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Asymptotic form of zero energy wave functions in supersymmetric matrix models

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Abstract

We derive the power law decay, and asymptotic form, of $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ invariant wavefunctions satisfying $Q_\beta \psi = 0$ for all $s_d = 2(d-1)$ supercharges of reduced $(d+1)$-dimensional supersymmetric SU(2) Yang–Mills theory, of, respectively, the SU(2) matrix model related to supermembranes in $d+2$ dimensions.

1. Introduction

During the past few years there has been renewed interest in matrix models, owing to some interesting developments in string and M-theory, in particular the discovery of D-branes.

Bosonic matrix models were originally introduced in the early eighties as regularizations of relativistic membrane dynamics; see Refs. [1–3]. (A particularly original feature of the work in [1–3] is the use of non-commutative parameter spaces approximating classical surfaces.) These models also arise as-dimensional reduction to $0+1$ dimensions of Yang–Mills theory. A few years later, supersymmetric matrix models were proposed and analyzed in [4–8]. There was comparatively little activity in the analysis of these models until, three years ago, they were proposed as models for the dynamics of D0-branes and of M-theory (with a flat, eleven-dimensional target space-time) in [9], following seminal work in [10]. This led to a reinterpretation of the physical significance of supersymmetric matrix models avoiding problems described in [8].

The question of whether the Hamiltonian of supersymmetric $\mathrm{SU}(N)$ matrix models has a normalizable, unique, gauge-invariant ground state, for arbitrary $N = 2,3,\ldots$ and in different dimensions $d = 2,3,5$ and 9, where $d+2$ is the dimension of space-time, has emerged as one of the fundamental issues in the study of these models and has

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attracted a lot of interest. Early negative results for $d < 9$ can be found in [11–13], at least for $N = 2$. Different approaches to establishing properties of normalizable ground states for various values of N and $d = 9$ have been developed in [14–21]. The approach in [14–16,18] (see also Ref. [22] and references therein) is based on a calculation of the Witten index. In [19], the asymptotic form of the ground state wave function for the $N = 2$, $d = 9$ model is derived with the help of Hamiltonian Born–Oppenheimer methods. A noteworthy feature of [19] is that the analysis applies to possible ground states which are not Spin(9) singlets. In [13,23] a Born–Oppenheimer method involving an explicit use of the supercharges is described. This note is an elaboration of the methods proposed there. With the help of Born–Oppenheimer-type calculations with supercharges, we find that asymptotically normalizable SU(2)- and Spin(d)-invariant ground state wave functions do not exist for $d = 2,3$, and 5, while in $d = 9$ dimensions precisely one such wave function appears to exist (in agreement with Ref. [19]).

The paper is organized as follows. In Section 2 we recall the definition of the models, and in Section 3 we state our main result about zero-modes. The proof is given in Section 4 and Appendix A. We suggest to skip Subsection 4.5 and Appendix A at a first reading. As a warm-up the reader is advised to read Appendix B, where a simpler model is treated by the same method.

2. The models

The configuration space of the bosonic degrees of freedom is $X = \mathbb{R}^{3d}$ with coordinates

$$q = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_d) = (q_{sA})_{s=1,\ldots,d,\ A=1,2,3}.$$

To describe the fermionic degrees of freedom let, as a preliminary,

$$\gamma^i = \left(\gamma^i_{\alpha\beta}\right)_{\alpha,\beta=1,\ldots,s_d} \quad (i = 1, \ldots, d), \tag{1}$$

be the *real* representation of smallest dimension, called s_d, of the Clifford algebra with d generators: $\{\gamma^s, \gamma^t\} = 2\,\delta^{st}\mathbf{1}$. On the representation space, Spin(d) is realized through matrices $R \in \mathrm{SO}(s_d)$, so that we may view

$$\mathrm{Spin}(d) \hookrightarrow \mathrm{SO}(s_d), \tag{2}$$

as a simply connected subgroup. We recall that

$$s_d = \begin{cases} 2^{[d/2]}, & d = 0,1,2 \bmod 8, \\ 2^{[d/2]+1} & \text{otherwise}, \end{cases}$$

where $[\cdot]$ denotes the integer part. We then consider the Clifford algebra with s_d generators and its irreducible representation on $\mathscr{C} = \mathbb{C}^{2^{s_d/2}}$. On $\mathscr{C}^{\otimes 3}$ the Clifford generators

$$(\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_{s_d}) = (\Theta_{\alpha A})_{\alpha=1,\ldots,s_d,\ A=1,2,3}$$

are defined, satisfying $\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\,\delta_{AB}$. The Hilbert space, finally, is

$$\mathscr{H} = \mathrm{L}^2(X, \mathscr{C}^{\otimes 3}). \tag{3}$$

There is a natural representation of $\mathrm{SU}(2)\times\mathrm{Spin}(d)\ni(U,R)$ on $\mathscr{H}$. In fact, the group acts naturally on X through its representation $\mathrm{SO}(3)\times\mathrm{SO}(d)$ (which we also denote by (U,R)). On $\mathscr{C}^{\otimes 3}$ we have the representation $\mathscr{R}$ of $\mathrm{Spin}(s_d)\ni R$

$$\mathscr{R}(R)^*\,\Theta_{\alpha A}\mathscr{R}(R)=\tilde{R}_{\alpha\beta}\Theta_{\beta A},\tag{4}$$

where $\tilde{R}=\tilde{R}(R)$ is its $\mathrm{SO}(s_d)$ representation. Through $\mathrm{SO}(s_d)=\mathrm{Spin}(s_d)/\mathbb{Z}_2$ and (2) we have

$$\mathrm{Spin}(d)\hookrightarrow\mathrm{Spin}(s_d),\tag{5}$$

and thus a representation $\mathscr{R}$ of $\mathrm{Spin}(d)$. The representation $\mathscr{U}$ of $\mathrm{SU}(2)\ni U$ on $\mathscr{C}^{\otimes 3}$ is characterized by $\mathscr{U}(U)^*\Theta_{\alpha A}\mathscr{U}(U)=U_{AB}\Theta_{\alpha B}$.

We shall now restrict to $d=2,3,5,9$, where $s_d=2,4,8,16$, the reason being that in these cases

$$s_d=2(d-1),\tag{6}$$

whereas s_d is strictly larger otherwise. Eq. (6) is essential for the algebra (7) below [6].

The supercharges, acting on $\mathscr{H}$, are given by the s_d hermitian operators

$$Q_\beta=\boldsymbol{\Theta}_\alpha\cdot\Big(-\mathrm{i}\gamma^t_{\alpha\beta}\boldsymbol{\nabla}_t+\tfrac{1}{2}\,\boldsymbol{q}_s\times\boldsymbol{q}_t\,\gamma^{st}_{\beta\alpha}\Big)\quad(\beta=1,\dots,s_d),$$

where $\gamma^{st}=\frac{1}{2}(\gamma^s\gamma^t-\gamma^t\gamma^s)$. These supercharges transform as scalars under SU(2) transformations generated by

$$J_{AB}=-\mathrm{i}(q_{sA}\partial_{sB}-q_{sB}\partial_{sA})-\frac{\mathrm{i}}{2}(\Theta_{\alpha A}\Theta_{\alpha B}-\Theta_{\alpha B}\Theta_{\alpha A})\equiv L_{AB}+M_{AB},$$

and as vectors in $\mathbb{R}^{s_d}$ under $\mathrm{Spin}(d)$ transformation generated by

$$J_{st}=-\mathrm{i}(\boldsymbol{q}_s\cdot\boldsymbol{\nabla}_t-\boldsymbol{q}_t\cdot\boldsymbol{\nabla}_s)-\frac{\mathrm{i}}{4}\,\boldsymbol{\Theta}_\alpha\gamma^{st}_{\alpha\beta}\boldsymbol{\Theta}_\beta\equiv L_{st}+M_{st}.$$

The anticommutation relations of the supercharges are

$$\{Q_\alpha,Q_\beta\}=\delta_{\alpha\beta}H+\gamma^t_{\alpha\beta}q_{tA}\varepsilon_{ABC}J_{BC}.\tag{7}$$

Here, H is the Hamiltonian

$$H=-\sum_{s=1}^{9}\boldsymbol{\nabla}_s^{\,2}+\sum_{s<t}(\boldsymbol{q}_s\times\boldsymbol{q}_t)^2+\mathrm{i}\,\boldsymbol{q}_s\cdot(\boldsymbol{\Theta}_\alpha\times\boldsymbol{\Theta}_\beta)\,\gamma^s_{\alpha\beta},\tag{8}$$

which commutes with both J_{AB} and J_{st}. The question we address is the possibility of a normalizable state $\psi\in\mathscr{H}$ with zero energy, i.e. with $H\psi=0$, which is a singlet with respect to both SU(2) and $\mathrm{Spin}(d)$. Note that on SU(2) invariant states $H=2Q_\beta^2\geqslant 0$ and in fact the energy spectrum is ([8]) $\sigma(H)=[0,\infty)$. Equivalently, we look for zero-modes

$$Q_\beta\psi=0\quad(\beta=1,\dots,s_d).$$

3. Results

The potential $\sum_{s<t}(\boldsymbol{q}_s\times\boldsymbol{q}_t)^2$ vanishes on the manifold

$$\boldsymbol{q}_s=r\boldsymbol{e}E_s$$

with $r>0$ and $\boldsymbol{e}^2=\sum_s E_s^2=1$. The dimension of the manifold is $1+2+(d-1)=3d$

$-2(d-1)$. Points in a conical neighborhood of the manifold can be expressed in terms of tubular (or ''end-point'') coordinates [25]

$$\boldsymbol{q}_s = r\boldsymbol{e}E_s + r^{-1/2}\boldsymbol{y}_s \tag{9}$$

with

$$\boldsymbol{y}_s \cdot \boldsymbol{e} = 0, \qquad \boldsymbol{y}_s E_s = \boldsymbol{0}. \tag{10}$$

A prefactor has been put explicitly in front of the transversal coordinates $\boldsymbol{y}_s$, so as to anticipate the length scale $r^{-1/2}$ of the ground state. The change

$$(\boldsymbol{e}, E, y) \mapsto (-\boldsymbol{e}, -E, y) \tag{11}$$

does not affect $\boldsymbol{q}_s$. Rather than identifying the two coordinates for $\boldsymbol{q}_s$, we shall look for states which are even under the antipode map (11).

We can now describe the structure of a putative ground state.

Theorem 1. Consider the equations $Q_\beta \psi = 0$ for a formal power series solution near $r = \infty$ of the form

$$\psi = r^{-\kappa} \sum_{k=0}^{\infty} r^{-\frac{3}{2}k} \psi_k, \tag{12}$$

where

$\psi_k = \psi_k(\boldsymbol{e}, E, y)$ is square integrable with respect to $de\, dE\, dy$;

ψ_k is $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ invariant;

$\psi_0 \neq 0$.

Then, up to linear combinations,
· $d = 9$: The solution is unique, and $\kappa = 6$;
· $d = 5$: There are three solutions with $\kappa = -1$ and one with $\kappa = 3$;
· $d = 3$: There are two solutions with $\kappa = 0$;
· $d = 2$: There are no solutions.
All solutions are even under the antipode map (11),

$$\psi_k(\boldsymbol{e}, E, y) = \psi_k(-\boldsymbol{e}, -E, y),$$

except for the state $d = 5$, $\kappa = 3$, which is odd.

Remark 2. The equation $Q_\beta \psi = 0$ can be viewed as an ordinary differential equation in $z = r^{3/2}$ for a function taking values in $\mathrm{L}^2(de\, dE\, dy, \mathscr{C}^{\otimes 3})$ (see Eq. (14) below). It turns out that $z = \infty$ is a singular point of the second kind [24]. In such a situation the series (12) is typically asymptotic to a true solution, but not convergent.

Remark 3. The integration measure is $dq = dr \cdot r^2 de \cdot r^{d-1} dE \cdot r^{-\frac{1}{2}\cdot 2(d-1)} dy = r^2 dr\, de\, dE\, dy$. The wave function (12) is square integrable at infinity if $\int^\infty dr\, r^2 (r^{-\kappa})^2 < \infty$, i.e. if $\kappa > 3/2$. The theorem is consistent with the statement according to which *only* for $d=9$ a (unique) normalizable ground state for (8) (which would have to be even) is possible.

Remark 4. Note that the connection of matrix models with supergravity requires the zero-energy solutions to be Spin(d) singlets only for $d=9$.

Remark 5. The result for $d=9$ agrees with the one found in [19] for the Spin(9)-singlet case.

The case $d=2$ can be dealt with immediately. We may assume $\gamma^2=\sigma_3$, $\gamma^1=\sigma_1$ (Pauli matrices), so that

$$M_{12}=\frac{\mathrm{i}}{2}\Theta_{1A}\Theta_{2A},$$

with commuting terms. Since, for each $A=1,2,3, (\Theta_{1A}\Theta_{2A})^2=-1/4$, we see that M_{12} has spectrum in $\mathbb{Z}/2+1/4$. Given that L_{12} has spectrum $\mathbb{Z}$, no state with $J_{12}\psi=0$ is possible. We mention [11] that, more generally, for $d=2$ no normalizable SU(2) invariant ground state exists.

The proof of the theorem will thus deal with $d=9,5,3$ only.

4. Proof

We shall first derive the power series expansion of the supercharges Q_β. To this end we note that

$$\begin{aligned}\frac{\partial}{\partial q_{tA}}&=r^{1/2}(\delta_{st}-E_sE_t)(\delta_{AB}-e_Ae_B)\frac{\partial}{\partial y_{sB}}\\&\quad+r^{-1}\left[e_AE_t\left(r\frac{\partial}{\partial r}+\tfrac{1}{2}y_{sB}\frac{\partial}{\partial y_{sB}}\right)+\mathrm{i}e_BE_tL_{BA}+\mathrm{i}e_AE_sL_{st}\right]+\mathrm{O}(r^{-5/2}),\end{aligned}\tag{13}$$

with the remainder not containing derivatives with respect to r (see Appendix A for derivation). This yields

$$Q_\beta=r^{1/2}Q^0_\beta+r^{-1}\left(\hat{Q}^1_\beta r\frac{\partial}{\partial r}+Q^1_\beta\right)+r^{-5/2}Q^2_\beta+\ldots\tag{14}$$

with r-independent operators

$$Q^0_\beta=-\mathrm{i}\Theta_{\alpha A}\gamma^t_{\alpha\beta}(\delta_{st}-E_sE_t)(\delta_{AB}-e_Ae_B)\frac{\partial}{\partial y_{sB}}+\boldsymbol{\Theta}_\alpha\cdot(\boldsymbol{e}\times\boldsymbol{y}_t)E_s\gamma^{st}_{\beta\alpha},$$

$$\hat{Q}^1_\beta=-\mathrm{i}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^t_{\alpha\beta}E_t,$$

$$Q^1_\beta=\Theta_{\alpha A}\gamma^t_{\alpha\beta}\left(e_BE_tL_{BA}+e_AE_sL_{st}-\frac{\mathrm{i}}{2}e_AE_ty_{sB}\frac{\partial}{\partial y_{sB}}\right)+\tfrac{1}{2}\boldsymbol{\Theta}_\alpha\cdot(\boldsymbol{y}_s\times\boldsymbol{y}_t)\gamma^{st}_{\beta\alpha}.$$

The explicit expressions of Q^n_β $(n \geqslant 2)$ will not be needed. We then equate coefficients of powers of $r^{-3/2}$ in the equation $Q_\beta \psi = 0$ with the result

$$Q^0_\beta \psi_n + \Big(-\big(\kappa + \tfrac{3}{2}(n-1)\big)\hat{Q}^1_\beta + Q^1_\beta\Big)\psi_{n-1} + Q^2_\beta \psi_{n-2} + \ldots + Q^n_\beta \psi_0 = 0$$
$$(n = 0,1,\ldots). \tag{15}$$

4.1. The equation at n = 0

The equation at $n = 0$,

$$Q^0_\beta \psi_0 = 0, \tag{16}$$

admits precisely the (not necessarily $SU(2) \times Spin(d)$ invariant) solutions

$$\psi_0(\boldsymbol{e}, E, y) = e^{-\sum_s y_s{}^2/2} |F(E,\boldsymbol{e})\rangle, \tag{17}$$

(with $\boldsymbol{y}$ restricted to (10)), where the fermionic states $|F(E,\boldsymbol{e})\rangle$ can be described as follows: Let $\boldsymbol{n}_\pm$ be two complex vectors satisfying $\boldsymbol{n}_+ \cdot \boldsymbol{n}_- = 1$, $\boldsymbol{e} \times \boldsymbol{n}_\pm = \mp \mathrm{i}\boldsymbol{n}_\pm$ (and hence $\boldsymbol{n}_\pm \cdot \boldsymbol{n}_\pm = 0$, $\boldsymbol{n}_+ \times \boldsymbol{n}_- = -\mathrm{i}\boldsymbol{e}$). For any vector $v \in \mathbb{R}^{s_d}$ we may introduce $\boldsymbol{\Theta}(v) = \boldsymbol{\Theta}_\alpha v_\alpha$, as well as fermionic operators $\boldsymbol{\Theta}(v) \cdot \boldsymbol{n}_\pm$ satisfying canonical anticommutation relations:

$$\{\boldsymbol{\Theta}(u) \cdot \boldsymbol{n}_+, \boldsymbol{\Theta}(v) \cdot \boldsymbol{n}_-\} = u_\alpha v_\alpha, \qquad \{\boldsymbol{\Theta}(u) \cdot \boldsymbol{n}_\pm, \boldsymbol{\Theta}(v) \cdot \boldsymbol{n}_\pm\} = 0.$$

Then, $|F(E,\boldsymbol{e})\rangle$ is required to obey

$$\boldsymbol{\Theta}(v) \cdot \boldsymbol{n}_\pm |F(E,\boldsymbol{e})\rangle = 0 \quad \text{for } E_s \gamma^s v = \pm v. \tag{18}$$

To prove the above, let us note that

$$\{Q^0_\alpha, Q^0_\beta\} = \delta_{\alpha\beta} H^0 + \gamma^t_{\alpha\beta} E_t \varepsilon_{ABC} M_{AB} e_C, \tag{19}$$

$$H^0 = \left[-(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sA}}\frac{\partial}{\partial y_{tB}} + \sum_s \boldsymbol{y}_s^2\right]$$
$$+ \mathrm{i} E_s \gamma^s_{\alpha\beta} \boldsymbol{e} \cdot (\boldsymbol{\Theta}_\alpha \times \boldsymbol{\Theta}_\beta) \equiv H^0_B + H^0_F.$$

By contracting Eq. (19) with $\delta_{\alpha\beta}$ and $\gamma^t_{\alpha\beta} E_t$ we see that Eqs. (16), respectively, are equivalent to the pair of equations

$$H^0 \psi_0 = 0, \qquad \varepsilon_{ABC} M_{AB} e_C \psi_0 = 0. \tag{20}$$

Here, H^0_B is a harmonic oscillator in $2(d-1)$ degrees of freedom, with orbital ground state wave function $e^{-\sum_s y_s^2/2}$ and energy $2(d-1)$. On the other hand,

$$H^0_F = -E_s \gamma^s_{\alpha\beta}\big((\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_+)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_-) - (\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_-)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_+)\big)$$
$$= -s_d + 2P^+_{\alpha\beta}(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_-)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_+) + 2P^-_{\alpha\beta}(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_+)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_-), \tag{21}$$

where we used the spectral decomposition $E_s\gamma^s = P^+ - P^-$. In view of (6), the equation $H^0\psi_0 = 0$ is fulfilled iff the fermionic state is annihilated by the last two positive terms in (21), i.e. if (18) holds. The second equation (20) is now also satisfied, since

$$\begin{aligned}\tfrac{1}{2}\varepsilon_{ABC}M_{AB}e_C &= -\frac{\mathrm{i}}{2}\boldsymbol{e}\cdot(\boldsymbol{\Theta}_\alpha\times\boldsymbol{\Theta}_\alpha)\\ &= \tfrac{1}{2}\big((\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_-)-(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_-)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)\big)\\ &= P^-_{\alpha\beta}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{n}_-)-P^+_{\alpha\beta}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_-)(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{n}_+)\end{aligned}\qquad(22)$$

annihilates $|F(E,\boldsymbol{e})\rangle$.

4.2. SU*(2)* $\times$ Spin*(d) invariant states*

We recall that the representation $\mathscr{R}[\cdot]$ of Spin(d) on $\mathscr{H}$ is $(\mathscr{R}[R]\psi)(q) = \mathscr{R}(R)(\psi(R^{-1}q))$, where $\mathscr{R}(R)$ acts on $\mathscr{C}^{\otimes 3}$. Similarly for SU(2). The invariant solutions among (17) are thus those which satisfy

$$\mathscr{U}(U)|F(E,\boldsymbol{e})\rangle = |F(E,U\boldsymbol{e})\rangle,\qquad \mathscr{R}(R)|F(E,\boldsymbol{e})\rangle = |F(RE,\boldsymbol{e})\rangle,\qquad(23)$$

for $(U,R)\in \mathrm{SU}(2)\times\mathrm{Spin}(d)$. These states are in bijective correspondence to states invariant under the 'little group' $(U,R)\in\mathrm{U}(1)\times\mathrm{Spin}(d-1)$, i.e. to states $|F(E,\boldsymbol{e})\rangle$ satisfying

$$\mathscr{U}(U)|F(E,\boldsymbol{e})\rangle = |F(E,\boldsymbol{e})\rangle,\qquad \mathscr{R}(R)|F(E,\boldsymbol{e})\rangle = |F(E,\boldsymbol{e})\rangle,\qquad(24)$$

for some arbitrary but fixed $(E,\boldsymbol{e})$ and all U, R with $U\boldsymbol{e}=\boldsymbol{e}$, $RE=E$. The first relation holds on all of (18). In fact the generator (22) of the group $\mathscr{U}(U)$ of rotations U about $\boldsymbol{e}$ annihilates $|F(E,\boldsymbol{e})\rangle$, as we just saw. To discuss the second relation (24) we note that the generators of Spin$(d-1)$ (i.e. of the fermionic rotations about E), are $M_{st}U_sV_t$ with $U_sE_s = V_sE_s = 0$. We write $M_{st} = M_{st}^{\perp} + M_{st}^{\parallel}$, where

$$M_{st}^{\perp} = -(\mathrm{i}/2)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)\gamma^{st}_{\alpha\beta}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{n}_-),\qquad M_{st}^{\parallel} = -(\mathrm{i}/4)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^{st}_{\alpha\beta}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e}),\qquad(25)$$

and remark that, by a computation similar to (22), $M_{st}^{\perp}U_sV_t$ annihilates $|F(E,\boldsymbol{e})\rangle$. As a result, we may study the representation $\mathscr{R}$ of the group Spin$(d-1)$ through its embedding in the Clifford algebra generated by the $\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e}$.

The operators $\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e}$ leave the space (18) invariant and act irreducibly on it. That space is thus isomorphic to $\mathscr{C}$, and Spin(s_d) acts according to (4) (with $\Theta_{\alpha A}$ replaced by $\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e}$). This representation decomposes (see e.g. Ref. [26]) as

$$\mathscr{C} = (2^{(s_d/2)-1})_+\oplus(2^{(s_d/2)-1})_-\qquad(26)$$

with respect to the subspaces where $\Theta\equiv 2^{s_d/2}\prod_{\alpha=1}^{s_d}\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e} = +1$, and -1, respectively. The embedding (5) and the corresponding branching of the representation (but not the statement of the theorem!) depend on the choice of the γ-matrices. In order to select a definite embedding, let

$$\gamma^d = \begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix},\qquad \gamma^{d-1} = \begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix},\qquad \gamma^j = \begin{pmatrix}0 & \mathrm{i}\Gamma^j\\ -\mathrm{i}\Gamma^j & 0\end{pmatrix}\qquad(27)$$

with Γ^j $(j=1,\dots,d-2)$ purely imaginary, antisymmetric, and $\{\Gamma^j,\Gamma^k\}=2\,\delta_{jk}\mathbf{1}_{s_d/2}$. Then (26) branches as (see Ref. [27], and Refs. [13,23], respectively)

$$\mathscr{C}=\begin{cases}(44\oplus 84)\oplus 128, & (d=9),\\ (5\oplus 1\oplus 1\oplus 1)\oplus(4\oplus 4), & (d=5),\\ 2\oplus(1\oplus 1), & (d=3),\end{cases}\tag{28}$$

when viewed as a representation of Spin(d). (The choice $\tilde{\gamma}^i_{\alpha\beta}=\tilde{R}_{\alpha'\alpha}\gamma^i_{\alpha'\beta'}\tilde{R}_{\beta'\beta}$ with $\tilde{R}\in \mathrm{O}(s_d)$, $\det\tilde{R}=-1$ would have inverted the branching of the representations on the r.h.s. of (26)). The case $d=3$ deserves a remark, as there are additional inequivalent embeddings Spin$(d=3)\hookrightarrow$ Spin$(s_d=4)$, and one has to consider the one appropriate to (5). In fact $R\in \mathrm{Spin}(3)=\mathrm{SU}(2)$ acts in the fundamental representation on $\mathbb{C}^2$, the irreducible representation space of the complex Clifford algebra with 3 generators. The real representation (27) is obtained by joining two complex representations, followed by an appropriate change T of basis. The embedding (5) is thus realized through $R\mapsto T^{-1}(R\otimes\mathbf{1}_2)T$ and the embedding $\mathrm{su}(2)_{\mathbb{C}}\hookrightarrow \mathrm{so}(4)_{\mathbb{C}}=\mathrm{su}(2)_{\mathbb{C}}\oplus\mathrm{su}(2)_{\mathbb{C}}$ is equivalent to $u\mapsto(u,0)$.

The further branching Spin$(d)\hookleftarrow$ Spin$(d-1)$ yields

$$\mathscr{C}=\begin{cases}(1\oplus 8_{\mathrm{v}}\oplus 35_{\mathrm{v}})\oplus(28\oplus 56_{\mathrm{v}})\oplus(8_{\mathrm{s}}\oplus 8_{\mathrm{c}}\oplus 56_{\mathrm{s}}\oplus 56_{\mathrm{c}}), & (d-1=8),\\ 1\oplus 1\oplus 1\oplus(1\oplus 4)\oplus(2_+\oplus 2_-)\oplus(2_+\oplus 2_-), & (d-1=4),\\ (1_1\oplus 1_{-1})\oplus 1_0\oplus 1_0, & (d-1=2).\end{cases}\tag{29}$$

The content of invariant states stated in the theorem is now manifest. One should notice that for $d=3$ the little group U(1) is abelian and the singlets $1_{\pm 1}$ do not correspond to invariant states. For later use we also retain the fermionic Spin(d) representation to which the remaining singlets are associated,

$$44\quad(d=9);\qquad 1,1,1,5\quad(d=5);\qquad 1,1\quad(d=3),\tag{30}$$

together with the corresponding eigenvalue of Θ:

$$\Theta=\quad 1\quad(d=9);\qquad 1,1,1,1\quad(d=5);\qquad -1,-1\quad(d=3).\tag{31}$$

4.3. Even states

It remains to check which of these states satisfy $|F(-E,-\boldsymbol{e})\rangle=|F(E,\boldsymbol{e})\rangle$. Let us begin by noting that by (23)

$$|F(-E,-\boldsymbol{e})\rangle=\mathrm{e}^{\mathrm{i}M_{AB}e_Au_B\pi}\mathrm{e}^{\mathrm{i}M_{st}E_sU_t\pi}|F(E,\boldsymbol{e})\rangle,$$

where $\boldsymbol{u}\in\mathbb{R}^3$ and $U\in\mathbb{R}^d$ are unit vectors orthogonal to $\boldsymbol{e}$ and E, respectively. The Spin(d) rotation can be factorized as $\mathrm{e}^{\mathrm{i}M_{st}E_sU_t\pi}=\mathrm{e}^{\mathrm{i}M^{\perp}_{st}E_sU_t\pi}\mathrm{e}^{\mathrm{i}M^{\parallel}_{st}E_sU_t\pi}$. We claim that $\mathrm{e}^{\mathrm{i}M^{\parallel}_{st}E_sU_t\pi}\,|F(E,\boldsymbol{e})\rangle=\sigma|F(E,\boldsymbol{e})\rangle$ with

$$\begin{aligned}&\sigma=1 && (d=9);\\ &\sigma=1,1,1,-1 && (d=5);\\ &\sigma=1,1 && (d=3).\end{aligned}\tag{32}$$

The operator represents a rotation $R\in\mathrm{Spin}(d)$ with $RE=-E$ in the representation (30). For $d=9$ the latter can be realized on symmetric traceless tensors T_{ij}, $(i,j=$

$1,\dots,9$), where the Spin(8)-singlet is $E_i E_j - (1/9)\delta_{ij}$, implying $\sigma = 1$. For $d = 5$, the last representation (30) is just the vector representation, where $\sigma = -1$. As the remaining cases are evident, Eq. (32) is proven. A computation using (27) and, without loss $E = (0,\dots,0,1), U = (0,\dots,1,0)$ shows

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}M^{\perp}_{d,d-1}\pi}|F(E,\boldsymbol{e})\rangle &= \prod_{\alpha=1}^{s_d/2} \mathrm{e}^{[(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_{\alpha+s_d/2}\cdot\boldsymbol{n}_-)-(\boldsymbol{\Theta}_{\alpha+s_d/2}\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_-)]\pi/2}|F(E,\boldsymbol{e})\rangle \\
&= \prod_{\alpha=1}^{s_d/2}(\boldsymbol{\Theta}_{\alpha+s_d/2}\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_-)|F(E,\boldsymbol{e})\rangle \equiv |\overline{F}(E,\boldsymbol{e})\rangle, \\
\mathrm{e}^{\mathrm{i}M_{AB}e_A u_B\pi}|\overline{F}(E,\boldsymbol{e})\rangle &= \prod_{\alpha=1}^{s_d} \mathrm{e}^{(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{u})\pi}|\overline{F}(E,\boldsymbol{e})\rangle \\
&= (-1)^{s_d/4}\Theta\prod_{\alpha=1}^{s_d/2}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{n}_+)(\boldsymbol{\Theta}_{\alpha+s_d/2}\cdot\boldsymbol{n}_-)|\overline{F}(E,\boldsymbol{e})\rangle \\
&= |F(E,\boldsymbol{e})\rangle,
\end{aligned}$$

where we used (31) in the last step. Together with (32) this proves the statement of theorem concerning the invariance under (11).

4.4. The equation at $n > 0$

We next discuss Eqs. $(15)_n$ with $n \geqslant 1$. Let P_0 be the orthogonal projection onto the states (17), i.e. onto the null space of Q^0_β. We replace them with an equivalent pair of equations, obtained by multiplication of $(15)_{n+1}$ with P_0, and $(15)_n$ with Q^0_β, respectively, which is injective on the range of the complementary projection $\overline{P}_0 = 1 - P_0$:

$$P_0\big(-\big(\kappa+\tfrac{3}{2}n\big)\big)\hat{Q}^1_\beta + Q^1_\beta)\,P_0\psi_n = -P_0\Big(Q^1_\beta\,\overline{P}_0\psi_n + Q^2_\beta\psi_{n-1} + \dots + Q^{n+1}_\beta\psi_0\Big) \qquad (n=0,1,\dots), \tag{33}$$

$$\big(Q^0_\beta\big)^2\psi_n = -Q^0_\beta\Big(\Big(-\big(\kappa+\tfrac{3}{2}(n-1)\big)\hat{Q}^1_\beta + Q^1_\beta\Big)\psi_{n-1} + Q^2_\beta\psi_{n-2} + \dots + Q^n_\beta\psi_0\Big) \qquad (n=1,2,\dots) \tag{34}$$

(we used $P_0\hat{Q}^1_\beta\overline{P}_0 = 0$). Here, and until the end of this subsection, no summation over β is understood. Eq. (33) at $n = 0$ reads

$$P_0 Q^1_\beta\psi_0 = \kappa P_0\hat{Q}^1_\beta\psi_0\ \Big(=\kappa\hat{Q}^1_\beta\psi_0\Big). \tag{35}$$

We shall verify this by explicit computation later on. Since a similar issue will show up in solving Eq. (33) at $n > 0$, let us also present a more general statement, whose proof is postponed to the next subsection.

Lemma 6. Let T_β be linear operators on the range of P_0, which transform as real spinors of Spin(d) and commute with the antipode map. Then, for each invariant state we have

$$T_\beta\psi_0 = \kappa\hat{Q}^1_\beta\psi_0, \tag{36}$$

with κ depending only on the associated representation (30).

We now assume having solved Eqs. (33), (34) up to $n-1$ for Spin(d) invariant $\psi_1, \dots \psi_{n-1}$ (which is true for $n-1=0$), and claim the same is possible for n. Since Q^0_β is invertible on the range of $\bar{P}_0$, Eq. $(34)_n$ determines $\bar{P}_0\psi_n$ uniquely. The fact that the solution so obtained is independent of β and is Spin(d) invariant may deserve a comment, because the equivalence of the equations $Q_\beta\psi = 0$ and $(Q_\beta)^2\psi = 0$, which holds on (3), does not apply in the sense of formal power series (12). Consider the expansion (14), i.e.

$$Q_\beta = r^{1/2}\sum_{k=0}^{\infty} r^{-\frac{3}{2}k}\left[Q_\beta\right]_k, \qquad \left[Q_\beta\right]_k = Q^k_\beta + \delta_{1k}\hat{Q}^1_\beta r\frac{\partial}{\partial r},$$

as well as its formal square

$$\left(Q_\beta\right)^2 = r\sum_{k=0}^{\infty} r^{-\frac{3}{2}k}\left[\left(Q_\beta\right)^2\right]_k.$$

Notice that $(Q_\beta)^2$ is, by (7), independent of β and Spin(d) invariant as an operator on SU(2) invariant power series. Similarly, let $[Q_\beta\psi]_k$ (given by the l.h.s. of (15)) and $[(Q_\beta)^2\psi]_k$ be the coefficients of the corresponding series. By induction assumption we have $[Q_\beta\psi]_k = 0$ for $k = 0, \dots, n-1$. Since $Q_\beta(Q_\beta\psi) = (Q_\beta)^2\psi$, we obtain

$$\left[\left(Q_\beta\right)^2\psi\right]_n = \sum_{k=0}^{n} Q^k_\beta\left[Q_\beta\psi\right]_{n-k} - \left(\kappa + \tfrac{3}{2}n - 2\right)\hat{Q}^1_\beta\left[Q_\beta\psi\right]_{n-1} = Q^0_\beta\left[Q_\beta\psi\right]_n,$$

$$\left[\left(Q_\beta\right)^2\psi\right]_n = \left(Q^0_\beta\right)^2\psi_n + \tilde{\psi}_{n-1},$$

where $\tilde{\psi}_{n-1}$ (determined by $\psi_0, \dots \psi_{n-1}$) has the desired properties. The Eq. $(34)_n$, i.e. $Q^0_\beta[Q_\beta\psi]_n = 0$ is thus equivalent to $(Q^0_\beta)^2\psi_n = -\tilde{\psi}_{n-1}$, which exhibits the claim.

On the other hand, invariance requires $P_0\psi_n$ to be a linear combination of invariant singlets. For the ansatz $P_0\psi_n = \lambda_n\psi_0$, Eq. $(33)_n$ reads

$$\tfrac{3}{2}n\lambda_n\hat{Q}^1_\beta\psi_0 = -P_0\left(Q^1_\beta\bar{P}_0\psi_n + Q^2_\beta\psi_{n-1} + \dots + Q^{n+1}_\beta\psi_0\right),$$

because of (35). Again, by the lemma, this holds true for suitable λ_n. Indeed, this solution for $P_0\psi_n$ is the only one.

4.5. Proof of the lemma

The vectors $T_\beta\psi_0$, $(\beta = 1, \dots, s_d)$ transform under Spin(d) as real spinors, although they might be linearly dependent. By reducing matters to the little group as before, any representation of that sort is specified by the values $|F^\beta(E,\boldsymbol{e})\rangle$ of its states (see (17)) at one point $(E,\boldsymbol{e})$, which are required to satisfy

$$\tilde{R}_{\beta\alpha}(R)|F^\alpha(E,\boldsymbol{e})\rangle = \mathscr{R}(R)|F^\beta(E,\boldsymbol{e})\rangle$$

for R with $RE = E$. Pretending the states $|F^\beta(E,\boldsymbol{e})\rangle$ to be linearly independent, the branching Spin$(d) \hookleftarrow$ Spin$(d-1)$ yields

$$16 = 8_s \oplus 8_c \quad (d=9); \qquad 4\oplus 4 = (2_+ \oplus 2_-)\oplus(2_+\oplus 2_-) \quad (d=5);$$

$$2\oplus 2 = (1_1 \oplus 1_{-1})\oplus(1_1\oplus 1_{-1}) \quad (d=3).$$

For $d=9,5$ each term on the r.h.s. occurs as often as in (29), and ψ_0 can indeed be chosen so that the s_d vectors $\hat{Q}^1_\beta\psi_0$ are independent. Not so in the last case, where the vectors $T_\beta\psi_0$ just belong to $1_1\oplus 1_{-1}$. We continue the discussion for different values of d separately.

● $d=9$. Any linear transformation K commuting with a Spin(9) representation as above is thus of the form $K=\kappa_s\oplus\kappa_c$. If K also commutes with the antipode map, then $\kappa_s=\kappa_c\equiv\kappa$. Applying this to the representation $\hat{Q}^1_\beta\psi_0$ and to the map $K\colon \hat{Q}^1_\beta\psi_0\mapsto T_\beta\psi_0$ yields the claim.

● $d=5$. Let us regroup $(2_+\oplus 2_-)\oplus(2_+\oplus 2_-)\cong(2_+\otimes\mathbf{1}_2)\oplus(2_-\otimes\mathbf{1}_2)$. Then any map K commuting with the representation is of the form

$$K=(\mathbf{1}\otimes K_+)\oplus(\mathbf{1}\otimes K_-),$$

where K_- is conjugate to K_+ if K commutes with the antipode map. This allows for a four-dimensional space of such maps K. To proceed further we shall again assume that $E=(0,\dots,0,1)$ and introduce creation operators

$$a^*_\alpha=\frac{1}{\sqrt{2}}\big[(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})+\mathrm{i}(\boldsymbol{\Theta}_{\alpha+4}\cdot\boldsymbol{e})\big],\qquad(\alpha=1,\dots 4)$$

which then define a vacuum through $a_\alpha|0\rangle=0$. We next choose an orthonormal basis $\{\psi^1_0,\dots,\psi^4_0\}$ for the 4-dimensional subspace of singlets in the range of P_0 by specifying the values of the corresponding fermionic parts (see (17)) at $(E,\boldsymbol{e})$:

$$|F^4_0(E,\boldsymbol{e})\rangle=\frac{1}{\sqrt{2}}\big(|0\rangle-a^*_1a^*_2a^*_3a^*_4|0\rangle\big),$$

$$|F^i_0(E,\boldsymbol{e})\rangle=\frac{1}{2\sqrt{2}}\tilde{\Gamma}^i_{\alpha\beta}a^*_\alpha a^*_\beta|0\rangle=\frac{\mathrm{i}}{4}\big(\gamma^4\tilde{\gamma}^i\big)_{\alpha\beta}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e})|F^4_0(E,\boldsymbol{e})\rangle,$$

$$(i=1,2,3),$$

where

$$\tilde{\gamma}^i=\begin{pmatrix}0 & \mathrm{i}\tilde{\Gamma}^i\\ -\mathrm{i}\tilde{\Gamma}^i & 0\end{pmatrix}=\sigma^{-1}\gamma^i\sigma,\qquad \sigma=\begin{pmatrix}\Sigma & 0\\ 0 & \Sigma\end{pmatrix}$$

with $\Sigma\in\mathrm{O}(4)$ and $\det\Sigma=-1$. Note that ψ^4_0 is the singlet belonging to the 5-dimensional fermionic representation of Spin(5). One can verify that the four maps

$$K^i\colon \hat{Q}^1_\beta\psi^1_0\mapsto\begin{cases}\hat{Q}^1_\beta\psi^i_0, & (i=1,2,3),\\ \gamma^t_{\beta\alpha}E_t\hat{Q}^1_\alpha\psi^4_0, & (i=4),\end{cases}$$

besides being of the kind just discussed, are linearly independent. Therefore any map K of the above form is a linear combination thereof. In particular this applies, for any $(\underline{x},x_4)\in\mathbb{R}^{3+1}$, to the map $K\colon\hat{Q}^1_\beta\psi^1_0\mapsto x_iT_\beta\psi^i_0+x_4\gamma^t_{\beta\alpha}E_tT_\alpha\psi^4_0$, hence

$$x_iT_\beta\psi^i_0+x_4\gamma^t_{\beta\alpha}E_tT_\alpha\psi^4_0=y_i\hat{Q}^1_\beta\psi^i_0+y_4\gamma^t_{\beta\alpha}E_t\hat{Q}^1_\alpha\psi^4_0.$$

This defines a linear map $\kappa\colon(\underline{x},x_4)\mapsto(\underline{y},y_4)$ on $\mathbb{R}^{3+1}$. We claim that

$$\kappa\colon(R\underline{x},x_4)\mapsto(R\underline{y},y_4)\tag{37}$$

for $R\in\mathrm{SO}(3)$, which implies $\kappa=\mathrm{diag}(\kappa_1=\kappa_2=\kappa_3,\kappa_4)$ and hence (36). Eq. (37) can be proven using $R_{ij}\psi_0^i = \mathscr{R}\psi_0^j$ for $\mathscr{R} \in \mathrm{Spin}(8)$ projecting to $R \in \mathrm{Spin}(3) \subset \mathrm{Spin}(5) \hookrightarrow \mathrm{SO}(8)$. This in turn follows from (4) and from $\mathscr{R}\psi_0^4 = \psi_0^4$.

● $d = 3$. Analogously to $d = 9$.

4.6. Determination of κ

Since $J_{AB}\psi_0 = J_{st}\psi_0 = 0$ we may replace Q_β^1 by

$$Q_\beta^1 = \Theta_{\alpha A}\gamma_{\alpha\beta}^t\left(-e_B E_t M_{BA} - e_A E_s M_{st} - \frac{\mathrm{i}}{2} e_A E_t y_{sB}\frac{\partial}{\partial y_{sB}}\right) + \tfrac{1}{2}\boldsymbol{\Theta}_\alpha \cdot (\boldsymbol{y}_s \times \boldsymbol{y}_t)\gamma_{\beta\alpha}^{st}. \tag{38}$$

We discuss the contributions to (35) of these four terms separately.

(i) With

$$e_B M_{BA} = -\frac{\mathrm{i}}{2}\big((\boldsymbol{\Theta}_\beta \cdot \boldsymbol{e})\Theta_{\beta A} - \Theta_{\beta A}(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{e})\big)$$

we find

$$\Theta_{\alpha A} e_B M_{BA} = \mathrm{i}\big((\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_+)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_-) + (\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{n}_-)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{n}_+)\big)(\boldsymbol{\Theta}_\beta \cdot \boldsymbol{e}),$$

$$P_0\Theta_{\alpha A} e_B M_{BA}\psi_0 = \mathrm{i}(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\psi_0,$$

since only the term with $\beta = \alpha$ survives the projection P_0. Hence

$$-P_0\Theta_{\alpha A}\gamma_{\alpha\beta}^t e_B E_t M_{BA}\psi_0 = \hat{Q}_\beta^1\psi_0 \tag{39}$$

contributes 1 to κ.

(ii) Similarly,

$$-P_0(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\gamma_{\alpha\beta}^t E_s M_{st}\psi_0 = -(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\gamma_{\alpha\beta}^t E_s M_{st}^{\|}\psi_0,$$

where $M_{st}^{\|}$ is given in (31). For the r.h.s. we then claim

$$-(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\gamma_{\alpha\beta}^t E_s M_{st}^{\|}\psi_0 = \kappa'\hat{Q}_\beta^1\psi_0 \tag{40}$$

with

$$\kappa' = \begin{cases} 9, & (d = 9), \\ 0,0,0,4, & (d = 5), \\ 0,0, & (d = 3). \end{cases} \tag{41}$$

This is clear in the cases where the representation in (30) is already a singlet, i.e. when $\kappa' = 0$. To prove the two remaining cases we first establish

$$-(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\gamma_{\alpha\beta}^t E_s M_{st}^{\|}\psi_0 = -\frac{\mathrm{i}}{2}\gamma_{\alpha\beta}^s E_s\big[\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e}, M_{ut}^{\|}M_{ut}^{\|}\big]\psi_0 - \mathrm{i}\frac{d^2 - d}{8}(\boldsymbol{\Theta}_\alpha \cdot \boldsymbol{e})\gamma_{\alpha\beta}^s E_s\psi_0, \tag{42}$$

or the equivalent equation obtained by multiplication from the right with $E_u\gamma^u$:

$$-(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})(\gamma^t\gamma^u)_{\alpha\beta}E_uE_sM^{\parallel}_{st}\psi_0=-\frac{\mathrm{i}}{2}\left[\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e},M^{\parallel}_{ut}M^{\parallel}_{ut}\right]\psi_0 -\mathrm{i}\frac{d^2-d}{8}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e})\psi_0. \quad (43)$$

To this end we note that, by the invariance of ψ_0, its fermionic part $|F(E,\boldsymbol{e})\rangle$ at $E\in S^{d-1}$ is invariant under rotations of $\mathrm{Spin}(d)$ leaving E fixed: $(\delta_{us}-E_uE_s)M^{\parallel}_{sv}(\delta_{vt}-E_vE_t)\psi_0=0$, i.e.

$$\left(M^{\parallel}_{st}E_uE_s+M^{\parallel}_{uv}E_vE_t\right)\psi_0=M^{\parallel}_{ut}\psi_0. \quad (44)$$

Using $\gamma^t\gamma^u=-\gamma^{ut}+\delta^{ut}\mathbf{1}$ and the observation just made we rewrite the l.h.s. of (43) as

$$\begin{aligned}-(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})(\gamma^t\gamma^u)_{\alpha\beta}E_uE_sM^{\parallel}_{st}\psi_0&=(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^{ut}_{\alpha\beta}E_uE_sM^{\parallel}_{st}\psi_0\\&=\tfrac{1}{2}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^{ut}_{\alpha\beta}\left(E_uE_sM^{\parallel}_{st}-E_tE_sM^{\parallel}_{su}\right)\psi_0\\&=\tfrac{1}{2}(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^{ut}_{\alpha\beta}M^{\parallel}_{ut}\psi_0.\end{aligned}$$

The commutation relation

$$\mathrm{i}\left[\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e},M^{\parallel}_{ut}\right]=\tfrac{1}{2}\gamma^{ut}_{\alpha\beta}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e})$$

follows from (4) or by direct computation. It implies

$$\begin{aligned}\mathrm{i}\left[\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e},M^{\parallel}_{ut}M^{\parallel}_{ut}\right]&=\tfrac{1}{2}\gamma^{ut}_{\alpha\beta}\left\{\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e},M^{\parallel}_{ut}\right\}=\gamma^{ut}_{\alpha\beta}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e})M^{\parallel}_{ut}-\tfrac{1}{2}\gamma^{ut}_{\alpha\beta}\left[\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e},M^{\parallel}_{ut}\right]\\&=\gamma^{ut}_{\alpha\beta}(\boldsymbol{\Theta}_\beta\cdot\boldsymbol{e})M^{\parallel}_{ut}-\mathrm{i}\frac{d^2-d}{4}\,\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e}.\end{aligned}$$

Solving for the first term on the r.h.s. proves (43) and hence (42). Let us now note that for $d=9$ the fermionic part of ψ_0 and $(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\psi_0$ belongs to the 44 and 128 representation respectively of Spin(9) (see (28)). Eq. (42) then implies

$$-(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^t_{\alpha\beta}E_sM^{\parallel}_{st}\psi_0=(C(44)-C(128)+9)\hat{Q}^1_\beta\psi_0=9\hat{Q}^1_\beta\psi_0,$$

where we used the values [27] of the Casimir: $C(44)=C(128)=18$. In the case $d=5$ the fermionic part of ψ_0 and $(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\psi_0$ belongs to the representation 5 and $4\oplus4$, respectively. We conclude that

$$-(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^t_{\alpha\beta}E_sM^{\parallel}_{st}\psi_0=\left(C(5)-C(4)+\tfrac{5}{2}\right)\hat{Q}^1_\beta\psi_0=4\hat{Q}^1_\beta\psi_0,$$

given that $C(5)=4, C(4)=5/2$.

We remark that the proof of (41) can be shortened by using the lemma, according to which (40) holds true for some κ'. Thus, contracting with $\hat{Q}^1_\beta\psi_0$ and summing over β, we find

$$\begin{aligned}-\kappa'\left(\psi_0,\hat{Q}^1_\beta\hat{Q}^1_\beta\psi_0\right)&=-\mathrm{i}\left(\psi_0,(\boldsymbol{\Theta}_\gamma\cdot\boldsymbol{e})\gamma^u_{\gamma\beta}E_u(\boldsymbol{\Theta}_\alpha\cdot\boldsymbol{e})\gamma^t_{\alpha\beta}E_sM^{\parallel}_{st}\psi_0\right)\\&=4\left(\psi_0,E_uM^{\parallel}_{ut}M^{\parallel}_{st}E_s\psi_0\right)\\&=2\left(\psi_0,M^{\parallel}_{ut}\left(M^{\parallel}_{st}E_uE_s+M^{\parallel}_{uv}E_vE_t\right)\psi_0\right)=2\left(\psi_0,M^{\parallel}_{ut}M^{\parallel}_{ut}\psi_0\right).\end{aligned}$$

In the step before last we relabeled indices in half the expression; in the last one we used (44). Using $\hat{Q}^1_\beta\hat{Q}^1_\beta=-s_d/2$ we obtain $(s_d/2)\kappa'=2\cdot2\cdot C$, i.e. $\kappa'=8C/s_d$, where C

is the Casimir in the representation (30). The above values of $C(44)(d=9)$ and of $C(5)(d=5)$ yield again (41).

(iii) Using $d\mathrm{e}^{-y^2/2}/dy = -y\mathrm{e}^{-y^2/2}$ we get

$$\tfrac{1}{2} y_{sB} \frac{\partial}{\partial y_{sB}} \psi_0 = -\tfrac{1}{2} y_{sB} y_{sB} \psi_0 = -\tfrac{1}{2} \sum_{sB} \left(y_{sB}^2 - \tfrac{1}{2} \right) \psi_0 - \tfrac{1}{4} \cdot 2(d-1)\psi_0, \tag{45}$$

where the sum, consisting of second Hermite functions, is annihilated by P_0.

(iv) The last term in (38), when acting on ψ_0, is similarly annihilated by P_0.

Collecting terms (39, 41, 45) we find

$$\kappa = 1 + \kappa' - \tfrac{1}{2}(d-1) = \begin{cases} 6, & (d=9), \\ -1,-1,-1,3, & (d=5), \\ 0,0, & (d=3). \end{cases}$$

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Appendix A

To prove (13) we shall compute the partial derivatives in

$$\frac{\partial}{\partial q_{tA}} = \frac{\partial r}{\partial q_{tA}} \frac{\partial}{\partial r} + \frac{\partial e_B}{\partial q_{tA}} \frac{\partial}{\partial e_B} + \frac{\partial E_s}{\partial q_{tA}} \frac{\partial}{\partial E_s} + \frac{\partial y_{sB}}{\partial q_{tA}} \frac{\partial}{\partial y_{sB}}. \tag{A.1}$$

We regard $r, \boldsymbol{e}, E, y$ as functions of q defined by $\boldsymbol{e}^2 = \sum_s E_s^2 = 1$ and (9, 10) and solve for their differentials by taking different contractions of

$$dq_{tA} = \left(e_A E_t - \tfrac{1}{2} r^{-3/2} y_{tA} \right) dr + r E_t de_A + r e_A dE_t + r^{-1/2} dy_{tA}.$$

Using that

$$e_A dy_{tA} + y_{tA} de_A = 0, \qquad E_t dy_{tA} + y_{tA} dE_t = 0, \qquad e_A de_A = 0, \qquad E_t dE_t = 0,$$

the contractions are

$$e_A E_t dq_{tA} = dr,$$

$$(\delta_{BA} - e_B e_A) E_t dq_{tA} = r de_B - r^{-1/2} y_{tA} dE_t, \tag{A.2}$$

$$e_A(\delta_{st} - E_s E_t) dq_{tA} = r dE_s - r^{-1/2} y_{sA} de_A, \tag{A.3}$$

$$(\delta_{BA} - e_B e_A)(\delta_{st} - E_s E_t) dq_{tA} = -\tfrac{1}{2} r^{-3/2} y_{sB} dr + r^{-1/2}(dy_{sB} + e_B y_{sA} de_A + E_s y_{tB} dE_t).$$

We solve (A.2), (A.3) for de_B, dE_s:

$$dr = e_A E_t dq_{tA},$$

$$de_B = (m^{-1})_{BC}\big(r^{-1}(\delta_{CA} - e_C e_A) E_t + r^{-5/2} y_{tC} e_A\big) dq_{tA}$$

$$= \big(r^{-1}(\delta_{BA} - e_B e_A) E_t + \mathrm{O}(r^{-5/2})\big) dq_{tA},$$

$$dE_s = (M^{-1})_{su}\big(r^{-1}(\delta_{ut} - E_u E_t) e_A + r^{-5/2} y_{sA} E_t\big) dq_{tA}$$

$$= \big(r^{-1}(\delta_{st} - E_s E_t) e_A + \mathrm{O}(r^{-5/2})\big) dq_{tA},$$

$$dy_{sB} = \big[r^{1/2}(\delta_{BA} - e_B e_A)(\delta_{st} - E_s E_t) + \tfrac{1}{2} r^{-1} e_A E_t y_{sB}\big] dq_{tA} - e_B y_{sA} de_A - E_s y_{tB} dE_t,$$

where m, M are the matrices

$$m_{AB} = \delta_{AB} - r^{-3} y_{tA} y_{tB}, \qquad M_{st} = \delta_{st} - r^{-3} y_{sA} y_{tA}.$$

We can now read off the partial derivatives appearing in (A.1) and obtain

$$\frac{\partial}{\partial q_{tA}} = r^{1/2}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} + r^{-1}\left[e_A E_t\left(r\frac{\partial}{\partial r} + \tfrac{1}{2} y_{sB}\frac{\partial}{\partial y_{sB}}\right)\right]$$
$$+ r^{-1}(\delta_{AC} - e_A e_C) E_t\left(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}\right)$$
$$+ r^{-1}(\delta_{ut} - E_u E_t) e_A\left(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}\right) + \mathrm{O}(r^{-5/2}), \qquad \text{(A.4)}$$

with the remainder not containing derivatives with respect to r. Finally, we insert this expression into

$$\mathrm{i} L_{BA} = q_{sB}\frac{\partial}{\partial q_{sA}} - q_{sA}\frac{\partial}{\partial q_{sB}}$$
$$= \big[(\delta_{AC} - e_A e_C) y_{sB} - (\delta_{BC} - e_B e_C) y_{sA}\big]\frac{\partial}{\partial y_{sC}} + e_B\left(\delta_{AC}\frac{\partial}{\partial e_C} - e_C y_{sA}\frac{\partial}{\partial y_{sC}}\right)$$
$$- e_A\left(\delta_{BC}\frac{\partial}{\partial e_C} - e_C y_{sB}\frac{\partial}{\partial y_{sC}}\right),$$

(with no higher order corrections, as L_{AB} is of exact order $\mathrm{O}(r^0)$) and then into

$$\mathrm{i} r^{-1} e_B E_t L_{BA} = r^{-1}(\delta_{AC} - e_A e_C) E_t\left(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}\right).$$

Similarly, we have

$$\mathrm{i} r^{-1} e_A E_s L_{st} = r^{-1}(\delta_{ut} - E_u E_t) e_A\left(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}\right).$$

Together with (A.4), this proves (13).

Appendix B

Consider

$$H=\left(-\partial_x^2-\partial_y^2+x^2y^2\right)\mathbf{1}+\begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}, \tag{B.1}$$

which is the square of

$$Q=\mathrm{i}\begin{pmatrix} \partial_x & \partial_y+xy \\ \partial_y-xy & -\partial_x \end{pmatrix}.$$

Just as in (8), the bosonic potential $V\ (=x^2y^2)$ is non-negative, but vanishing in regions of the configuration space that extend to infinity (causing the classical partition function to diverge). Quantum mechanically, just as in (8), the bosonic system is stabilized by the zero point energy of fluctuations transverse to the flat directions; the fermionic matrix part in (B.1) exactly cancels this effect, causing the spectrum to cover the whole positive real axis [8]. As simple as it is, it has remained an open question (for now more than ten years) whether (B.1) admits a normalizable zero energy solution, or not. The argument, derived in a few lines below, gives 'no' as an answer and provides the simplest illustration of our method: as $x\to+\infty$, $Q\Psi=0$ has two approximate solutions,

$$\Psi_+=e^{-\frac{1}{2}xy^2}\begin{pmatrix}0\\1\end{pmatrix} \quad\text{and}\quad \Psi_-=e^{+\frac{1}{2}xy^2}\begin{pmatrix}1\\0\end{pmatrix}, \tag{B.2}$$

the first of which should be chosen for Ψ_0 in the asymptotic expansions

$$\Psi=x^{-\kappa}\left(\Psi_0+\Psi_1+\dots\right). \tag{B.3}$$

In this simple example, the sum $Q=\sum_{n=0}^{\infty}Q^{(n)}$ terminates after the first two terms, and

$$0\overset{!}{=}Q\Psi=\left(\begin{pmatrix} 0 & \partial_y+xy \\ \partial_y-xy & 0 \end{pmatrix}+\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}\right)\left(x^{-\kappa}\left(\Psi_0+\Psi_1+\dots\right)\right),$$

yields (as already anticipated, cf. (B.2))

$$\begin{pmatrix} 0 & \partial_y+xy \\ \partial_y-xy & 0 \end{pmatrix}\Psi_0=0$$

and

$$\begin{pmatrix} 0 & \partial_y+xy \\ \partial_y-xy & 0 \end{pmatrix}\Psi_n+x^{\kappa}\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}x^{-\kappa}\Psi_{n-1}=0,\quad n=1,2,\dots \tag{B.4}$$

Multiplying (B.4) by $\Psi_0^{\dagger}$ and integrating over y one sees that

$$\int_{-\infty}^{+\infty}e^{-\frac{1}{2}xy^2}x^{\kappa}\left(0,-\partial_x\right)x^{-\kappa}\Psi_{n-1}\,dy$$

has to vanish, implying in particular

$$0=\int_{-\infty}^{+\infty}\left(\frac{y^2}{2}+\frac{\kappa}{x}\right)e^{-xy^2}dy,$$

$$\kappa=-\tfrac{1}{4},$$

which proves that (B.1) does not admit any square-integrable solution of the form (B.3).

A different approach has recently been undertaken by Avramidi [28]. Finally note that, calculating the $\Psi_{n>0}$ from (B.4), yields the asymptotic expansion, $x \to +\infty$,

$$\Psi(x,y) = x^{\frac{1}{4}} e^{-\frac{1}{2}xy^2} \sum_{n=0}^{\infty} x^{-\frac{3n}{2}} \begin{pmatrix} \frac{y}{4x} f_n(xy^2) \\ g_n(xy^2) \end{pmatrix},$$

where $f_0 = 1 = g_0$, $f_1 = 0 = g_1$, and the $f_n(s)$, $g_n(s)$ are the (unique) polynomial solutions

$$f_n(s) = \sum_{i=0}^{n} f_{n,i} s^i, \qquad g_n(s) = \sum_{i=0}^{n} g_{n,i} s^i$$

of

$$2sf_n' + (1-2s) f_n = (1-2s-6n) g_n + 4sg_n',$$

$$8g_{n+2}' = \left(\tfrac{3}{4} + \frac{s}{2} + \frac{3n}{2} \right) f_n - sf_n'.$$

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