Asymptotic Form of Zero Energy Wave Functions in Supersymmetric Matrix Models

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Abstract

We derive the power law decay, and asymptotic form, of SU(2)×Spin(d) invariant wave–functions satisfying $Q_\beta \psi = 0$ for all $s_d = 2(d-1)$ supercharges of reduced $(d+1)$–dimensional supersymmetric SU(2) Yang Mills theory, resp. of the SU(2)–matrix model related to supermembranes in $d+2$ dimensions.

1 Introduction

It is generally believed that supersymmetric SU(N) matrix models in $d = 9$ dimensions admit exactly one normalizable zero-energy solution for each $N > 1$, while they admit none for all other dimensions for which the models can be formulated, i.e., for $d = 2, 3, 5$. For various approaches to this problem see e.g. [1]–[13].

In this article, we would like to summarize (and slightly modify/extend) what is known about the behaviour of SU(2) zero–energy solutions far out at infinity in (and near) the space of configurations where the bosonic potential (the trace of all commutator–squares) vanishes. Based on some early 'negative' result concerning $N = 2$, $d = 2$ (that used rather different techniques/arguments; see [1, 18]) we started our investigation of the asymptotic behaviour, in the fall of 1997, with a Hamiltonian Born–Oppenheimer analysis of that $N = 2$, $d = 2$ case. Some months later, we realized that the rather complicated Hamiltonian analysis (Halpern and Schwartz [8] had, in the meantime, derived the form of the wave function for $d = 9$ near ∞, by Hamiltonian Born–Oppenheimer methods) can be replaced by a simple first order analysis, using only the first order operators Q, and first order perturbation theory. One finds that asymptotically normalizable, SU(2) and SO(d) invariant, wave functions do not exist for $d = 2, 3$, and 5, in contrast to $d = 9$, where there is exactly one.

We close these introductory words by recalling that the models discussed below arise in at least 3 somewhat different ways: As supersymmetric extensions of regulated membrane theories in $d+2$ space–time dimensions [14, 18], as reductions (to $0+1$ dimension) of $d+1$ dimensional Super Yang Mills theories [15]–[17], and, for $d = 9$, as a description of the dynamics of D–0 branes in superstring theory, [20, 21]. In this physical interpretation, the existence of a normalizable zero–energy solution is an important consistency requirement.

The paper is organized as follows. In Section 2 we recall the definition of the models, and in Section 3 we state our main result about zero–modes. The proof is given in Section 4 and Appendix 1. We suggest to skip Subsection 4.5 and Appendix 1 at a first reading. As a warm–up the reader is advised to read Appendix 2, where a simpler model is treated by the same method.

2 The models

The configuration space of the bosonic degrees of freedom is $X = \mathbb{R}^{3d}$ with coordinates

$$q = (\vec{q}_1, \dots , \vec{q}_d) = (q_{sA})_{\substack{s=1,\dots,d \\ A=1,2,3}} .$$

To describe the fermionic degrees of freedom let, as a preliminary,

$$\gamma^i = (\gamma^i_{\alpha\beta})_{\alpha,\beta=1,\dots,s_d} , \qquad (i = 1, \dots , d) , \tag{1}$$

be the *real* representation of smallest dimension, called s_d, of the Clifford algebra with d generators: $\{\gamma^s, \gamma^t\} = 2\delta^{st}\mathbb{1}$. On the representation space, $\mathrm{Spin}(d)$ is realized through matrices $R \in \mathrm{SO}(s_d)$, so that we may view

$$\mathrm{Spin}(d) \hookrightarrow \mathrm{SO}(s_d) , \tag{2}$$

as a simply connected subgroup. We recall that

$$s_d = \begin{cases} 2^{[d/2]} , & d = 0, 1, 2 \mod 8 , \\ 2^{[d/2]+1} , & \text{otherwise} , \end{cases}$$

where $[\cdot]$ denotes the integer part. We then consider the Clifford algebra with s_d generators and its irreducible representation on $\mathcal{C} = \mathbb{C}^{2^{s_d/2}}$. On $\mathcal{C}^{\otimes 3}$ the Clifford generators

$$(\vec{\Theta}_1, \dots , \vec{\Theta}_{s_d}) = (\Theta_{\alpha A})_{\substack{\alpha=1,\dots,s_d \\ A=1,2,3}}$$

are defined, satisfying $\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\,\delta_{AB}$. The Hilbert space, finally, is

$$\mathcal{H} = \mathrm{L}^2(X, \mathcal{C}^{\otimes 3}) . \tag{3}$$

There is a natural representation of $\mathrm{SU}(2) \times \mathrm{Spin}(d) \ni (U, R)$ on $\mathcal{H}$. In fact, the group acts naturally on X through its representation $\mathrm{SO}(3) \times \mathrm{SO}(d)$ (which we also denote by (U, R)). On $\mathcal{C}^{\otimes 3}$ we have the representation $\mathcal{R}$ of $\mathrm{Spin}(s_d) \ni R$

$$\mathcal{R}(R)^* \Theta_{\alpha A} \mathcal{R}(R) = \widetilde{R}_{\alpha\beta} \Theta_{\beta A} , \tag{4}$$

where $\widetilde{R} = \widetilde{R}(R)$ is its $\mathrm{SO}(s_d)$ representation. Through $\mathrm{SO}(s_d) = \mathrm{Spin}(s_d)/\mathbb{Z}_2$ and (2) we have

$$\mathrm{Spin}(d) \hookrightarrow \mathrm{Spin}(s_d) , \tag{5}$$

and thus a representation $\mathcal{R}$ of Spin(d). The representation $\mathcal{U}$ of SU(2) $\ni U$ on $\mathcal{C}^{\otimes 3}$ is characterized by $\mathcal{U}(U)^* \Theta_{\alpha A} \mathcal{U}(U) = U_{AB} \Theta_{\alpha B}$.

We shall now restrict to $d = 2, 3, 5, 9$, where $s_d = 2, 4, 8, 16$, the reason being that in these cases

$$s_d = 2(d-1) \ , \tag{6}$$

whereas s_d is strictly larger otherwise. Eq. (6) is essential for the algebra (7) below [17].

The supercharges, acting on $\mathcal{H}$, are given by the s_d hermitian operators

$$Q_\beta = \vec{\Theta}_\alpha \cdot \left(-\mathrm{i} \gamma^t_{\alpha\beta} \vec{\nabla}_t + \frac{1}{2} \vec{q}_s \times \vec{q}_t \, \gamma^{st}_{\beta\alpha} \right) , \qquad (\beta = 1, \ldots, s_d) \ ,$$

where $\gamma^{st} = (1/2)(\gamma^s \gamma^t - \gamma^t \gamma^s)$. These supercharges transform as scalars under SU(2) transformations generated by

$$J_{AB} = -\mathrm{i}(q_{sA} \partial_{sB} - q_{sB} \partial_{sA}) - \frac{\mathrm{i}}{2} (\Theta_{\alpha A} \Theta_{\alpha B} - \Theta_{\alpha B} \Theta_{\alpha A}) \equiv L_{AB} + M_{AB} \ ,$$

resp. as vectors in $\mathbb{R}^{s_d}$ under Spin(d) transformation generated by

$$J_{st} = -\mathrm{i}(\vec{q}_s \cdot \vec{\nabla}_t - \vec{q}_t \cdot \vec{\nabla}_s) - \frac{\mathrm{i}}{4} \vec{\Theta}_\alpha \gamma^{st}_{\alpha\beta} \vec{\Theta}_\beta \equiv L_{st} + M_{st} \ .$$

The anticommutation relations of the supercharges are

$$\left\{ Q_\alpha, Q_\beta \right\} = \delta_{\alpha\beta} H + \gamma^t_{\alpha\beta} q_{tA} \varepsilon_{ABC} J_{BC} \ . \tag{7}$$

Here, H is the Hamiltonian

$$H = -\sum_{s=1}^{9} \vec{\nabla}_s^2 + \sum_{s<t} \left(\vec{q}_s \times \vec{q}_t \right)^2 + \mathrm{i} \vec{q}_s \cdot \left(\vec{\Theta}_\alpha \times \vec{\Theta}_\beta \right) \gamma^s_{\alpha\beta} \ , \tag{8}$$

which commutes with both J_{AB} and J_{st}. The question we address is the possibility of a normalizable state $\psi \in \mathcal{H}$ with zero energy, i.e., with $H\psi = 0$, which is a singlet w.r.t. both SU(2) and Spin(d). Note that on SU(2) invariant states $H = 2Q_\beta^2 \geq 0$ and in fact the energy spectrum is ([19]) $\sigma(H) = [0, \infty)$. Equivalently, we look for zero-modes

$$Q_\beta \psi = 0 \ , \qquad (\beta = 1, \ldots, s_d) \ .$$

3 Results

The potential $\sum_{s<t} (\vec{q}_s \times \vec{q}_t)^2$ vanishes on the manifold

$$\vec{q}_s = r \vec{e} E_s$$

with $r > 0$ and $\vec{e}^{\,2} = \sum_s E_s^2 = 1$. The dimension of the manifold is $1 + 2 + (d-1) = 3d - 2(d-1)$. Points in a conical neighborhood of the manifold can be expressed in terms of tubular (or "end–point") coordinates [23]

$$\vec{q}_s = r \vec{e} E_s + r^{-1/2} \vec{y}_s \tag{9}$$

with

$$\vec{y}_s \cdot \vec{e} = 0 \;, \qquad \vec{y}_s E_s = \vec{0} \;. \tag{10}$$

A prefactor has been put explicitely in front of the transversal coordinates $\vec{y}_s$, so as to anticipate the length scale $r^{-1/2}$ of the ground state. The change

$$(\vec{e}, E, y) \mapsto (-\vec{e}, -E, y) \tag{11}$$

does not affect $\vec{q}_s$. Rather than identifying the two coordinates for $\vec{q}_s$, we shall look for states which are even under the antipode map (11).

We can now describe the structure of a putative ground state.

Theorem *Consider the equations $Q_\beta \psi = 0$ for a formal power series solution near $r = \infty$ of the form*

$$\psi = r^{-\kappa} \sum_{k=0}^{\infty} r^{-\frac{3}{2}k} \psi_k \;, \tag{12}$$

where: $\psi_k = \psi_k(\vec{e}, E, y)$ is square integrable w.r.t. $\mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y$;
ψ_k is $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ invariant;
$\psi_0 \neq 0$.
Then, up to linear combinations,

- *d=9: The solution is unique, and $\kappa = 6$;*
- *d=5: There are three solutions with $\kappa = -1$ and one with $\kappa = 3$;*
- *d=3: There are two solutions with $\kappa = 0$;*
- *d=2: There are no solutions.*

All solutions are even under the antipode map (11),

$$\psi_k(\vec{e}, E, y) = \psi_k(-\vec{e}, -E, y) \;,$$

except for the state $d = 5$, $\kappa = 3$, which is odd.

Remarks 1. The equation $Q_\beta \psi = 0$ can be viewed as an ordinary differential equation in $z = r^{3/2}$ for a function taking values in $\mathrm{L}^2(\mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y, \mathcal{C}^{\otimes 3})$ (see eq. (14) below). It turns out that $z = \infty$ is a singular point of the second kind [22]. In such a situation the series (12) is typically asymptotic to a true solution, but not convergent.
2. The integration measure is $\mathrm{d}q = \mathrm{d}r \cdot r^2 \mathrm{d}e \cdot r^{d-1} \mathrm{d}E \cdot r^{-\frac{1}{2} \cdot 2(d-1)} \mathrm{d}y = r^2 \mathrm{d}r\, \mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y$. The wave function (12) is square integrable at infinity if $\int^\infty \mathrm{d}r\, r^2 (r^{-\kappa})^2 < \infty$, i.e., if $\kappa > 3/2$. The theorem is consistent with the statement according to which **only** for $d = 9$ a (unique) normalizable ground state for (8) (which would have to be even) is possible.
3. Note that the connection of matrix models with supergravity requires the zero–energy solutions to be $\mathrm{Spin}(d)$ singlets only for $d = 9$.

The case $d = 2$ can be dealt with immediately. We may assume $\gamma^2 = \sigma_3$, $\gamma^1 = \sigma_1$ (Pauli matrices), so that

$$M_{12} = \frac{\mathrm{i}}{2} \Theta_{1A} \Theta_{2A} \;,$$

with commuting terms. Since, for each $A = 1, 2, 3$, $(\Theta_{1A}\Theta_{2A})^2 = -1/4$, we see that M_{12} has spectrum in $\mathbb{Z}/2 + 1/4$. Given that L_{12} has spectrum $\mathbb{Z}$, no state with $J_{12}\psi = 0$ is possible. We mention [1] that, more generally, for $d = 2$ no normalizable SU(2) invariant ground state exists.

The proof of the theorem will thus deal with $d = 9, 5, 3$ only.

4 Proof

We shall first derive the power series expansion of the supercharges Q_β. To this end we note that

$$\begin{aligned}\frac{\partial}{\partial q_{tA}} &= r^{1/2}(\delta_{st} - E_sE_t)(\delta_{AB} - e_Ae_B)\frac{\partial}{\partial y_{sB}} \\ &\quad + r^{-1}[e_AE_t(r\frac{\partial}{\partial r} + \frac{1}{2}y_{sB}\frac{\partial}{\partial y_{sB}}) + \mathrm{i}e_BE_tL_{BA} + \mathrm{i}e_AE_sL_{st}] + \mathrm{O}(r^{-5/2}) \ ,\end{aligned} \tag{13}$$

with the remainder not containing derivatives w.r.t. r (see Appendix 1 for derivation). This yields

$$Q_\beta = r^{1/2}Q^0_\beta + r^{-1}(\widehat{Q}^1_\beta r\frac{\partial}{\partial r} + Q^1_\beta) + r^{-5/2}Q^2_\beta + \dots \tag{14}$$

with r–independent operators

$$\begin{aligned}Q^0_\beta &= -\mathrm{i}\Theta_{\alpha A}\gamma^t_{\alpha\beta}(\delta_{st} - E_sE_t)(\delta_{AB} - e_Ae_B)\frac{\partial}{\partial y_{sB}} + \vec{\Theta}_\alpha\cdot(\vec{e}\times\vec{y}_t)E_s\gamma^{st}_{\beta\alpha} \ , \\ \widehat{Q}^1_\beta &= -\mathrm{i}(\vec{\Theta}_\alpha\cdot\vec{e})\gamma^t_{\alpha\beta}E_t \ , \\ Q^1_\beta &= \Theta_{\alpha A}\gamma^t_{\alpha\beta}\big(e_BE_tL_{BA} + e_AE_sL_{st} - \frac{\mathrm{i}}{2}e_AE_ty_{sB}\frac{\partial}{\partial y_{sB}}\big) + \frac{1}{2}\vec{\Theta}_\alpha\cdot(\vec{y}_s\times\vec{y}_t)\gamma^{st}_{\beta\alpha} \ .\end{aligned}$$

The explicit expressions of Q^n_β, $(n \geq 2)$ will not be needed. We then equate coefficients of powers of $r^{-3/2}$ in the equation $Q_\beta\psi = 0$ with the result

$$\begin{aligned}Q^0_\beta\psi_n + \big(-(\kappa + \frac{3}{2}(n-1))\widehat{Q}^1_\beta + Q^1_\beta\big)\psi_{n-1} + Q^2_\beta\psi_{n-2} + \dots + Q^n_\beta\psi_0 &= 0 \ , \\ (n &= 0, 1, \dots) \ .\end{aligned} \tag{15}$$

4.1 The equation at $n = 0$

The equation at $n = 0$,

$$Q^0_\beta\psi_0 = 0 \ , \tag{16}$$

admits precisely the (not necessarily SU(2) × Spin(d) invariant) solutions

$$\psi_0(\vec{e}, E, y) = \mathrm{e}^{-\sum_s \vec{y}_s{}^2/2}|F(E, \vec{e})\rangle \ , \tag{17}$$

(with $\vec{y}$ restricted to (10)), where the fermionic states $|F(E, \vec{e})\rangle$ can be described as follows: Let $\vec{n}_\pm$ be two complex vectors satisfying $\vec{n}_+\cdot\vec{n}_- = 1$, $\vec{e}\times\vec{n}_\pm = \mp\mathrm{i}\vec{n}_\pm$ (and hence

$\vec{n}_\pm \cdot \vec{n}_\pm = 0$, $\vec{n}_+ \times \vec{n}_- = -\mathrm{i}\vec{e}$). For any vector $v \in \mathbb{R}^{s_d}$ we may introduce $\vec{\Theta}(v) = \vec{\Theta}_\alpha v_\alpha$, as well as fermionic operators $\vec{\Theta}(v) \cdot \vec{n}_\pm$ satisfying canonical anticommutation relations:

$$\{\vec{\Theta}(u) \cdot \vec{n}_+, \vec{\Theta}(v) \cdot \vec{n}_-\} = u_\alpha v_\alpha \ , \qquad \{\vec{\Theta}(u) \cdot \vec{n}_\pm, \vec{\Theta}(v) \cdot \vec{n}_\pm\} = 0 \ .$$

Then, $|F(E, \vec{e})\rangle$ is required to obey

$$\vec{\Theta}(v) \cdot \vec{n}_\pm |F(E, \vec{e})\rangle = 0 \qquad \text{for} \qquad E_s \gamma^s v = \pm v \ . \tag{18}$$

To prove the above, let us note that

$$\{Q^0_\alpha, Q^0_\beta\} = \delta_{\alpha\beta} H^0 + \gamma^t_{\alpha\beta} E_t \varepsilon_{ABC} M_{AB} e_C \ , \tag{19}$$

$$\begin{aligned} H^0 &= \left[-(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B) \frac{\partial}{\partial y_{sA}} \frac{\partial}{\partial y_{tB}} + \sum_s \vec{y}_s^2\right] + \mathrm{i} E_s \gamma^s_{\alpha\beta} \vec{e} \cdot (\vec{\Theta}_\alpha \times \vec{\Theta}_\beta) \\ &\equiv H^0_B + H^0_F \ . \end{aligned}$$

By contracting eq. (19) against $\delta_{\alpha\beta}$, resp. $\gamma^t_{\alpha\beta} E_t$ we see that the equations (16) are equivalent to the pair of equations

$$H^0 \psi_0 = 0 \ , \qquad \varepsilon_{ABC} M_{AB} e_C \psi_0 = 0 \ . \tag{20}$$

Here, H^0_B is a harmonic oscillator in $2(d-1)$ degrees of freedom, with orbital ground state wave function $\mathrm{e}^{-\sum_s \vec{y}_s^2/2}$ and energy $2(d-1)$. On the other hand,

$$\begin{aligned} H^0_F &= -E_s \gamma^s_{\alpha\beta} \big((\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) - (\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+)\big) \\ &= -s_d + 2P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) + 2P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) \ , \end{aligned} \tag{21}$$

where we used the spectral decomposition $E_s \gamma^s = P^+ - P^-$. In view of (6), the equation $H^0 \psi_0 = 0$ is fulfilled iff the fermionic state is annihilated by the last two positive terms in (21), i.e., if (18) holds. The second equation (20) is now also satisfied, since

$$\begin{aligned} \frac{1}{2} \varepsilon_{ABC} M_{AB} e_C &= -\frac{\mathrm{i}}{2} \vec{e} \cdot (\vec{\Theta}_\alpha \times \vec{\Theta}_\alpha) \\ &= \frac{1}{2}\big((\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\alpha \cdot \vec{n}_-) - (\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\alpha \cdot \vec{n}_+)\big) \\ &= P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) - P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) \end{aligned} \tag{22}$$

annihilates $|F(E, \vec{e})\rangle$.

4.2 $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ invariant states

We recall that the representation $\mathcal{R}[\cdot]$ of $\mathrm{Spin}(d)$ on $\mathcal{H}$ is $(\mathcal{R}[R]\psi)(q) = \mathcal{R}(R)(\psi(R^{-1}q))$, where $\mathcal{R}(R)$ acts on $\mathcal{C}^{\otimes 3}$. Similarly for SU(2). The invariant solutions among (17) are thus those which satisfy

$$\mathcal{U}(U)|F(E, \vec{e})\rangle = |F(E, U\vec{e})\rangle \ , \qquad \mathcal{R}(R)|F(E, \vec{e})\rangle = |F(RE, \vec{e})\rangle \ , \tag{23}$$

for $(U, R) \in \mathrm{SU}(2) \times \mathrm{Spin}(d)$. These states are in bijective correspondence to states invariant under the 'little group' $(U, R) \in \mathrm{U}(1) \times \mathrm{Spin}(d-1)$, i.e., to states $|F(E, \vec{e})\rangle$ satisfying

$$\mathcal{U}(U)|F(E, \vec{e})\rangle = |F(E, \vec{e})\rangle \ , \qquad \mathcal{R}(R)|F(E, \vec{e})\rangle = |F(E, \vec{e})\rangle \ , \tag{24}$$

for some arbitrary but fixed $(E,\vec{e})$ and all U, R with $U\vec{e}=\vec{e}$, $RE=E$. The first relation holds on all of (18). In fact the generator (22) of the group $\mathcal{U}(U)$ of rotations U about $\vec{e}$ annihilates $|F(E,\vec{e})\rangle$, as we just saw. To discuss the second relation (24) we note that the generators of $\mathrm{Spin}(d-1)$ (i.e., of the fermionic rotations about E), are $M_{st}U_sV_t$ with $U_sE_s=V_sE_s=0$. We write $M_{st}=M^{\perp}_{st}+M^{\parallel}_{st}$, where

$$M^{\perp}_{st}=-(\mathrm{i}/2)(\vec{\Theta}_\alpha\cdot\vec{n}_+)\gamma^{st}_{\alpha\beta}(\vec{\Theta}_\beta\cdot\vec{n}_-)\ ,\qquad M^{\parallel}_{st}=-(\mathrm{i}/4)(\vec{\Theta}_\alpha\cdot\vec{e})\gamma^{st}_{\alpha\beta}(\vec{\Theta}_\beta\cdot\vec{e})\ , \tag{25}$$

and remark that, by a computation similar to (22), $M^{\perp}_{st}U_sV_t$ annihilates $|F(E,\vec{e})\rangle$. As a result, we may study the representation $\mathcal{R}$ of the group $\mathrm{Spin}(d-1)$ through its embedding in the Clifford algebra generated by the $\vec{\Theta}_\alpha\cdot\vec{e}$.

The operators $\vec{\Theta}_\alpha\cdot\vec{e}$ leave the space (18) invariant and act irreducibly on it. That space is thus isomorphic to $\mathcal{C}$, and $\mathrm{Spin}(s_d)$ acts according to (4) (with $\Theta_{\alpha A}$ replaced by $\vec{\Theta}_\alpha\cdot\vec{e}$). This representation decomposes (see e.g. [24]) as

$$\mathcal{C}=(2^{(s_d/2)-1})_+\oplus(2^{(s_d/2)-1})_- \tag{26}$$

w.r.t. the subspaces where $\Theta\equiv 2^{s_d/2}\prod_{\alpha=1}^{s_d}\vec{\Theta}_\alpha\cdot\vec{e}=+1$, resp. -1. The embedding (5) and the corresponding branching of the representation (but not the statement of the theorem!) depend on the choice of the γ–matrices. In order to select a definite embedding, let

$$\gamma^d=\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix},\quad \gamma^{d-1}=\begin{pmatrix}0&\mathbb{1}\\\mathbb{1}&0\end{pmatrix},\quad \gamma^j=\begin{pmatrix}0&\mathrm{i}\Gamma^j\\-\mathrm{i}\Gamma^j&0\end{pmatrix} \tag{27}$$

with Γ^j, $(j=1,\dots,d-2)$ purely imaginary, antisymmetric, and $\{\Gamma^j,\Gamma^k\}=2\delta_{jk}\mathbb{1}_{s_d/2}$. Then (26) branches as (see [25], resp. [12, 13])

$$\mathcal{C}=\begin{cases}(44\oplus 84)\oplus 128\ , & (d=9)\ ,\\ (5\oplus 1\oplus 1\oplus 1)\oplus(4\oplus 4)\ , & (d=5)\ ,\\ 2\oplus(1\oplus 1)\ , & (d=3)\ ,\end{cases} \tag{28}$$

when viewed as a representation of $\mathrm{Spin}(d)$. (The choice $\widetilde{\gamma}^i_{\alpha\beta}=\widetilde{R}_{\alpha'\alpha}\gamma^i_{\alpha'\beta'}\widetilde{R}_{\beta'\beta}$ with $\widetilde{R}\in\mathrm{O}(s_d)$, $\det\widetilde{R}=-1$ would have inverted the branching of the representations on the r.h.s. of (26)). The case $d=3$ deserves a remark, as there are additional inequivalent embeddings $\mathrm{Spin}(d=3)\hookrightarrow\mathrm{Spin}(s_d=4)$, and one has to consider the one appropriate to (5). In fact $R\in\mathrm{Spin}(3)=\mathrm{SU}(2)$ acts in the fundamental representation on $\mathbb{C}^2$, the irreducible representation space of the complex Clifford algebra with 3 generators. The real representation (27) is obtained by joining two complex representations, followed by an appropriate change T of basis. The embedding (5) is thus realized through $R\mapsto T^{-1}(R\otimes\mathbb{1}_2)T$ and the embedding $\mathrm{su}(2)_{\mathbb{C}}\hookrightarrow\mathrm{so}(4)_{\mathbb{C}}=\mathrm{su}(2)_{\mathbb{C}}\oplus\mathrm{su}(2)_{\mathbb{C}}$ is equivalent to $u\mapsto(u,0)$.

The further branching $\mathrm{Spin}(d)\hookleftarrow\mathrm{Spin}(d-1)$ yields

$$\mathcal{C}=\begin{cases}(1\oplus 8_{\mathrm{v}}\oplus 35_{\mathrm{v}})\oplus(28\oplus 56_{\mathrm{v}})\oplus(8_{\mathrm{s}}\oplus 8_{\mathrm{c}}\oplus 56_{\mathrm{s}}\oplus 56_{\mathrm{c}})\ , & (d-1=8)\ ,\\ 1\oplus 1\oplus 1\oplus(1\oplus 4)\oplus(2_+\oplus 2_-)\oplus(2_+\oplus 2_-)\ , & (d-1=4)\ ,\\ (1_1\oplus 1_{-1})\oplus 1_0\oplus 1_0\ , & (d-1=2)\ .\end{cases} \tag{29}$$

The content of invariant states stated in the theorem is now manifest. One should notice that for $d = 3$ the little group U(1) is abelian and the singlets $1_{\pm 1}$ do not correspond to invariant states. For later use we also retain the fermionic Spin(d) representation to which the remaining singlets are associated,

$$44 \quad (d=9)\ ; \qquad 1,1,1,5 \quad (d=5)\ ; \qquad 1,1 \quad (d=3)\ , \tag{30}$$

together with the corresponding eigenvalue of Θ:

$$\Theta = \quad 1 \quad (d=9)\ ; \qquad 1,1,1,1 \quad (d=5)\ ; \qquad -1,-1 \quad (d=3)\ . \tag{31}$$

4.3 Even states

It remains to check which of these states satisfy $|F(-E,-\vec{e})\rangle = |F(E,\vec{e})\rangle$. Let us begin by noting that by (23)

$$|F(-E,-\vec{e})\rangle = \mathrm{e}^{\mathrm{i}M_{AB}e_A u_B \pi}\mathrm{e}^{\mathrm{i}M_{st}E_s U_t \pi}|F(E,\vec{e})\rangle\ ,$$

where $\vec{u} \in \mathbb{R}^3$, resp. $U \in \mathbb{R}^d$ are unit vectors orthogonal to $\vec{e}$, resp. E. The Spin(d) rotation can be factorized as $\mathrm{e}^{\mathrm{i}M_{st}E_sU_t\pi} = \mathrm{e}^{\mathrm{i}M^{\perp}_{st}E_sU_t\pi}\mathrm{e}^{\mathrm{i}M^{\parallel}_{st}E_sU_t\pi}$. We claim that $\mathrm{e}^{\mathrm{i}M^{\parallel}_{st}E_sU_t\pi}$ $|F(E,\vec{e})\rangle = \sigma|F(E,\vec{e})\rangle$ with

$$\sigma = \quad 1 \quad (d=9)\ ; \qquad 1,1,1,-1 \quad (d=5)\ ; \qquad 1,1 \quad (d=3)\ . \tag{32}$$

The operator represents a rotation $R \in$ Spin(d) with $RE = -E$ in the representation (30). For $d = 9$ the latter can be realized on symmetric traceless tensors T_{ij}, $(i,j = 1,\dots,9)$, where the Spin(8)–singlet is $E_iE_j - (1/9)\delta_{ij}$, implying $\sigma = 1$. For $d = 5$, the last representation (30) is just the vector representation, where $\sigma = -1$. As the remaining cases are evident, eq. (32) is proven. A computation using (27) and, without loss $E = (0,\dots,0,1)$, $U = (0,\dots,1,0)$ shows

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}M^{\perp}_{d,d-1}\pi}|F(E,\vec{e})\rangle &= \prod_{\alpha=1}^{s_d/2}\mathrm{e}^{[(\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_{\alpha+s_d/2}\cdot\vec{n}_-)-(\vec{\Theta}_{\alpha+s_d/2}\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-)]\pi/2}|F(E,\vec{e})\rangle \\
&= \prod_{\alpha=1}^{s_d/2}(\vec{\Theta}_{\alpha+s_d/2}\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-)|F(E,\vec{e})\rangle \equiv |\overline{F}(E,\vec{e})\rangle\ , \\
\mathrm{e}^{\mathrm{i}M_{AB}e_Au_B\pi}|\overline{F}(E,\vec{e})\rangle &= \prod_{\alpha=1}^{s_d}\mathrm{e}^{(\vec{\Theta}_\alpha\cdot\vec{e})(\vec{\Theta}_\alpha\cdot\vec{u})\pi}|\overline{F}(E,\vec{e})\rangle \\
&= (-1)^{s_d/4}\Theta\prod_{\alpha=1}^{s_d/2}(\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_{\alpha+s_d/2}\cdot\vec{n}_-)|\overline{F}(E,\vec{e})\rangle = |F(E,\vec{e})\rangle\ ,
\end{aligned}$$

where we used (31) in the last step. Together with (32) this proves the statement of theorem concerning the invariance under (11).

4.4 The equation at $n > 0$

We next discuss the equations $(15)_n$ with $n \geq 1$. Let P_0 be the orthogonal projection onto the states (17), i.e., onto the null space of Q^0_β. We replace them with an equivalent pair of equations, obtained by multiplication of $(15)_{n+1}$ with P_0, resp. of $(15)_n$ with Q^0_β, which is injective on the range of the complementary projection $\overline{P}_0 = 1 - P_0$:

$$P_0\big(-(\kappa + \frac{3}{2}n))\widehat{Q}^1_\beta + Q^1_\beta\big)P_0\psi_n = -P_0\big(Q^1_\beta \overline{P}_0\psi_n + Q^2_\beta\psi_{n-1} + \ldots + Q^{n+1}_\beta\psi_0\big) \;, \qquad (n = 0, 1, \ldots) \;, \tag{33}$$

$$(Q^0_\beta)^2\psi_n = -Q^0_\beta\Big(\big(-(\kappa + \frac{3}{2}(n-1))\widehat{Q}^1_\beta + Q^1_\beta\big)\psi_{n-1} + Q^2_\beta\psi_{n-2} + \ldots + Q^n_\beta\psi_0\Big) \;, \qquad (n = 1, 2, \ldots) \tag{34}$$

(we used $P_0\widehat{Q}^1_\beta\overline{P}_0 = 0$). Here, and until the end of this subsection, no summation over β is understood. The equation (33) at $n = 0$ reads

$$P_0 Q^1_\beta \psi_0 = \kappa P_0 \widehat{Q}^1_\beta \psi_0 \; (= \kappa \widehat{Q}^1_\beta \psi_0) \; . \tag{35}$$

We shall verify this by explicit computation later on. Since a similar issue will show up in solving the equation (33) at $n > 0$, let us also present a more general statement, whose proof is postponed to the next subsection.

Lemma *Let T_β be linear operators on the range of P_0, which transform as real spinors of* Spin(d) *and commute with the antipode map. Then, for each invariant state we have*

$$T_\beta \psi_0 = \kappa \widehat{Q}^1_\beta \psi_0 \;, \tag{36}$$

with κ depending only on the associated representation (30).

We now assume having solved the equations (33, 34) up to $n-1$ for Spin(d) invariant $\psi_1, \ldots \psi_{n-1}$ (which is true for $n - 1 = 0$), and claim the same is possible for n. Since Q^0_β is invertible on the range of $\overline{P}_0$, eq. $(34)_n$ determines $\overline{P}_0\psi_n$ uniquely. The fact that the solution so obtained is independent of β and is Spin(d) invariant may deserve a comment, because the equivalence of the equations $Q_\beta\psi = 0$ and $(Q_\beta)^2\psi = 0$, which holds on (3), does not apply in the sense of formal power series (12). Consider the expansion (14), i.e.,

$$Q_\beta = r^{1/2}\sum_{k=0}^{\infty} r^{-\frac{3}{2}k}[Q_\beta]_k \;, \qquad [Q_\beta]_k = Q^k_\beta + \delta_{1k}\widehat{Q}^1_\beta r\frac{\partial}{\partial r} \;,$$

as well as its formal square

$$(Q_\beta)^2 = r\sum_{k=0}^{\infty} r^{-\frac{3}{2}k}[(Q_\beta)^2]_k \; .$$

Notice that $(Q_\beta)^2$ is, by (7), independent of β and Spin(d) invariant as an operator on SU(2) invariant power series. Similarly, let $[Q_\beta\psi]_k$ (given by the l.h.s. of (15)) and $[(Q_\beta)^2\psi]_k$ be the coefficients of the corresponding series. By induction assumption we have $[Q_\beta\psi]_k = 0$ for $k = 0, \ldots, n-1$. Since $Q_\beta(Q_\beta\psi) = (Q_\beta)^2\psi$, we obtain

$$[(Q_\beta)^2\psi]_n = \sum_{k=0}^{n} Q^k_\beta[Q_\beta\psi]_{n-k} - (\kappa + \frac{3}{2}n - 2)\widehat{Q}^1_\beta[Q_\beta\psi]_{n-1} = Q^0_\beta[Q_\beta\psi]_n \;,$$

$$[(Q_\beta)^2\psi]_n = (Q^0_\beta)^2\psi_n + \widetilde{\psi}_{n-1} \;,$$

where $\widetilde{\psi}_{n-1}$ (determined by $\psi_0, \ldots \psi_{n-1}$) has the desired properties. The equation $(34)_n$, i.e., $Q^0_\beta [Q_\beta \psi]_n = 0$ is thus equivalent to $(Q^0_\beta)^2 \psi_n = -\widetilde{\psi}_{n-1}$, which exhibits the claim.

On the other hand, invariance requires $P_0 \psi_n$ to be a linear combination of invariant singlets. For the ansatz $P_0 \psi_n = \lambda_n \psi_0$, eq. $(33)_n$ reads

$$\frac{3}{2} n \lambda_n \widehat{Q}^1_\beta \psi_0 = -P_0 \big(Q^1_\beta \overline{P}_0 \psi_n + Q^2_\beta \psi_{n-1} + \ldots + Q^{n+1}_\beta \psi_0 \big) \ ,$$

because of (35). Again, by the lemma, this holds true for suitable λ_n. Indeed, this solution for $P_0 \psi_n$ is the only one.

4.5 Proof of the lemma

The vectors $T_\beta \psi_0$, $(\beta = 1, \ldots, s_d)$ transform under Spin(d) as real spinors, although they might be linearly dependent. By reducing matters to the little group as before, any representation of that sort is specified by the values $|F^\beta(E, \vec{e})\rangle$ of its states (see (17)) at one point $(E, \vec{e})$, which are required to satisfy

$$\widetilde{R}_{\beta\alpha}(R) |F^\alpha(E, \vec{e})\rangle = \mathcal{R}(R) |F^\beta(E, \vec{e})\rangle$$

for R with $RE = E$. Pretending the states $|F^\beta(E, \vec{e})\rangle$ to be linearly independent, the branching Spin(d) $\hookleftarrow$ Spin($d-1$) yields

$$16 = 8_{\rm s} \oplus 8_{\rm c} \quad (d = 9) \ ; \qquad\qquad 4 \oplus 4 = (2_+ \oplus 2_-) \oplus (2_+ \oplus 2_-) \quad (d = 5) \ ;$$
$$2 \oplus 2 \ = \ (1_1 \oplus 1_{-1}) \oplus (1_1 \oplus 1_{-1}) \quad (d = 3) \ .$$

For $d = 9, 5$ each term on the r.h.s. occurs as often as in (29), and ψ_0 can indeed be chosen so that the s_d vectors $\widehat{Q}^1_\beta \psi_0$ are independent. Not so in the last case, where the vectors $T_\beta \psi_0$ just belong to $1_1 \oplus 1_{-1}$. We continue the discussion for different values of d separately.

• $d = 9$. Any linear transformation K commuting with a Spin(9) representation as above is thus of the form $K = \kappa_{\rm s} \oplus \kappa_{\rm c}$. If K also commutes with the antipode map, then $\kappa_{\rm s} = \kappa_{\rm c} \equiv \kappa$. Applying this to the representation $\widehat{Q}^1_\beta \psi_0$ and to the map $K : \ \widehat{Q}^1_\beta \psi_0 \mapsto T_\beta \psi_0$ yields the claim.

• $d = 5$. Let us regroup $(2_+ \oplus 2_-) \oplus (2_+ \oplus 2_-) \cong (2_+ \otimes \mathbb{1}_2) \oplus (2_- \otimes \mathbb{1}_2)$. Then any map K commuting with the representation is of the form

$$K = (\mathbb{1} \otimes K_+) \oplus (\mathbb{1} \otimes K_-) \ ,$$

where K_- is conjugate to K_+ if K commutes with the antipode map. This allows for a four dimensional space of such maps K. To proceed further we shall again assume that $E = (0, \ldots, 0, 1)$ and introduce creation operators

$$a^*_\alpha = \frac{1}{\sqrt{2}} [(\vec{\Theta}_\alpha \cdot \vec{e}) + {\rm i} (\vec{\Theta}_{\alpha+4} \cdot \vec{e})] \ , \qquad (\alpha = 1, \ldots 4)$$

which then define a vacuum through $a_\alpha |0\rangle = 0$. We next choose an orthonormal basis $\{\psi^1_0, \ldots, \psi^4_0\}$ for the 4-dimensional subspace of singlets in the range of P_0 by specifying the values of the corresponding fermionic parts (see (17)) at $(E,\vec{e})$:

$$
\begin{aligned}
|F_0^4(E,\vec{e})\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - a_1^* a_2^* a_3^* a_4^*|0\rangle) \ , \\
|F_0^i(E,\vec{e})\rangle &= \frac{1}{2\sqrt{2}}\widetilde{\Gamma}^i_{\alpha\beta} a_\alpha^* a_\beta^* |0\rangle = \frac{\mathrm{i}}{4}(\gamma^4\widetilde{\gamma}^i)_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{e})(\vec{\Theta}_\beta\cdot\vec{e})|F_0^4(E,\vec{e})\rangle \ , \qquad (i=1,2,3) \ ,
\end{aligned}
$$

where

$$
\widetilde{\gamma}^i = \begin{pmatrix} 0 & \mathrm{i}\widetilde{\Gamma}^i \\ -\mathrm{i}\widetilde{\Gamma}^i & 0 \end{pmatrix} = \sigma^{-1}\gamma^i\sigma \ , \qquad \sigma = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}
$$

with $\Sigma \in \mathrm{O}(4)$ and $\det\Sigma = -1$. Note that ψ_0^4 is the singlet belonging to the 5–dimensional fermionic representation of Spin(5). One can verify that the four maps

$$
K^i : \widehat{Q}^1_\beta \psi_0^1 \mapsto \begin{cases} \widehat{Q}^1_\beta \psi_0^i \ , & (i=1,2,3) \ , \\ \gamma^t_{\beta\alpha} E_t \widehat{Q}^1_\alpha \psi_0^4 \ , & (i=4) \ , \end{cases}
$$

besides being of the kind just discussed, are linearly independent. Therefore any map K of the above form is a linear combination thereof. In particular this applies, for any $(\underline{x}, x_4) \in \mathbb{R}^{3+1}$, to the map $K : \widehat{Q}^1_\beta\psi_0^1 \mapsto x_i T_\beta \psi_0^i + x_4 \gamma^t_{\beta\alpha} E_t T_\alpha \psi_0^4$, hence

$$
x_i T_\beta \psi_0^i + x_4 \gamma^t_{\beta\alpha} E_t T_\alpha \psi_0^4 = y_i \widehat{Q}^1_\beta \psi_0^i + y_4 \gamma^t_{\beta\alpha} E_t \widehat{Q}^1_\alpha \psi_0^4 \ .
$$

This defines a linear map $\kappa : (\underline{x}, x_4) \mapsto (\underline{y}, y_4)$ on $\mathbb{R}^{3+1}$. We claim that

$$
\kappa : (R\underline{x}, x_4) \mapsto (R\underline{y}, y_4) \tag{37}
$$

for $R \in \mathrm{SO}(3)$, which implies $\kappa = \mathrm{diag}(\kappa_1 = \kappa_2 = \kappa_3, \kappa_4)$ and hence (36). Eq. (37) can be proven using $R_{ij}\psi_0^i = \mathcal{R}\psi_0^j$ for $\mathcal{R} \in \mathrm{Spin}(8)$ projecting to $R \in \mathrm{Spin}(3) \subset \mathrm{Spin}(5) \hookrightarrow \mathrm{SO}(8)$. This in turn follows from (4) and from $\mathcal{R}\psi_0^4 = \psi_0^4$.

- $d = 3$. Analogously to $d = 9$.

4.6 Determination of κ

Since $J_{AB}\psi_0 = J_{st}\psi_0 = 0$ we may replace Q^1_β by

$$
Q^1_\beta = \Theta_{\alpha A}\gamma^t_{\alpha\beta}\big(-e_B E_t M_{BA} - e_A E_s M_{st} - \frac{\mathrm{i}}{2}\, e_A E_t y_{sB}\frac{\partial}{\partial y_{sB}}\big) + \frac{1}{2}\vec{\Theta}_\alpha\cdot(\vec{y}_s\times\vec{y}_t)\gamma^{st}_{\beta\alpha} \ . \tag{38}
$$

We discuss the contributions to (35) of these four terms separately.

i) With

$$
e_B M_{BA} = -\frac{\mathrm{i}}{2}\big((\vec{\Theta}_\beta\cdot\vec{e})\Theta_{\beta A} - \Theta_{\beta A}(\vec{\Theta}_\beta\cdot\vec{e})\big)
$$

we find

$$
\begin{aligned}
\Theta_{\alpha A} e_B M_{BA} &= \mathrm{i}\big((\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_\beta\cdot\vec{n}_-) + (\vec{\Theta}_\alpha\cdot\vec{n}_-)(\vec{\Theta}_\beta\cdot\vec{n}_+)\big)(\vec{\Theta}_\beta\cdot\vec{e}\,) \ , \\
P_0\Theta_{\alpha A} e_B M_{BA}\psi_0 &= \mathrm{i}(\vec{\Theta}_\alpha\cdot\vec{e}\,)\psi_0 \ ,
\end{aligned}
$$

since only the term with $\beta = \alpha$ survives the projection P_0. Hence

$$-P_0 \Theta_{\alpha A} \gamma^t_{\alpha\beta} e_B E_t M_{BA} \psi_0 = \widehat{Q}^1_\beta \psi_0 \tag{39}$$

contributes 1 to κ.

ii) Similarly,

$$-P_0(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M_{st} \psi_0 = -(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M^{\|}_{st} \psi_0 \ ,$$

where $M^{\|}_{st}$ is given in (31). For the r.h.s. we then claim

$$-(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M^{\|}_{st} \psi_0 = \kappa' \widehat{Q}^1_\beta \psi_0 \tag{40}$$

with

$$\kappa' = \begin{cases} 9\,, & (d=9)\ , \\ 0,0,0,4\,, & (d=5)\ , \\ 0,0\,, & (d=3)\ . \end{cases} \tag{41}$$

This is clear in the cases where the representation in (30) is already a singlet, i.e., when $\kappa' = 0$. To prove the two remaining cases we first establish

$$-(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M^{\|}_{st} \psi_0 = -\frac{\mathrm{i}}{2}\gamma^s_{\alpha\beta} E_s [\vec{\Theta}_\alpha \cdot \vec{e}, M^{\|}_{ut} M^{\|}_{ut}]\psi_0 - \mathrm{i}\frac{d^2-d}{8}(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^s_{\alpha\beta} E_s \psi_0 \ , \tag{42}$$

or the equivalent equation obtained by multiplication from the right with $E_u \gamma^u$:

$$-(\vec{\Theta}_\alpha \cdot \vec{e})(\gamma^t\gamma^u)_{\alpha\beta} E_u E_s M^{\|}_{st} \psi_0 = -\frac{\mathrm{i}}{2}[\vec{\Theta}_\beta \cdot \vec{e}, M^{\|}_{ut} M^{\|}_{ut}]\psi_0 - \mathrm{i}\frac{d^2-d}{8}(\vec{\Theta}_\beta \cdot \vec{e})\psi_0 \ . \tag{43}$$

To this end we note that, by the invariance of ψ_0, its fermionic part $|F(E,\vec{e})\rangle$ at $E \in S^{d-1}$ is invariant under rotations of $\mathrm{Spin}(d)$ leaving E fixed: $(\delta_{us} - E_u E_s) M^{\|}_{sv} (\delta_{vt} - E_v E_t)\psi_0 = 0$, i.e.,

$$(M^{\|}_{st} E_u E_s + M^{\|}_{uv} E_v E_t)\psi_0 = M^{\|}_{ut} \psi_0 \ . \tag{44}$$

Using $\gamma^t \gamma^u = -\gamma^{ut} + \delta^{ut}\mathbb{1}$ and the observation just made we rewrite the l.h.s. of (43) as

$$\begin{aligned} -(\vec{\Theta}_\alpha \cdot \vec{e})(\gamma^t\gamma^u)_{\alpha\beta} E_u E_s M^{\|}_{st} \psi_0 &= (\vec{\Theta}_\alpha \cdot \vec{e})\gamma^{ut}_{\alpha\beta} E_u E_s M^{\|}_{st} \psi_0 \\ &= \frac{1}{2}(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^{ut}_{\alpha\beta}(E_u E_s M^{\|}_{st} - E_t E_s M^{\|}_{su})\psi_0 \\ &= \frac{1}{2}(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^{ut}_{\alpha\beta} M^{\|}_{ut} \psi_0 \ . \end{aligned}$$

The commutation relation

$$\mathrm{i}[\vec{\Theta}_\alpha \cdot \vec{e}, M^{\|}_{ut}] = \frac{1}{2}\gamma^{ut}_{\alpha\beta}(\vec{\Theta}_\beta \cdot \vec{e})$$

follows from (4) or by direct computation. It implies

$$\mathrm{i}[\vec{\Theta}_\alpha \cdot \vec{e}, M^{\|}_{ut} M^{\|}_{ut}] = \frac{1}{2}\gamma^{ut}_{\alpha\beta}\{\vec{\Theta}_\beta \cdot \vec{e}, M^{\|}_{ut}\} = \gamma^{ut}_{\alpha\beta}(\vec{\Theta}_\beta \cdot \vec{e}) M^{\|}_{ut} - \frac{1}{2}\gamma^{ut}_{\alpha\beta}[\vec{\Theta}_\beta \cdot \vec{e}, M^{\|}_{ut}]$$

$$= \gamma^{ut}_{\alpha\beta}(\vec{\Theta}_\beta \cdot \vec{e}) M^{\|}_{ut} - \mathrm{i}\frac{d^2-d}{4}\vec{\Theta}_\alpha \cdot \vec{e} \ .$$

Solving for the first term on the r.h.s. proves (43) and hence (42). Let us now note that for $d = 9$ the fermionic part of ψ_0, resp. of $(\vec{\Theta}_\alpha \cdot \vec{e})\psi_0$ belongs to the 44, resp. 128 representation of Spin(9) (see (28)). Eq. (42) then implies

$$-(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M^{\|}_{st}\psi_0 = (C(44) - C(128) + 9)\widehat{Q}^1_\beta\psi_0 = 9\widehat{Q}^1_\beta\psi_0 \ ,$$

where we used the values [25] of the Casimir: $C(44) = C(128) = 18$. In the case $d = 5$ the fermionic part of ψ_0, resp. of $(\vec{\Theta}_\alpha \cdot \vec{e})\psi_0$ belongs to the representation 5, resp. $4 \oplus 4$. We conclude that

$$-(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta} E_s M^{\|}_{st}\psi_0 = (C(5) - C(4) + \frac{5}{2})\widehat{Q}^1_\beta\psi_0 = 4\widehat{Q}^1_\beta\psi_0 \ ,$$

given that $C(5) = 4$, $C(4) = 5/2$.

We remark that the proof of (41) can be shortened by using the lemma, according to which (40) holds true for some κ'. Thus, contracting with $\widehat{Q}^1_\beta\psi_0$ and summing over β, we find

$$\begin{aligned}
-\kappa'(\psi_0, \widehat{Q}^1_\beta\widehat{Q}^1_\beta\psi_0) &= -\mathrm{i}(\psi_0, (\vec{\Theta}_\gamma \cdot \vec{e})\gamma^u_{\gamma\beta}E_u(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta}E_s M^{\|}_{st}\psi_0) \\
&= 4(\psi_0, E_u M^{\|}_{ut} M^{\|}_{st} E_s \psi_0) \\
&= 2(\psi_0, M^{\|}_{ut}(M^{\|}_{st}E_u E_s + M^{\|}_{uv}E_v E_t)\psi_0) = 2(\psi_0, M^{\|}_{ut}M^{\|}_{ut}\psi_0) \ .
\end{aligned}$$

In the step before last we relabeled indices in half the expression; in the last one we used (44). Using $\widehat{Q}^1_\beta\widehat{Q}^1_\beta = -s_d/2$ we obtain $(s_d/2)\kappa' = 2 \cdot 2 \cdot C$, i.e., $\kappa' = 8C/s_d$, where C is the Casimir in the representation (30). The above values of $C(44)$ ($d = 9$) and of $C(5)$ ($d = 5$) yield again (41).

iii) Using $\mathrm{d}e^{-y^2/2}/\mathrm{d}y = -y e^{-y^2/2}$ we get

$$\frac{1}{2}y_{sB}\frac{\partial}{\partial y_{sB}}\psi_0 = -\frac{1}{2}y_{sB}y_{sB}\psi_0 = -\frac{1}{2}\sum_{sB}(y^2_{sB} - \frac{1}{2})\psi_0 - \frac{1}{4}\cdot 2(d-1)\psi_0 \ , \qquad (45)$$

where the sum, consisting of second Hermite functions, is annihilated by P_0.

iv) The last term in (38), when acting on ψ_0, is similarly annihilated by P_0.

Collecting terms (39, 41, 45) we find

$$\kappa = 1 + \kappa' - \frac{1}{2}(d-1) = \begin{cases} 6\,, & (d=9)\ , \\ -1, -1, -1, 3\,, & (d=5)\ , \\ 0, 0\,, & (d=3)\ . \end{cases}$$

Appendix 1

To prove (13) we shall compute the partial derivatives in

$$\frac{\partial}{\partial q_{tA}} = \frac{\partial r}{\partial q_{tA}}\frac{\partial}{\partial r} + \frac{\partial e_B}{\partial q_{tA}}\frac{\partial}{\partial e_B} + \frac{\partial E_s}{\partial q_{tA}}\frac{\partial}{\partial E_s} + \frac{\partial y_{sB}}{\partial q_{tA}}\frac{\partial}{\partial y_{sB}} \ . \qquad (46)$$

We regard r, $\vec{e}$, E, y as functions of q defined by $\vec{e}^2 = \sum_s E_s^2 = 1$ and (9, 10) and solve for their differentials by taking different contractions of

$$\mathrm{d}q_{tA} = (e_A E_t - \frac{1}{2} r^{-3/2} y_{tA})\mathrm{d}r + rE_t \mathrm{d}e_A + re_A \mathrm{d}E_t + r^{-1/2}\mathrm{d}y_{tA} \ .$$

Using that

$$\begin{aligned} e_A \mathrm{d}y_{tA} + y_{tA}\mathrm{d}e_A = 0 \ , &\qquad E_t \mathrm{d}y_{tA} + y_{tA}\mathrm{d}E_t = 0 \ , \\ e_A \mathrm{d}e_A = 0 \ , &\qquad E_t \mathrm{d}E_t = 0 \ , \end{aligned}$$

the contractions are:

$$\begin{aligned} e_A E_t \mathrm{d}q_{tA} &= \mathrm{d}r \ , \\ (\delta_{BA} - e_B e_A)E_t \mathrm{d}q_{tA} &= r\mathrm{d}e_B - r^{-1/2} y_{tA}\mathrm{d}E_t \ , \qquad (47) \\ e_A(\delta_{st} - E_s E_t)\mathrm{d}q_{tA} &= r\mathrm{d}E_s - r^{-1/2} y_{sA}\mathrm{d}e_A \ , \qquad (48) \\ (\delta_{BA} - e_B e_A)(\delta_{st} - E_s E_t)\mathrm{d}q_{tA} &= -\frac{1}{2} r^{-3/2} y_{sB}\mathrm{d}r + r^{-1/2}(\mathrm{d}y_{sB} + e_B y_{sA}\mathrm{d}e_A + E_s y_{tB}\mathrm{d}E_t) \ . \end{aligned}$$

We solve (47, 48) for $\mathrm{d}e_B$, $\mathrm{d}E_s$:

$$\begin{aligned} \mathrm{d}r &= e_A E_t \mathrm{d}q_{tA} \ , \\ \mathrm{d}e_B &= (m^{-1})_{BC}(r^{-1}(\delta_{CA} - e_C e_A)E_t + r^{-5/2} y_{tC} e_A)\mathrm{d}q_{tA} \\ &= (r^{-1}(\delta_{BA} - e_B e_A)E_t + \mathrm{O}(r^{-5/2}))\mathrm{d}q_{tA} \ , \\ \mathrm{d}E_s &= (M^{-1})_{su}(r^{-1}(\delta_{ut} - E_u E_t)e_A + r^{-5/2} y_{sA} E_t)\mathrm{d}q_{tA} \\ &= (r^{-1}(\delta_{st} - E_s E_t)e_A + \mathrm{O}(r^{-5/2}))\mathrm{d}q_{tA} \ , \\ \mathrm{d}y_{sB} &= [r^{1/2}(\delta_{BA} - e_B e_A)(\delta_{st} - E_s E_t) + \frac{1}{2} r^{-1} e_A E_t y_{sB}]\mathrm{d}q_{tA} - e_B y_{sA}\mathrm{d}e_A - E_s y_{tB}\mathrm{d}E_t \ , \end{aligned}$$

where m, M are the matrices

$$m_{AB} = \delta_{AB} - r^{-3} y_{tA} y_{tB} \ , \qquad M_{st} = \delta_{st} - r^{-3} y_{sA} y_{tA} \ .$$

We can now read off the partial derivatives appearing in (46) and obtain

$$\begin{aligned} \frac{\partial}{\partial q_{tA}} &= r^{1/2}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} + r^{-1}[e_A E_t (r\frac{\partial}{\partial r} + \frac{1}{2} y_{sB}\frac{\partial}{\partial y_{sB}})] \\ &\quad + r^{-1}(\delta_{AC} - e_A e_C)E_t(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}) \\ &\quad + r^{-1}(\delta_{ut} - E_u E_t)e_A(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}) + \mathrm{O}(r^{-5/2}) \ , \qquad (49) \end{aligned}$$

with the remainder not containing derivatives w.r.t. r. Finally, we insert this expression into

$$\begin{aligned} \mathrm{i}L_{BA} &= q_{sB}\frac{\partial}{\partial q_{sA}} - q_{sA}\frac{\partial}{\partial q_{sB}} \\ &= [(\delta_{AC} - e_A e_C)y_{sB} - (\delta_{BC} - e_B e_C)y_{sA}]\frac{\partial}{\partial y_{sC}} \\ &\quad + e_B(\delta_{AC}\frac{\partial}{\partial e_C} - e_C y_{sA}\frac{\partial}{\partial y_{sC}}) - e_A(\delta_{BC}\frac{\partial}{\partial e_C} - e_C y_{sB}\frac{\partial}{\partial y_{sC}}) \ , \end{aligned}$$

(with no higher order corrections, as L_{AB} is of exact order $\mathrm{O}(r^0)$) and then into

$$\mathrm{i}r^{-1}e_B E_t L_{BA} = r^{-1}(\delta_{AC} - e_A e_C)E_t(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}) \; .$$

Similarly, we have

$$\mathrm{i}r^{-1}e_A E_s L_{st} = r^{-1}(\delta_{ut} - E_u E_t)e_A(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}) \; .$$

Together with (49), this proves (13).

Appendix 2

Consider

$$H = (-\partial_x{}^2 - \partial_y{}^2 + x^2y^2)\mathbb{1} + \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix} \; , \tag{50}$$

which is the square of

$$Q = \mathrm{i}\begin{pmatrix} \partial_x & \partial_y + xy \\ \partial_y - xy & -\partial_x \end{pmatrix} .$$

Just as in (8), the bosonic potential V $(= x^2y^2)$ is non-negative, but vanishing in regions of the configuration space that extend to infinity (causing the classical partition function to diverge). Quantum–mechanically, just as in (8), the bosonic system is stabilized by the zero point energy of fluctuations transverse to the flat directions; the fermionic matrix part in (50) exactly cancels this effect, causing the spectrum to cover the whole positive real axis [19]. As simple as it is, it has remained an open question (for now more than 10 years) whether (50) admits a normalizable zero energy solution, or not. The argument, derived in a few lines below, gives 'no' as an answer and provides the simplest illustration of our method: as $x \to +\infty$, $Q\Psi = 0$ has two approximate solutions,

$$\Psi_+ = e^{-\frac{xy^2}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \Psi_- = e^{+\frac{xy^2}{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \tag{51}$$

the first of which should be chosen for Ψ_0 in the asymptotic expansions

$$\Psi = x^{-\kappa}(\Psi_0 + \Psi_1 + ...) \; . \tag{52}$$

In this simple example, the sum $Q = \sum_{n=0}^{\infty} Q^{(n)}$ terminates after the first two terms, and

$$0 \stackrel{!}{=} Q\Psi = \left(\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix} + \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}\right)\left(x^{-\kappa}(\Psi_0 + \Psi_1 + ...)\right) \; ,$$

yields (as already anticipated, cp. (51))

$$\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix}\Psi_0 = 0$$

and

$$\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix} \Psi_n + x^\kappa \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} x^{-\kappa}\Psi_{n-1} = 0 \,, \qquad n = 1, 2, \dots \,. \tag{53}$$

Multiplying (53) by $\Psi_0^\dagger$ and integrating over y one sees that

$$\int_{-\infty}^{+\infty} e^{-\frac{xy^2}{2}} x^\kappa (0, -\partial_x) x^{-\kappa} \Psi_{n-1} dy$$

has to vanish, implying in particular

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} \left(\frac{y^2}{2} + \frac{\kappa}{x} \right) e^{-xy^2} dy \,, \\ \kappa &= -\frac{1}{4} \,, \end{aligned}$$

which proves that (50) does not admit any square–integrable solution of the form (52). A different approach has recently been undertaken by Avramidi [26]. Finally note that, calculating the $\Psi_{n>0}$ from (53), yields the asymptotic expansion, $x \to +\infty$,

$$\Psi(x, y) = x^{\frac{1}{4}} e^{-\frac{xy^2}{2}} \sum_{n=0}^{\infty} x^{-\frac{3n}{2}} \begin{pmatrix} \frac{y}{4x} f_n(xy^2) \\ g_n(xy^2) \end{pmatrix} \,,$$

where $f_0 = 1 = g_0$, $f_1 = 0 = g_1$, and the $f_n(s)$, $g_n(s)$ are the (unique) polynomial solutions

$$f_n(s) = \sum_{i=0}^{n} f_{n,i} s^i \,, \qquad g_n(s) = \sum_{i=0}^{n} g_{n,i} s^i$$

of

$$\begin{aligned} 2sf_n' + (1 - 2s) f_n &= (1 - 2s - 6n)\, g_n + 4sg_n' \,, \\ 8g_{n+2}' &= \left(\frac{3}{4} + \frac{s}{2} + \frac{3n}{2} \right) f_n - sf_n' \,. \end{aligned}$$

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