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Nuclear Physics B 568 [PM] (2000) 475–502

NUCLEAR
PHYSICS B

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Superconformal hypermultiplets

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Received 19 October 1999; accepted 2 November 1999

Abstract

We present theories of $N = 2$ hypermultiplets in four space-time dimensions that are invariant under rigid or local superconformal symmetries. The target spaces of theories with rigid superconformal invariance are $(4n)$ -dimensional *special* hyper-Kähler manifolds. Such manifolds can be described as cones over tri-Sasakian metrics which itself are $\text{Sp}(1)$ fibrations over quaternionic manifolds. The latter manifolds appear in the coupling of hypermultiplets to $N = 2$ supergravity. We employ local sections of an $\text{Sp}(n) \times \text{Sp}(1)$ bundle in the formulation of the Lagrangian and transformation rules, thus allowing for arbitrary coordinatizations of the hyper-Kähler and quaternionic manifolds. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 04.65.+e; 11.30.Pb

Keywords: $N = 2$ Conformal supergravity; Hypermultiplets; Hyper-Kähler and quaternionic geometry

1. Introduction

It is well known that in theories with rigid $N = 2$ supersymmetry the hypermultiplet action takes the form of a supersymmetric sigma model with scalars that parametrize a hyper-Kähler manifold [1]. In the case of local supersymmetry the scalar fields parametrize a quaternionic manifold of negative curvature [2]. In this paper we study actions for hypermultiplets invariant under rigid or local $N = 2$ superconformal symmetries. This study is both motivated by recent interest in superconformal theories [3] and

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by efforts to find alternative and hopefully more convenient formulations of the hypermultiplet actions. The $N = 2$ superconformal algebra in four dimensions contains the bosonic subalgebra associated with $SO(4,2) \times SU(2) \times U(1)$, together with 8 real supersymmetry and 8 real ‘special’ supersymmetry transformations, called Q - and S -supersymmetry, respectively. Requiring that the action is invariant under rigid superconformal transformations leads to extra constraints on the target-space geometry [4]. For instance, these manifolds admit a so-called hyper-Kähler potential whose derivative defines a conformal homothetic Killing vector and the three complex structures rotate under the action of the $SU(2)$ group, which must be contained as a factor in the isometry group of the manifold. Spaces that satisfy these constraints will be called *special hyper-Kähler manifolds*¹. By using the superconformal multiplet calculus [5,6] one can then obtain corresponding quaternionic sigma models coupled to $N = 2$ supergravity. Because of the gauge degrees of freedom associated with the dilatations and the $SU(2)$ transformations, a $(4n)$ -dimensional special hyper-Kähler manifold leads to a $(4n - 4)$ -dimensional quaternionic manifold. At the time this construction was applied to only flat hyper-Kähler spaces or hyper-Kähler quotients thereof. The coupling to supergravity then leads to a quaternionic projective space and its quaternionic hyper-Kähler quotients [6]. But it has been known for some time that there exist quaternionic spaces that can couple to supergravity which are not in this class but can be described in the context of the formalism of [2]. Some of them have also been obtained explicitly in the context of harmonic superspace [7]. Therefore it is imperative to apply the superconformal approach to more general special hyper-Kähler sigma models. This application is the main topic of our paper, where, in order to avoid introducing an infinite number of fields, we will no longer insist on off-shell supersymmetry for the hypermultiplets.

Already quite some time ago the very same option was discussed by Galicki [8]. Rather than starting from the superconformally invariant hypermultiplets, he described the geometry of these more general hyper-Kähler spaces using a result of Swann [9], who has proven that any quaternionic manifold has a corresponding special hyper-Kähler manifold which admits a quaternionically extended homothety and which has three complex structures that rotate under an isometric $SU(2)$ action. And indeed, the hyper-Kähler manifolds that he discusses have many properties in common with the hypermultiplet actions discussed in [4]. Moreover it is known that a special hyper-Kähler manifold is a cone over a so-called tri-Sasakian manifold, so that there exists a beautiful relation between quaternionic manifolds, tri-Sasakian manifolds and special hyper-Kähler manifolds (for a recent review, see Ref. [10]). The tri-Sasakian manifolds have also appeared recently in the context of supergravity compactifications and the ADS/CFT correspondence [11].

In this paper we follow the original superconformal approach and start with the $(4n)$ -dimensional special hyper-Kähler manifolds as they were formulated in [4]. We establish that these spaces are indeed cones over $(4n - 1)$ -dimensional tri-Sasakian spaces (this feature was also discussed in [12]). The special hyper-Kähler manifolds have

¹ Note that hyper-Kähler manifolds that are in the image of the c -map are sometimes called special, because of the underlying special geometry features. We stress that the usage of the term *special hyper-Kähler* in this paper has no relationship to special geometry.

only a restricted holonomy group contained in $\mathrm{Sp}(n-1)$; locally they are a product of a flat four-dimensional space and a $(4n-4)$ -dimensional quaternionic space. After gauging away the degrees of freedom associated with the dilatations and the $\mathrm{SU}(2)$ transformations, the quaternionic space remains when coupling to supergravity. We present the full Lagrangian and transformation rules for the supersymmetric non-linear sigma models based on special hyper-Kähler spaces, including the option of gauged isometries. Furthermore we construct local $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ sections of the so-called associated quaternionic bundle which is known to exist for any special hyper-Kähler manifold [9]. It turns out that the use of these sections greatly simplifies the formulation of the transformation rules and the Lagrangian. In this way our general results remain closely in line with the results of Ref. [6]; the formulae are identical up to modifications by connections and covariant tensors. When the sections are trivial, so that the connections can be put to zero and the tensors become constant, they can be identified with the hypermultiplet scalar fields and one directly recovers the results of Ref. [6]. Guided by supersymmetry we thus make contact with the mathematical results quoted above and we construct the general action and transformation rules in a new form.

The last topic is to couple these supersymmetric non-linear sigma models to supergravity, using the conformal multiplet calculus. In addition to presenting the corresponding field theory, we exhibit how the quaternionic manifold emerges in the coupling. This manifold can now be encoded in terms of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ sections that are projective with respect to quaternionic multiplication.

Our results could facilitate the study of type-II string compactifications on Calabi–Yau three-folds. These lead to four-dimensional models with both vector multiplets and hypermultiplets. While the moduli space of the vector multiplet scalars is described in terms of a special Kähler geometry and is well understood, much less is known about the full quaternionic hypermultiplet moduli space. It is known that at string tree level the quaternionic manifolds are obtained from a special Kähler manifold via the \mathbf{c} -map [13]. One would like to understand what the corrections are to the classical hypermultiplet moduli space coming from both string perturbation theory and non-perturbative effects [14–17]. With rigid conformal symmetry, the results of this paper could also be helpful in the description of cone branes [11].

This paper is organized as follows. In Section 2 we briefly summarize some essential features of hypermultiplet Lagrangians with gauged target-space isometries. For hypermultiplets there exists no unconstrained off-shell formulation in terms of a finite number of degrees of freedom, hence the supersymmetry algebra will only be realized up to the field equations of the hypermultiplet fermions. This is in contrast with the vector multiplets, introduced to gauge the isometries, and the superconformal theory itself, for which off-shell formulations exist. As a result of the latter, the algebra of gauged isometries and of the superconformal transformations, including certain field-dependent structure constants, is completely fixed and not affected by the presence of hypermultiplets. Section 3 deals with rigidly superconformal hypermultiplets, where we find the constraints on the hyper-Kähler manifold imposed by superconformal invariance. Subsection 3.1 defines the superconformal transformation rules, the second one deals with the hyper-Kähler potential and the construction of local $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ sections, and the third one gives the Lagrangian and the transformation rules. The geometry of special hyper-Kähler manifolds is explained in Section 4. We first discuss the cone structure of

these hyper-Kähler manifolds which lead to a tri-Sasakian space. The latter is indeed an $\text{Sp}(1)$ fibration over a smaller space, which we prove to be quaternionic. This quaternionic space couples to supergravity, as we then show in Section 5. Here we present the action for the hypermultiplets associated with a special hyper-Kähler target space coupled to conformal supergravity and exhibit how the target-space metric becomes quaternionic.

2. Preliminaries

In this section we summarize hypermultiplet Lagrangians in flat space-time. As is well known, these constitute $N = 2$ supersymmetric non-linear sigma models with a hyper-Kähler target space [2]. The holonomy group is contained in $\text{Sp}(n)$ and it is this group that is relevant for the hypermultiplet fermions. In Subsection 2.1 we discuss the supersymmetry transformations, the Lagrangian and the target-space geometry. In Subsection 2.2 we present possible extensions related to gauged target-space isometries, which will involve couplings to vector multiplets associated with the gauge algebra.

2.1. Hypermultiplet non-linear sigma models

We will base ourselves on the formulation of hypermultiplet Lagrangians of [18]. With respect to the results of [2] this formulation differs in that it incorporates both a metric g_{AB} for the hyper-Kähler target space and a metric $G_{\bar{\alpha}\beta}$ for the fermions. Here we assume that the n hypermultiplets are described by $4n$ real scalars ϕ^A , $2n$ positive-chirality spinors $\zeta^{\bar{\alpha}}$ and $2n$ negative-chirality spinors ζ^{α} . Hence target-space indices A, B, \dots take values $1, 2, \dots, 4n$, and the indices α, β, \dots and $\bar{\alpha}, \bar{\beta}, \dots$ run from 1 to $2n$. The chiral and antichiral spinors are related by complex conjugation (so that we have $2n$ Majorana spinors) under which indices are converted according to $\alpha \leftrightarrow \bar{\alpha}$, while $\text{SU}(2)$ indices $i, j, \dots = 1, 2$ are raised and lowered. An explicit fermionic metric $G_{\bar{\alpha}\beta}$ can be avoided as it can always be converted to a constant diagonal matrix by a similarity transformation. But retaining a fermionic metric is, for example, important in obtaining transparent transformation rules under symplectic transformations induced by the so-called \mathbf{e} -map from the electric-magnetic duality transformations on a corresponding theory of vector multiplets. In formulations based on $N = 1$ superfields (such as in [19]) one naturally has a fermionic metric but of a special form.

The Lagrangian and transformation rules are subject to a number of equivalence transformations, two of which are associated with the target space. One set consists of the target-space diffeomorphisms $\phi \rightarrow \phi'(\phi)$. The other refers to reparametrizations of the fermion ‘frame’ of the form $\zeta^{\alpha} \rightarrow S^{\alpha}_{\beta}(\phi) \zeta^{\beta}$, and corresponding redefinitions of other quantities carrying indices α or $\bar{\alpha}$. For example, the fermionic metric transforms as $G_{\bar{\alpha}\beta} \rightarrow [\bar{S}^{-1}]^{\bar{\gamma}}_{\bar{\alpha}} [S^{-1}]^{\delta}_{\beta} G_{\bar{\gamma}\delta}$. There are connections, $\Gamma_A^{\alpha}_{\beta}$, associated with these fermionic redefinitions, which appear in the Lagrangian and supersymmetry transformation rules. Finally, there are chiral $\text{SU}(2) \cong \text{Sp}(1)$ redefinitions of the supercharges, which in the rigidly supersymmetric case must be constant and are therefore trivial. In the locally supersymmetric case this will be different and in the latter part of this paper we will have to deal with local $\text{SU}(2)$.

The supersymmetry transformations are parametrized in terms of certain ϕ -dependent quantities γ^A and V_A according to

$$\begin{aligned} \delta_Q \phi^A &= 2 \left(\gamma_{i\bar{\alpha}}^A \bar{\epsilon}^i \zeta^{\bar{\alpha}} + \bar{\gamma}_\alpha^{Ai} \bar{\epsilon}_i \zeta^\alpha \right), \\ \delta_Q \zeta^\alpha &= V_{A i}^\alpha \not{d}\phi^A \epsilon^i - \delta_Q \phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta, \\ \delta_Q \zeta^{\bar{\alpha}} &= \bar{V}_A^{i\bar{\alpha}} \not{d}\phi^A \epsilon_i - \delta_Q \phi^A \bar{\Gamma}_A^{\bar{\alpha}}{}_\beta \zeta^{\bar{\beta}}. \end{aligned} \tag{2.1}$$

In principle γ^A and V_A each denote $(4n) \times (4n)$ complex quantities, but as we shall see below, these quantities are related and satisfy a pseudoreality condition. As it turns out they will play the role of the quaternionic (inverse) vielbeine of the target space. Observe that the supersymmetry variations are consistent with a $U(1)$ chiral invariance under which the scalars remain invariant, while the fermion fields and the supersymmetry transformation parameters transform nontrivially. This group will be denoted by $U(1)_R$ to indicate that it is a subgroup of the automorphism group of the supersymmetry algebra. In Section 3 we will see that this $U(1)$ will correspond to one of the conformal gauge transformations. However, for generic γ^A and V_A , the $SU(2)_R \cong Sp(1)$ part of the automorphism group cannot be realized consistently on the fields. This would require the presence of an $SU(2)$ isometry in the target space. In the above, we merely used that ζ^α and $\zeta^{\bar{\alpha}}$ are related by complex conjugation.

The Lagrangian takes the following form:

$$\mathcal{L} = -\frac{1}{2} g_{AB} \partial_\mu \phi^A \partial^\mu \phi^B - G_{\bar{\alpha}\beta} \left(\bar{\zeta}^{\bar{\alpha}} \not{D} \zeta^\beta + \bar{\zeta}^{\bar{\beta}} \not{D} \zeta^{\bar{\alpha}} \right) - \frac{1}{4} W_{\bar{\alpha}\beta\bar{\gamma}\delta} \bar{\zeta}^{\bar{\alpha}} \gamma_\mu^{\bar{\beta}} \zeta^\beta \bar{\zeta}^{\bar{\gamma}} \gamma^{\mu\delta}, \tag{2.2}$$

where we employed the covariant derivatives

$$D_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta, \quad D_\mu \zeta^{\bar{\alpha}} = \partial_\mu \zeta^{\bar{\alpha}} + \partial_\mu \phi^A \bar{\Gamma}_A^{\bar{\alpha}}{}_\beta \zeta^{\bar{\beta}}. \tag{2.3}$$

Besides the Riemann curvature R_{ABCD} we will be dealing with another curvature $R_{AB}{}^\alpha{}_\beta$ associated with the connections $\Gamma_A^\alpha{}_\beta$, which takes its values in $sp(n) \cong usp(2n; \mathbb{C})$. The tensor W is defined by

$$W_{\bar{\alpha}\beta\bar{\gamma}\delta} = R_{AB}{}^\epsilon{}_\gamma \gamma_{i\bar{\alpha}}^A \bar{\gamma}_\beta^{iB} G_{\epsilon\delta} = \frac{1}{2} R_{ABCD} \gamma_{i\bar{\alpha}}^A \bar{\gamma}_\beta^{iB} \gamma_{j\bar{\gamma}}^C \bar{\gamma}_\delta^{jD}, \tag{2.4}$$

and will be discussed shortly in more detail.

The target-space metric g_{AB} , the tensors γ^A , V_A and the fermionic hermitian metric $G_{\bar{\alpha}\beta}$ (i.e. satisfying $(G_{\bar{\alpha}\beta})^* = G_{\bar{\beta}\alpha}$) are all covariantly constant with respect to the Christoffel connection and the connections $\Gamma_A^\alpha{}_\beta$. Furthermore we note the following relations:

$$\begin{aligned} \gamma_{i\bar{\alpha}}^A \bar{V}_B^{j\bar{\alpha}} + \bar{\gamma}_\alpha^{Aj} V_{Bi}^\alpha &= \delta_i^j \delta_B^A, \\ g_{AB} \gamma_{i\bar{\alpha}}^B &= G_{\bar{\alpha}\beta} V_{Ai}^\beta, \quad \bar{V}_A^{i\bar{\alpha}} \gamma_{j\bar{\beta}}^A = \delta_j^i \delta^{\bar{\alpha}}_{\bar{\beta}}. \end{aligned} \tag{2.5}$$

From them one derives a number of useful relations, such as

$$\bar{\gamma}_{A\alpha}^j V_{Bi}^\alpha = \gamma_{B i \bar{\alpha}} \bar{V}_A^{j\bar{\alpha}} = -\bar{\gamma}_{B\alpha}^j V_{iA}^\alpha + \delta_i^j g_{AB}. \tag{2.6}$$

The following three bilinears define antisymmetric covariantly constant target-space tensors:

$$J_{AB}^{ij} = \gamma_{Ak\bar{\alpha}} \varepsilon^{k(i} \bar{V}_B^{j)\bar{\alpha}}, \tag{2.7}$$

that span the complex structures of the hyper-Kähler target space. They satisfy

$$(J_{ij})_{AB} \equiv (J_{AB}^{ij})^* = \varepsilon_{ik} \varepsilon_{jl} J_{AB}^{kl}, \quad J_A^{ijC} J_{CB}^{kl} = \frac{1}{2} \varepsilon^{i(k} \varepsilon^{l)j} g_{AB} + \varepsilon^{i(k} J_{AB}^{l)j}). \tag{2.8}$$

In addition we note the following useful identities:

$$\gamma_{Ai\bar{\alpha}} \bar{V}_B^{j\bar{\alpha}} = \varepsilon_{ik} J_{AB}^{kj} + \frac{1}{2} g_{AB} \delta_i^j, \quad J_{AB}^{ij} \gamma_{\alpha k}^B = -\delta_k^{(i} \varepsilon^{j)l} \gamma_{Al\bar{\alpha}}. \tag{2.9}$$

We also note the existence of covariantly constant antisymmetric tensors,

$$\Omega_{\bar{\alpha}\bar{\beta}} = \frac{1}{2} \varepsilon^{ij} g_{AB} \gamma_{i\bar{\alpha}}^A \gamma_{j\bar{\beta}}^B, \quad \bar{\Omega}^{\bar{\alpha}\bar{\beta}} = \frac{1}{2} \varepsilon_{ij} g^{AB} \bar{V}_A^{i\bar{\alpha}} \bar{V}_B^{j\bar{\beta}}, \tag{2.10}$$

satisfying $\Omega_{\bar{\alpha}\bar{\gamma}} \bar{\Omega}^{\bar{\gamma}\bar{\beta}} = -\delta_{\bar{\alpha}}^{\bar{\beta}}$. Their complex conjugates satisfy

$$\bar{\Omega}_{\alpha\beta} \equiv (\Omega_{\bar{\alpha}\bar{\beta}})^* = G_{\bar{\gamma}\alpha} \bar{\Omega}^{\bar{\gamma}\bar{\delta}} G_{\bar{\delta}\beta}. \tag{2.11}$$

The tensor Ω can be used to define a reality condition on V and γ ,

$$\varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}} \bar{V}_A^{j\bar{\beta}} = g_{AB} \gamma_{i\bar{\alpha}}^B = G_{\bar{\alpha}\beta} V_{Ai}^\beta. \tag{2.12}$$

This equation leads to

$$g^{AB} V_{Ai}^\alpha V_{Bj}^\beta = \varepsilon_{ij} \Omega^{\alpha\beta}, \quad g_{AB} \gamma_{i\bar{\alpha}}^A \gamma_{j\bar{\beta}}^B = \varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}}. \tag{2.13}$$

Another convenient identity is given by

$$\bar{V}_A^{i\bar{\alpha}} \Omega_{\bar{\alpha}\bar{\beta}} \bar{V}_B^{j\bar{\beta}} = \frac{1}{2} \varepsilon^{ij} g_{AB} - J_{AB}^{ij}. \tag{2.14}$$

The existence of the covariantly constant tensors implies a variety of integrability conditions which have a number of consequences for the various curvature tensors [2,18]. First of all the covariant constancy of $\bar{\gamma}^A$ implies

$$R_{ABCD} \bar{\gamma}_\alpha^{Ci} \bar{\gamma}_\beta^{Dj} = -\varepsilon^{ij} \bar{\Omega}_{\alpha\gamma} R_{AB}{}^\gamma{}_\beta. \tag{2.15}$$

Observe that the right-hand side is manifestly antisymmetric in $[ij]$ and symmetric in $(\alpha\beta)$. This implies that the Riemann tensor can be written with tangent-space indices according to

$$R_{ABCD} \bar{\gamma}_\alpha^{Ai} \bar{\gamma}_\beta^{Bj} \bar{\gamma}_\gamma^{Ck} \bar{\gamma}_\delta^{Dl} = \frac{1}{2} \varepsilon^{ij} \varepsilon^{kl} W_{\alpha\beta\gamma\delta}, \tag{2.16}$$

where, as a result of the cyclicity property of the Riemann tensor, $W_{\alpha\beta\gamma\delta}$ is symmetric in all four indices. This tensor is linearly related to the tensor (2.4) upon multiplication with the tensors G and Ω . In terms of $W_{\alpha\beta\gamma\delta}$ the curvatures read

$$R_{ABCD} = \frac{1}{2} \varepsilon^{ij} \varepsilon^{kl} V_{Ai}^\alpha V_{Bj}^\beta V_{Ck}^\gamma V_{Dl}^\delta W_{\alpha\beta\gamma\delta},$$

$$\bar{\Omega}_{\alpha\epsilon} R_{AB}{}^\epsilon{}_\beta = -\frac{1}{2} \varepsilon^{ij} V_{Ai}^\gamma V_{Bj}^\delta W_{\alpha\beta\gamma\delta}. \tag{2.17}$$

The above results are all derived from the requirement of supersymmetry. To characterize the geometry of the target space, one could start from the non-singular $\bar{V}_A^{i\bar{\alpha}}$ and a non-singular skew-symmetric tensor $\bar{\Omega}_{\bar{\alpha}\bar{\beta}}$ that is covariantly constant with respect to a symplectic connection $\bar{T}_A{}^{\bar{\alpha}}{}_{\bar{\beta}}$. Subsequently one notes that $\varepsilon_{ij} \bar{\Omega}_{\bar{\alpha}\bar{\beta}} \bar{V}_A^{j\bar{\beta}}$ and the inverse of \bar{V}_A , denoted by $\gamma_{i\bar{\alpha}}^B$, are linearly related by a symmetric matrix g_{AB} . Requiring that this matrix is real we can identify it with the target-space metric while the ensuing reality constraint on the V_A enables their identification as the corresponding quaternionic vielbeine. This information is sufficient for deriving all the algebraic identities listed above. The vielbeine and the symplectic connection then allow the definition of an affine target-space connection, with respect to which the vielbeine are covariantly constant thus leading to a generalized vielbein postulate. All of above results then follow upon assuming that the target space has no torsion so that the affine connection and the Christoffel connection coincide.

2.2. Gauged target-space isometries

The equivalence transformations of the fermions and the target-space diffeomorphisms do not constitute invariances of the theory. This is only the case when the metric g_{AB} and the $\text{Sp}(n) \times \text{Sp}(1)$ one-form V_i^α (and thus the related geometric quantities) are left invariant under (a subset of) them. Therefore these are related to isometries of the hyper-Kähler space. We can then elevate such invariances to a group of local (i.e. space-time-dependent) transformations, by introducing the required gauge fields in the form of vector multiplets. Such gauged isometries have been studied earlier in the literature [7,19–22] and the purpose of our discussion here is to incorporate them into the formulation used in this paper.

We consider scalar fields transforming under a certain isometry (sub)group G characterized by a number of Killing vectors $k_I^A(\phi)$, with parameters θ^I . Hence under infinitesimal transformations,

$$\delta_G \phi^A = g \theta^I k_I^A(\phi), \tag{2.18}$$

where g is the coupling constant and the $k_I^A(\phi)$ satisfy the Killing equation,

$$D_A k_{IB} + D_B k_{IA} = 0. \tag{2.19}$$

The isometries constitute an algebra with structure constants $f_{IJ}{}^K$,

$$k_I^B \partial_B k_J^A - k_J^B \partial_B k_I^A = -f_{IJ}{}^K k_K^A. \tag{2.20}$$

Our definitions are such that the gauge fields that are needed once the θ^I become space-time dependent, transform according to $\delta_G W_\mu^I = \partial_\mu \theta^I - g f_{JK}{}^I W_\mu^J \theta^K$. The Killing equation generally implies the following property:

$$D_A D_B k_{IC} = R_{BCAE} k_I^E. \quad (2.21)$$

Quantities that carry $\text{Sp}(n)$ indices, such as V_{Ai}^α , are only required to be invariant under isometries up to fermionic equivalence transformations. Thus $-g(k_I^B \partial_B V_{Ai}^\alpha + \partial_A k_I^B V_{Bi}^\alpha)$ must be cancelled by a suitable infinitesimal rotation on the index α . Here we assume that the effect of the diffeomorphism is entirely compensated by a rotation that affects the indices α . In principle, one can also allow for a compensating $\text{Sp}(1)$ transformation acting on the indices i, j, \dots . However, the latter transformations must be constant, so they will generically not appear here. This is equivalent to requiring that the isometry group will commute with supersymmetry.

Let us parametrize the compensating transformation acting on the $\text{Sp}(n)$ indices by $\delta_G \zeta^\alpha = g[t_I - k_I^A \Gamma_A]^\alpha{}_\beta \zeta^\beta$, where the (ϕ -dependent) matrices $t_I(\phi)$ remain to be determined,

$$-k_I^B \partial_B V_{Ai}^\alpha - \partial_A k_I^B V_{Bi}^\alpha + (t_I - k_I^B \Gamma_B)^\alpha{}_\beta V_{Ai}^\beta = 0. \quad (2.22)$$

Obviously similar equations apply to the other geometric quantities, but as those are not independent we do not need to consider them. Using the covariant constancy of V_A , we derive from (2.22),

$$(t_I)^\alpha{}_\beta V_{Ai}^\beta = D_A k_I^B V_{Bi}^\alpha, \quad (2.23)$$

so that

$$(t_I)^\alpha{}_\beta = \frac{1}{2} V_{Ai}^\alpha \bar{\gamma}_\beta^{Bi} D_B k_I^A. \quad (2.24)$$

Target-space scalars will satisfy algebraic identities, such as

$$(\bar{t}_I)^\gamma{}_{\bar{\alpha}} G_{\bar{\gamma}\beta} + (t_I)^\gamma{}_\beta G_{\bar{\alpha}\gamma} = (t_I)^\gamma{}_{[\bar{\alpha}} \Omega_{\beta]\bar{\gamma}} = 0. \quad (2.25)$$

This establishes that the field-dependent matrices t_I take values in $\text{sp}(n)$. From (2.19) and (2.21), it easily follows that

$$D_A t_I{}^\alpha{}_\beta = k_I^B R_{AB}{}^\alpha{}_\beta, \quad (2.26)$$

for any infinitesimal isometry. From the group property of the isometries it follows that the matrices t_I satisfy the commutation relation

$$[t_I, t_J]^\alpha{}_\beta = f_{IJ}{}^K (t_K)^\alpha{}_\beta + k_I^A k_J^B R_{AB}{}^\alpha{}_\beta, \quad (2.27)$$

which takes values in $\text{sp}(n)$. The apparent lack of closure represented by the presence of the curvature term is related to the fact that the coordinates ϕ^A on which the matrices depend, transform under the action of the group. One can show that this result is consistent with the Jacobi identity.

Furthermore we derive from (2.22) that the complex structures J_{AB}^{ij} are invariant under the isometries,

$$k_I^C \partial_C J_{AB}^{ij} - 2 \partial_{[A} k_I^C J_{B]C}^{ij} = 0. \quad (2.28)$$

This means that the isometries are *tri-holomorphic*. From (2.28) one shows that $\partial_A(J_{BC}^{ij} k_I^C) - \partial_B(J_{AC}^{ij} k_I^C) = 0$, so that, locally, one can associate three Killing potentials (or moment maps) P_I^{ij} to every Killing vector, according to

$$\partial_A P_I^{ij} = J_{AB}^{ij} k_I^B. \tag{2.29}$$

Observe that this condition determines the moment maps up to a constant. Up to constants one can also derive the equivariance condition,

$$J_{AB}^{ij} k_I^A k_J^B = -f_{IJ}{}^K P_K^{ij}, \tag{2.30}$$

which implies that the moment maps transform covariantly under the isometries,

$$\delta_G P_I^{ij} = \theta^J k_J^A \partial_A P_I^{ij} = -f_{JI}{}^K P_K^{ij} \theta^J. \tag{2.31}$$

Summarizing, the invariance group of the isometries acts as follows:

$$\delta_G \phi = g \theta^I k_I^A, \quad \delta_G \zeta^\alpha = g (\theta^I t_I)^\alpha{}_\beta \zeta^\beta - \delta_G \phi^A \Gamma_A{}^\alpha{}_\beta \zeta^\beta. \tag{2.32}$$

When the parameters of these isometries become space-time dependent we introduce corresponding gauge fields and fully covariant derivatives,

$$\mathcal{D}_\mu \phi^A = \partial_\mu \phi^A - g W_\mu^I k_I^A, \quad \mathcal{D}_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_A{}^\alpha{}_\beta \zeta^\beta - g W_\mu{}^\alpha{}_\beta \zeta^\beta, \tag{2.33}$$

where $W_\mu{}^\alpha{}_\beta = W_\mu^I (t_I)^\alpha{}_\beta$. The covariance of $\mathcal{D}_\mu \zeta^\alpha$ depends crucially on (2.26) and (2.27); after some calculation one finds

$$\delta_G \mathcal{D}_\mu \zeta^\alpha = g (\theta^I t_I)^\alpha{}_\beta \mathcal{D}_\mu \zeta^\beta - \delta_G \phi^A \Gamma_A{}^\alpha{}_\beta \mathcal{D}_\mu \zeta^\beta. \tag{2.34}$$

The gauge fields W_μ^I are accompanied by complex scalars X^I , spinors Ω_i^I and auxiliary fields Y_{ij}^I , constituting off-shell $N = 2$ vector multiplets. For our notation of vector multiplets, the reader may consult [18].

The minimal coupling to the gauge fields requires extra terms in the supersymmetry transformation rules for the hypermultiplet spinors as well as in the Lagrangian, in order to regain $N = 2$ supersymmetry. The extra terms in the transformation rules are

$$\delta_Q \zeta^\alpha = 2 g X^I k_I^A V_{Ai}^\alpha \varepsilon^{ij} \epsilon_j, \quad \delta'_Q \zeta^{\bar{\alpha}} = 2 g \bar{X}^I k_I^A \bar{V}_A^{\bar{\alpha}i} \varepsilon_{ij} \epsilon^j. \tag{2.35}$$

These terms can be conveniently derived by imposing the commutator of two supersymmetry transformations on the scalars, as this commutator should yield the correct field-dependent gauge transformation.

We distinguish three additional couplings to the Lagrangian. The first one is quadratic in the hypermultiplet spinors and reads

$$\mathcal{L}_g^{(1)} = g \bar{X}^I \bar{\gamma}_\alpha^{Ai} \varepsilon_{ij} \bar{\gamma}_\beta^{Bj} D_B k_{AI} \bar{\zeta}^{\alpha\gamma\beta} + \text{h.c.} = 2 g \bar{X}^I t_I^\gamma{}_\alpha \bar{\Omega}_{B\gamma} \bar{\zeta}^{\alpha\gamma\beta} + \text{h.c.} \tag{2.36}$$

The second one is proportional to the vector multiplet spinor Ω^I and takes the form

$$\mathcal{L}_g^{(2)} = -2 g k_I^A V_{Ai}^\alpha \bar{\Omega}_{\alpha\beta} \bar{\zeta}^\beta \Omega^{Ii} + \text{h.c.} = 2 g k_I^A \bar{\gamma}_{A\alpha}^i \varepsilon_{ij} \bar{\zeta}^\alpha \Omega^{Ij} + \text{h.c.} \tag{2.37}$$

Finally there is a potential given by

$$\mathcal{L}_g^{\text{scalar}} = -2 g^2 k_I^A k_J^B g_{AB} X^I \bar{X}^J + g P_I^{ij} Y_{ij}^I, \tag{2.38}$$

where P_i^{jj} is the triplet of moment maps on the hyper-Kähler space. These terms were determined both from imposing the supersymmetry algebra and from the invariance of the action. To prove (2.38), one has to make use of the equivariance condition (2.30). Actually, gauge invariance, which is prerequisite to supersymmetry, already depends on (2.31).

3. Rigidly superconformal hypermultiplets

In this section we determine the restrictions on the hyper-Kähler geometry that follow from imposing invariance under rigid superconformal transformations. As we already mentioned in Section 1, the corresponding spaces, called *special* hyper-Kähler manifolds, have an intriguing geometrical structure. In Section 5 we will obtain the coupling of hypermultiplets to conformal supergravity. A crucial element in the construction of this coupling is that the full superconformal theory is known in an off-shell form, so that the superconformal algebra remains unaffected in the presence of matter fields. Our goal is more modest in this section where we only consider rigid superconformal transformations. This aspect does not play a role for the derivation of the superconformal transformations on the hypermultiplets and the results of this section describe the situation that would arise when freezing all the fields of conformal supergravity to zero in a flat space-time metric. In that case the superconformal transformations acquire an explicit but fixed dependence on the space-time coordinates parametrized by a finite number of space-time-independent parameters (this is explained, for instance, in [23]).

In Subsection 3.1 we impose the superconformal algebra on the fields and find the transformation rules as well as a number of important results for the complex structures and the moment maps associated with possible isometries. In Subsection 3.2 we derive the existence of a hyper-Kähler potential and reformulate the theory in terms of local sections of an $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ bundle. Then, in Subsection 3.3 we present the Lagrangian and the transformation rules in terms of these local sections.

3.1. Superconformal transformations

We start by implementing the $N = 2$ superconformal algebra [5] on the hypermultiplet fields. We assume that the scalars are invariant under special conformal and special supersymmetry transformations, but they transform under Q -supersymmetry and under the additional bosonic symmetries of the superconformal algebra, namely chiral $[\mathrm{SU}(2) \times \mathrm{U}(1)]_{\mathbb{R}}$ and dilatations denoted by D . At this point we do not assume that these transformations are symmetries of the action and we simply parametrize them as follows:

$$\delta\phi^A = \theta_D k_D^A(\phi) + \theta_{\mathrm{U}(1)} k_{\mathrm{U}(1)}^A(\phi) + (\theta_{\mathrm{SU}(2)})^i_k \varepsilon^{jk} k_{ij}^A(\phi), \quad (3.1)$$

where the k^A are left arbitrary. Note that $k_{ij}^A(\phi)$ is assigned to the same symmetric pseudoreal representation of $\mathrm{SU}(2)$ as the complex structures, while $\theta_{\mathrm{SU}(2)}$ is antihermitian and traceless.

An important difference with the situation described in Section 2, is that in the conformal superalgebra the dilatations and chiral transformations do not appear in the

commutator of two Q -supersymmetries, but in the commutator of a Q - and an S -supersymmetry. To evaluate the S -supersymmetry variation of the fermions, we assume that $\delta_S \phi^A = \delta_K \zeta^\alpha = 0$ and covariantize the derivative in the fermionic transformations with respect to dilatations. Subsequently we impose the commutator, $[\delta_K(\Lambda_K), \delta_Q(\epsilon)] = -\delta_S(\Lambda_K \epsilon)$ on the spinors. This expresses the S -supersymmetry variations in terms of k_D^A ,

$$\delta_S(\eta) \zeta^\alpha = V_{iA}^\alpha k_D^A \eta^i, \quad \delta_S(\eta) \zeta^{\bar{\alpha}} = \bar{V}_A^{i\bar{\alpha}} k_D^A \eta_i. \tag{3.2}$$

With this result we first evaluate the commutator of an S - and a Q -supersymmetry transformation on the scalars. This yields

$$[\delta_S(\eta), \delta_Q(\epsilon)] \phi^A = (\bar{\epsilon}^i \eta_i + \bar{\epsilon}_i \eta^i) k_D^A + 2 J_{ik}{}^A{}_B \epsilon^{kj} (\bar{\epsilon}^i \eta_j - \bar{\epsilon}_j \eta^i) k_D^B. \tag{3.3}$$

This result can be confronted with the corresponding expression from the $N = 2$ superconformal algebra, which reads

$$\begin{aligned} [\delta_S(\eta), \delta_Q(\epsilon)] &= \delta_M(2\bar{\eta}^i \sigma^{ab} \epsilon_i + \text{h.c.}) + \delta_D(\bar{\eta}_i \epsilon^i + \text{h.c.}) + \delta_{U(1)}(i\bar{\eta}_i \epsilon^i + \text{h.c.}) \\ &\quad + \delta_{SU(2)}(-2\bar{\eta}^i \epsilon_j - \text{h.c.}; \text{traceless}). \end{aligned} \tag{3.4}$$

Comparison thus shows that $k_{U(1)}^A$ vanishes and that the $SU(2)$ vectors satisfy

$$k_{ij}^A = J_{ij}{}^A{}_B k_D^B. \tag{3.5}$$

Now we proceed to impose the same commutator on the fermions, where on the right-hand side we find a Lorentz transformation, a $U(1)$ transformation and a dilatation, if and only if we assume the following condition on k_D^A :

$$D_A k_D^B = \delta_A^B. \tag{3.6}$$

The geometric significance of these results will be discussed in later subsections. Here we note that (3.6) suffices to show that the kinetic term of the scalar fields is invariant under dilatations, provided one includes a space-time metric or, in flat space-time, includes corresponding scale transformations of the space-time coordinates. Nevertheless, observe that k_D^A is *not* a Killing vector of the hyper-Kähler space, although it still satisfies (2.21), but an example of a conformal homothetic Killing vector. Another consequence is that the $SU(2)$ vectors k_{ij}^A , as expressed by (3.5), are themselves Killing vectors, because their derivative is proportional to the corresponding antisymmetric complex structure

$$D_A k_B^{ij} = -J_{AB}^{ij}. \tag{3.7}$$

From this it follows that the Kähler two-forms are exact, provided that the Killing vectors are globally defined. The product rule of the $SU(2)$ Killing vectors can now be worked out and one finds

$$k^B{}^{ij} \partial_B k^{Akl} - k^{Bkl} \partial_B k^{Aij} = 2 k^{A(i(k} \epsilon^{l)j)}, \tag{3.8}$$

which is indeed in accord with the $SU(2)$ structure constants.

From the $[\delta_S, \delta_Q]$ commutator we also establish the fermionic transformation rules under the chiral transformations and the dilatations,

$$\begin{aligned} \delta_{\text{SU}(2)} \zeta^\alpha + \delta_{\text{SU}(2)} \phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta &= 0, \\ \delta_{\text{U}(1)} \zeta^\alpha + \delta_{\text{U}(1)} \phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta &= -\frac{1}{2} i \theta_{\text{U}(1)} \zeta^\alpha, \\ \delta_D \zeta^\alpha + \delta_D \phi^A \Gamma_A^\alpha{}_\beta \zeta^\beta &= \frac{3}{2} \theta_D \zeta^\alpha. \end{aligned} \tag{3.9}$$

Note that the U(1) transformation further simplifies because $\delta_{\text{U}(1)} \phi^A = 0$.

To establish that the model as a whole is now invariant under the superconformal transformations it remains to be shown that the tensor V_{Ai}^α is invariant under the diffeomorphisms generated by k_{ij}^A , $k_{\text{U}(1)}^A$ and k_D^A up to compensating transformations that act on the $\text{Sp}(n) \times \text{Sp}(1)$ indices in accordance with the transformations of the ζ^α given above and the symmetry assignments of the supersymmetry parameters ϵ^i . To emphasize the systematics we ignore the fact that $k_{\text{U}(1)}^A$ actually vanishes and we write

$$\begin{aligned} -k_{kl}^B \partial_B V_{Ai}^\alpha - \partial_A k_{kl}^B V_{Bi}^\alpha - k_{kl}^B \Gamma_B^\alpha{}_\beta V_{Ai}^\beta + [-\delta_{(k}^j \epsilon_{l)i}] V_{Aj}^\alpha &= 0, \\ -k_{\text{U}(1)}^B \partial_B V_{Ai}^\alpha - \partial_A k_{\text{U}(1)}^B V_{Bi}^\alpha + \left[-\frac{1}{2} i \delta_\beta^\alpha - k_{\text{U}(1)}^B \Gamma_B^\alpha{}_\beta\right] V_{Ai}^\beta + \left[\frac{1}{2} i \delta_i^j\right] V_{Aj}^\alpha &= 0, \\ -k_D^B \partial_B V_{Ai}^\alpha - \partial_A k_D^B V_{Bi}^\alpha + \left[\frac{3}{2} \delta_\beta^\alpha - k_D^B \Gamma_B^\alpha{}_\beta\right] V_{Ai}^\beta + \left[-\frac{1}{2} \delta_i^j\right] V_{Aj}^\alpha &= 0. \end{aligned} \tag{3.10}$$

In these equations the first two terms on the left-hand side represent the effect of the isometry or dilatation, the third term represents a uniform scale and chiral U(1) transformation on the indices associated with the $\text{Sp}(n)$ tangent space, and the last terms represent an SU(2), a U(1) and a scale transformation, respectively, on the indices associated with $\text{Sp}(1)$. Eq. (3.10) should be regarded as a direct extension of (2.22).

We close with a few comments. First of all, the SU(2) isometries induce a rotation on the complex structures,

$$k_{kl}^C \partial_C J_{AB}^{ij} - 2 \partial_{[A} k_{kl}^C J_{B]C}^{ij} = -2 J_{kl[A} J_{B]}^{ijC} = 2 \delta_{(k}^{(i} \epsilon_{l)m} J_{AB}^{j)m}, \tag{3.11}$$

as should be expected. Under dilatations, the Kähler two-forms J_{AB} scale with weight two, whereas the complex structures $J^A{}_B$ are invariant.

Secondly, one can verify that the isometries introduced in Subsection 2.2 commute with scale transformations, provided that

$$k_I^A = k_D^B D_B k_I^A. \tag{3.12}$$

This leads to another identity,

$$g_{AB} k_I^A k_D^B = 0. \tag{3.13}$$

In particular these results hold for the SU(2) Killing vectors and imply, in addition, that the latter commute with the tri-holomorphic isometries. To see this, one writes $k_{ij}^B D_B k_{IA}$ as $k_D^B D_A \partial_B P_{Iij}$ using (3.5), (2.29) and the fact that the complex structures are

covariantly constant. Interchanging the order of the derivatives and extracting the complex structure then gives

$$k_{ij}^B D_B k_I^A = J_{ijB}^A k_I^B, \tag{3.14}$$

which implies that the tri-holomorphic Killing vectors commute with SU(2). From the above equations one can derive the following result for the variation of the moment maps under a dilatation:

$$k_D^A \partial_A P_I^{ij} = J_{AB}^{ij} k_D^A k_I^B = -k_A^{ij} k_I^A = 2 P_I^{ij}, \tag{3.15}$$

i.e. they scale with conformal weight 2. Here we have adjusted an integration constant in P_I^{ij} in the last equation. Combining the above equation with previous results, one establishes that the moment maps transform under SU(2) according to

$$k_{kl}^A \partial_A P_I^{ij} = 2 \delta_{(k}^{(i} \varepsilon_{l)m} P_I^{j)m}. \tag{3.16}$$

The latter expresses the fact that the moment maps form a triplet under SU(2). It is then easy to check that the action is invariant under dilatations, U(1) and SU(2).

3.2. Hyper-Kähler potential and Sp(n) × Sp(1) sections

The existence of the homothetic Killing vector satisfying (3.6) has important consequences for the geometry. First of all (3.6) implies that k_D^A can (locally) be expressed in terms of a potential χ , according to $k_{DA} = \partial_A \chi$. Up to a suitable additive integration constant, one can then show that [12]

$$\chi(\phi) = \frac{1}{2} g_{AB} k_D^A k_D^B. \tag{3.17}$$

Observe that χ is positive for a space of positive signature. A second (covariant) derivative acting on χ yields the metric, and therefore a third derivative vanishes,

$$D_A D_B \chi = g_{AB}, \quad D_A D_B D_C \chi = 0. \tag{3.18}$$

The first condition expresses the fact that the metric is the second (covariant) derivative of some function, somewhat analogous to the Kähler potential in Kähler metrics, but now written in real coordinates. A Kähler potential is guaranteed to exist for any hyper-Kähler space, but the potential χ does not always exist. In the literature χ is sometimes called the hyper-Kähler potential (see, e.g., Refs. [8,9]). This means that χ serves as a Kähler potential for each of the three complex structures, as follows from

$$\frac{1}{2} (\delta_A^C + J^A{}_A{}^C) (\delta_B^D - J^A{}_B{}^D) D_C D_D \chi = J_{AB}^A, \tag{3.19}$$

where $J^A = (\sigma_2 \sigma^A)_{ij} J^{ij}$ and $A = 1,2,3$ is kept fixed.

The hyper-Kähler potential χ is invariant under isometries, as follows directly from (3.13). In particular it is invariant under the SU(2) isometry; explicitly,

$$\delta \chi = (\theta_{\text{SU}(2)})^i{}_k \varepsilon^{jk} k_{ij}^B k_{ij}^B \partial_B \chi = (\theta_{\text{SU}(2)})^i{}_k \varepsilon^{jk} J_{ijAB} k_D^B k_D^A = 0, \tag{3.20}$$

where we made use of (3.5). However, it is not invariant under dilatations,

$$\delta \chi = k_D^B \partial_B \chi = 2 \chi. \tag{3.21}$$

Another interesting consequence of the homothety is that it enables a reformulation in terms of local sections of an $\text{Sp}(n) \times \text{Sp}(1)$ bundle. The existence of such a so-called associated quaternionic bundle is known from general arguments [9]. These sections are defined from the S -supersymmetry variation of the hypermultiplet spinors (c.f. (3.2)),

$$A_i^\alpha(\phi) \equiv k_D^B(\phi) V_{B i}^\alpha(\phi). \tag{3.22}$$

They satisfy a quaternionic pseudo-reality condition

$$A^{i\bar{\alpha}} \equiv (A_i^\alpha)^* = \varepsilon^{ij} \bar{\Omega}^{\bar{\alpha}\bar{\beta}} G_{\bar{\beta}\gamma} A_j^\gamma, \tag{3.23}$$

as follows from (2.12). Using (3.6) one proves that the covariant derivative of A_i^α reproduces the quaternionic vielbeine,

$$D_B A_i^\alpha = V_{B i}^\alpha, \quad \varepsilon_{ij} \Omega_{\bar{\alpha}\bar{\beta}} D_B A^{j\bar{\beta}} = g_{BC} \gamma_{i\bar{\alpha}}^C. \tag{3.24}$$

One easily verifies that the hyper-Kähler potential χ can be written as

$$\chi = \frac{1}{2} g_{AB} k_D^A k_D^B = \frac{1}{2} G_{\bar{\beta}\alpha} A_i^\alpha A^{i\bar{\beta}} = \frac{1}{2} \varepsilon^{ij} \bar{\Omega}_{\alpha\beta} A_i^\alpha A_j^\beta, \tag{3.25}$$

or

$$\bar{\Omega}_{\alpha\beta} A_i^\alpha A_j^\beta = \varepsilon_{ij} \chi. \tag{3.26}$$

We also note the following identity:

$$J^{ij}{}_B{}^C D_C A_k^\alpha = -\delta_k^{(i} \varepsilon^{j)l} D_B A_l^\alpha. \tag{3.27}$$

Furthermore we have

$$R_{AB}{}^\alpha{}_\beta A_i^\beta = R_{AB}{}^\alpha{}_\beta \bar{\Omega}_{\alpha\gamma} A_i^\gamma = 0, \tag{3.28}$$

which is a consequence of $D_A D_B A_i^\alpha = 0$ and the symplectic nature of the curvature $R_{AB}{}^\alpha{}_\beta$. This implies that the generic holonomy group is now reduced from $\text{Sp}(n)$ to $\text{Sp}(n-1)$. Also, using (3.5), (3.26) and (3.27), one finds

$$k_D^B D_B A_i^\alpha = A_i^\alpha, \quad k^{ijB} D_B A_k^\alpha = \delta_k^{(i} \varepsilon^{j)l} A_l^\alpha, \tag{3.29}$$

so that

$$\bar{\Omega}_{\alpha\beta} A_i^\alpha D_B A_j^\beta = \frac{1}{2} \varepsilon_{ij} k_{DB} + k_{ijB}. \tag{3.30}$$

Applying a second derivative D_A to the above relation gives

$$\bar{\Omega}_{\alpha\beta} D_A A_i^\alpha D_B A_j^\beta = \frac{1}{2} \varepsilon_{ij} g_{AB} - J_{ijAB}. \tag{3.31}$$

Note that the quantities in (3.31) have weight 2 under the homothety. For future use we also recall some earlier results, but now expressed in terms of the local sections,

$$\begin{aligned} g^{AB} D_A A_i^\alpha D_B A_j^\beta &= \varepsilon_{ij} \Omega^{\alpha\beta}, \\ g^{AB} D_A A_i^\alpha D_B A^{j\bar{\beta}} &= \delta_i^j G^{\alpha\bar{\beta}}, \\ R_{AB}{}^\gamma{}_\alpha \bar{\Omega}_{\gamma\beta} D_C A_i^\alpha D_D A_j^\beta \varepsilon^{ij} &= R_{ABCD}. \end{aligned} \tag{3.32}$$

3.3. The hypermultiplet action and transformation rules

In this subsection, we write the hypermultiplet action and transformation rules in terms of the sections $A_i^\alpha(\phi)$ introduced in (3.22). The complete Lagrangian, including the terms associated with gauged isometries, can be written as

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} G_{\alpha\beta} \mathcal{D}_\mu A_i^\beta \mathcal{D}^\mu A^{i\bar{\alpha}} - G_{\alpha\beta} \left(\bar{\zeta}^{\bar{\alpha}} \mathcal{D}^j \zeta^\beta + \bar{\zeta}^{\beta} \mathcal{D}^j \zeta^{\bar{\alpha}} \right) \\ & - \frac{1}{4} W_{\alpha\beta\gamma\delta} \bar{\zeta}^{\bar{\alpha}} \gamma_\mu \zeta^\beta \bar{\zeta}^{\bar{\gamma}} \gamma^\mu \zeta^\delta \\ & + \left[2g \bar{X}^\gamma_\alpha \bar{\Omega}_{\beta\gamma} \bar{\zeta}^{\bar{\alpha}} \zeta^\beta + 2g \bar{\zeta}^{\bar{\alpha}} \bar{\Omega}_{\alpha\beta} \Omega^{i\beta}_\gamma A_i^\gamma + \text{h.c.} \right] \\ & + 2g^2 G_{\alpha\beta} A^{i\bar{\alpha}} \bar{X}^\beta_\gamma X^\gamma_\delta A_i^\delta + \frac{1}{2} g A_i^\alpha \bar{\Omega}_{\alpha\beta} Y^{ij\beta}_\gamma A_j^\gamma, \end{aligned} \quad (3.33)$$

where the covariant derivatives are defined by

$$\begin{aligned} \mathcal{D}_\mu A_i^\alpha &= \partial_\mu A_i^\alpha + \partial_\mu \phi^A \Gamma_{A\beta}^\alpha A_i^\beta - g W_\mu^\alpha{}_\beta A_i^\beta, \\ \mathcal{D}_\mu \zeta^\alpha &= \partial_\mu \zeta^\alpha + \partial_\mu \phi^A \Gamma_{A\beta}^\alpha \zeta^\beta - g W_\mu^\alpha{}_\beta \zeta^\beta, \end{aligned} \quad (3.34)$$

and we have used Lie-algebra valued vector multiplet fields associated with gauged isometries, $W_\mu^\alpha{}_\beta$, $X^\alpha{}_\beta$, $Y^{ij\alpha}{}_\beta$ and $\Omega^{i\alpha}{}_\beta$ (for the precise definition, see below), In addition to the equation in Subsection 3.2 we made use of the identities,

$$\begin{aligned} k_I^A V_{Ai}^\alpha &= k_I^A D_A A_i^\alpha = t_I^\alpha{}_\beta A_i^\beta, \\ P_{Iij} &= -\frac{1}{2} k_{Aij} k_I^A = -\frac{1}{2} \bar{\Omega}_{\alpha\beta} A_i^\alpha (t_I)^\beta{}_\gamma A_j^\gamma. \end{aligned} \quad (3.35)$$

The first relation follows from (2.23) and (3.12), and for the second equation we made use of the last equality in (3.15).

The action may be compared to the one in [6] (more precisely, to the part that pertains to the rigidly supersymmetric Lagrangian). However, in that reference, the A_i^α are identical to the coordinate fields, whereas in the present more general case they are local sections as explained in Subsection 3.2. Because the target-space manifold is not flat, we encounter a non-trivial metric in (3.33) as well as non-trivial $\text{Sp}(n)$ connections in the covariant derivatives (3.34). Furthermore, the generators $t_I(\phi)$ associated with the isometries are not constant, but depend on the scalar fields as we indicated before. This means that the Lie-algebra valued vector multiplet fields associated with the gauged isometries depend also on the hypermultiplet scalars. Their definitions are

$$\begin{aligned} W_\mu^\alpha{}_\beta &= W_\mu^I [t_I(\phi)]^\alpha{}_\beta, \\ X^\alpha{}_\beta &= X^I [t_I(\phi)]^\alpha{}_\beta, \quad \bar{X}^\alpha{}_\beta = \bar{X}^I [t_I(\phi)]^\alpha{}_\beta, \\ Y^{ij\alpha}{}_\beta &= Y^{Iij} [t_I(\phi)]^\alpha{}_\beta, \\ \Omega^{i\alpha}{}_\beta &= \Omega^{Ii} [t_I(\phi)]^\alpha{}_\beta, \quad \Omega_i^\alpha{}_\beta = \Omega_i^I [t_I(\phi)]^\alpha{}_\beta. \end{aligned} \quad (3.36)$$

Nevertheless, the correspondence with the formulation in [6] will be helpful later on when evaluating the coupling to conformal supergravity.

In order to obtain the transformation rules of the $\text{Sp}(n) \times \text{Sp}(1)$ sections under dilations, $\text{SU}(2)$ and isometry transformations, we use the general relation

$$\delta A_i^\alpha = \delta\phi^B \partial_B A_i^\alpha = \delta\phi^B V_{Bi}^\alpha - \delta\phi^B \Gamma_B^\alpha A_i^\beta. \tag{3.37}$$

Using (2.9), (2.23) and (3.12), we then find for a combined dilatation, chiral transformation and target-space isometry, that

$$\delta A_i^\alpha = \theta_D A_i^\alpha + (\theta_{\text{SU}(2)})_i^j A_j^\alpha + g\theta^I t_I^\alpha A_i^\beta - \delta\phi^A \Gamma_A^\alpha A_i^\beta. \tag{3.38}$$

This result should be combined with that for the fermions, derived in Section 2,

$$\delta\zeta^\alpha = \frac{3}{2}\theta_D \zeta^\alpha - \frac{1}{2}i\theta_{\text{U}(1)} \zeta^\alpha + g\theta^I t_I^\alpha \zeta^\beta - \delta\phi^A \Gamma_A^\alpha \zeta^\beta. \tag{3.39}$$

Similarly we determine the transformations under Q - and S -supersymmetry,

$$\begin{aligned} \delta A_i^\alpha &= 2\bar{\epsilon}_i \zeta^\alpha + 2\epsilon_{ij} G^{\alpha\bar{\beta}} \Omega_{\bar{\beta}\bar{\gamma}} \bar{\epsilon}^j \zeta^{\bar{\gamma}} - \delta_Q \phi^B \Gamma_B^\alpha A_i^\beta, \\ \delta\zeta^\alpha &= \not{D} A_i^\alpha \epsilon^i - \delta_Q \phi^B \Gamma_B^\alpha \zeta^\beta + 2gX^\alpha_\beta A_i^\beta \epsilon^{ij} \epsilon_j + A_i^\alpha \eta^i, \\ \delta\zeta^{\bar{\alpha}} &= \not{D} A^i \bar{\alpha}_i \epsilon_i - \delta_Q \phi^B \Gamma_B^{\bar{\alpha}} \zeta^{\bar{\beta}} + 2g\bar{X}^{\bar{\alpha}}_{\bar{\beta}} A^i \bar{\beta} \epsilon_{ij} \epsilon^j + A^i \bar{\alpha}_i \eta_i. \end{aligned} \tag{3.40}$$

Again, we stress that, apart from the $\text{Sp}(n)$ connection (and a slight change in notation), these transformation rules are identical to the ones specified for a flat target space [6], where the local sections can be identified directly with the target-space coordinates.

Finally, we recall that it is straightforward to write down actions for the vector multiplets that are invariant under rigid $N=2$ superconformal transformations. Those are based on a holomorphic function that is homogeneous of degree two [24].

4. Cone structure and quaternionic geometry

In this section we discuss the properties of the special hyper-Kähler space. We will show how this space can be described as a cone over a tri-Sasakian manifold. The latter spaces (which are of dimension $4n - 1$) are characterized by the existence of three (1,1) tensors and three Killing vectors that are subject to certain conditions. A manifold is tri-Sasakian if and only if its cone is hyper-Kähler. Tri-Sasakian spaces are Einstein and take the form of an $\text{Sp}(1)$ fibration over a quaternionic space. This quaternionic space is the one that appears in the coupling of hypermultiplets to supergravity (for more details, see Ref. [10], where the relation between special hyper-Kähler, tri-Sasakian and quaternionic spaces is reviewed from a more mathematical viewpoint).

We start by noting that the Riemann tensor vanishes upon contraction with any one of the four vectors (k_D^A, k_{ij}^A) , i.e.

$$R_{ABCE} k_D^E = 0, \quad R_{ABCE} k_{ij}^E = 0. \tag{4.1}$$

The first equation (4.1) is derived by antisymmetrizing the second equation (3.18) in the indices $[AB]$. The second (4.1) follows from inserting (3.7) into (2.21). Incidentally,

(4.1) implies that the Ricci tensor has at least four null vectors. However, in the case at hand this poses no extra restrictions as hyper-Kähler spaces are Ricci-flat. The above results can also be derived from the fact that the $\text{Sp}(n)$ holonomy group is reduced to $\text{Sp}(n-1)$, c.f. (3.28). This follows from applying (3.30).

We recall that these four vectors are orthogonal (cf. (3.5), (3.17)),

$$k_D^A k_{DA} = 2\chi, \quad k_{ij}^A k_A^{kl} = \delta_{(i}^k \delta_{j)}^l \chi, \quad k_D^A k_A^{ij} = 0. \tag{4.2}$$

This implies that the hyper-Kähler manifold is locally a product $\mathbf{R}^4 \times \mathbf{Q}^{4n-4}$, where \mathbf{R}^4 denotes a flat four-dimensional space. By decomposing \mathbf{R}^4 as $R^+ \times S^3$, we can write the hyper-Kähler manifold as a cone over a so-called tri-Sasakian manifold; the latter is then a fibration of $\text{Sp}(1)$ over \mathbf{Q}^{4n-4} . Hence the manifold can be written as $R^+ \times [\text{Sp}(1) \times \mathbf{Q}^{4n-4}]$. Spaces with a homothety can always be described as a cone. This becomes manifest when decomposing the coordinates ϕ^A into coordinates tangential and orthogonal to the $(4n-1)$ -dimensional hypersurface defined by setting χ to a constant. The line element can then be written in the form [12],

$$ds^2 = \frac{d\chi^2}{2\chi} + 2\chi h_{ab}(x) dx^a dx^b, \tag{4.3}$$

where the x^a are the coordinates associated with the hypersurface³. In the present case this hypersurface must be a tri-Sasakian space and the hyper-Kähler space is therefore a cone over the tri-Sasakian space.

The purpose of the remainder of this section is to establish that \mathbf{Q}^{4n-4} is a quaternionic manifold. In Section 5 we show how \mathbf{Q}^{4n-4} arises in the coupling of hypermultiplets to supergravity. The tangent space of the hyper-Kähler space can be decomposed into the four directions along (k_D^A, k_{ij}^A) , and a $(4n-4)$ -dimensional space \mathbf{Q}^{4n-4} that is locally orthogonal to that. Tensors that vanish upon contraction with (k_D^A, k_{ij}^A) will be called *horizontal*.

Let us introduce a vector field \mathcal{V}_{Aij} which will serve as a connection for $\text{Sp}(1)$ in a way that will become clear shortly,

$$\mathcal{V}_{Aij} = \frac{k_{ijA}}{\chi} = J_{ijA}{}^B \partial_B \ln \chi. \tag{4.4}$$

This vector field is invariant under target-space dilatations and gauge isometries, i.e.

$$\begin{aligned} \delta_D \mathcal{V}_{Aij} &= k_D^B \partial_B \mathcal{V}_{Aij} + \partial_A k_D^B \mathcal{V}_{Bij} = 0, \\ \delta_G \mathcal{V}_{Aij} &= k_I^B \partial_B \mathcal{V}_{Aij} + \partial_A k_I^B \mathcal{V}_{Bij} = 0, \end{aligned} \tag{4.5}$$

and rotates under target-space $\text{SU}(2)$, as follows from

$$\delta \mathcal{V}_A^{ij} = k^{Bkl} \partial_B \mathcal{V}_A^{ij} + \partial_A k^{Bkl} \mathcal{V}_B^{ij} = 2\varepsilon^{(ik} \mathcal{V}_A^{l)j}. \tag{4.6}$$

² Strictly speaking it is $\text{Sp}(1)/\mathbf{Z}_2$ where $\text{Sp}(1)$ is the group that acts on the quaternionic vielbeine and on the sections introduced in the previous chapter.

³ In terms of a radial variable $r^2 = 2\chi$, this yields the usual form of a cone metric $ds^2 = dr^2 + r^2 h_{ab}(x) dx^a dx^b$.

With \mathcal{Z}_{Aij} we associate an $\text{Sp}(1)$ curvature tensor,

$$\begin{aligned} R_{ABij} &\equiv \partial_A \mathcal{Z}_{Bij} - \partial_B \mathcal{Z}_{Aij} - \varepsilon^{kl} (\mathcal{Z}_{Aik} \mathcal{Z}_{Bjl} + \mathcal{Z}_{Ajk} \mathcal{Z}_{Bil}) \\ &= \chi^{-1} \Delta_{\alpha\beta} \left[D_A A_i^\alpha D_B A_j^\beta + D_A A_j^\alpha D_B A_i^\beta \right], \end{aligned} \tag{4.7}$$

where we have used the definition

$$\Delta_{\alpha\beta} = \bar{\Omega}_{\alpha\beta} - \frac{1}{\chi} \varepsilon^{kl} (\bar{\Omega}_{\alpha\gamma} A_k^\gamma) (\bar{\Omega}_{\beta\delta} A_l^\delta). \tag{4.8}$$

Observe that $\Delta_{\alpha\beta}$ is a projection operator, i.e. it satisfies $\Delta_{\alpha\beta} \Omega^{\beta\gamma} \Delta_{\gamma\delta} = -\Delta_{\alpha\delta}$, and it projects onto the $(2n - 2)$ -dimensional subspace orthogonal to the A_i^α ,

$$\Delta_{\alpha\beta} A_i^\beta = 0. \tag{4.9}$$

Note that we have $k_D^B D_B \Delta_{\alpha\beta} = k_{ij}^B D_B \Delta_{\alpha\beta} = 0$, so that $\Delta_{\alpha\beta}$ is invariant under dilations and $\text{SU}(2)$ transformations. One can also show that $\Delta_{\alpha\beta} D_B A_i^\beta$ is horizontal, i.e.

$$k_D^B \Delta_{\alpha\beta} D_B A_i^\beta = k_{ij}^B \Delta_{\alpha\beta} D_B A_i^\beta = 0. \tag{4.10}$$

The identity (4.7) can be generalized to

$$\chi^{-1} \Delta_{\alpha\beta} D_A A_i^\alpha D_B A_j^\beta = \frac{1}{2} \varepsilon_{ij} G_{AB} + \frac{1}{2} R_{ABij}, \tag{4.11}$$

where

$$G_{AB} = \chi^{-1} \varepsilon^{ij} \Delta_{\alpha\beta} D_A A_i^\alpha D_B A_j^\beta. \tag{4.12}$$

Observe that both G_{AB} and R_{ABij} are of zero weight under the homothety and are horizontal, i.e. they vanish upon contraction with any of the four vectors (k_D^A, k_{ij}^A) , and are thus orthogonal to the corresponding (local) four-dimensional subspace.

The tensor G_{AB} will provide a metric for \mathbb{Q}^{4n-4} . The relation between G_{AB} and the hyper-Kähler metric g_{AB} is given by

$$\begin{aligned} g_{AB} &= \frac{1}{2\chi} k_{DA} k_{DB} + \frac{1}{\chi} k_{Aij} k_B^{ij} + \chi G_{AB} \\ &= \frac{1}{2\chi} k_{DA} k_{DB} + \chi [\mathcal{Z}_{Aij} \mathcal{Z}_B^{ij} + G_{AB}], \end{aligned} \tag{4.13}$$

where we have used (3.30) and (3.31). Observe that this relation reflects both the cone structure of the hyper-Kähler space and the $\text{Sp}(1)$ fibration of the tri-Sasakian space. It is not possible to give an explicit expression for the inverse metric, at least not in general, but this is not really needed in view of the horizontality of G_{AB} . When acting on horizontal tensors, χg^{AB} acts as the inverse metric in view of the identity

$$G_{AC} g^{CD} G_{DB} = \chi^{-1} G_{AB}. \tag{4.14}$$

We already showed that $\Delta_{\alpha\beta} D_B A_i^\beta$ was horizontal, and conversely, the horizontal projection $G_{AB} g^{BC} D_C A_i^\alpha$ is in the $(2n - 2)$ -dimensional eigenspace projected onto by $\Delta_{\alpha\beta}$. Therefore $\Delta_{\alpha\beta} D_B A_i^\beta$ is a candidate for the quaternionic vielbein associated with

\mathbf{Q}^{4n-4} and $\Delta_{\alpha\beta}$ projects onto the tangent space of \mathbf{Q}^{4n-4} . More precisely, we introduce the following related sets of $4n - 4$ vectors:

$$\begin{aligned} \hat{V}_{Ai}^\alpha &\equiv -\frac{1}{\sqrt{\chi}} \Omega^{\alpha\beta} \Delta_{\beta\gamma} V_{Ai}^\gamma = -\frac{1}{\sqrt{\chi}} \Omega^{\alpha\beta} \Delta_{\beta\gamma} D_A A_i^\gamma, \\ \hat{\gamma}_{Ai\bar{\alpha}} &\equiv \frac{1}{\sqrt{\chi}} G_{AB} \gamma_{i\bar{\alpha}}^B = \varepsilon_{ij} \bar{\Delta}_{\bar{\alpha}\bar{\beta}} \bar{V}_A^{j\bar{\beta}}, \end{aligned} \tag{4.15}$$

which satisfy algebraic relations that are completely analogous to those satisfied by the quaternionic vielbeine of the hyper-Kähler space. In particular we note that \hat{V}_A and $\hat{\gamma}_A$ are each other's inverse in the reduced $(4n - 4)$ -dimensional space,

$$\chi g^{AB} \bar{\gamma}_{A\alpha}^i \hat{V}_{Bj}^\beta = \delta_j^i \Delta_{\alpha\gamma} \Omega^{\beta\gamma}, \quad \varepsilon^{ij} \Delta_{\alpha\beta} \hat{V}_{Ai}^\alpha \hat{V}_{Bj}^\beta = G_{AB}, \tag{4.16}$$

where $\Delta_{\alpha\gamma} \Omega^{\beta\gamma}$ is the identity matrix projected onto the $(2n - 2)$ -dimensional subspace. The significance of these results will become clear in due course.

Subsequently we note that there exists an identity similar to (4.13) which relates the complex structures to the field strength R_{ABij} ,

$$J_{ijAB} = -\frac{1}{\chi} \left[k_{D[A} k_{ijB]} + \varepsilon^{kl} k_{ki[A} k_{ljB]} \right] - \frac{1}{2} \chi R_{ABij}. \tag{4.17}$$

This motivates us to introduce the following tensors:

$$\mathcal{F}_{AB}^{ij} = J_A^{ijC} G_{CB}. \tag{4.18}$$

A straightforward calculation using (4.17) shows that they satisfy

$$\mathcal{F}_{ABij} = -\frac{1}{2} R_{ABij}, \tag{4.19}$$

so that the \mathcal{F}_{ABij} are antisymmetric, horizontal and scale invariant. Furthermore these tensors satisfy the product rule

$$\chi \mathcal{F}_{AC}^{ij} g^{CD} \mathcal{F}_{DB}^{kl} = \frac{1}{2} \varepsilon^{i(k} \varepsilon^{l)j} G_{AB} + \varepsilon^{(ik} \mathcal{F}_{AB}^{l)j}), \tag{4.20}$$

which is similar to (2.8). The tensors \mathcal{F}_{ABij} are candidate almost-complex structures in the horizontal subspace \mathbf{Q}^{4n-4} . Under $SU(2)$ target-space transformations they rotate into each other according to

$$k_{kl}^C \partial_C \mathcal{F}_{ABij} + \partial_A k_{kl}^C \mathcal{F}_{CBij} + \partial_B k_{kl}^C \mathcal{F}_{ACij} = 2 \varepsilon_{(ik} \mathcal{F}_{AB)lj}). \tag{4.21}$$

Given a horizontal tensor $H_{AB\dots}$ that is invariant under the homothety and the $SU(2)$ target-space transformations, then the covariant derivative of such a tensor is no longer horizontal. This can be cured by making use of a modified covariant derivative \hat{D}_A , defined so that the following properties hold:

$$\begin{aligned} k_D^A \hat{D}_C H_{AB\dots} &= k_D^C \hat{D}_C H_{AB\dots} = 0, \\ k_{ij}^A \hat{D}_C H_{AB\dots} &= k_{ij}^C \hat{D}_C H_{AB\dots} = 0. \end{aligned} \tag{4.22}$$

The modified derivative is obtained by using a modified target-space connection,

$$\hat{\Gamma}_{AB}^C = \Gamma_{AB}^C - \delta_{(A}^C \partial_{B)} \ln \chi + 2 \mathcal{V}_{(Aij} J_{B)}^{ijC}. \tag{4.23}$$

Because the modification is symmetric in (A, B) , the connection remains torsion free. Observe that $\hat{D}_A(\chi g^{BC})$, $\hat{D}_A k_D^B$ and $\hat{D}_A k_{ij}^B$ should be zero when contracted with a horizontal tensor. This is obviously the case as can be seen from the formulae

$$\begin{aligned} \hat{D}_A(\chi g^{BC}) &= -\delta_A^{(B} k_D^{C)} + 2 J_A^{ij(B} k_{ij}^{C)}, \\ \hat{D}_A k_D^B &= \chi^{-1} \left(-\frac{1}{2} k_{DA} k_D^B + k_{ijA} k^{ijB} \right), \\ \hat{D}_A k_{ij}^B &= \chi^{-1} \left(-\frac{1}{2} k_{DA} k_{ij}^B + \frac{1}{2} k_{ijA} k_D^B - k_A^{kl} \varepsilon_{k(i} k_{j)l}^B \right). \end{aligned} \tag{4.24}$$

The above construction can be generalized to tensors H that carry also $SU(2)$ indices, indicating that they transform covariantly under target-space $SU(2)$ transformations, e.g. as in $k_{kl}^A \partial_A H^i = \delta_{(k}^i \varepsilon_{l)j} H^j$ in the simplest case. Then one can show that the derivatives of these tensors are still horizontal, provided one covariantizes \hat{D}_A and includes an $SU(2)$ connection \mathcal{V}_{Aij} . The crucial identity for showing this is $k_{ij}^A \mathcal{V}_A^{kl} = \delta_{(i}^k \delta_{j)l}^l$.

With respect to the new connection, G_{AB} is covariantly constant,

$$\hat{D}_C G_{AB} = 0, \tag{4.25}$$

so that the new connection must be just the Christoffel connection associated with G_{AB} . Likewise the tensors \mathcal{F}_{ABij} are covariantly constant modulo a rotation that involves the $Sp(1)$ connection,

$$\hat{D}_C \mathcal{F}_{ABij} = 2 \mathcal{V}_{Ck(i} \mathcal{F}_{ABj)l} \varepsilon^{kl}. \tag{4.26}$$

Note that the terms on the right-hand side covariantize the derivative on the left-hand side with respect to $SU(2)$. Hence the tensors \mathcal{F}_{ABij} define three almost-complex structures in the space \mathbf{Q}^{4n-4} which are covariantly constant up to an $Sp(1)$ rotation proportional to the $Sp(1)$ connections. This implies that \mathbf{Q}^{4n-4} is a quaternionic space (see, e.g., Refs. [25–27]).

To verify this result, let us compute the Riemann tensor associated with the new connection (4.23).

$$\hat{R}_{ABC}{}^D = R_{ABC}{}^D - G_{C[A} \delta_{B]}^D + R_{ABij} J_C^{ijD} - R_{C[Aij} J_{B]}^{ijD}. \tag{4.27}$$

Observe that the right-hand side is not horizontal, but by construction (via the Ricci identity) is horizontal when acting on a horizontal tensor with lower index D . Hence, when lowering the index by contraction with the metric G_{DE} one must obtain a horizontal tensor. This is confirmed by explicit construction,

$$\hat{R}_{ABCD} \equiv \hat{R}_{ABC}{}^E G_{ED} = \chi^{-1} R_{ABCD} + G_{D[A} G_{B]C} + R_{ABij} \mathcal{F}_{CD}^{ij} - R_{C[Aij} \mathcal{F}_{B]D}^{ij}. \tag{4.28}$$

By virtue of (4.19) \hat{R}_{ABCD} has all the symmetry properties of a Riemann tensor. Observe that the explicit factor of χ^{-1} arises because the original curvature of the

hyper-Kähler manifold is defined by lowering the upper index by means of the metric g_{DE} . Furthermore it satisfies the Bianchi identity $\hat{D}_{[A}\hat{R}_{BC]DE} = 0$.

Let us now calculate the Ricci tensor, which is symmetric by virtue of (4.25),

$$\hat{R}_{AB} = \chi \hat{R}_{ACBD} g^{CD} = -2(n + 1) G_{AB}. \tag{4.29}$$

Observe that we used that the original hyper-Kähler manifold was Ricci flat and that $G_{AB} g^{AB} \chi = 4(n - 1)$. We may also verify the expressions for the Sp(1) holonomy

$$\hat{R}_{ABCD} g^{CE} g^{DF} \chi^2 \mathcal{F}_{EF}^{ij} = -4(n - 1) \mathcal{F}_{AB}^{ij}, \tag{4.30}$$

where we used that the original hyper-Kähler manifold has zero Sp(1) holonomy. These are the expected results [28,29] for a $(4n - 4)$ -dimensional quaternionic manifold with Sp(1) curvature given by (4.19).

This completes the discussion of target-space properties. We now return to aspects related to the Sp(n) bundle over the special hyper-Kähler space. First of all we consider a modification of the connection $\Gamma_A^\alpha{}_\beta$ such that the modified derivative of a tensor that is orthogonal to A_i^α remains orthogonal. This requires that this derivative acting on A_i^α must be proportional to A_i^α itself. When combining this with a few other obvious requirements⁴,

$$\hat{I}_A^\alpha{}_\beta = \Gamma_A^\alpha{}_\beta - \frac{2}{\chi} \left[\varepsilon^{ij} A_i^{(\alpha} D_A A_j^{\gamma)} + A_i^\alpha A_j^\gamma \mathcal{V}_A^{ij} \right] \bar{\Omega}_{\gamma\beta}. \tag{4.31}$$

With this modification, the tensors $\bar{\Omega}_{\alpha\beta}$ and $G_{\bar{\alpha}\beta}$ remain covariantly constant. The presence of the term proportional to \mathcal{V}_A^{ij} is required to preserve covariance with respect to target-space SU(2) transformations. This term also ensures that the modification is horizontal. With the modified connection we establish the required result,

$$\hat{D}_A A_i^\alpha = \frac{1}{2} \partial_A \ln \chi A_i^\alpha + \mathcal{V}_{Aik} A_l^\alpha \varepsilon^{kl}, \tag{4.32}$$

where the last term can be interpreted as an SU(2) covariantization of the derivative on the left-hand side. The result (4.32) suffices to show that the modified derivative of a tensor that is orthogonal to A_i^α , will remain orthogonal. It is now obvious that the projection operator $\Delta_{\alpha\beta}$ is covariantly constant under the modified derivative

$$\hat{D}_A \Delta_{\alpha\beta} = 0. \tag{4.33}$$

Including the modified connections \hat{I}_{AB}^C and $\hat{I}_A^\alpha{}_\beta$ as well as the SU(2) connection \mathcal{V}_A^{ij} , one can explicitly verify that $\hat{D}_A V_{Bi}^\alpha$ is equal to $\frac{1}{2} \partial_A \ln \chi V_{Bi}^\alpha$, up to terms that

⁴ In determining the precise modifications of the various connections, we were also guided to some extent by supersymmetry. However, this aspect is postponed to Section 5, where we outline the significance of the results of this section in the context of the coupling of hypermultiplets to supergravity.

are proportional to A_k^α . This implies that the quaternionic vielbeine introduced in (4.15) are covariantly constant with respect to the new connections, so that we have

$$\hat{D}_A(A_i^\alpha/\sqrt{\chi}) = \hat{D}_A \hat{V}_{Bi}^\alpha = \hat{D}_A \hat{\gamma}_{Bi\bar{\alpha}} = 0. \tag{4.34}$$

This result leads to two integrability relations

$$\begin{aligned} \hat{R}_{AB}{}^\alpha{}_\beta A_i{}^\beta - R_{ABik} A_j{}^\alpha \varepsilon^{kj} &= 0, \\ \hat{R}_{ABCD} \bar{\gamma}_\alpha{}^{Di} + \hat{R}_{AB}{}^\beta{}_\alpha \bar{\gamma}_{C\beta}{}^i + R_{AB}^{ik} \bar{\gamma}_{C\alpha}{}^l \varepsilon_{kl} &= 0. \end{aligned} \tag{4.35}$$

Here $\hat{R}_{AB}{}^\alpha{}_\beta$ is the curvature associated with the new connection (4.31). We can explicitly evaluate this tensor,

$$\hat{R}_{AB}{}^\alpha{}_\beta = R_{AB}{}^\alpha{}_\beta + \frac{2}{\chi} \Omega^{\alpha\gamma} \Delta_{\gamma\delta} \varepsilon^{ij} D_A A_i{}^{(\delta} D_B A_j{}^{\epsilon)} \Delta_{\epsilon\beta} - \frac{1}{\chi} R_{AB}^{ij} A_i{}^\alpha A_j{}^\gamma \bar{\Omega}_{\gamma\beta}, \tag{4.36}$$

which indeed satisfies the first integrability relation. Note that all expressions appearing in (4.36) are horizontal.

Now we recall that for a special hyper-Kähler manifold the tensor $W_{\alpha\beta\gamma\delta}$ defined in (2.16) satisfies the constraint

$$W_{\alpha\beta\gamma\delta} A_i{}^\delta = 0. \tag{4.37}$$

With this in mind we write the new curvature tensors as follows:

$$\begin{aligned} \hat{R}_{ABCD} &= \frac{1}{2} \varepsilon^{ij} \varepsilon^{kl} \hat{V}_{Ai}^\alpha \hat{V}_{Bj}^\beta \hat{V}_{Ck}^\gamma \hat{V}_{Dl}^\delta \hat{W}_{\alpha\beta\gamma\delta} + G_{D[A} G_{B]C} \\ &\quad - 2 \mathcal{F}_{AB}^{ij} \mathcal{F}_{CDij} + 2 \mathcal{F}_{C[A}^ij \mathcal{F}_{B]Dij}, \\ \bar{\Omega}_{\alpha\epsilon} \hat{R}_{AB}{}^\epsilon{}_\beta &= -\varepsilon^{ij} \hat{V}_{Ai}^\gamma \hat{V}_{Bj}^\delta \left[\frac{1}{2} \hat{W}_{\alpha\beta\gamma\delta} + 2 \Delta_{\alpha(\gamma} \Delta_{\delta)\beta} \right] \\ &\quad + \chi^{-1} R_{AB}^{ij} A_i{}^\gamma A_j{}^\delta \bar{\Omega}_{\gamma\alpha} \bar{\Omega}_{\delta\beta}, \end{aligned} \tag{4.38}$$

where

$$\hat{W}_{\alpha\beta\gamma\delta} \equiv \chi W_{\alpha\beta\gamma\delta}. \tag{4.39}$$

One can now verify that these curvatures satisfy also the second integrability condition (4.35). We will return to this and related issues in Section 5.

We close this section with a brief discussion of the isometries. For every tri-holomorphic Killing vector of the special hyper-Kähler manifold we construct a corresponding vector in the horizontal manifold \mathbf{Q}^{4n-4} by the projection

$$\hat{k}_{IA} = G_{AB} k_I^B. \tag{4.40}$$

By explicit calculation one can then show that $\hat{D}_A \hat{k}_{IB} + \hat{D}_B \hat{k}_{IA} = 0$, so that we have a corresponding Killing vector in the horizontal space and thus an isometry. Observe that the SU(2) isometries of the special hyper-Kähler manifold do not generalize in this way, because the corresponding \hat{k}_{IA} would simply vanish. This is not so surprising, as the SU(2) acts on the corresponding tri-Sasakian space through its Sp(1) fibre.

To study whether these isometries are tri-holomorphic in the horizontal subspace, we first raise the index according to

$$\hat{k}_I^A = \chi g^{AB} G_{BC} k_I^C = k_I^A + 2 \hat{P}_I^{ij} k_{ij}^A. \tag{4.41}$$

where $\hat{P}_{Iij} = \chi^{-1} P_{Iij}$. The transformation of the almost complex structures in the horizontal subspace is then governed by the expression,

$$\begin{aligned} \hat{k}_I^C \partial_C \mathcal{F}_{ABij} - 2 \partial_{[A} \hat{k}_I^C \mathcal{F}_{B]Cij} &= k_I^C \partial_C \mathcal{F}_{ABij} - 2 \partial_{[A} k_I^C \mathcal{F}_{B]Cij} \\ &+ 2 \hat{P}_I^{kl} (k_{kl}^C \partial_C \mathcal{F}_{ABij} - 2 \partial_{[A} k_{kl}^C \mathcal{F}_{B]Cij}). \end{aligned} \tag{4.42}$$

The first line on the right-hand side is zero, as follows from (4.17) and the fact that the isometries are tri-holomorphic and commute with dilatations and SU(2) in the special hyper-Kähler space. The second line is equal to $4 \mathcal{F}_{ABk(i} \varepsilon_{j)l} \hat{P}_I^{kl}$ by virtue of (4.21). We can now elevate the derivatives on the left-hand side to SU(2) covariant derivatives. In this way we find

$$\hat{D}_A (\mathcal{F}_{BCij} \hat{k}_I^C) - \hat{D}_B (\mathcal{F}_{ACij} \hat{k}_I^C) = -2 R_{ABk(i} \varepsilon_{j)l} \hat{P}_I^{kl}, \tag{4.43}$$

where we used the horizontality of \hat{k}_I^C and the Bianchi identity for (or the covariant constancy of) $R_{ABij} \propto \mathcal{F}_{ABij}$. The solution is given by

$$\mathcal{F}_{ABij} \hat{k}_I^B = \hat{D}_A \hat{P}_{Iij}, \tag{4.44}$$

which can also be verified by explicit calculation. By substituting previous results one verifies directly the modified equivariance condition,

$$\mathcal{F}_{ABij} \hat{k}_I^A \hat{k}_J^B = -f_{IJ}{}^K \hat{P}_{Kij} + 4 \varepsilon^{kl} \hat{P}_{Ik(i} \hat{P}_{Jj)l}. \tag{4.45}$$

The above results are in complete agreement with the moment map construction for quaternionic manifolds [21,30]. The fact that the isometries generated by \hat{k}_I^A act consistently on horizontal tensors is ensured by the following identities which follow from explicit calculation:

$$k_D^B \hat{D}_B \hat{k}_I^A = k_{ij}^B \hat{D}_B \hat{k}_I^A = 0. \tag{4.46}$$

Finally the algebra of the isometries is governed by

$$\hat{k}_I^B \partial_B \hat{k}_J^A - \hat{k}_J^B \partial_B \hat{k}_I^A = -f_{IJ}{}^K \hat{k}_K^A + 2 \mathcal{F}_{BCij} \hat{k}_I^B \hat{k}_J^C k^{Aij}. \tag{4.47}$$

Hence the algebra of isometries is satisfied up to SU(2).

5. Locally superconformal hypermultiplets

In this last section we consider the coupling of the hypermultiplets to superconformal gravity. To that order we introduce the Weyl multiplet, which contains the gauge fields associated with the superconformal symmetries as well as some extra matter fields [5]. The bosonic gauge fields are the vielbeine e_μ^a , the spin-connection ω_μ^{ab} , the dilatational gauge field b_μ , the gauge field associated with special conformal boosts f_μ^a and the

gauge fields associated with $SU(2) \times U(1)$, denoted by V_{μ}^i (antihermitian) and A_{μ} . The fermionic gauge fields are the gravitino fields ψ_{μ}^i and the fields ϕ_{μ}^i associated with S -supersymmetry. Finally, the matter fields are T_{abij} (antisymmetric and self-dual in Lorentz indices and antisymmetric in $SU(2)$ indices), a spinor χ^i and a real scalar D . The fields ω_{μ}^{ab} , f_{μ}^a and ϕ_{μ}^i are not independent and can be expressed in terms of the other fields. We refer to [5,6] for more details on the notation and conventions.

The transformation rules have been given in previous sections, but will change in the context of local supersymmetry. The most obvious change concerns the replacement of the derivatives by derivatives that are covariant with respect to the additional gauge symmetries. The derivatives covariant with respect to the bosonic gauge symmetries for the scalar fields, the sections and the fermion fields, read

$$\begin{aligned} \mathcal{D}_{\mu} \phi^A &= \partial_{\mu} \phi^A - b_{\mu} k_D^A + \frac{1}{2} V_{\mu k}^i \varepsilon^{jk} k_{ij}^A - g W_{\mu}^I k_I^A, \\ \mathcal{D}_{\mu} A_i^{\alpha} &= \partial_{\mu} A_i^{\alpha} - b_{\mu} A_i^{\alpha} + \frac{1}{2} V_{\mu i}^j A_j^{\alpha} - g W_{\mu}^{\alpha \beta} A_i^{\beta} + \partial_{\mu} \phi^A \Gamma_A^{\alpha \beta} A_i^{\beta}, \\ \mathcal{D}_{\mu} \zeta^{\alpha} &= \partial_{\mu} \zeta^{\alpha} - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \zeta^{\alpha} - \frac{3}{2} b_{\mu} \zeta^{\alpha} + \frac{1}{2} i A_{\mu} \zeta^{\alpha} \\ &\quad - g W_{\mu}^{\alpha \beta} \zeta^{\beta} + \partial_{\mu} \phi^A \Gamma_A^{\alpha \beta} \zeta^{\beta}, \end{aligned} \tag{5.1}$$

where we have also included the terms related to possible gauged isometries. All covariantizations follow straightforwardly from the formulae presented in Subsection 3.3 and from the gauge field conventions given in [5,6]. Observe that the derivative in $\partial_{\mu} \phi^A$ multiplying the connection $\Gamma_A^{\alpha \beta}$ does *not* require an additional covariantization.

The transformation rules under Q - and S -supersymmetry are now as follows:

$$\begin{aligned} \delta \phi^A &= 2 \left(\gamma_{i\bar{\alpha}}^A \bar{\varepsilon}^i \zeta^{\bar{\alpha}} + \bar{\gamma}_{\alpha}^{Ai} \bar{\varepsilon}_i \zeta^{\alpha} \right), \\ \delta A_i^{\alpha} &= 2 \bar{\varepsilon}_i \zeta^{\alpha} + 2 \varepsilon_{ij} G^{\alpha\bar{\beta}} \Omega_{\bar{\beta}\bar{\gamma}} \bar{\varepsilon}^j \zeta^{\bar{\gamma}} - \delta_Q \phi^B \Gamma_B^{\alpha \beta} A_i^{\beta}, \\ \delta \zeta^{\alpha} &= \not{D} A_i^{\alpha} \varepsilon^i - \delta_Q \phi^B \Gamma_B^{\alpha \beta} \zeta^{\beta} + 2 g X^{\alpha \beta} A_i^{\beta} \varepsilon^{ij} \varepsilon_j + A_i^{\alpha} \eta^i, \\ \delta \zeta^{\bar{\alpha}} &= \not{D} A_i^{\bar{\alpha}} \bar{\varepsilon}_i - \delta_Q \phi^B \Gamma_B^{\bar{\alpha} \bar{\beta}} \zeta^{\bar{\beta}} + 2 g \bar{X}^{\bar{\alpha} \bar{\beta}} A_i^{\bar{\beta}} \varepsilon_{ij} \varepsilon^j + A_i^{\bar{\alpha}} \eta_i, \end{aligned} \tag{5.2}$$

where we have made use of the supercovariant derivatives (we also give the supercovariant derivative of ζ^{α} which is not needed above),

$$\begin{aligned} D_{\mu} \phi^A &= \mathcal{D}_{\mu} \phi^A - \gamma_{i\bar{\alpha}}^A \bar{\psi}_{\mu}^i \zeta^{\bar{\alpha}} - \bar{\gamma}_{\alpha}^{Ai} \bar{\psi}_{\mu i} \zeta^{\alpha}, \\ D_{\mu} A_i^{\alpha} &= \mathcal{D}_{\mu} A_i^{\alpha} - \bar{\psi}_{\mu i} \zeta^{\alpha} - \varepsilon_{ij} G^{\alpha\bar{\beta}} \Omega_{\bar{\beta}\bar{\gamma}} \bar{\psi}_{\mu}^j \zeta^{\bar{\gamma}}, \\ D_{\mu} \zeta^{\alpha} &= \mathcal{D}_{\mu} \zeta^{\alpha} - \frac{1}{2} \not{D} A_i^{\alpha} \psi_{\mu}^i - \frac{1}{2} A_i^{\alpha} \phi_{\mu}^i. \end{aligned} \tag{5.3}$$

We have verified that no further modifications of the fermionic transformation rules beyond those given above are possible, assuming that the bosonic transformation rules remain the same. One of the underlying reasons for the absence of additional terms may be that the above rules are already consistent with rigid supersymmetry and with the case of a flat hyper-Kähler manifold which was taken as a starting point in [6]. All additional modifications would thus have to vanish in the corresponding limits, while at

the same time one must preserve covariance under target-space diffeomorphisms and fermionic frame reparametrizations. Therefore the possible modifications should be proportional to the target-space curvature times the superconformal fields and, as it turns out, it is difficult if not impossible to see how such terms could emerge. Given the fact that the transformation rules take the same form, we expect the same situation for the Lagrangian, where, again, it is difficult to construct suitable modifications that would vanish in the appropriate limits.

Motivated by these considerations, we write down the Lagrangian by converting and covariantizing the relevant equation (3.28) in Ref. [6]. Here we suppress the hypermultiplet auxiliary fields, as we no longer insist on off-shell supersymmetry for the hypermultiplets. The result reads as follows, where the derivatives are all fully covariantized:

$$\begin{aligned}
 e^{-1} \mathcal{L} = & \frac{1}{2} \varepsilon^{ij} \bar{\Omega}_{\alpha\beta} A_i^\alpha \left(D^a D_a + \frac{3}{2} D \right) A_j^\beta \\
 & + \bar{\Omega}_{\alpha\beta} \left[2 g^2 \varepsilon^{ij} A_i^\alpha \bar{X}^\beta_\gamma X^\gamma_\delta A_j^\delta + \frac{1}{2} g A_i^\alpha Y^{ij\beta}_\gamma A_j^\gamma \right] \\
 & - \left[\left(\bar{\zeta}^\alpha - \frac{1}{2} \bar{\psi}_\mu^i \gamma^\mu A_i^\alpha \right) \left(G_{\beta\alpha} \not{D} \bar{\zeta}^\beta \right. \right. \\
 & + \bar{\Omega}_{\alpha\beta} \left(\frac{3}{2} \varepsilon^{jk} \chi_j A_k^\beta - \frac{1}{4} \varepsilon^{jk} T_{abjk} \gamma^{ab} \zeta^\beta \right) \\
 & \left. \left. - g \bar{\Omega}_{\alpha\beta} \left(\Omega^{i\beta}_\gamma A_i^\gamma + 2 \bar{X}^\beta_\gamma \zeta^\gamma \right) \right) + \text{h.c.} \right] \\
 & + \frac{1}{2} g \left[\bar{\Omega}_{\alpha\beta} A_i^\alpha \bar{\Omega}^{i\beta}_\gamma \zeta^\gamma + \text{h.c.} \right]. \tag{5.4}
 \end{aligned}$$

After substituting the expressions for the dependent gauge fields ϕ_μ^i and f_μ^a in terms of the other fields and dropping a total derivative, we write the Lagrangian as follows:

$$\begin{aligned}
 e^{-1} \mathcal{L} = & -\frac{1}{2} G_{\bar{\alpha}\beta} \mathcal{D}_\mu A_i^\beta \mathcal{D}^\mu A^{i\bar{\alpha}} + \frac{1}{12} R G_{\bar{\alpha}\beta} A_i^\beta A^{i\bar{\alpha}} + \frac{1}{4} D G_{\bar{\alpha}\beta} A_i^\beta A^{i\bar{\alpha}} \\
 & - G_{\bar{\alpha}\beta} \left(\bar{\zeta}^{\bar{\alpha}} \not{D} \zeta^\beta + \bar{\zeta}^\beta \not{D} \zeta^{\bar{\alpha}} \right) - \frac{1}{4} W_{\bar{\alpha}\beta\gamma\delta} \bar{\zeta}^{\bar{\alpha}} \gamma_\mu \zeta^\beta \bar{\zeta}^{\bar{\gamma}} \gamma^\mu \zeta^\delta \\
 & + \left[G_{\bar{\alpha}\beta} \left(-\frac{1}{12} A_i^\beta A^{i\bar{\alpha}} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu j} \gamma_\nu \mathcal{D}_\rho \psi_\sigma^j + \frac{1}{8} A_i^\beta A^{i\bar{\alpha}} \bar{\psi}_{j\mu} \gamma^\mu \chi^j \right. \right. \\
 & - \frac{1}{48} A_k^\beta A^{k\bar{\alpha}} \bar{\psi}_\mu^i \psi_\nu^j T_{ij}^{\mu\nu} - A^{i\bar{\alpha}} \bar{\zeta}^\beta \chi_i + \frac{1}{16} \bar{\Omega}^{\bar{\alpha}\bar{\gamma}} G_{\bar{\gamma}\lambda} \bar{\zeta}^\beta \gamma^{ab} T_{abij} \varepsilon^{ij} \zeta^\lambda \\
 & + \bar{\zeta}^\beta \gamma^\mu \not{D} A^{i\bar{\alpha}} \bar{\psi}_{\mu i} - \frac{2}{3} A^{i\bar{\alpha}} \bar{\zeta}^\beta \gamma^{\mu\nu} \mathcal{D}_\mu \psi_{\nu i} + \frac{1}{24} A^{i\bar{\alpha}} \bar{\zeta}^\beta \gamma^{ab} T_{abij} \gamma^\mu \psi_\mu^j \\
 & \left. \left. - \frac{1}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu^i \gamma_\nu \psi_{\rho j} A_i^\beta \mathcal{D}_\sigma A^{j\bar{\alpha}} \right. \right. \\
 & \left. \left. - \frac{1}{2} \bar{\zeta}^\beta \gamma^\mu \gamma^\nu \psi_{\mu i} \left(\bar{\psi}_\nu^i \zeta^{\bar{\alpha}} + \varepsilon^{ij} \bar{\Omega}^{\bar{\alpha}\bar{\rho}} G_{\rho\lambda} \bar{\psi}_{\nu j} \zeta^\lambda \right) + \text{h.c.} \right] \right]. \tag{5.5}
 \end{aligned}$$

Here we did not include the terms related to gauged isometries. To incorporate those one includes the relevant terms into the covariant derivatives and adds the following g -dependent terms to the Lagrangian:

$$\begin{aligned}
 e^{-1} \mathcal{L}_g = & 2 g^2 G_{\bar{\alpha}\beta} A^{i\bar{\alpha}} \bar{X}^\beta{}_\gamma X^\gamma{}_\delta A_i^\delta + \frac{1}{2} g A_i^\alpha \bar{\Omega}_{\alpha\beta} Y^{ij\beta}{}_\gamma A_j{}^\gamma \\
 & + g \left[2 \bar{X}^\gamma{}_\alpha \bar{\zeta}^{\alpha\gamma\beta} \bar{\Omega}_{\beta\gamma} + 2 \bar{\Omega}_{\alpha\beta} \bar{\zeta}^\alpha \Omega^{i\beta}{}_\delta A_i{}^\delta - 2 \bar{\psi}_\mu^i \gamma^{\mu\gamma\beta} \bar{X}^\alpha{}_\beta \bar{\Omega}_{\alpha\gamma} A_i{}^\gamma \right. \\
 & \left. - \frac{1}{2} \bar{\psi}_\mu^i \gamma^\mu \Omega^{k\alpha}{}_\beta \bar{\Omega}_{\alpha\gamma} A_i{}^\gamma A_k{}^\beta - \frac{1}{2} \bar{\psi}_\mu^i \gamma^{\mu\nu} \psi_\nu^k A_k{}^\beta \bar{\Omega}_{\alpha\gamma} A_i{}^\gamma \bar{X}^\alpha{}_\beta + \text{h.c.} \right].
 \end{aligned} \tag{5.6}$$

As mentioned above, these results are in agreement with the action presented in Subsection 2.2 as well as with the results of [6] in the appropriate limits. In addition we performed a number of independent checks on (5.5) and (5.6). For instance, because the superalgebra closes only modulo the field equations for the fermion fields ζ^α and $\zeta^{\bar{\alpha}}$, we have calculated these field equations from the supersymmetry transformation rules (5.2). As it turns out the result is in agreement with the field equations derived from the action.

The above action is invariant under all superconformal symmetries. In particular the scalar fields are subject to dilatations and to $SU(2)$ transformations. Ignoring the contributions from the vector multiplets, which are essential for obtaining the complete and consistent action for Poincaré supergravity coupled to vector multiplets and hypermultiplets, but which do not affect the target-space geometry of the hypermultiplets, we express the bosonic terms in scale-invariant quantities, by introducing a normalized section

$$\hat{A}_i{}^\alpha = \chi^{-1/2} A_i{}^\alpha, \tag{5.7}$$

which satisfies $\Omega_{\alpha\beta} \hat{A}_i{}^\alpha \hat{A}_j{}^\beta = \varepsilon_{ij}$. Similarly we redefine the various other fields, such as the vierbeine, spin connection, etc., by a χ -dependent scale transformation. The result for the bosonic terms then takes the form

$$\mathcal{L} = -\frac{1}{2} e \left[\bar{\Omega}_{\alpha\beta} \varepsilon^{ij} \mathcal{D}_\mu \hat{A}_i{}^\alpha \mathcal{D}^\mu \hat{A}_j{}^\beta - \frac{1}{3} R - D \right], \tag{5.8}$$

where R is the Ricci scalar of the space-time. Suppressing possible gauged isometries for convenience, this results in

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{2} e \bar{\Omega}_{\alpha\beta} \varepsilon^{ij} \left(\partial_\mu \phi^A D_A \hat{A}_i{}^\alpha + \frac{1}{2} V_{\mu i}{}^k \hat{A}_k{}^\alpha \right) \left(\partial^\mu \phi^B D_B \hat{A}_j{}^\beta + \frac{1}{2} V^{\mu j}{}^l \hat{A}_l{}^\beta \right) \\
 & + \frac{1}{6} e R + \frac{1}{2} e D.
 \end{aligned} \tag{5.9}$$

The field equations for the $SU(2)$ gauge fields $V_{\mu i}{}^j$ yield,

$$V_{\mu i}{}^j = -2 \partial_\mu \phi^A \mathcal{F}_{A ik} \varepsilon^{kj}. \tag{5.10}$$

This result can be substituted back into the Lagrangian, which then reads

$$\mathcal{L} = -\frac{1}{2} e G_{AB} \partial_\mu \phi^A \partial^\mu \phi^B + \frac{1}{6} e R + \frac{1}{2} e D, \tag{5.11}$$

so that the target-space metric G_{AB} corresponds indeed to the quaternionic space which we constructed in Section 4. The terms with the Ricci scalar and the auxiliary field D combine with similar terms from the Lagrangian of the vector multiplets to give the Einstein–Hilbert action.

The material derived in Section 4 now fits in nicely with what is known about the general coupling of hypermultiplets to supergravity [2]. First of all, the quantity $\Delta_{\alpha\beta}$ projects out precisely the S -invariant hypermultiplet spinors which thus describe $2n - 2$ physical spinors after modding out the S -supersymmetry. Hence, the non-linear sigma model comprises precisely the expected $4n - 4$ scalars and $2n - 2$ spinors. The relevant quaternionic vielbeine have already been defined in (4.15), but can equally well be obtained from working out the above Lagrangian after removing the appropriate gauge degrees of freedom. We will list a number of relevant identities, which all follow from Section 4,

$$\begin{aligned} \Delta_{\alpha\beta} \hat{V}_{Ai}^\alpha \hat{V}_{Bj}^\beta &= \frac{1}{2} \varepsilon_{ij} G_{AB} + \frac{1}{2} R_{ABij}, \\ G_{AB} \bar{\gamma}_\alpha^{Ai} \bar{\gamma}_\beta^{Bj} &= \varepsilon^{ij} \Delta_{\alpha\beta}, \\ R_{ABij} \bar{\gamma}_\alpha^{Ak} \bar{\gamma}_\beta^{Bl} &= 2 \delta_i^{(k} \delta_j^{l)} \Delta_{\alpha\beta}. \end{aligned} \tag{5.12}$$

The second integrability condition (4.35) can be rewritten as

$$\hat{R}_{ABCD} \bar{\gamma}_\alpha^{Ci} \bar{\gamma}_\beta^{Dj} = -\varepsilon^{ij} \Delta_{\alpha\gamma} \hat{R}_{AB}{}^\gamma{}_\beta - \Delta_{\alpha\beta} R_{AB}^{ij}, \tag{5.13}$$

which gives the decomposition of the Riemann tensor into an $\text{Sp}(n - 1)$ and an $\text{Sp}(1)$ curvature. Of course, this relation is already incorporated into the expression (4.38) and its correctness can also be verified directly. The curvature $\hat{R}_{AB}{}^\alpha{}_\beta$ satisfies (c.f. (4.36)),

$$\bar{\Omega}_{\alpha\gamma} \hat{R}_{AB}{}^\gamma{}_\beta = \bar{\Omega}_{\alpha\gamma} R_{AB}{}^\gamma{}_\beta - 2 \varepsilon_{ij} \bar{\gamma}_{A(\alpha}^i \bar{\gamma}_{B\beta)}^j + \bar{\Omega}_{\alpha\gamma} \bar{\Omega}_{\beta\delta} \hat{A}_i{}^\gamma A_j{}^\delta R_{AB}^{ij}. \tag{5.14}$$

Upon projection with Δ , the last term vanishes and one finds an identity that is well known from the literature.

Hence we see that all aspects of quaternionic geometry that arise in the coupling of hypermultiplets to supergravity are correctly reproduced. Our results provide an elegant extension of the work reported in [6] and give a unified prescription for all hypermultiplet couplings to supergravity. Although this is in principle straightforward, it remains to work out the details of the Lagrangian and transformation rules after removing the gauge degrees of freedom associated with S -supersymmetry.

Acknowledgements

We acknowledge useful discussions with J. Figueroa-O’Farrill, G. Gibbons and A. Swann. B.d.W. is grateful to the Alexander von Humboldt-Stiftung for supporting his stay at the AEI as part of the Humboldt Award program. S. V. thanks PPARC for financial support during his stay at the University of Swansea, and the Institute for

Theoretical Physics in Utrecht and the AEI for their hospitality. This work is supported in part by the European Commission TMR programmes FMRX-CT96-0012, in which the Albert Einstein Institut and the University of Wales in Swansea participate, and ERBFMRX-CT96-0045, in which Utrecht University participates.

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