2-Form Gravity of the Lorentzian Signature

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Abstract

We introduce a new spinorial, BF-like action for the Einstein gravity. This is a first, up to our knowledge, 2-form action which describes the real, Lorentzian gravity and uses only the self-dual connection. In the generic case, the corresponding classical canonical theory is equivalent to the Einstein-Ashtekar theory plus the reality conditions.

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Introduction. It is well known that, in four dimensions, a metric tensor can be derived from a suitably normalized triad of self-dual 2-forms [1, 2, 3, 4, 6, 5, 11]. Given a complex 4 dimensional vector space V equipped with a complex valued metric tensor, one decomposes the space $\wedge^2 V^\star$ of two forms into the direct, wedge-orthogonal sum

$$\wedge^2 V^\star = \Omega^- \oplus \Omega^+$$

of the spaces of anti-self-dual and self-dual 2-forms, respectively. The space Ω^+ (as well as Ω^-), determines the metric tensor up to a conformal factor. The missing information about the metric tensor can be provided by indicating an element ${\Sigma^0}_0 \in \Omega^+$ such that

$${\Sigma^0}_0 \wedge {\Sigma^0}_0 = i\,\mathrm{vol},$$

where 'vol' stands for the volume 4-form defined by the metric tensor. This property gave motivation to formulate Einstein's gravity as a theory of the 2-forms rather then the metric tensors. In the cases of the complex gravity, and the real gravity of the Euclidean signature, there is known an action, introduced by Capovilla, Dell and Jacobson (CDJ), which is written purely in terms of the self-dual connection and self-dual 2-forms. In the real Lorentzian case, however, the actions which are known, are either Plebański's action which involves both self-dual and anti-self-dual connections and 2-forms or one uses the complex version of the CDJ action plus extra reality conditions imposed on the solutions. The canonical theory based on the CDJ action is equivalent to the Ashtekar [7, 8] theory, where again the reality conditions are taken into account as some extra conditions. Recently, the 2-form gravity is often viewed as a close neighbour of the so called BF theory and is applied in the spin-foam quantization [9, 10]. There are also attempts to use the 2-form approach of general relativity in a construction of a holographic formulation of gravity [11].

In this letter, we focus on the Einstein gravity of the Lorentzian signature. We introduce an action which involves only two forms and self-dual connections and incorporates all the reality conditions. To our knowledge this is the first action of those properties, although it seems quite geometric and natural. The canonical theory derived from this action [13] will be described in detail in a subsequent paper [14]. Here, we will briefly outline the results.

The action. Let M be a four dimensional real manifold. The main variables of our theory are a 1-form A and a 2-form Σ both defined on

M and taking values in the Lie algebra $sl(2,\mathbb{C})$ of the group $SL(2,\mathbb{C})$. The group acts in a certain abstract spinor space S and preserves the antisymmetric (symplectic) 2-form ϵ defined therein. The complex conjugation of spinors is an anti-linear isomorphisms that carries S into another spinor space S'

$$S \ni \mu \ \mapsto \ \mu^* \in S',$$

and the symplectic form ϵ into the symplectic form ϵ' of S'.

The action is defined as follows[3]:

$$\begin{aligned} S[\Sigma^{AB},\, A_{AB},\, \Psi_{ABCD},\, \Phi_{ABC'D'},\, R] = \\ = \int \Sigma^{AB} \wedge F_{AB} - \frac{1}{2}\Psi_{ABCD}\Sigma^{AB} \wedge \Sigma^{CD} - \\ - \Phi_{ABC'D'}\Sigma^{AB} \wedge \Sigma^{*C'D'} - \frac{1}{2}R(\Sigma^{AB} \wedge \Sigma_{AB} + \Sigma^{*A'B'} \wedge \Sigma^{*}_{A'B'}) \end{aligned} \quad (1)$$

where

$$F_{AB} := dA_{AB} + A_A{}^C \wedge A_{CB},$$

and the extra spinor fields Ψ and Φ, and the scalar field R play the role of the Lagrange multipliers which satisfy[4]

$$\Psi_{ABCD} \ = \ \Psi_{(ABCD)},\ \Phi_{ABC'D'} \ = \ \Phi_{(AB)(C'D')}.$$

The vanishing of the variations of the action S with respect to Ψ, Φ, R is, generically (we indicate the limitations below), equivalent to the existence and uniqueness of the corresponding metric tensor, real and of the Lorentzian signature $\pm(+---)$. Indeed:

$$\partial_\Psi S = 0 \ \ \Rightarrow \ \ \Sigma^{(AB} \wedge \Sigma^{CD)} = 0; \quad (2)$$

$$\partial_\Phi S = 0 \ \ \Rightarrow \ \ \Sigma^{AB} \wedge \Sigma^{*C'D'} = 0; \quad (3)$$

$$\partial_R S = 0 \ \ \Rightarrow \ \ \Sigma^{AB} \wedge \Sigma_{AB} + \Sigma^{*A'B'} \wedge \Sigma^{*}_{A'B'} = 0. \quad (4)$$

[3]We use here the standard notation [12], expect indication of complex conjugation of spinors — we mark it by means of $*$. The spinors are represented by their components, with respect to basis $o,\, \iota \in S$, and $o^*,\, \iota^* \in S'$, that is $\mu \ = \ \mu^0 o \ + \ \mu^1 \iota \in S$, and $\nu' \ = \ \nu^{0'} o^* + \nu^{1'} \iota^* \in S'$. The indices are lowered and raised by the symplectic forms, that is $\mu_A = \mu^B \epsilon_{BA},\ \mu^A = \mu_B \epsilon^{AB},\ \nu_{A'} = \nu^{B'} \epsilon'_{B'A'},\ \nu^{A'} = \nu_{B'} \epsilon'^{A'B'}$ and finally $\epsilon^{AC}\epsilon_{BC} = \delta^A_B,\ \epsilon'^{A'C'}\epsilon_{B'C'} = \delta^{A'}_{B'}$.

[4]Since A and Σ take values in $sl(2,\mathbb{C})$, $A_{AB} = A_{(AB)}$ and $\Sigma^{AB} = \Sigma^{(AB)}$.

The first condition, generically, implies the existence of four linearly independent complex valued 1-forms $\theta^{AB'}$, $A, B = 0, 1$ tangent to M, such that

$$\Sigma^{AB} = \theta^{AA'} \wedge \theta^{BB'} \epsilon_{B'A'}.$$

Due to the conditions (3) and (4) the metric tensor defined by the above null frame, that is

$$g \ := \ \theta^{AC'} \otimes \theta^{BD'} \epsilon_{AB} \epsilon_{C'D'}$$

is real and its signature is $(+---)$ or $(-+++)$. The above conditions, and the reconstruction of the metric tensor come from Plebański [1] (see also [4]).

Given the metric tensor, the vanishing of the variation of S with respect to A implies the metricity of A,

$$\partial_A S \ = \ 0 \ \Rightarrow \ d\Sigma^{AB} \ - A^A{}_C \wedge \Sigma^{CB} \ - A^B{}_C \wedge \Sigma^{AC} = 0. \qquad (5)$$

From this equation, the 1-forms $A^A{}_B$ are determined completely by Σ and set the spin connection corresponding to the spin structure defined locally by

$$S \otimes S' \ni \mu \otimes \nu' \ \mapsto \ \mu_A \nu_{B'} \theta^{AB'}.$$

At this point, the meaning of the spinors Ψ and Φ and the scalar R becomes clear after taking the variation of S with respect to Σ, namely[5]:

$$\partial_\Sigma S \ = \ 0 \ \Rightarrow \ F_{AB} = \Psi_{ABCD} \Sigma^{CD} + \Phi_{ABC'D'} \Sigma^{*C'D'} + R\Sigma_{AB}.$$

This is exactly the familiar spin decomposition of the spinorial curvature, where Ψ is the Weyl spinor, Φ represents the traceless part of the Ricci tensor and $12R$ is the Ricci scalar.

The vacuum Einstein equations (imposed on the derived metric) follow from the vanishing of the variation with respect to Σ^*, namely

$$\partial_{\Sigma^*} S \ = \ 0 \ \Rightarrow \ \Phi_{ABC'D'} \Sigma^{AB} \ + \ R\Sigma^*_{C'D'} \ = \ 0. \qquad (6)$$

Since in the generic case, the six 2-forms $\Sigma^{(AB)}$, $\Sigma^*_{(C'D')}$ form a basis of the complexified space of 2-forms, the above equation implies

$$\Phi_{ABC'D'} \ = \ 0 \ = \ R.$$

[5]This is a holomorphic variation, that is $\partial_\Sigma \Sigma^* = 0$.

As emphasized, the above reconstruction of Einstein's theory applies to the generic case. The genericity condition is

$$\Sigma^{00} \wedge \Sigma^{11} \;\neq\; 0. \tag{7}$$

Otherwise, for every values of A, B, C, D

$$\Sigma^{AB} \wedge \Sigma^{CD} \;=\; 0.$$

Every degenerate triad of 2-forms Σ^{AB} is generated by the wedge product and the linear combinations from certain triad of 1-forms. Farther degeneracy takes place if the triad of 1-forms is not linearly independent, the most degenerate case being just

$$\Sigma \;=\; 0.$$

Given degenerate Σ, the equations $(5, 6)$ do not determine A, Φ and R. The freedom depends on the degeneracy. For example, in the most degenerate case, the general solution is: a flat connection A and arbitrary Φ, Ψ and R.

Hamiltonian and the constraints. For the construction of the canonical theory corresponding to the action (1) let us introduce a variable t and a coordinates system $(x^0 = t, x^i)$. Then, the configuration variables q^r and the corresponding canonically conjugate momenta $\widetilde{p}_r$ are

$$\{q^r\} = \{A_{\alpha AB}, \Sigma^{AB}_{0i}, \Sigma^{AB}_{ij}, \Sigma^{*A'B'}_{0i}, \Sigma^{*A'B'}_{ij}, \Psi_{ABCD}, \Phi_{ABC'D'}, R\},$$
$$\{\widetilde{p}_r\} \;=\; \{\widetilde{p}^{\alpha AB}, \widetilde{p}^{0i}_{AB}, \widetilde{p}^{ij}_{AB}, \widetilde{p}^{*0i}_{A'B'}, \widetilde{p}^{*ij}_{A'B'}, \widetilde{p}^{ABCD}, \widetilde{p}^{ABC'D'}, \widetilde{p}_R\},$$

where we take into account only independent components ($(A, B) = (0,0)$, $(0,1)$, $(1,1)$ in $A_{\alpha AB}$, etc.). Obviously, the Legendre transform $(q^r, \partial_t q^r) \mapsto (q^r, \widetilde{p}_r)$ is not invertible and gives the primary constraints:

$$\begin{cases} \widetilde{\phi}^{iAB} := \widetilde{p}^{iAB} - \widetilde{\sigma}^{iAB} = 0 & \\ \widetilde{\phi}_r := \widetilde{p}_r = 0 & \text{otherwise,} \end{cases} \tag{8}$$

where[6]

$$\widetilde{\sigma}^{iAB} := \widetilde{\epsilon}^{ijk}\Sigma^{AB}_{jk}; \qquad \widetilde{\epsilon}^{ijk} := \widetilde{\epsilon}^{0ijk}$$

[6] $\widetilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the Levi-Civita density of weight 1 on $\mathcal{M}$.

Therefore, the Hamiltonian is defined up to an additive term, a combination of the constraints and Lagrange multipliers, that is

$$H(q,\widetilde{p},u) = \int_S \mathrm{Tr}[\,\widetilde{\sigma}^i D_i A_0 - 2\Sigma_{0i}\widetilde{\epsilon}^{ijk}F_{jk} + 2\Sigma_{0i}\Psi\,\widetilde{\sigma}^i + \\ + 2\Sigma_{0i}\Phi\,\widetilde{\sigma}^{*i} + 2\,\widetilde{\sigma}^i\Phi\Sigma^*_{0i} + 2R(\Sigma_{0i}\,\widetilde{\sigma}^i + \Sigma^*_{0i}\,\widetilde{\sigma}^{*i})\,] + \int_S u^r\widetilde{\phi}_r \quad (9)$$

where:

$$D_i A_{0AB} := \partial_i A_{AB} + A_{iA}{}^C A_{0CB} + A_{iB}{}^C A_{0AC}$$

and the Lagrange multipliers u^r are tensors on S which have the same (anti)symmetries in the indices as the corresponding primary constraint functions $\widetilde{\phi}_r$.

Demanding that the primary constraints $\widetilde{\phi}$ are preserved during the time evolution, that is:

$$\{...\{\{\widetilde{\phi},H\},H\},...,H\} \;=\; 0, \quad (10)$$

one gets secondary constraints and restrictions on the Lagrange multipliers. The complete set of the constraints for the Hamiltonian (9) is known in the case, when the triad ($\widetilde{\sigma}^{i00}$, $\widetilde{\sigma}^{i01}$, $\widetilde{\sigma}^{i11}$) of the vector densities is linearly independent, that is, when the 2-forms Σ^{AB} give rise to a Lorentzian metric tensor in the space-time M and the three-metric induced on the $t=$ const surfaces is not degenerate (the signature of the induced three-metric is $\pm(+++)$ or $\pm(-++)$). Then the set of the constraints consists of the primary ones (8) and of the following secondary constraints:

$$\begin{gathered}
\Sigma_{0i}^{(AB}\widetilde{\sigma}^{iCD)} = 0;\;\; \Phi_{ABC'D'} = 0;\;\; R = 0 \\
D_i\widetilde{\sigma}^{iAB} = 0;\;\; \epsilon^{ijk}F_{jkAB} - \Psi_{ABCD}\widetilde{\sigma}^{iCD} = 0 \\
\Sigma_{0i}^{AB}\widetilde{\sigma}^{*iC'D'} + \Sigma_{0i}^{*C'D'}\widetilde{\sigma}^{iAB} = 0 \\
\Sigma_{0i}^{AB}\widetilde{\sigma}^{i}{}_{AB} + \Sigma_{0i}^{*A'B'}\widetilde{\sigma}^{*i}{}_{A'B'} = 0 \\
D_k(\widetilde{\sigma}^{kCA}\widetilde{\sigma}^{(iB}{}_C)\widetilde{\sigma}^{j)}{}_{AB} + [D_k(\widetilde{\sigma}^{kCA}\widetilde{\sigma}^{(iB}{}_C)\widetilde{\sigma}^{j)}{}_{AB}]^* = 0
\end{gathered} \quad (11)$$

Finally, the conditions on the Lagrange multipliers u implied by

(10) are:

$$
\begin{gathered}
A_{0C}{}^{(A}\widetilde{\sigma}^{iB)C} + D_j(\widetilde{\epsilon}^{ijk}\Sigma^{AB}_{0k}) - \widetilde{\epsilon}^{ijk}u^{AB}_{jk} = 0 \\
D_k A_{0AB} + 2\Psi_{ABCD}\Sigma^{CD}_{0k} - u_{kAB} = 0 \\
\widetilde{\epsilon}^{ijk}(D_j u_{kAB} - 2\Psi_{ABCD}u^{CD}_{jk}) - u_{ABCD}\widetilde{\sigma}^{iCD} = 0 \\
u_R = 0; \;\; u_{ABC'D'} = 0; \quad u^{(AB}_{0i}\widetilde{\sigma}^{iCD)} + 2\Sigma^{(AB}_{0i}\widetilde{\epsilon}^{ijk}u^{CD)}_{jk} = 0 \\
u^{AB}_{0i}\widetilde{\sigma}^{*iC'D'} + 2\Sigma^{AB}_{0i}\widetilde{\epsilon}^{ijk}u^{*C'D'}_{jk} + u^{*C'D'}_{0i}\widetilde{\sigma}^{iAB} + 2\Sigma^{*C'D'}_{0i}\widetilde{\epsilon}^{ijk}u^{AB}_{jk} = 0 \\
u^{AB}_{0i}\widetilde{\sigma}^{i}{}_{AB} + 2\Sigma^{AB}_{0i}\widetilde{\epsilon}^{ijk}u_{jkAB} + u^{*A'B'}_{0i}\widetilde{\sigma}^{*i}{}_{A'B'} + 2\Sigma^{*A'B'}_{0i}\widetilde{\epsilon}^{ijk}u^{*}_{jkA'B'} = 0
\end{gathered}
$$

Concluding, the classical theory given by our action is equivalent to the real section of the Ashtekar theory as long as the triads ($\widetilde{\sigma}^{i00}$, $\widetilde{\sigma}^{i01}$, $\widetilde{\sigma}^{i11}$) are linearly independent. In the degenerate case, however, the theories are different [13, 14]. Our formulation provides a new starting point for the quantization. Then, the differences in the degenerate sector may become relevant, because the quantum geometry [15] is degenerate in most of the space points.

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