ possess no continuous moduli.

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1 Introduction

It has been known for a long time [1, 2, 3] that certain rational $\mathcal{N} = 2$ conformal field theories, the so-called Gepner models, can be used to describe closed strings in the small volume regime of geometrical Calabi-Yau compactifications. More recently, it was attempted to extend this correspondence to the open string sector. This discussion was initiated in [4] through a comparison of D-branes on the quintic to the rational boundary theories for the Gepner model $(k = 3)^5$ obtained in [5]. The results were then extended to other models in [6, 7, 8, 9, 10]. Significant progress has been made in matching B-type boundary states with the brane spectra at large volume [11, 10, 12]. Related work using Landau-Ginzburg and gauged linear sigma model methods can be found in [13, 14, 15, 16].

Until now, most of the conformal field theory results deal with the construction of boundary states at Gepner points [5, 17, 18, 19, 14]. These states allow to compute the open string partition functions and thereby to determine the spectrum of operators in the small volume limit. However, we can only get insight into the world-volume theory on the brane if we know how the world-volume fields interact. This information is encoded in operator product expansions (or correlation functions) of vertex operators at the boundary of the 2-dimensional world-sheet. The aim of the present paper is to provide the tools that are needed to compute correlation functions of open string vertex operators for a large set of rational branes in Gepner models. We are going to concentrate on A-type boundary states, but our method can be extended to the B-type situation as well.

Superpotentials for the massless fields are of particular interest because they determine the moduli spaces of branes. There exists a powerful theorem due to MacLean, which states that at the large volume point, all moduli of A-type branes are unrestricted. When rephrased in the framework of CFT the theorem claims that all marginal boundary operators are in fact truly marginal. According to [20, 21], the theorem breaks down once we are forced to take world-sheet instantons into account. Hence, when we pass to smaller volumes we expect to find restricted moduli. Clearly, this is to be expected from a general world-sheet point of view [22]. Below we shall demonstrate such a restriction through an explicit example. In fact, there exist boundary theories for the quintic supporting a massless field at the Gepner point which is not present at large volume [4]. We shall apply our general techniques to these boundary theories and show that there is no continuous modulus associated with this massless mode.

Beyond the investigation of truly marginal operators and continuous moduli, computations of correlation functions are also essential for the analysis of unstable brane configurations and the identification of their bound states. Decays caused by tachyon condensation have been discussed a lot in the recent literature (see e.g. [23, 24]). There exist other scenarios where gauge fields on a stack of branes condense [25]. Our understanding of such condensation phenomena in non-trivial compactifications is still very limited (but see [26, 10, 12] for some results in this direction).

This paper is organized as follows: We are starting out with a brief review of the general structure of open string vertex operators and their correlation functions in $\mathcal{N} = 2$ superconformal field theories (Section 2). Then we turn to a more detailed analysis of boundary conformal field theories describing the internal sector of a Gepner compactification. The first step is performed in Section 3 and it involves some general results dealing with correlation functions of open string vertex operators on a particular class of orbifolds. Section 4 contains a more technical computation of the fusing matrix of $\mathcal{N} = 2$ minimal models. The results of Section 4 are then combined with the general analysis from Section 3 to determine the operator product expansion of open string vertex operators for A-type branes in Gepner models. After a brief background review on branes in Gepner models we will present the main formulas of this text in Section 5.3. Finally, in Section 6, we shall apply our formalism to certain boundary states of the quintic. The aim here is to show that the moduli spaces at large and small radii do agree after superpotential terms have been taken into account. Note that this is not guaranteed by the decoupling conjecture of [4] because the latter only makes statements about the (in)dependence of B-type moduli spaces on the Kähler modulus.

2 Open string correlators in $\mathcal{N} = 1$ theories.

Let us consider a BPS brane placed in a $\mathcal{N} = 2$ compactification of type II string theory. It is well known that this scenario leads to a 4-dimensional $\mathcal{N} = 1$ supersymmetric field theory on the world-volume of the brane, provided the brane extends in all the non-compact directions. The effective field theories may contain a number of massless chiral and vector superfields. The former are composed from a scalar ϕ, a fermionic component ψ and an auxiliary field F. Vector superfields consist of a vector v, a fermionic field λ and an auxiliary component D. In principle, both D- and F-terms can contribute to the potential for scalar fields. Here we are interested in computing superpotentials for the chiral world volume fields. Hence, the leading non-linear term in the corresponding superpotential is of the form $\phi\psi\psi$ or, equivalently, $F\phi\phi$.

To compute these terms from conformal field theory, we have to assign vertex operators to all the fields we are interested in. In particular, the world-volume scalars ϕ^a [1] are represented by vertex operators in the Neveu-Schwarz (NS) sector. Within the (-1) picture, they are given by

$$\mathcal{V}^{(-1)}_{\phi^a}(x) \;=\; e^{-\Phi(x)}\,\psi^a_\phi(x) \quad .$$

Here, Φ denotes the bosonized superconformal ghost and ψ^a_ϕ is a NS operator of the internal theory. Both factors have dimension 1/2 in the case that the scalar is massless.

For world-volume fermions one uses Ramond vertex operators. When we write them in their canonical $(-1/2)$ picture, they are of the form

$$\mathcal{V}^{(-1/2)}_{\psi^a}(x) \;=\; \xi^{a,\alpha}_i\, S_\alpha(x) e^{-\frac{1}{2}\Phi(x)}\,\Sigma^{a,i}(x) \quad ,$$

where $\xi^{a,\alpha}_i$ describes the polarization, S^α is the space-time part of the spin field which has dimension 1/4. The dimension 3/8 fields $\Sigma^i(x)$ are associated to Ramond ground states of the internal sector.

Vector fields v come with the NS-sector again. Their vertex operators have the identity field in the internal part,

$$\mathcal{V}^{(-1)}_{v}(x) \;=\; \xi_\mu\, e^{-\Phi(x)}\,\psi^\mu(x) \quad .$$

ψ^μ is a vector of world-sheet fermions and ξ describes the polarization. One may add Chan-Paton matrices to all three vertex operators we have described. After this extension, the vector fields can give rise to non-abelian gauge fields.

The factors $\psi^a, \Sigma^{a,i}$ which the internal part contributes to the vertex operators, carry the label a depending on the pair of boundary conditions we consider for the two ends of open strings. We shall be much more specific about these labels later when we turn to concrete models.

Before spelling out which correlators we want to compute, we would like to construct the space-time supersymmetry generators $Q_\alpha(x)$. To this end, let us introduce a free bosonic field X for the internal $U(1)$ current $J(x)$ of the superconformal algebra,

$$J(x) \;=\; i\sqrt{\frac{c}{3}}\partial X(x) \quad ,$$

X appears together with the space-time spin field $S^\alpha(x)$ in the following formula for Q_α

$$Q^{(\pm 1/2)}_\alpha(x) \;=\; e^{\pm\frac{\Phi(x)}{2}} e^{\pm\frac{i}{2}\sqrt{\frac{c}{3}}X(x)} S_\alpha(x) \tag{1}$$

[1]We shall use the superscript a to distinguish between different massless chiral superfields

The exponential $\exp(\pm i\eta\sqrt{c/3}X(x))$ generates a spectral flow by η units in the internal part of the theory. Hence, the formula (1) for the supersymmetry generators Q_α contains the spectral flow operator for a flow by 1/2 units. Note that the application of Q_a to the vertex operator for scalars gives the vertex operator of a fermionic field because the spectral flow by 1/2 unit maps the NS- into the R-sector. One unit of spectral flow (and hence an action of two space-time supersymmetry generators) is needed to construct the vertex operator of the auxiliary field F^a from the vertex operator for the scalar ϕ^a.

In computing the world-sheet correlators we have to respect two simple rules. First the total Φ-charge of fields inside the correlation function must add up to -2. If necessary, we have to use the vertex operators in pictures different from the canonical ones we spelled out above. The second rule states that three of the vertex operators should be multiplied with the anti-commuting ghost fields c. After this, one has to integrate over the world-sheet arguments of the boundary fields and to sum over arbitrary permutations, as usual.

Let us give an explicit formula for the correlator that computes the leading non-linear contribution $\phi\psi\psi$ to the superpotential for the scalar ϕ. In this case, we have to compute a 3–point function and we can use all three vertex operators in their canonical pictures since their Φ charge add up to -2,

$$\phi\psi\psi \ : \qquad \langle c(x_1)\mathcal{V}_\phi^{(-1)}(x_1)\ c(x_2)\mathcal{V}_\psi^{(-1/2)}(x_2)\ c(x_3)\mathcal{V}_\psi^{(-1/2)}(x_3)\rangle \ + \ 2 \leftrightarrow 3 \ . \qquad (2)$$

Similarly, one could compute the $F\phi\phi$ term by applying one full unit of spectral flow (and of Φ-charge) to the vertex operator for scalars. The computation of such correlation functions factors into several independent parts. Contributions from the ghosts c, the space-time spin fields S_α and the superconformal ghosts Φ are evaluated easily since this involves only calculations in a free field theory. For this reason, we shall focus mainly on the correlation functions of the fields $\psi^a, \Sigma^{a,i}, X$ in the internal sector.

Most Calabi-Yau compactifications are described by an interacting internal conformal field theory which makes the calculations rather difficult. But we can at least illustrate the computations we have in mind through one simple example where the internal part is given by a free field theory, too. Namely, we consider D3 branes in a torus compactification. Their world-volume theory is an $\mathcal{N} = 4$ gauge theory. In $\mathcal{N} = 1$ language, it contains three chiral superfields Z^a. In this case, the superpotential is known to be of the form $W = \mathrm{tr}\ Z^1[Z^2, Z^3]$. Here the trace is over the Chan-Paton matrices which are non-trivial if we stack several D3 branes. The superpotential W can easily be reproduced by a world-sheet computation. To this end, we take three vertex operators for world volume scalars:

$$\mathcal{V}^{(-1)}_{\phi^a}(x) \;=\; e^{-\Phi(x)}\psi^a_\phi(x) \quad , \tag{3}$$

where the $\psi^a_\phi = \psi^a$ are the three (fermionic) chiral primary fields of charge one that exist on a 3 complex dimensional torus. According to our general discussion we need to add one unit of spectral flow to one of the fields in the internal theory (or, alternatively, two 1/2 units to two fields). In the case at hand, the spectral flow operator is $(\psi^a\psi^b\psi^c)^\dagger$. This leads to a three-point function that is proportional to the antisymmetric tensor ϵ_{abc} reproducing the known result. Note that the string amplitude vanishes for a single brane.

The last observation raises the question, whether the allowed superpotential terms on a single brane vanish generically after summing over all permutations of the insertion points at the boundary of the world-sheet. But we learn from the relation between open string theory and non-commutative geometry [27, 28, 29, 30] that the products of boundary operators in a conformal field theory (such as the Gepner models) can depend very much on the order in which fields are multiplied. Only a small number of fields, namely those which can be analytically continued into the bulk, are protected against such effects. This happens, for instance, in case of the free fermionic fields ψ^a which appeared in our discussion of $D3$ branes in a torus compactification. Similar arguments may apply to marginal operators built from fermionic fields in the large volume limit, but they are expected to break down when we get into the small volume regime. The aim of this paper is to substantiate such expectations by rigorous conformal field theory computations at the Gepner point.

3 Boundary conformal field theory on orbifolds

In this section we present some basic material relevant to the investigation of D-branes on orbifolds. Under certain simplifying assumptions on the nature of the orbifold action and on the D-branes under consideration, we shall present a general formula for the operator product expansion of boundary operators.

3.1 Boundary conformal field theory on the covering space

To begin with, let us review the necessary input from Cardy's work. Suppose we are given some bulk conformal field theory with chiral algebra $\mathcal{A}$ and a modular invariant partition function of the form

$$Z(q,\bar{q}) \;=\; \sum_j \chi_j(q)\,\chi_{\bar{j}}(\bar{q}) \quad . \tag{4}$$

Here j runs through the sectors of $\mathcal{A}$ and $\bar{j}$ is just another sector that appears together with j in the partition function. $\chi_i(q)$ denotes the character that comes associated with the sector i of the chiral algebra.

It was explained in [5] that the construction of boundary theories involves picking some automorphism Ω of the chiral algebra. This appears in the boundary conditions to describe how left- and right movers are glued along the boundary. Any such automorphism Ω induces a map ω that acts on sectors i of the chiral algebra. Cardy's analysis of boundary conditions applies whenever $\omega(j)^\vee = \bar{j}$. Here $i^\vee$ denotes the sector conjugate to i, i.e. the unique label with the property that its fusion product with j contains the vacuum representation 0 of the chiral algebra. We will call such a modular invariant (Ω)-diagonal.

Under this condition, Cardy provides us with a list of boundary theories. Their number agrees with the number of sectors of $\mathcal{A}$. We will use labels $I, J, K, \dots$ to distinguish between boundary conditions and sectors but it should be kept in mind that small and capital letters run through the same index set. The spectrum of open strings that stretch between the branes that are associated with the labels I and J is given by

$$Z_{IJ}(q) \;=\; \sum_j {N_{Ij}}^J\,\chi_j(q) \quad . \tag{5}$$

Obviously, this tells us how the state space $\mathcal{H}_{IJ}$ of the boundary theory is built up from sectors of the chiral algebra. For a much more detailed explanation of these results the reader is referred to [5].

There is a version of the state-field correspondence in boundary conformal field theory that assigns a boundary field to each state in $\mathcal{H}_{IJ}$. Hence we can read off from (5) that the boundary primary field ψ_j appears with multiplicity ${N_{Ij}}^J$ in the boundary theory. The operator product expansion for two such primary fields is given by

$$\psi_i^{LM}(x_1)\,\psi_j^{MN}(x_2) \;=\; \sum_k (x_1-x_2)^{h_i+h_j-h_k}\,\psi_k^{LN}(x_2)\,F_{Mk}\big[\begin{smallmatrix} i & j \\ L & N\end{smallmatrix}\big] + \dots \quad \text{for } x_1 < x_2, \tag{6}$$

where F stands for the fusing matrix of the chiral algebra $\mathcal{A}$. It is defined as a linear transformation that relates two different orthonormal bases in the space of conformal blocks (see [31] and Subsection 4.2 below) and it can be visualized through our Figure 1.

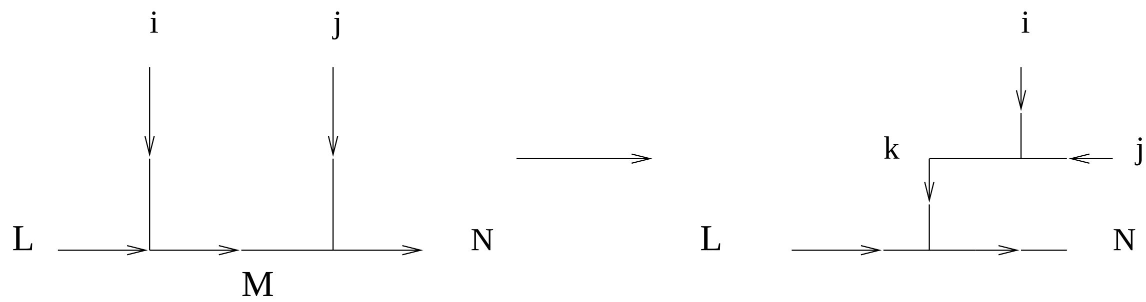

Figure 1: Graphical description of the fusing matrix.

The formula (6) was originally found for minimal models by Runkel [32] and extended to more general cases in [33, 34, 35]. Geometrically, boundary operator products describes the scattering of two open strings which are stretched between the branes L, M and M, N, respectively, into an open string that stretches between L and N.

Note that for the relation between the coefficients of the boundary OPE and the fusing matrix it is crucial that boundary conditions and boundary fields are labeled with elements from the same set. This is no longer true for models with a 'non-diagonal' (in the sense specified above) bulk modular invariant partition function. We shall see below how this can affect the boundary operator product expansions. The first examples of boundary OPEs for non-diagonal modular invariants were studied by Runkel in [36].

3.2 Boundary conditions for the orbifold

Suppose now that we want to discuss D-branes on an orbifold of the original conformal field theory. Geometrically, one would like to understand these branes on the orbifold space through D-branes on the covering space. In such an approach, a brane on the orbifold gets represented by several pre-images on the covering space which are mapped onto each other by the action of the orbifold group. As we discussed in [18], there is a large class of cases in which these geometric ideas carry over to the construction of branes in exactly solvable conformal field theories.

Our main assumption is that the orbifold action is induced by simple currents of the conformal field theory. Before we make this more precise, let us introduce some notations. Primaries (or the associated conformal families) of a conformal field theory form a set $\mathcal{J}$. Within this set $\mathcal{J}$ there can be non-trivial elements $g \in \mathcal{J}$ such that the fusion product of g with any other $j \in \mathcal{J}$ gives again a single primary $g \times j = gj \in \mathcal{J}$. Such elements g are called *simple currents* and the set $\mathcal{C}$ of all these simple currents forms an abelian subgroup $\mathcal{C} \subset \mathcal{J}$. The product in $\mathcal{C}$ is inherited from the fusion product of representations. From now on, let us fix some subgroup $\Gamma \subset \mathcal{C}$.

Through the fusion of representations, the index set $\mathcal{J}$ comes equipped with an action $\Gamma \times \mathcal{J} \to \mathcal{J}$ of the group Γ on labels $j \in \mathcal{J}$. Under this action, $\mathcal{J}$ may be decomposed into orbits. The space of these orbits will be denoted by $\mathcal{J}/\Gamma$ and we use the symbol $[j]$ to denote the orbit represented by $j \in \mathcal{J}$. These orbits may have *fixed points*, i.e. there can be labels $j \in \mathcal{J}$ for which the following stabilizer subgroup $\mathcal{S}_j \subset \Gamma$

$$\mathcal{S}_j \;=\; \{\; g \in \Gamma \;\mid\; g \cdot j \;=\; j \;\} \tag{7}$$

is nontrivial. Up to isomorphism, the stabilizer subgroups depend only on the orbits $[j]$ not on the choice of a particular representative $j \in [j]$, i.e. $S_j = S_[J]$.

The last object we have to introduce is the so-called *monodromy charge* $Q_g(j)$ of a primary j with respect to the simple current g. To this end we consider the following special matrix elements

$$\Omega\left(\begin{smallmatrix} & i & \\ k & & j \end{smallmatrix}\right) \;=\; B_{ik}\left[\begin{smallmatrix} i & j \\ 0 & k \end{smallmatrix}\right]$$

of the braiding matrix $B = B^{(+)}$ (see [31] for details). From these elements of the braiding matrix we inherit a map $\hat{Q}_g(j)$ defined by

$$(-1)^{\hat{Q}_g(j)} \;:=\Omega\left(\begin{smallmatrix} & j & \\ gj & & g \end{smallmatrix}\right) \quad . \tag{8}$$

Note that this specifies $\hat{Q}_g(j)$ up to an even integer, i.e. $\hat{Q}_g(j) \in \mathbb{R}/2\mathbb{Z}$. The monodromy charge $Q_g(j) := \hat{Q}_g(j)$ mod 1 is only defined up to integers. Note that the latter is given by the standard formula

$$Q_g(j) \;=\; h_j + h_g - h_{gj} \mod 1 \quad .$$

Let us remark that Q is conserved under fusion while this may not be the true for $\hat{Q}$. In case the simple currents have integer conformal weight, the monodromy charge $Q_g(j)$ depends only on the equivalence class $[j]$ of $j \in J$. An orbit $[j]$ is said to be invariant, if $Q_g([j]) = Q_g(j) = 0$ for all $g \in \Gamma$.

After this preparation, we can give a precise formulation of our main assumption on the partition function $Z^{\text{orb}}(q)$ of the bulk theory that we want to study. We assume that there exists some bulk theory to which Cardy's theory applies, an orbifold group Γ within the group of all simple currents such that $Z^{\rm orb}$ is of the form

$$Z^{\rm orb}(q) \;=\; \sum_{j, Q_\Gamma([j])=0} |\,\mathcal{S}_{[j]}\,|\;|\sum_{g\in\Gamma/\mathcal{S}_{[j]}} \chi_{gj}\,|^2 \quad . \tag{9}$$

Note that this partition function does not have the simple form (4) so that Cardy's theory for the classification and construction of D-branes does not apply directly.

As we discussed in [18], an orbifold theory with bulk partition function of the form (9) possesses consistent boundary theories which are assigned to orbits $[I]$ of labels I that parametrize the boundary theories of the parent CFT. The open string spectra associated with a pair of such brane on the orbifold are given by

$$Z^{\rm orb}_{[I][J]}(q) \;=\; \sum_{g,k} N_{I\,k}^{\;gJ}\chi_k(q) \quad . \tag{10}$$

This agrees precisely with the prediction from the geometric picture of branes on orbifolds. In fact, the I, J can be considered as geometric labels specifying the position of the brane on the covering space. To compute spectrum of two branes $[I]$ and $[J]$ of the orbifold theory, we lift $[I]$ to one of its preimages I on the covering space and include all the open strings that stretch between this fixed brane I on the cover and an arbitrary preimage gJ of the second brane $[J]$.

We should remark that in many cases the boundary conditions $[I]$ can be further resolved, i.e. there exists a larger set of boundary theories such that $[I]$ can be written as a linear combination of boundary theories with integer coefficients. This happens whenever the stabilizer subgroup $S_{[I]}$ is non-trivial (see [37, 38, 19, 18] for more details). Geometrically this corresponds to the fact that the CP-factors of branes at orbifold fixed points can carry different representations of the stabilizer subgroup.

3.3 Boundary OPE for the orbifold

Restricting to unresolved D-branes, it is relatively easy to give explicit expressions for the operator products of boundary fields. Before we spell them out, let us have another look at eq. (10) and observe that for fixed I, J, k there can be several group elements $g \in \Gamma$ such that $N_{I\,k}^{gJ} \neq 0$. We label these elements by a subscript ϵ. While the range for ϵ depends only on k and the orbits $[I], [J]$, the definition of the group elements $g_\epsilon = g_\epsilon(\,_I\,{}^k\,_J\,) \in \Gamma$

requires to fix representatives $I \in [I]$ and $J \in [J]$. If we shift these representatives along their orbits, the group elements behave according to

$$g_\epsilon(\,_I\,{}^k\,_{gJ}\,) \;=\; g^{-1}\, g_\epsilon(\,_I\,{}^k\,_J\,) \quad \text{and} \quad g_\epsilon(\,_{gI}\,{}^k\,_J\,) \;=\; g\, g_\epsilon(\,_I\,{}^k\,_J\,) \;\; .$$

The group elements g_ϵ appear in the following formula for the boundary operator product expansions in our orbifold theory,

$$\Psi^{[L][M]}_{i,\epsilon_1}(x_1)\; \Psi^{[M][N]}_{j,\epsilon_2}(x_2) \;=\; \sum_k\, (x_1 - x_2)^{h_i + h_j - h_k} \Psi^{[L][N]}_{k,\epsilon_{12}}(x_2)\; F_{g_1 M k}\big[\begin{smallmatrix} i & j \\ L & g_{12}N \end{smallmatrix}\big] \;+\; \dots \tag{11}$$

for $x_1 < x_2$. Here, L, M, N are representatives of the orbits $[L], [M], [N]$ and the group elements g_1, g_{12} in the fusing matrix F are given by

$$g_1 \;=\; g_{\epsilon_1}(\,_L\,{}^i\,_M\,) \quad , \quad g_{12} \;=\; g_{\epsilon_{12}}(\,_L\,{}^k\,_N\,) \;=\; g_{\epsilon_1}(\,_L\,{}^i\,_M\,) g_{\epsilon_2}(\,_M\,{}^j\,_N\,) \;\; .$$

Obviously, the expansions (11) in the orbifold theory are inherited from the operator products (6) of the theory on the covering space. In geometric terms we have singled out one of the preimages L of the branes $[L]$ and then described the scattering of open strings between strings on the orbifold through strings stretching between various preimages of the other two branes $[M], [N]$. This prescription is independent of the choices we have made, provided that the charge $\hat{Q}$ is conserved in the sense

$$\hat{Q}_g(i) + \hat{Q}_g(j) - \hat{Q}_g(k) \;=\; 0 \bmod 2 \quad \text{for all} \quad i, j, k \;\; \text{with} \;\; N^k_{ij} \neq 0 \;\; . \tag{12}$$

More precisely, one can show that under the assumption (12) our operator product expansions (11) obey the usual factorization (or *sewing*) constrains [39, 40, 32]. In the derivation one uses the following invariance of the fusing matrix

$$F_{gg_1Mk}\big[\begin{smallmatrix} i & j \\ gL & gg_{12}N \end{smallmatrix}\big] \;=\; F_{g_1 k}\big[\begin{smallmatrix} i & j \\ L & g_{12}N \end{smallmatrix}\big](-1)^{\hat{Q}_g(i) + \hat{Q}_g(j) - \hat{Q}_g(k)} \;\; .$$

It is possible to relax the assumptions and to generalize the formula for the operator product expansions but for our purposes, eq. (11) will turn out to suffice.

4 The $\mathcal{N} = 2$ superconformal minimal models

The $\mathcal{N} = 2$ minimal models are the basic building blocks for Gepner models. More precisely, the latter are obtained from a product of $\mathcal{N} = 2$ minimal models by orbifold techniques. Following the general strategy of the previous section it is therefore essential to understand the open string sector of $\mathcal{N}=2$ minimal models, including the OPE of the open string operators. As we have seen, the operator products are determined by the fusing matrix. It is the main aim of this section to compute fusing matrix of $\mathcal{N}=2$ minimal models. This will be achieved through the coset construction.

4.1 The coset construction for $\mathcal{N}=2$ minimal models

The $\mathcal{N}=2$ minimal models MM_k have a coset realization of the following form

$$\mathrm{MM}_k \;=\; \frac{\widehat{SU}(2)_k \times U_4}{U_{2k+4}} \tag{13}$$

where $\widehat{SU}(2)_k$ is a level k affine current algebra and U_{2N} stands for an extension of the $U(1)$ current algebra that is generated by the exponentials $W^{(2N)}_{\pm} \;=\; \exp(\pm i\sqrt{2N}X(z))$ along with the current $J^{(2N)}(z) = J(z)$. We will denote the generators of the $\widehat{SU}(2)_k$ current algebra by $E(z), F(z)$ and $H(z)$. As usual, $H(z)$ is associated with the Cartan subalgebra of $su(2)$ while $E(z)$ and $F(z)$ come with the raising and lowering operators, respectively. The space of ground states of the $\widehat{SU}(2)_k$ representations is spanned by vectors $|l,n\rangle, |n| \leq l$, which are eigenstates of the zero mode H_0 with eigenvalue n. The coset construction of g/g' requires to embed the denominator g' into the nominator g. In the case at hand this is achieved through the identification

$$J^{(2k+4)}(z) \;=\; H(z) + J^{(4)}(z) \quad .$$

The Virasoro field of the coset theory is then given by the difference $T^{g/g'}(z) = T^g(z) - T^{g'}(z)$ which has central charge $c^{g/g'} = c^g - c^{g'}$. These formulas specialize to

$$T^{\mathrm{MM}_k}(z) \;=\; T^{\widehat{SU}(2)_k}(z) + T^{U_4}(z) - T^{U_{2k+4}}(z) \tag{14}$$

$$\text{with} \qquad c^{\mathrm{MM}_k} = c^{\widehat{SU}(2)_k} \;+1 \;-1 \;= \frac{3k}{k+2} \quad . \tag{15}$$

Representations of the coset algebra can be realized on the representation spaces $\mathcal{H}_\lambda$ of the theory g in the nominator. The embedding of g' into g defines an action of the denominator on $\mathcal{H}_\lambda$. With respect to this action $\mathcal{H}_\lambda$ decomposes according to

$$\mathcal{H}_\lambda \;=\; \bigoplus_{\lambda'} \mathcal{H}_{\lambda\lambda'} \otimes \mathcal{H}_{\lambda'} \tag{16}$$

where $\mathcal{H}_{\lambda'}$ are sectors of the denominator g'. The spaces $\mathcal{H}_{\lambda\lambda'}$ which appear in this decomposition are acted upon by all fields which commute with the fields of the denominator theory and hence they carry representations of the coset algebra g/g'. In general, it is difficult to decide, which representations λ' of g' occur for given λ an which pairs of λ, λ' give rise to the same representation for g/g'. To determine such selection rules and identifications is rather difficult, in general, but simple currents can help with this task. In this approach one constructs the largest group Γ_{id} of integer weight simple currents from the theory $g \oplus g'$ such that all the pairs λ, λ' that appear in the sum (16) have vanishing monodromy charge $Q_g(\lambda) + Q_g(\lambda') = 0 \bmod 1$. Elements of Λ_{id} are called *identification currents* since their action on pairs λ, λ' generates orbits of identical representation spaces $\mathcal{H}_{\lambda\lambda'}$ for the coset theory. Additional complications arise, if not all of these orbits have equal length. The fixed point resolutions required in such cases have been analyzed in [41, 42]. We will not discuss this process here, since there are no such fixed points for $\mathcal{N} = 2$ minimal models.

The representations of the nominator in the cosets (13) are labeled by pairs (l, s) with $l = 0, 1, \ldots, k$ and $s = -1, 0, 1, 2$. Under fusion, the labels l obey the usual $\widehat{SU}(2)_k$ fusion rules while the four choices for s get identified with elements of $\mathbb{Z}_4$. Likewise, we enumerate the sectors of the denominator U_{2k+4} by $m = -(k+1), -k, \ldots, k+1, k+2$. They form the abelian group $\mathbb{Z}_{2k+4}$. Among the triples (l, m, s) labeling representations of $\widehat{SU}(2)_k \oplus U_4 \oplus U^*_{2k+4}$ there is the generator $\gamma = (l = k, m = k+2, s = 2)$ of the identification group $\Gamma_{\text{id}} \cong \mathbb{Z}_2$. It maps the element (l, m, s) to $\gamma(l, m, s) = (k-l, m+k+2, s+2)$. The following formula for the monodromy charge

$$Q_\gamma(l, m, s) \;=\; h_{(k,2,k+2)} + h_{(l,s,m)} - h_{(k-l,m+k+2,s+2)} \;=\; \frac{l+m-s}{2} \bmod 1 \tag{17}$$

encodes the selection rules through the the requirement $Q_\gamma = 0$ and hence it leads us to the decomposition

$$\mathcal{H}^{\widehat{SU}(2)_k}_l \otimes \mathcal{H}^{U_4}_s \;=\; \bigoplus_{m, l+m+s \text{ even}} \mathcal{H}^{\text{MM}_k}_{(l,m,s)} \otimes \mathcal{H}^{U_{2k+4}}_m \quad . \tag{18}$$

In addition, we recover the well-known field identification of $\mathcal{N} = 2$ minimal models from the action of the identification current γ. It states that $\mathcal{H}_{(l,m,s)}$ and $\mathcal{H}_{(k-l,m+k+2,s+2)}$ carry the same representation of the coset MM_k.

So far, our remarks on the coset construction have been fairly standard. But our construction of the fusing matrix requires additional input. Namely, we have to understand in detail, how the ground states $|l, m, s\rangle \otimes |m\rangle$ of $\mathcal{H}_{(l,m,s)} \otimes \mathcal{H}_m$ are realized within the representation spaces $\mathcal{H}_l \otimes \mathcal{H}_s$ of the nominator theory. We know from (18) that each sector (l, s) of the nominator theory must contain $k+2$ such ground states. To find these ground states we fix l, s and choose m such that $l+m+2$ is even. Within the subspace

$$\mathcal{H}_{ls}^{(m)} = (\mathcal{H}_l \otimes \mathcal{H}_s)^{(m)} := \{\psi \in \mathcal{H}_l \otimes \mathcal{H}_s \mid e^{\frac{2\pi i}{\sqrt{2k+4}} J_0^{(2k+4)}} \psi = e^{\frac{2\pi i m}{2k+4}} \psi \} \tag{19}$$

we search then for eigenstates $\psi_{ls}^{(m)}$ of $L_0^{\widehat{SU}(2)_k} + L_0^{U_4}$ with minimal eigenvalue.

Some of these states are easily identified. These are the states which are realized in terms of ground states of the nominator theory,

$$\psi_{l\,s}^{(m)} = |l, m, s\rangle \otimes |m\rangle = |l, n = m - s\rangle \otimes |s\rangle \tag{20}$$

where m is restricted by $|m - s| \leq l$. In this way we have realized all fields from the so-called standard range of $\mathcal{N} = 2$ minimal models,

$$l \leq k \ , \qquad |m - s| \leq l, \ , \ l + m + s \text{ even} \ . \tag{21}$$

For these fields, the following formula for their conformal weights holds exactly (not just up to an integer),

$$h_{(l,m,s)} = \frac{l(l+1) - m^2}{4(k+2)} + \frac{s^2}{8} \ .$$

But the $l+1$ states we have found do not exhaust the ground states of the denominator theory. Additional states can be constructed with the help of

$$E_{-1}^{\nu}|l, l\rangle \ , \ F_{-1}^{\nu}|l, -l\rangle \in \mathcal{H}_l \quad \text{for} \quad \nu = 1, 2, \ldots \ .$$

These states carry the charge $n = l + 2\nu$ or $n = -l - 2\nu$, respectively. When combined with appropriate states from $\mathcal{H}_s$ (not necessarily the ground state) they furnish all the ground states for the denominator theory of the coset (13). Spelling out the complete list is somewhat involved and since we do not need the explicit formulas in the following, we shall content ourselves with these sketchy remarks.

Before we conclude this subsection, let us briefly discuss some important simple currents of the $\mathcal{N} = 2$ minimal model. The fusion rules for (l, m, s) which we have described above imply that e.g. $(0, 1, 1)$ and $(0, 0, 2)$ are both simple currents. They are of special interest in the context of Gepner models and will be used to generate our simple current group Γ. $(0, 1, 1)$ is the spectral flow by $1/2$ unit and $(0, 0, 2)$ the world-sheet supersymmetry generator. $(0, 0, 2)$ is a simple current of order 2 and can be used to combine the world-sheet fields into supermultiplets. The order of the simple current $(0, 1, 1)$ is model dependent. To see this, we apply the current $2k+4$ times to the identity. This will lead us back to the identity whenever the level k is even. Since $(0,0,2)$ is nowhere on this orbit, $(0,0,2)$ and $(0,1,1)$ together generate the simple current group $\Gamma = \Gamma_k = \mathbb{Z}_{2k+4} \times \mathbb{Z}_2$ for even level k. When k is odd, however, we reach the field $(0,0,2)$ after $2k+4$ applications of $(0,1,1)$ and hence we have to apply the simple current $(0,1,1)$ another $2k+4$ times. In this case, the orbit contains the label $(0,0,2)$ and hence our orbifold group is $\Gamma = \Gamma_k = \mathbb{Z}_{4k+8}$ for odd k.

The orbits for the action of Γ_k on the set $\mathcal{J}$ depend once more on the parity of k. If k is odd, the orbifold group Γ_k acts freely so that all orbits have length $4k+8$. For even level k, however, we generate short orbits of length $2k+4$ whenever we start from a field (l,m,s) with $l = k/2$ because the label $l = k/2$ is invariant under field identification. The stabilizer for these short orbits is a subgroup $\mathbb{Z}_2 \subset \Gamma_k$.

4.2 The fusing matrix

As discussed in the previous section, the coset construction we will be using below involves two basic building blocks, one of them being the $\widehat{SU}(2)_k$ Kac-Moody algebra while the other is simply some abelian U_{2N}-theory. Here we shall briefly describe the fusing matrices of these two theories before we turn to the fusing matrix of $\mathcal{N} = 2$ superconformal minimal models.

Let us start with the discussion of $\widehat{SU}(2)_k$ to illustrate the basic steps that go into the construction of fusing matrices. The reader may profit from the more comprehensive treatment in [31].

To any three given labels l, l_s, l_t such that l_t is contained in the fusion product of l and l_s there is assigned an intertwiner

$$\phi(\,{}_{l_t}{}^{l}{}_{l_s})(z) : \; \mathcal{H}_l \otimes \mathcal{H}_{l_s} \; \rightarrow \; \mathcal{H}_{l_t}$$

that intertwines the natural action of the affine Kac-Moody algebra on the involved spaces. For $\mathcal{H}_l \otimes \mathcal{H}_{l_s}$ this action is through the so-called fusion product and it depends on the co-ordinate z. Let us pick an orthonormal basis $\{|l,\nu\rangle; \nu \in -2k+l+2\mathbb{N}^0\}$ of vectors in $\mathcal{H}_l$ such that $|l\rangle = |l,-l\rangle$ is primary. It will also be convenient below to reserve

$|l,n\rangle, |n| < 2k-l$ for the vectors

$$|l,\nu\rangle \;=\; \begin{cases} F_{-1}^{-\frac{1}{2}(l+\nu)}\,|l,-l\rangle & \text{for } \;\nu = -2k+l\;\ldots,-l-2 \\ |l,\nu\rangle & \text{for } \;\nu = -l,\ldots,l \\ E_{-1}^{\frac{1}{2}(\nu+l)}\;|l,\;l\rangle & \text{for } \;\nu = l+2,\ldots,2k+l \end{cases} \quad .$$

To each state $|l,\nu\rangle$ from the basis we assign a vertex operators by

$$\phi\big(\begin{smallmatrix} & l,\nu & \\ l_t & & l_s\end{smallmatrix}\big)(z) \;:=\; \phi\big(\begin{smallmatrix} & l,\nu & \\ l_t & & l_s\end{smallmatrix}\big)[|l,\nu\rangle;.](z):\;\mathcal{H}_{l_s}\;\rightarrow\;\mathcal{H}_{l_t}\quad .$$

These vertex operators are uniquely determined by their intertwining properties, up to a common normalization that we fix by requiring that

$$\langle l_t|\;\phi\big(\begin{smallmatrix} & l,l_s-l_t & \\ l_t & & l_s\end{smallmatrix}\big)(1)\;|l_s\rangle \;=\; 1 \quad . \tag{22}$$

Now we can finally define the fusing matrix through an operator product expansion of chiral vertex operators,

$$\phi\big(\begin{smallmatrix} & l_3,\mu & \\ l & & l_{12}\end{smallmatrix}\big)(z_1)\;\phi\big(\begin{smallmatrix} & l_2,\nu & \\ l_{12} & & l_1\end{smallmatrix}\big)(z_2) \;=\; \sum_{l_{23},\rho} F_{l_{12}l_{23}}\big[\begin{smallmatrix} l_2 & l_3 \\ l_1 & l\end{smallmatrix}\big]\;\phi\big(\begin{smallmatrix} & l_{23},\rho & \\ l & & l_1\end{smallmatrix}\big)(z_2)\;\langle l_{23},\rho|\phi\big(\begin{smallmatrix} & l_2,\nu & \\ l_{23} & & l_3\end{smallmatrix}\big)(z_{12})|l_3,\mu\rangle \quad ,$$

where $z_{12} = z_1 - z_2$. Explicit formulas for the fusing matrix can be found in the literature. One can also compute the 3-point functions on the right hand side. When all the involved states $|l_2,\nu\rangle, |l_3,\mu\rangle$ and $|l_{23},\rho\rangle$ are taken from the lowest energy subspaces(i.e. $|\nu| \leq l_2, |\mu| \leq l_3$ and $|\rho| \leq l_{23}$), these amplitudes can be expressed in terms of Clebsch-Gordan coefficients for the finite dimensional Lie algebra $su(2)$,

$$\langle l_{23},n_{23}|\;\phi\big(\begin{smallmatrix} & l_2,n_2 & \\ l_{23} & & l_3\end{smallmatrix}\big)\;|l_3,n_3\rangle \;=\; \big[\begin{smallmatrix} l_3 & l_2 & l_{23} \\ n_3 & n_2 & n_{23}\end{smallmatrix}\big] \qquad \text{with} \qquad \big[\begin{smallmatrix} l_3 & l_2 & l_{23} \\ -l_3 & l_3-l_{23} & -l_{23}\end{smallmatrix}\big] \;=\; 1 \quad . \tag{23}$$

The normalization condition for the Clebsch-Gordan coefficients is required to match with the corresponding condition (22) for chiral vertex operators. Relation (23) implies that the coefficient of the leading singularity in the operator product expansion of two primary fields is given by the fusing matrix.

The algebras U_{2N} are much simpler to describe. They have central charge $c = 1$ and sectors labeled by $m = -N+1,\ldots,N$ with primaries of conformal dimension $h_m = m^2/4N$. The fusion rules are just given by the composition in $\mathbb{Z}_{2N}$. We will usually assume that our labels m run through the allowed range $-N+1 \leq m \leq N$. When two such integers m_1 and m_2 are added, their sum does not necessarily lie in the same range.

To cure this problem it will be useful to work with another sum $\hat{+}$ which is defined such that $m_1 \hat{+} m_2$ is the unique integer between $-N+1$ and N such that $m_1 \hat{+} m_2 = m_1 + m_2$ mod $2N$.

Again, we are mainly interested in the fusing matrix of U_{2N}. This time we can compute it in detail using the general recipes we have sketched before. The basic fields in the chiral algebra of the U_{2N}-theory may be obtained from a single free bosonic field through the expression

$$W_\nu(z) \ := \ :e^{i\nu\sqrt{2N}X(z)}: \qquad \text{where} \qquad \nu \in \mathbb{Z}$$

Using the familiar OPE of such normal ordered exponentials it is easy to see that these fields are local with respect to each other, i.e. that their correlation functions are meromorphic throughout the complex plane.

Similarly, we would like to realize the vertex operators in terms of normal ordered exponentials. This is possible by means of the formula

$$\phi\left(\begin{smallmatrix} & m,\mu & \\ m_t & & m_s \end{smallmatrix}\right)(z) \ := \ :e^{i\frac{m+\mu 2N}{\sqrt{2N}}X(z)}: \ (-1)^{\frac{m(\mathsf{p}-m_s)}{2N}}|_{\mathcal{H}_{m_s}} \quad , \tag{24}$$

where $\mathsf{p} = J_0\sqrt{2N}$ is obtained from the zero mode of the current $J(z) = i\partial X(z)$ by a simple rescaling. The vertex operator is restricted to the subspace $\mathcal{H}_{m_s}$ in the full state space of the free bosonic field which carries the m_s-representation of the U_{2N} algebra. Formally, it can be defined on an eigenspace of the operator $\exp(2\pi i\mathsf{p}/2N)$ with the eigenvalue $\exp(2\pi i m_s/2N)$. Note that by construction the operators defined in eq. (24) provide maps between different sectors of the U_{2N}-theory. The p-dependent factor was introduced to guarantee that the operators (24) have trivial braiding with respect to all the chiral fields $W_\nu(z)$, thereby turning them into true vertex operators. Setting $\mu = 0$ in eq. (24), we obtain primaries with respect to the extended chiral algebra. All other vertex operators in eq. (24) are U_{2N}-descendants of primary fields. Finally, we have subtracted the integer m_s from p to ensure that the transition amplitudes satisfy the normalization condition

$$\langle m_t|\ \phi\left(\begin{smallmatrix} & m,\mu & \\ m_t & & m_s \end{smallmatrix}\right)(1)\ |m_s\rangle \ = \ \delta_{m_t, m+\mu 2N+m_s} \quad .$$

This is analogous to the normalization condition we have imposed on the 3-point functions in the $SU(2)$-theory above.

The computation of the operator product expansion of any two such vertex operators (24) is rather straightforward and it gives the following fusing matrix,

$$F_{m_{12}m_{23}}\left[\begin{smallmatrix} m_2 & m_3 \\ m_1 & m \end{smallmatrix}\right] \ = \ (-1)^{\frac{m_3}{2N}(m_1+m_2-m_{12})}\ \delta_{m_{12},m_1\hat{+}m_2}\ \delta_{m_{23},m_2\hat{+}m_3}\ \delta_{m,m_{12}\hat{+}m_3} \quad . \tag{25}$$

One can also use the vertex operators (24) to compute the braiding matrix matrix of U_{2N}-theory and then check that these objects satisfy all the polynomial equations of [31]. Our solution of these equations actually differs from the one described in Appendix E of [31] by a 'gauge transformation' with $\lambda(i,j) = (-1)^{ij}$.

Let us now turn towards our ultimate goal to determine the fusing matrix of the $\mathcal{N} = 2$ superconformal algebra. For the latter, we will be using its $\widehat{SU}(2)_k \times U_4/U_{2k+4}$-coset presentation discussed before.

The construction requires to find the vertex operators of the $\mathcal{N} = 2$ minimal models. Their products with the vertex operators of the U_{2k+4} theory in the denominator can be realized on the spaces $\mathcal{H}^{(m)}_{ls}$ introduced in eq. (19),

$$\phi(\,{}^{(l,s,m)}_{(l_t,s_t,m_t)\;(l_s,s_s,m_s)}\,)\;\phi(\,{}^{\;\;m}_{m_t\;m_s}\,)\;:\;\mathcal{H}^{(m_s)}_{l_s\,s_s}\;\to\;\mathcal{H}^{(m_t)}_{l_t\,s_t}\quad . \tag{26}$$

Using the explicit embedding of ground states of $\mathrm{MM}_k \oplus U_{2k+4}$ into the representation space $\mathcal{H}_l \otimes \mathcal{H}_s$ that we sketched in the previous subsection, we decompose the operators (26) into products of vertex operators for the nominator theory,

$$\phi(\,{}^{(l,s,m)}_{(l_t,s_t,m_t)\;(l_s,s_s,m_s)}\,)\phi(\,{}^{\;\;m}_{m_t\;m_s}\,) \;=\; \begin{vmatrix} l_s & l & l_t \\ m_s & m & m_t \\ s_s & s & s_t \end{vmatrix}^{-1} \phi(\,{}^{\;\psi^{(m)}_{ls;1}}_{l_t\;\;\;l_s}\,)\phi(\,{}^{\;\psi^{(m)}_{ls;2}}_{s_t\;\;\;s_s}\,) \tag{27}$$

$$\text{where}\qquad \begin{vmatrix} l_s & l & l_t \\ m_s & m & m_t \\ s_s & s & s_t \end{vmatrix} \;:=\; \langle\psi^{(m_t)}_{l_t s_t}|\phi(\,{}^{\;\psi^{(m)}_{ls;1}}_{l_t\;\;\;l_s}\,)\phi(\,{}^{\;\psi^{(m)}_{ls;2}}_{s_t\;\;\;s_s}\,)|\psi^{(m_s)}_{l_s s_s}\rangle\quad .$$

Here, $\psi^{(m)}_{l\,s} \in \mathcal{H}_l \otimes \mathcal{H}_s$ denotes the vector introduced in the text after eq. (19). This vector can be split into a product $\psi^{(m)}_{ls;1} \in \mathcal{H}_l$ and $\psi^{(m)}_{ls;2} \in \mathcal{H}_s$ of components in the representations spaces of the $\widehat{SU}(2)_k$ and the U_4 algebras, respectively. If all the three triples $(l,m,s), (l_s,m_s,s_s), (l_t,m_t,s_t)$ are in the standard range (21), the normalization in eq. (27) can be spelled out more explicitly with the help of eq. (23),

$$\begin{vmatrix} l_s & l & l_t \\ m_s & m & m_t \\ s_s & s & s_t \end{vmatrix} = \left[\begin{smallmatrix} l_s & l & l_t \\ n_s & n & n_t \end{smallmatrix}\right]\quad . \tag{28}$$

Here $n_\alpha = m_\alpha - s_\alpha$. We can now employ the formula (27) to compute the fusing matrix of the superconformal algebra in terms of the fusing matrices of the building blocks of the coset construction (13). It is given by

$$F_{(l_{12},m_{12},s_{12})(l_{23},m_{23},s_{23})}\left[\begin{smallmatrix} (l_2,m_2,s_2) & (l_3,m_3,s_3) \\ (l_1,m_1,s_1) & (l,m,s) \end{smallmatrix}\right] \;=\; (-1)^{\frac{s_3}{4}(s_1+s_2-s_{12})}\;\delta_{s_{12},s_1\hat{+}s_2}\;\delta_{s_{23},s_2\hat{+}s_3}\;\delta_{s,s_{12}\hat{+}s_3}$$

$$(-1)^{-\frac{m_3}{2k+4}(m_1+m_2-m_{12})}\ \delta_{m_{12},m_1\hat{+}m_2}\ \delta_{m_{23},m_2\hat{+}m_3}\ \delta_{m,m_{12}\hat{+}m_3}\ F_{l_{12}l_{23}}\left[\begin{smallmatrix} l_2 & l_3 \\ l_1 & l \end{smallmatrix}\right] \frac{\begin{vmatrix} l_3 & l_2 & l_{23} \\ m_3 & m_2 & m_{23} \\ s_3 & s_2 & s_{23} \end{vmatrix}\begin{vmatrix} l_1 & l_{23} & l \\ m_1 & m_{23} & m \\ s_1 & s_{23} & s \end{vmatrix}}{\begin{vmatrix} l_1 & l_2 & l_{12} \\ m_1 & m_2 & m_{12} \\ s_1 & s_2 & s_{12} \end{vmatrix}\begin{vmatrix} l_{12} & l_3 & l \\ m_{12} & m_3 & m \\ s_{12} & s_3 & s \end{vmatrix}}.$$

If all the triples are in the standard range, then one can use eq. (28) to simplify this expression. However, in general it is not possible to bring all the involved triples into the standard range in a way that is consistent with the fusion rules and the factorization into the $SU(2)$ and $U(1)$ contributions. This is because the triples $(l, m_1\hat{+}m_2, s_1\hat{+}s_2)$ with $l = |l_1 - l_2|, \ldots, \min(l_1 + l_2, 2k - l_1 - l_2)$ need not be in the standard range even if (l_1, m_1, s_1) and (l_2, m_2, s_2) are.

4.3 The OPE in a single $\mathcal{N} = 2$ minimal model

In passing we would like to write down the boundary operator product expansions for a single minimal model for the case where Cardy's analysis applies, i.e. with the bulk modular invariant given by charge conjugation. As pointed out in Section 3, there exists a scaling of the boundary fields such that the OPE coefficients of the boundary OPE are equal to the fusing matrix. Hence, together with the results of the previous subsection we know all boundary structure constants. The expressions simplify, if we rescale all the boundary fields to absorb some of the normalizations defined in eq. (27). In formulas, the appropriate rescaling of the boundary states is given by

$$\psi^{(LMS)(L'M'S')}_{(l,m,s)} \mapsto \psi^{(LMS)(L'M'S')}_{(l,m,s)} \begin{vmatrix} L' & l & L \\ M' & m & M \\ S' & s & S \end{vmatrix}^{-1} \tag{29}$$

For these rescaled vertex operators, the operator product expansion (6) then takes the form

$$\psi^{(LMS)(L'M'S')}_{(lms)}(x_1)\ \psi^{(L'M'S')(L''M''S'')}_{(l'm's')}(x_2) \ = \sum_{l'',m'',s''} (x_1 - x_2)^{h+h'-h''} \begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ s & s' & s'' \end{vmatrix} \times$$
$$F_{l''L}\left[\begin{smallmatrix} l & l' \\ L & L'' \end{smallmatrix}\right] F_{s''S}\left[\begin{smallmatrix} s & s' \\ S & S'' \end{smallmatrix}\right] \left(F_{m''M}\left[\begin{smallmatrix} m & m' \\ M & M'' \end{smallmatrix}\right]\right)^{-1} \psi^{(LMS)(L''M''S'')}_{(l''m''s'')}(x_2) \tag{30}$$

whenever $x_1 < x_2$. This means that the structure constants of the operator product expansion equal a product of F-matrices times a normalization factor. The latter has no dependence on the boundary conditions.

5 Boundary OPE branes in Gepner models

This section contains our main result, namely the boundary operator product expansions for arbitrary boundary fields in a theory describing untwisted A-type branes in Gepner models. After a very brief introduction to Gepner models we will recall the untwisted boundary states of [5]. The boundary operator product expansions are presented in the last subsection.

5.1 The Gepner model in the bulk

The plan is to apply the general theory outlined above to an important class of examples, namely to Gepner models [2, 3] (see also [43] for a review). These are exactly solvable CFTs which are used to study strings moving on a Calabi-Yau manifold at small radius [1]. Their construction employs an orbifold-like projection starting from a tensor products of r $\mathcal{N} = 2$ minimal models. In our presentation we shall assume that there are $d = 2$ complex, transverse, external dimensions with the appropriate ghost sector. In order to get a consistent string background a GSO projection has to be performed. This means that we project on odd integer $U(1)$ charge in the full theory. In the internal part we are projecting on integer charges. The GSO projected partition function is of simple current type, where the simple current is given by the spectral flow operator. To describe this more explicitly we need some further notation.

Let us introduce the following vectors

$$\lambda := (l_1, \ldots, l_r) \quad \text{and} \quad \mu := (s_0; m_1, \ldots, m_r; s_1, \ldots, s_r)$$

to label the representations $(l_j,\, m_j,\, s_j)$ of the individual minimal models and the representations $s_0 = 0, 2, \pm 1$ of a single $SO(2)$ current algebra at level $k = 1$ that comes with the two complex fermions in the non-compact directions. The associated product of characters $\chi^{l_i}_{m_i,s_i}$ and χ_{s_0} is denoted by $\chi^{\lambda}_{\mu}(q)$.

Next, we introduce the special $(2r+1)$-dimensional vectors β_0 with all entries equal to 1, and β_j, $j = 1, \ldots, r$, having zeroes everywhere except for the 1st and the $(r+1+j)$th entry which are equal to 2. These vectors stand for particular elements in the group $\mathbb{Z}_4 \times \prod_i \Gamma_{k_i}$. It is easily seen that they generate a subgroup $\Gamma = \mathbb{Z}_K \times \mathbb{Z}_2^r$ where $K := \mathrm{lcm}(2k_j + 4)$. Elements of this subgroup will be denoted by $\boldsymbol{\nu} = (\nu, \nu_1, \ldots, \nu_r)$. The monodromy charge of a pair (λ, μ) is

$$Q_{\boldsymbol{\nu}}(\lambda, \mu) \;=\; \nu\beta_0 \cdot \mu + \sum_{i=1}^{r} \nu_i\beta_i \cdot \mu \mod 1 \tag{31}$$

$$\text{where} \qquad \beta_0 \cdot \mu \;:=\; -\frac{s_0}{4} - \sum_{j=1}^{r} \frac{s_j}{4} + \sum_{j=1}^{r} \frac{m_j}{2k_j+4} \;, \tag{32}$$

$$\beta_j \cdot \mu \;:=\; -\frac{s_0}{2} - \frac{s_j}{2} \;. \tag{33}$$

The orbifold group Γ acts on the labels λ and μ in the obvious way. There appear orbits of maximal length $K2^r$ and short orbits of length $K2^{r-1}$. The latter are characterized by the property that $\lambda = (l_1, \ldots, l_r)$ satisfy $l_i = k_i/2$ for all i such that $2k_i + 4$ is not a factor in $K/2$.

When we consider the full theory, the field generating the $\mathbb{Z}_K$-symmetry contains a factor from the ghost sector and the space-time part of the spin field S^α. In the $\pm 1/2$ picture, the operator was spelled out in eq. (1) above. It is a simple current and its internal part agrees with the simple current β_0 in the tensor product considered above. Since the operator (1) has total weight one, it can be added to the chiral algebra and we can use the formula (9) to determine the partition function of the orbifold theory. The formula requires to determine invariant orbits, i.e. orbits with vanishing monodromy charge. Taking the OPE of the spectral flow (1) with a vertex operator that represent space-time scalars of the theory gives the monodromy charge

$$\tilde{Q}_{\boldsymbol{\nu}}(\lambda, \mu) \;=\; \nu \left(\frac{\beta_0 \cdot \mu}{2} + \frac{1}{2} \right) + \sum_{i=1}^{r} \nu_i \beta_i \cdot \mu \mod 1 \quad .$$

$\tilde{Q}$ effectively replaces the monodromy charge Q introduced in eq. (31). The orbits of vanishing monodromy charge are those of odd integer $U(1)$ charge.

To write a partition function of physical states one has to extract the physical degrees of freedom. This can be done by a projection onto light-cone variables which removes, in particular, the ghost sector. In practice, the light-cone degrees of freedom may be read off directly in the canonical ghost pictures.

An important reason to include ghosts is that the fields of the theory acquire the right commutation properties. One would like to incorporate this feature in the physical theory. Therefore, in the partition function, the fields are counted with a ghost-charge dependent phase factor $\exp(2\pi i q_{ghost})$. This means that the states with half-integer ghost charge, i.e. the RR-states, contribute with a negative sign.

The partition function in the light cone gauge is then given by

$$Z_G^{(r)}(\tau, \bar{\tau}) = \frac{1}{2} \frac{(\operatorname{Im} \tau)^{-2}}{|\eta(q)|^2} \sum_{\lambda, \mu; \tilde{Q}(\lambda,\mu)=0} \; \sum_{\nu, \nu_j} \; (-1)^\nu \;\; \chi^\lambda_\mu(q) \;\; \chi^\lambda_{\mu + \nu\beta_0 + \nu_1\beta_1 + \ldots + \nu_r\beta_r}(\bar{q}) \quad .$$

The sign is the usual one occurring in (space-time) fermion one-loop diagrams. The τ-dependent factor in front of the sum accounts for the free bosons associated to the 2 physical transversal dimensions of flat external space-time, while the 1/2 is simply due to the field identification mentioned above. Except for these modifications, the formula for Z_G is the same as eq. (9). Elements $g = \nu\beta_0 + \ldots \nu_r\beta_r$ of the orbifold group Γ are labeled by ν, ν_i so that the second sum is over the full group Γ. Short orbits appear twice in the summation and give rise to an extra factor of 2 which is the order of the corresponding stabilizer subgroup. Since our orbifold group Γ is abelian, we used additive notation for the action of elements $g \in \Gamma$ on the labels λ, μ.

5.2 Boundary states in Gepner models

Quite generally, we can construct a set of D-branes on orbifolds by projecting down D-branes on the covering space [44]. Let us explain how to apply these methods in the context of string compactification based on $\mathcal{N} = 2$ superconformal field theories. Gepner models can then be discussed as a very special case (the underlying $\mathcal{N} = 2$ CFT is rational) of such theories. We will point out a few features which are common to branes in arbitrary $\mathcal{N} = 2$ theories, in particular they are not restricted to the Gepner point of a given compactification. To construct a theory with space-time supersymmetry, we have to perform a GSO-projection, projecting out all fields which are not local with respect to the supersymmetry operator (1). We will mainly concentrate on the internal part of (1), which consists of the spectral flow operator in the internal dimensions.

To discuss D-branes which preserve some of the space-time supersymmetry, we have to impose boundary conditions on the spectral flow. They read

$$\begin{aligned} \mathrm{A-type:} \quad \partial X_L &= -\partial X_R, \quad e^{iX_L} = e^{-iX_R} \\ \mathrm{B-type:} \quad \partial X_L &= \partial X_R, \quad e^{iX} = e^{iX_R} \quad . \end{aligned} \tag{34}$$

This means that they are just Dirichlet/Neumann (for A-type/B-type) conditions along the spectral flow. In the following, we discuss A-type/Dirichlet conditions. Let us assume that we do know how to construct D-branes on the covering space and ask how to construct D-branes on the orbifold, i.e. the GSO-projected theory. The orbifold action on states of charge q is given by

$$|q\rangle \to e^{2\pi i q}|q\rangle \quad . \tag{35}$$

In this way, we project on a discrete set of charges. In a geometrical context, the charges can be interpreted as momenta, and we are effectively compactifying on a circle. To describe D-branes in that situation, we can make use of methods developed in [45].

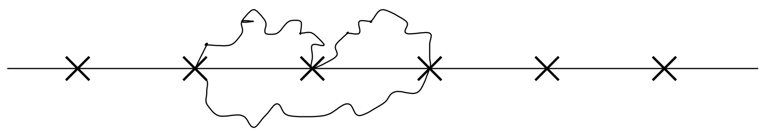

Figure 2: Pre-images of branes related by the spectral flow operator

Here, the D-branes on the circle are described by images of the translation operator along the real line. In our situation, this translation operator is given by spectral flow. To describe D-branes, we have to sum over images of D-branes in the non-GSO projected theory under spectral flow.

Moving to the Gepner point of the compactification means that we enhance the chiral algebra by so many operators that we only have a finite number of primary fields. Most of the branes would not preserve the full symmetry group. Those branes which do preserve the enhanced symmetry have only finitely many images under the spectral flow operator. In this case, the spectral flow generates a finite group Γ, and we can directly apply the methods developed in Section 3. The rationality of the model enables us to describe the D-branes on the covering space as boundary states and one can find a large set of boundary states [5] which respect the $\mathcal{N}=2$ world-sheet algebras of each minimal model factor of the Gepner model separately.

To this end we start with Cardy boundary states of the tensor product theory. They are given by the expression that involves the S-matrices of minimal models and the $SO(2)$ theory (see e.g. [5]). Cardy's boundary states belong to some gluing condition $W(z) = \Omega\bar{W}(\bar{z}), z = \bar{z}$, which becomes $J_i(z) = -\bar{J}_i(\bar{z})$ on the $U(1)$-currents of the individual theories. This means that they are A-type boundary conditions in the sense of [46].

The boundary states $|I\rangle =: |\Lambda, \Xi\rangle$ we have just described depend on a spin vector $\Lambda = (L_1, \dots, L_r)$ and a charge vector $\Xi = (S_0; M_1, \dots, M_r; S_1, \dots, S_r)$. From these states in the tensor product theory we can pass to boundary states of the Gepner model using the general strategy explained above. The projected boundary states in the orbifold theory are given by

$$|\Lambda,\Xi\rangle_{\rm proj} \;=\; \frac{1}{\sqrt{K2^r}}\sum_{\nu,\nu_i}(-1)^\nu(-1)^{\frac{\hat{s}_0^2}{2}}\;|\Lambda,\Xi+\nu\beta_0+\nu_1\beta_1+\ldots+\nu_r\beta_r\rangle\;\;. \tag{36}$$

Here, the element $\hat{s}_0$ is an operator acting on closed string states which measures the value s_0. The whole factor $(-1)^{\hat{s}_0^2/2}$ is needed to guarantee that in the open string partition function (similar to the closed string partition function) fields are counted with a phase factor referring to their ghost charge. Making use of Cardy's formalism we obtain the expressions established in [5],

$$|\alpha\rangle\rangle \;:=\; |\Lambda,\Xi\rangle\rangle_{\rm proj} \;=\; \sum_{\lambda,\mu;\tilde{Q}(\lambda,\mu)=0}(-1)^{\frac{s_0^2}{2}}\;\;B_\alpha^{\lambda,\mu}\;|\lambda,\mu\rangle\rangle\;. \tag{37}$$

with the coefficients:

$$B_\alpha^{\lambda,\mu} \;=\; \frac{\sqrt{K2^r}}{2}\,e^{-i\pi\frac{s_0S_0}{2}}\;\prod_{j=1}^r\frac{1}{\sqrt{\sqrt{2}(k_j+2)}}\frac{\sin(l_j,L_j)_{k_j}}{\sqrt{\sin(l_j,0)_{k_j}}}\;e^{i\pi\frac{m_jM_j}{k_j+2}}\;\;e^{-i\pi\frac{s_jS_j}{2}}\;\;. \tag{38}$$

Here $(l,l')_k\;=\;\pi(l+1)(l'+1)/(k+2)$. For these A-type boundary states the Ishibashi states are built on diagonal primary states, i.e. states in the untwisted sector, in accordance with our general theory in Section 2. The associated partition functions (10) acquire the following form (see also [5]):

$$\begin{aligned} Z^A_{\tilde{\alpha}\alpha}(q) \;\;&=\;\; \frac{1}{2}\;\sum_{\lambda',\mu'}\sum_{\nu=0}^{K-1}\sum_{\nu_i=0,1}\;(-1)^{s_0'+S_0-\tilde{S}_0}\;\;\delta^{(4)}_{s_0'+\tilde{S}_0-S_0+\nu+2\sum\nu_i-2} \\ &\qquad\times\;\prod_{j=1}^r N^{L_j}_{l_j',\tilde{L}_j}\;\;\delta^{(2k_j+4)}_{\nu-M_j+\tilde{M}_j+m_j'}\;\;\delta^{(4)}_{s_j'+\tilde{S}_j-S_j+\nu+2\nu_j}\;\;\chi^{\lambda'}_{\mu'}(q)\;. \end{aligned} \tag{39}$$

The factor $1/2$ in front of the right hand side accounts for the fact that field identification causes each character to appear twice when we sum over λ',μ' taken from the extended range.

There is one important difference between the orbits in equation (36) and the summation over D0-brane states on the real line. While the latter are infinite (due to the infinite extension of the real line), the former have finite length. In particular, there can be orbits of different length in a Gepner model. The identity orbit has length K, but in some cases there can be orbits of length $K/2$. This means that the boundary state is invariant under an application of the spectral flow to the power $K/2$. This situation can be treated similarly to the case that there is a fix-point under a geometrical $\mathbb{Z}_2$. The boundary states get labeled by an orbit label and a representation of the $\mathbb{Z}_2$, which is a sign. They cannot be directly obtained from the covering space. However, the sum of an orbit labeled "+" and an orbit labeled "−" has again an interpretation on the covering space. (In the language of [44] the resulting brane corresponds to choosing the regular representation.) This has been worked out in [18] (see also [19]). In the rest of this paper, we will restrict ourselves to the case of branes that can be obtained as invariant objects that come down from the covering space. That means that we are choosing the regular representation in the case that there is a non-trivial stabilizer.

If the two boundary conditions $\alpha, \tilde{\alpha}$ appearing in eq. (39) are both labeled by monodromy invariant orbits, they give rise to a monodromy invariant open string spectrum, i.e. to a spectrum that contains only odd-integer charges. One should recall, however, that non-invariant orbits of Γ are also admissible as labels for boundary conditions. The condition for a supersymmetric open string spectrum consisting of monodromy invariant states is that the U(1) charge of the two orbits labeling the boundary conditions α and $\tilde{\alpha}$ differs by an even integer.

5.3 The boundary OPE in Gepner models

We have seen that there is a set of Gepner model boundary states (36) which can be understood as invariant linear combinations of boundary states on the covering space as in our general discussion in Subsection 3.2. Each boundary operator $\phi_{\lambda,\mu}^{[\Lambda,\Xi],[\Lambda',\Xi']}$, where $[\Lambda,\Xi],[\Lambda',\Xi']$ are orbit labels, can be understood as coming from an operator on the covering space, i.e. we can identify

$$\Psi_{\lambda,\mu}^{[\Lambda\Xi],[\Lambda'\Xi']} \;:=\; \Psi_{\lambda,\mu}^{(\Lambda,\Xi+\nu_0\beta_0+\nu_i\beta_i)(\Lambda',\Xi'+\nu_0'\beta_0+\nu_r'\beta_r)} \tag{40}$$

Here, $(\Lambda, \Xi+\nu_0\beta_0+\nu_i\beta_i)$ and $(\Lambda', \Xi'+\nu_0'\beta_0+\nu_i'\beta_i)$ label branes on the covering space such that the field (λ,μ) propagates between them.

When we want to multiply boundary operators in the Gepner model, we can use the identification (40) and then interpret the right hand side as a product of boundary operators in the individual models, i.e.

$$\Psi_{\lambda,\mu}^{(\Lambda,\Xi)(\Lambda',\Xi')}(x) \;:=\; \psi_{s_0\,;\,0}^{S_0S_0'}(x)\,\prod_{i=1}^{r}\,\psi_{(l_im_is_i)\,;\,i}^{(L_iM_iS_i)(L_i'M_i'S_i')}(x) \tag{41}$$

where $\Lambda = (L_1, \ldots, L_r), \Xi = (S_0; M_1, \ldots, M_r, S_1, \ldots, S_r)$ etc. and $\psi^{\cdot\cdot}_{\cdot;i}, i = 1, \ldots, r$, denote boundary operators within the i^{th} minimal model. These operators are multiplied using eqs. (30) with the appropriate structure constants depending on the level k_i of the i^{th} minimal model.

In eq. (41) there appears one more set of boundary operators, namely the operators $\psi^{\cdot\cdot}_{\cdot;0}$ which come with the external fermions. They are multiplied according to

$$\psi^{S_0,S_0'}_{s_0\,;0}(x_1)\;\psi^{S_0',S_0''}_{s_0'\,;0}(x_2) \;=\; (-1)^{\frac{s_0'}{8}(S_0+s_0-S_0')}\;\delta_{S_0',s_0\hat{+}S_0}\;\delta_{S_0'',s_0'\hat{+}S_0'}\;\delta_{S_0'',s_0''\hat{+}S_0}\;\psi^{S_0,S_0''}_{s_0''\,;0}(x_2) \qquad (42)$$

for $x_1 < x_2$, as usual. The rules (40,41) along with the expansions (30) and (42) allow to compute the operator products of arbitrary boundary operators for untwisted A-type branes in Gepner models.

According to Subsection 3.3, the results one obtains from these four equations do not depend on the choice of the representative $(\Lambda, \Xi+\nu_0\beta_0+\nu_i\beta_i) \in [\Lambda, \Xi]$ that one has to make in eq. (40). Geometrically, the freedom is associated with the choice of a particular brane on the cover that is used to represent the brane $[\Lambda, \Xi]$ in the orbifold. If some open string operator (λ, μ) propagates between two pre-images $(\Lambda, \Xi), (\Lambda', \Xi')$ on the covering space, then is also propagates between two pre-images $(\Lambda, \Xi + \nu_0\beta_0 + \nu_r\beta_r), (\Lambda', \Xi' + \nu_0\beta_0 + \nu_r\beta_r)$ which are obtained from the first two by applying the simple current $\nu_0\beta_0 + \nu_r\beta_r$. In Figure 2 this is reflected by the fact that the starting point of a string can be moved from one brane to the next, if the endpoint is moved accordingly.

6 Superpotential for A-type branes on the quintic

We are finally able to combine the discussion of Section 2 with the explicit formulas (40,41,30, 42) for computations in the internal CFT to calculate terms in the superpotential of untwisted A-type branes in Gepner models. We shall illustrate these calculations through one example of an A-type brane in the quintic.

In the example we shall discuss, the boundary state and its geometrical interpretation is known. A set of special Lagrangian submanifolds on the quintic given by Im $\omega_j z_j$ with $\omega_j^5 = 1$ was described in [4]. Topologically, these submanifolds are the real projective space $\mathbb{RP}^3$. In particular, $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$ and therefore there are no continuous moduli in the geometric (large volume) regime. On the other hand, one can identify a set of boundary states which is believed to correspond to the geometric cycles. This was checked [4] by comparing the intersection numbers which can be computed both for the geometric cycles and the boundary states. It was observed in [4] that there exits one massless mode in the boundary CFT description of these cycles. We will show below that this massless operator is not truly marginal, or, in other words, that there is a superpotential term for this operator.

The Gepner model for the quintic is the model $(k = 3)^5$ and the boundary states corresponding to the cycles we consider are those for which $L_i = 1$, where $i = 1, \dots, 5$. In this case, there appears a single marginal operator in the spectrum whose internal part is given by

$$\psi(x) \;=\; \Psi^{[\mathbf{1},\Xi][\mathbf{1},\Xi]}_{(1,1,0)^{\times_i}}(x) \tag{43}$$

where $\mathbf{1}$ denotes the vector $\Lambda = (1,1,1,1,1) = \mathbf{1}$ and we assign the same label $(1,1,0)$ to all the minimal models. We shall also need the field that is obtained from Ψ by applying one unit of spectral flow, corresponding to the auxiliary F-field in the same multiplet. In the internal sector, this amounts to a fusion with the anti-chiral field of highest charge. The resulting internal part of the auxiliary field is

$$\psi_F(x) \;=\; \Psi^{[\mathbf{1},\Xi][\mathbf{1},\Xi]}_{(2,-2,0)^{\times_i}}(x) \quad . \tag{44}$$

The external and superghost contributions to the full vertex operators were described in Section 2. Let us also recall from there that the correlation function has to include three conformal ghost c.

Because of the $SL(2,\mathbb{R})$ invariance of the theory, the ghost contribution cancels the dependence on world-sheet coordinates in a 3-point function. The value of the correlator can be determined by successive OPEs of the vertex operators discussed in Section 2. Since the external part of the scalar is the identity field, the full amplitude is essentially a product of OPE coefficients of the fields in the internal sector.

According to our formulas in Section 5.3, these OPE coefficients can be obtained from the OPE on the 'covering space'. Therefore we just have to choose pre-images on the covering space, such that the fields with subscripts $(1,1,0)$ and $(2,-2,0)$ propagate. We start out by multiplying ψ with itself. The two operators in the theory for the covering space are given by

$$\prod_i \psi^{(1,M_i-2,S_i)(2,M_i-1,S_i)}_{(1,1,0)} \qquad \text{and} \qquad \prod_i \psi^{(2,M_i-1,S_i)(1,M_i,S_i)}_{(1,1,0)} \quad . \tag{45}$$

Note that we have applied a field identification on two of the superscripts labeling boundary conditions to replace $L = 1$ by $L = 2$. When we calculate the OPE of the two fields in (45), there appear several contributions out of which only one term can contribute to a 2-point function with ψ_F. This term is proportional to the field

$$\prod_i \psi^{(1,M_i-2,S_i)(1,M_i,S_i)}_{(2,2,0)} \quad . \tag{46}$$

In fact, if we choose to represent ψ_F on the covering space by a field of the form

$$\prod_i \psi^{(1,M_i,S_i)(1,M_i-2,S_i)}_{(2,-2,0)} \tag{47}$$

then the OPE between the operators (46) and (47) gets a contribution from the identity field. From the OPE of the two operators in (45) and a consecutive OPE of the fields (46), (47) we obtain the following coefficient of the correlation function

$$C_3 \;=\; \Big(F_{(2,M-1,S)(2,2,0)}\big[\begin{smallmatrix}(1,1,0) & (1,1,0)\\ (1,M-2,S) & (1,M,S)\end{smallmatrix}\big]F_{(1,M-2,S)(0,0,0)}\big[\begin{smallmatrix}(2,-2,0) & (2,2,0)\\ (1,M,S) & (1,M,S)\end{smallmatrix}\big]\big[\begin{smallmatrix}2 & & 2 & 0\\ -2 & & 2 & 0\end{smallmatrix}\big]\Big)^5 \tag{48}$$

The 5^{th} power comes from the five identical factors that contribute to our correlator. We can decompose the fusing matrices further into WZW fusing matrices and phase factors coming from the two $U(1)$ theories. Actually, the latter do not appear in this special case so that we obtain

$$C_3 \;=\; \Big(F_{22}\big[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\big]F_{10}\big[\begin{smallmatrix}2 & 2\\ 1 & 1\end{smallmatrix}\big]\big[\begin{smallmatrix}2 & & 2 & 0\\ -2 & & 2 & 0\end{smallmatrix}\big]\Big)^5 \quad .$$

Here, F and [.] denote the fusing matrix of $\widehat{SU}(2)_k$ and the Clebsch-Gordan coefficients of $su(2)$, respectively. To obtain the full correlation function in the Gepner model, we finally have to take into account the expectation value of the identity with boundary conditions $(\mathbf{1}, \Xi)$. It is given by

$$\langle \mathbf{1} \rangle_{(\mathbf{1},\Xi)} \;=\; \left(\frac{S_1{}^0}{S_0{}^0}\right)^5 \quad . \tag{49}$$

Here, the matrix elements $S_l{}^0$ are obtained directly from the S-matrix of the $\widehat{SU}(2)_k$ theory since the phase factors from the $U(1)$ theory drop out. The resulting expression for the correlator with one particular ordering of operators is

$$\Big(F_{22}\big[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\big]F_{10}\big[\begin{smallmatrix}2 & 2\\ 1 & 1\end{smallmatrix}\big]\big[\begin{smallmatrix}2 & & 2 & 0\\ -2 & & 2 & 0\end{smallmatrix}\big]\Big)^5 \left(\frac{S_1{}^0}{S_0{}^0}\right)^5 \quad . \tag{50}$$

This result is symmetric with respect to exchanging the last two fields. Hence, summation over inequivalent orderings of the three insertion points simply produces a factor 2. In conclusion, we have shown that the relevant cubic term in the superpotential of our brane is given by

$$\sim \; 2 \left(F_{22} \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right] F_{10} \left[\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} 2 & & 2 & 0 \\ & -2 & 2 & 0 \end{smallmatrix} \right] \right)^5 \left(\frac{S_1{}^0}{S_0{}^0} \right)^5 \; F \, \phi \, \phi \quad . \tag{51}$$

From the fact that our result does not vanish, we conclude that in first order perturbation theory, the modulus generated by Ψ gets lifted so that there is no associated flat direction in the world-volume theory.

7 Conclusions and outlook

In this paper we have computed the boundary operator product expansions for untwisted (or *projected*) A-type boundary states in Gepner models. The main idea was to compute the expansions on a 'covering space', i.e. for products of minimal models, first and then to (GSO) project them to the Gepner model. We applied our results to one particular example in which we computed the cubic term of the superpotential for a massless field.

There are a number of possible extension. First of all, we are not restricted to the evaluation of 3-point functions or, equivalently, third order terms of the superpotential. As is well known in conformal field theory, all correlation functions can be recovered from the operator product expansions with the help of Ward identities. In this sense, we have completely solved the boundary correlators by giving expressions for the coefficients of the OPE. Of course, the computations may still be rather involved.

Furthermore, we can compute correlators for fields of arbitrary masses, including tachyonic fields. This makes it possible to study bound states of unstable brane configurations in Gepner models. From a CFT point of view, one has to perturb the open string action by a (marginally) relevant operator. Knowledge of the boundary OPE enables us to compute the β- function along the RG trajectory generated by this perturbation. The bound state appears at a point where the β-function vanishes, giving rise to a new conformal field theory. Since the structure constants in our Gepner model correlators are essentially given by data of the $\widehat{SU}(2)_k$ WZW model, the results of [25] on bound states in WZW models may partly be carried over to an $\mathcal{N} = 2$ minimal model (see also [47] for an analysis of related problems in Virasoro minimal models). This suggests that arbitrary branes in Gepner models can be obtained as bound states of branes with $L = 0$.

Similar phenomena occur for B-type boundary conditions in Gepner models where all the known boundary theories appear as bound states of fractional branes, as was argued in [12]. The work by Douglas and Diaconescu contains terms of the superpotential for B-type branes. It would be interesting to verify their proposal through exact world-sheet computations. The main part of our analysis, namely Sections 2-4, provide a solid basis for extending our constructions so that they incorporate projected B-type boundary states.

Throughout this work we focused our attention on boundary correlators and omitted all questions related to bulk-boundary couplings. In the context of Gepner models it would be of particular interest to compute couplings of boundary fields to the bulk fields which generate deformations of the complex- and Kähler structure. Such couplings encode the dependence of brane moduli spaces on the Kähler/complex structure. A better understanding of these couplings could provide more insight into the decoupling conjecture of [4]. We plan to come back to some of these issues in the future.

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