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Correlation functions of conserved currents in $\mathcal{N} = 2$ superconformal theory

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Abstract. Using a manifestly supersymmetric formalism, we determine the general structure of two- and three-point functions of the supercurrent and the flavour current of $\mathcal{N} = 2$ superconformal field theories. We also express them in terms of $\mathcal{N} = 1$ superfields and compare to the generic $\mathcal{N} = 1$ correlation functions. A general discussion of the $\mathcal{N} = 2$ supercurrent superfield and the multiplet of anomalies and their definition as derivatives with respect to the supergravity prepotentials is also included.

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1. Introduction

Superconformal field theories in various dimensions have been intensively studied for many years. The conjecture of Maldacena [1], which in its simplest form relates $\mathcal{N} = 4$ super-Yang–Mills theory in four-dimensional Minkowski space to $\mathcal{N} = 8$ supergravity in five-dimensional anti-de Sitter space has led to a renewed interest in superconformal field theories in diverse dimensions with maximal and less than maximal supersymmetry. Here we will be interested in $\mathcal{N} = 2$ *generic* superconformally invariant theories. Particular examples can be realized as world-volume theories on D3 branes in the presence of D7 branes [2]. These theories have also been studied in the context of the Maldacena conjecture [3]. A more general interest in $\mathcal{N} = 2$ supersymmetric theories, not necessarily conformally invariant, arises within the context of Seiberg–Witten theory and its string/M-theory realization. For reviews, see, e.g., [4–6].

A general efficient formalism to analyse correlation functions of quasi-primary fields has been developed since the early days of conformal field theory. Some important recent contributions have been provided by Osborn and collaborators. We refer to their papers: to [7, 8] for the non-supersymmetric case in an arbitrary number of dimensions. A complete analysis of the $\mathcal{N} = 1$ supersymmetric case in $d = 4$ was presented in [9] (see also [10]). In [11] Park constructed the building blocks of correlators of quasi-primary fields for arbitrary $\mathcal{N}$ in four dimensions and for $(p, 0)$ superconformal symmetry in $d = 6$. The formalism is powerful for applications whenever there exist off-shell superfield formulations for superconformal theories, and such formulations are known in four dimensions for $\mathcal{N} = 1, 2, 3$.

In this paper we are going to analyse correlation functions of conserved currents in $\mathcal{N} = 2$, $d = 4$ superconformal field theory in a manifestly $\mathcal{N} = 2$ supersymmetric language. To this end we review in section 2 the formalism of Osborn and Park, specializing to the case of $\mathcal{N} = 2$.

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In section 3 we apply this to the computation of various two- and three-point correlation functions, involving the $\mathcal{N} = 2$ supercurrent $\mathcal{J}$ and flavour currents $\mathcal{L}^{ij}$. The three-point function of the supercurrent is shown to be the sum of two linearly independent superconformal structures whose coefficients are related to the anomaly coefficients, denoted by a and c in [12]. Whereas for $\mathcal{N} = 1$ there exist two independent structures for the three-point function of the flavour current, there is only one for $\mathcal{N} = 2$. This is a consequence of the fact that $\mathcal{N} = 2$ theories are non-chiral. We also analyse mixed three-point functions and, in particular, show that the three-point function $\langle \mathcal{J}\mathcal{J}\mathcal{L}^{ij} \rangle$ vanishes, as a consequence of $\mathcal{N} = 2$ superconformal symmetry. In section 4 we describe the reduction of our results to $\mathcal{N} = 1$ superfields. The main body of the paper ends with a brief discussion. We have included a few technical appendices to make the paper self-contained. In appendix A we review the Weyl and the minimal $\mathcal{N} = 2$ supergravity multiplets in harmonic superspace and present a new parametrization of the supergravity prepotential (it was sketched already in part by Siegel [13]) which is most convenient for any consideration involving the supercurrent and the multiplet of anomalies. In appendix B we describe the procedure to generate the supercurrent and the multiplet of anomalies as functional derivatives with respect to supergravity prepotentials. In appendix C we compute the supercurrent and the multiplet of anomalies for general renormalizable $\mathcal{N} = 2$ super-Yang–Mills models.

The multiplets of currents and anomalies for $\mathcal{N} = 2$ extended supersymmetry in four spacetime dimensions were introduced by Sohnius [14] 20 years ago. He considered the simplest $\mathcal{N} = 2$ supersymmetric model—the hypermultiplet with 8 + 8 off-shell degrees of freedom [15], and showed that the energy–momentum tensor Θ_{mn} belongs to a supermultiplet (called, by analogy with $\mathcal{N} = 1$ SUSY [16], the $\mathcal{N} = 2$ supercurrent) which (a) in addition, contains the $SU(2)$ R-current $j_m^{(ij)}$, the axial current $j_m^{(R)}$, the $\mathcal{N} = 2$ supersymmetry currents $j^i_{m\hat{\alpha}}$, where $\hat{\alpha} = \alpha, \dot{\alpha}$, the central charge current c_m as well as some auxiliary components of lower dimension; (b) is described by a real scalar superfield $\mathcal{J}(z)$ of mass dimension two. The central charge current is also part of the multiplet of anomalies which contains in addition $\Theta^m{}_m$, $\partial^m j_m^{(R)}$ and $(\gamma^m j^i_m)_{\hat{\alpha}}$ along with an auxiliary triplet. The multiplet of anomalies is described by a real isotriplet superfield $\mathcal{T}^{(ij)}(z)$, $\overline{\mathcal{T}^{ij}} = \mathcal{T}_{ij}$, which is subject to the constraint

$$D_\alpha^{(i}\mathcal{T}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i}\mathcal{T}^{jk)} = 0 \tag{1.1}$$

where $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i)$ are the $N = 2$ supersymmetric covariant derivatives, $i = \underline{1}, \underline{2}$. Both $\mathcal{J}$ and $\mathcal{T}^{ij}$ turn out to be invariant with respect to the central charge transformations. The supercurrent conservation law reads

$$\tfrac{1}{4}D^{ij}\mathcal{J} + \mathrm{i}\mathcal{T}^{ij} = 0 \quad \Longleftrightarrow \quad \tfrac{1}{4}\bar{D}^{ij}\mathcal{J} - \mathrm{i}\mathcal{T}^{ij} = 0 \tag{1.2}$$

where $D^{ij} = D^{\alpha(i}D^{j)}_\alpha$, $\bar{D}^{ij} = \bar{D}^{(i}_{\dot{\alpha}}\bar{D}^{j)\dot{\alpha}}$. The constraint (1.1) means that $\mathcal{T}^{ij}$ is a so-called $\mathcal{N} = 2$ linear multiplet. Such a multiplet contains a conserved vector and the reality condition for $\mathcal{T}^{ij}$ is equivalent to the absence of the second (fundamental) central charge (which is the case for all $\mathcal{N} = 2$ irreducible supermultiplets).

A nice feature of the $\mathcal{N} = 2$ multiplet of anomalies is that its supersymmetric structure is completely analogous to that of a $\mathcal{N} = 2$ superfield containing a conserved flavour current of a $\mathcal{N} = 2$ supersymmetric field theory. Such a flavour current superfield $\mathcal{L}^{(ij)}(z)$, $\overline{\mathcal{L}^{ij}} = \mathcal{L}_{ij}$ satisfies the same constraint,

$$D_\alpha^{(i}\mathcal{L}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i}\mathcal{L}^{jk)} = 0. \tag{1.3}$$

The similarity is not accidental. The point is that $\mathcal{L}^{ij}$ is generated by coupling matter hypermultiplets to a gauge vector supermultiplet. On the other hand, the source for $\mathcal{T}^{ij}$ is a vector multiplet which gauges the central charge and belongs to the $\mathcal{N} = 2$ supergravity multiplet.

The structure of the $\mathcal{N} = 2$ supercurrent has been used by Sohnius and West [17, 18] in their proof of finiteness of the $\mathcal{N} = 4$ SYM theory which was based on anomaly considerations. It is worth pointing out that the supercurrent conservation law in quantum $\mathcal{N} = 2$ super-Yang–Mills theories [19] (see also [20])

$$D^{ij}\mathcal{J} = -\frac{1}{3}\frac{\beta(g)}{g}\bar{D}^{ij}\bar{W}^2 \tag{1.4}$$

can be brought to the form (1.2) by a finite local shift of $\mathcal{J}$, resulting in

$$D^{ij}\mathcal{J} = \frac{1}{3}\frac{\beta(g)}{g}\left(D^{ij}W^2 - \bar{D}^{ij}\bar{W}^2\right). \tag{1.5}$$

Here W is the $\mathcal{N} = 2$ Yang–Mills field strength, and $\beta(g)$ is the beta-function of the gauge coupling constant.

Another consequence of the structure of the $\mathcal{N} = 2$ supercurrent follows from the fact that $\mathcal{J}$ presents itself as the multiplet of superconformal currents. Then, Noether's procedure tells us that $\mathcal{N} = 2$ conformal supergravity should be described by a real scalar prepotential $\boldsymbol{G}(z)$ [21, 22] to which the matter supercurrent is coupled. In appendix A.1 we will show how such a prepotential arises in the harmonic superspace approach to $\mathcal{N} = 2$ conformal supergravity [23, 24]. This point requires some comments. Many years ago, Gates and Siegel [25] showed that the first minimal $\mathcal{N} = 2$ Poincaré supergravity (in the terminology of the third reference in [26]) is described, at the linearized level, by a single unconstrained spinor superfield $\Psi_{\alpha i}(z)$†. Their conclusion is in perfect agreement with the fact that (a) the corresponding superspace differential geometry [27] contains two independent strengths—a covariantly chiral symmetric bi-spinor $W_{\alpha\beta}$ ($\mathcal{N} = 2$ super-Weyl tensor) and a spinor $T_{\alpha i}$; (b) the supergravity equation of motion reads

$$\frac{\delta S_{\text{sugra}}}{\delta\Psi_{\alpha i}} \propto T^{\alpha i} = 0. \tag{1.6}$$

In [25] it was argued that $\mathcal{N} = 2$ conformal supergravity should be described by the same prepotential $\Psi_{\alpha i}$ but with a larger gauge freedom. This led Gates *et al* [30] to postulate that the $\mathcal{N} = 2$ supercurrent be a spinor superfield

$$\mathcal{J}^i_\alpha = \frac{\delta S_{\text{matter}}}{\delta\Psi^\alpha_i}. \tag{1.7}$$

As will be described below, this puzzle can be resolved in the harmonic superspace approach to $\mathcal{N} = 2$ supergravity [23, 24]. There, the prepotential $\boldsymbol{G}$ is part of a larger harmonic multiplet $G(z, u)$ with a huge gauge symmetry. The gauge freedom can be fixed in part either to leave a single real unconstrained $\boldsymbol{G}(z)$, the leading component of $G(z, u)$ in its harmonic Fourier expansion, or to bring $G(z, u)$ to the form

$$G(z, u) = D^{\alpha i}\Psi^j_\alpha(z)u^+_i u^-_j + \text{conjugate} \tag{1.8}$$

with $\Psi_{\alpha i}(z)$ the Gates–Siegel prepotential. Therefore, we have $\mathcal{J}^i_\alpha = D^i_\alpha\mathcal{J}$ for all (renormalizable) $\mathcal{N} = 2$ matter systems. The details of this discussion are provided in appendices A and C.

Manifestly supersymmetric techniques to study the quantum dynamics and to compute the superconformal anomalies for $\mathcal{N} = 2$ matter systems in a supergravity background are not yet

† The harmonic superspace origin of this prepotential has been revealed recently by Zupnik [29].

available. In x-space, there exists an exhaustive description of general $\mathcal{N}=2$ supergravity–matter systems [26, 28]. In superspace, there exist elaborated differential geometry formalisms [27, 31, 32] corresponding to $\mathcal{N}=2$ conformal supergravity and the three known versions of $\mathcal{N}=2$ Poincaré supergravity. Moreover, the unconstrained prepotentials and the gauge group of $\mathcal{N}=2$ conformal supergravity were found in harmonic superspace [23, 24], and this analysis was extended to describe different versions of $\mathcal{N}=2$ Poincaré supergravity [24, 33] and most general supersymmetric sigma models in curved harmonic superspace [34]. What is still missing is the detailed relationship between the differential superspace geometry of $\mathcal{N}=2$ supergravity [27, 31, 32] and its description in terms of the unconstrained prepotentials given in [23, 24]. Another missing prerequisite is the definition of the $\mathcal{N}=2$ supercurrent and multiplet of anomalies as the response of the $\mathcal{N}=2$ matter action (in the full nonlinear theory) to small disturbances in supergravity prepotentials, similar to what is well known in $\mathcal{N}=1$ supersymmetry (see [35] for a review)

$$J_{\alpha\dot\alpha}=\frac{\delta S}{\delta H^{\alpha\dot\alpha}},\qquad T=\frac{\delta S}{\delta\varphi}; \tag{1.9}$$

here $H^{\alpha\dot\alpha}$ and φ are the $\mathcal{N}=1$ gravitational superfield and chiral compensator, respectively. Such a definition is of primary importance, since it allows us to compute correlators with supercurrent insertions simply as functional derivatives of the renormalized effective action with respect to supergravity prepotentials. In the appendices we will close some of these gaps. In particular, using the harmonic superspace approach to $\mathcal{N}=2$ supergravity [23, 24], which we briefly review, we introduce a new parametrization of the supergravity prepotentials which allows us to easily obtain the $\mathcal{N}=2$ analogue of (1.9).

Before closing this introductory section, we would like to comment on the $\mathcal{N}=1$ multiplets contained in $\mathcal{J}$ and $\mathcal{T}^{ij}$ (see also [30]). For that purpose we introduce the $\mathcal{N}=1$ spinor covariant derivatives $D_\alpha\equiv D^{\underline{1}}_\alpha$, $\bar D^{\dot\alpha}\equiv\bar D^{\dot\alpha}_{\underline{1}}$ and define the $\mathcal{N}=1$ projection $U|\equiv U(x,\theta^\alpha_i,\bar\theta^j_{\dot\alpha})|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}$ of an arbitrary $\mathcal{N}=2$ superfield U. It follows from (1.1) and (1.2) that $\mathcal{J}$ is composed of three independent $\mathcal{N}=1$ multiplets

$$J\equiv\mathcal{J}|=\bar J,\qquad J_\alpha\equiv D^{\underline{2}}_\alpha\mathcal{J}|,\qquad J_{\alpha\dot\alpha}\equiv\tfrac{1}{2}[D^{\underline{2}}_\alpha,\bar D_{\dot\alpha\underline{2}}]\mathcal{J}|-\tfrac{1}{6}[D^{\underline{1}}_\alpha,\bar D_{\dot\alpha\underline{1}}]\mathcal{J}|=\bar J_{\alpha\dot\alpha} \tag{1.10}$$

while $\mathcal{T}$ contains two independent $\mathcal{N}=1$ components

$$\begin{aligned} T&\equiv\mathrm{i}\mathcal{T}^{\underline{22}}|, & \bar D_{\dot\alpha}T&=0\\ L&\equiv\mathrm{i}\mathcal{T}^{\underline{12}}|=\bar L, & \bar D^2L&=0\end{aligned} \tag{1.11}$$

where T is a chiral superfield and L is a real linear superfield. It is easy to find the equations for J, J_α and $J_{\alpha\dot\alpha}$:

$$\begin{aligned} &\tfrac{1}{4}\bar D^2J=T\\ &\tfrac{1}{4}D^\alpha J_\alpha=-L,\qquad \bar D^2J_\alpha=0\\ &\bar D^{\dot\alpha}J_{\alpha\dot\alpha}=\tfrac{2}{3}\,D_\alpha T.\end{aligned} \tag{1.12}$$

The latter equation shows that $J_{\alpha\dot\alpha}$ is the $\mathcal{N}=1$ supercurrent and T the corresponding multiplet of anomalies. The spinor object J_α contains the second supersymmetry current, the central charge current and two of the three $SU(2)$ currents, namely those which correspond to the symmetries belonging to $SU(2)/U(1)$. Finally, the scalar J contains the current corresponding to the special combination of the $\mathcal{N}=2$ $U(1)$ R-transformation and $SU(2)$ z-rotation which leaves $\theta_{\underline{1}}$ and $\bar\theta^{\underline{1}}$ invariant. The central charge current is also contained in L, which is no accident. In $\mathcal{N} = 1$ supersymmetry, associated with any internal symmetry is a real linear superfield containing the corresponding conserved current; L is such a superfield for the central charge. Similarly, in a superconformal theory ($\mathcal{T}^{ij} = 0$) the real scalar J becomes a linear superfield and, hence, contains a conserved current.

2. Superconformal building blocks

2.1. Superconformal Killing vectors

In $\mathcal{N}$-extended global superspace $\mathbb{R}^{4|4\mathcal{N}}$ parametrized by $z^A = (x^a, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})$, infinitesimal superconformal transformations

$$z^A \quad \longrightarrow \quad z^A + \xi^A \tag{2.1}$$

are generated by superconformal Killing vectors [19, 35–37]

$$\xi = \overline{\xi} = \xi^a(z)\partial_a + \xi^\alpha_i(z)D^i_\alpha + \bar\xi^i_{\dot\alpha}(z)\bar D^{\dot\alpha}_i \tag{2.2}$$

defined to satisfy

$$[\xi, D^i_\alpha] \propto D^j_\beta. \tag{2.3}$$

From here one obtains

$$\xi^\alpha_i = -\tfrac{1}{8}\mathrm{i}\bar D_{\dot\beta i}\xi^{\dot\beta\alpha}, \qquad \bar D_{\dot\beta j}\xi^\alpha_i = 0 \tag{2.4}$$

while the vector parameters satisfy the master equation

$$D^i_{(\alpha}\xi_{\beta)\dot\beta} = \bar D_{i(\dot\alpha}\xi_{\beta\dot\beta)} = 0 \tag{2.5}$$

implying, in turn, the conformal Killing equation

$$\partial_a\xi_b + \partial_b\xi_a = \tfrac{1}{2}\eta_{ab}\partial_c\xi^c. \tag{2.6}$$

The general solution of equation (2.5) was given in [35] for $\mathcal{N} = 1$ and in [11] for $\mathcal{N} > 1$. From equations (2.4) and (2.5) it follows

$$[\xi, D^i_\alpha] = -(D^i_\alpha\xi^\beta_j)D^j_\beta = \hat\omega_\alpha{}^\beta D^i_\beta - \frac{1}{\mathcal{N}}\big((\mathcal{N}-2)\sigma + 2\bar\sigma\big)D^i_\alpha - \mathrm{i}\hat\Lambda_j{}^i D^j_\alpha. \tag{2.7}$$

Here the parameters of 'local' Lorentz $\hat\omega$ and scale-chiral σ transformations are

$$\hat\omega_{\alpha\beta}(z) = -\frac{1}{\mathcal{N}}D^i_{(\alpha}\xi_{\beta)i}, \qquad \sigma(z) = \frac{1}{\mathcal{N}(\mathcal{N}-4)}\left(\tfrac{1}{2}(\mathcal{N}-2)D^i_\alpha\xi^\alpha_i - \bar D^{\dot\alpha}_i\bar\xi^i_{\dot\alpha}\right) \tag{2.8}$$

and turn out to be chiral

$$\bar D_{\dot\alpha i}\hat\omega_{\alpha\beta} = 0, \qquad \bar D_{\dot\alpha i}\sigma = 0. \tag{2.9}$$

The parameters $\hat\Lambda_j{}^i$

$$\hat\Lambda_j{}^i(z) = -\frac{1}{32}\left([D^i_\alpha, \bar D_{\dot\alpha j}] - \frac{1}{\mathcal{N}}\delta_j{}^i[D^k_\alpha, \bar D_{\dot\alpha k}]\right)\xi^{\dot\alpha\alpha}, \qquad \hat\Lambda^\dagger = \hat\Lambda, \qquad \mathrm{tr}\,\hat\Lambda = 0 \tag{2.10}$$

correspond to 'local' $SU(\mathcal{N})$ transformations. One can readily check the identity

$$D^k_\alpha\hat\Lambda_j{}^i = 2\mathrm{i}\left(\delta^k_j D^i_\alpha - \frac{1}{\mathcal{N}}\delta^i_j D^k_\alpha\right)\sigma. \tag{2.11}$$

For $\mathcal{N} = 2$ it leads to the analyticity condition

$$D_\alpha^{(i}\hat{\Lambda}^{jk)} = \bar{D}_{\dot{\alpha}}^{(i}\hat{\Lambda}^{jk)} = 0, \qquad (\mathcal{N} = 2). \tag{2.12}$$

As is seen from (2.8), the above formalism cannot be directly applied to the case of $\mathcal{N} = 4$ which is treated in more detail, for example, in [11]. In what follows, our considerations will be restricted to $\mathcal{N} < 4$, with a special emphasis on the choice $\mathcal{N} = 2$ later on.

The superalgebra of $\mathcal{N}$-extended superconformal Killing vectors is isomorphic to the superalgebra $su(2, 2|\mathcal{N})$ spanned by elements of the form

$$\boldsymbol{g} = \begin{pmatrix} \omega_\alpha{}^\beta - \Delta\delta_\alpha{}^\beta & -\mathrm{i}b_{\alpha\dot{\beta}} & 2\eta_\alpha{}^j \\ -\mathrm{i}a^{\dot{\alpha}\beta} & -\bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}} + \bar{\Delta}\delta^{\dot{\alpha}}{}_{\dot{\beta}} & 2\bar{\epsilon}^{\dot{\alpha}j} \\ 2\epsilon_i{}^\beta & 2\bar{\eta}_{i\dot{\beta}} & \dfrac{2}{\mathcal{N}}(\bar{\Delta} - \Delta)\delta_i{}^j + \mathrm{i}\Lambda_i{}^j \end{pmatrix} \tag{2.13}$$

which satisfy the conditions

$$\mathrm{str}\,\boldsymbol{g} = 0, \qquad B\boldsymbol{g}^\dagger B = -\boldsymbol{g}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.14}$$

Here the matrix elements correspond to a Lorentz transformation $(\omega_\alpha{}^\beta, \bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}})$, translation $a^{\dot{\alpha}\alpha}$, special conformal transformation $b_{\alpha\dot{\alpha}}$, Q-supersymmetry $(\epsilon_i^\alpha, \bar{\epsilon}^{\dot{\alpha}i})$, S-supersymmetry $(\eta_\alpha^i, \bar{\eta}_{i\dot{\alpha}})$, combined scale and chiral transformation Δ and chiral $SU(\mathcal{N})$ transformation $\Lambda_i{}^j$. They are related to the parameters of the superconformal Killing vector as follows:

$$\begin{aligned} &\omega_\alpha{}^\beta = \hat{\omega}_\alpha{}^\beta(z = 0), \qquad \Delta = \sigma(z = 0), \qquad \Lambda_i{}^j = \hat{\Lambda}_i{}^j(z = 0), \\ &a^m = \xi^m(z = 0), \qquad \epsilon_i^\alpha = \xi_i^\alpha(z = 0), \end{aligned} \tag{2.15}$$

and so on. For such a correspondence, $\xi \longrightarrow \boldsymbol{g}$, we have

$$[\xi_1, \xi_2] \quad\longrightarrow\quad -[\boldsymbol{g}_1, \boldsymbol{g}_2]. \tag{2.16}$$

It is useful to identify Minkowski superspace as a homogeneous space of the superconformal group $SU(2, 2|\mathcal{N})$ using the above matrix realization

$$\Omega(z) = \exp\mathrm{i}\left\{-x^a P_a + \theta_i^\alpha Q_\alpha^i + \bar{\theta}_{\dot{\alpha}}^i \bar{Q}_i^{\dot{\alpha}}\right\} = \begin{pmatrix} \delta_\alpha{}^\beta & 0 & 0 \\ -\mathrm{i}x_+^{\dot{\alpha}\beta} & \delta^{\dot{\alpha}}{}_{\dot{\beta}} & 2\bar{\theta}^{\dot{\alpha}j} \\ 2\theta_i{}^\beta & 0 & \delta_i{}^j \end{pmatrix} \tag{2.17}$$

where $x_\mp$ denote ordinary (anti-)chiral bosonic variables

$$x_\pm^a = x^a \pm \mathrm{i}\theta_i\sigma^a\bar{\theta}^i. \tag{2.18}$$

One verifies that

$$\boldsymbol{g}\Omega(z) = \xi\Omega(z) + \Omega(z)\boldsymbol{h}(z),$$

where

$$\boldsymbol{h}(z) = \begin{pmatrix} \hat{\omega}_\alpha{}^\beta - \sigma\delta_\alpha{}^\beta & -\mathrm{i}b_{\alpha\dot{\beta}} & 2\hat{\eta}_\alpha{}^j \\ 0 & -\hat{\bar{\omega}}^{\dot{\alpha}}{}_{\dot{\beta}} + \bar{\sigma}\delta^{\dot{\alpha}}{}_{\dot{\beta}} & 0 \\ 0 & 2\hat{\bar{\eta}}_{i\dot{\beta}} & \dfrac{2}{\mathcal{N}}(\bar{\sigma} - \sigma)\delta_i{}^j + \mathrm{i}\hat{\Lambda}_i{}^j \end{pmatrix} \tag{2.19}$$

belongs to the Lie algebra of the stability group. Here $\hat{\eta}$ is

$$\hat{\eta}_\alpha{}^i(z) = \tfrac{1}{2}D_\alpha^i\sigma(z). \tag{2.20}$$

This should be interpreted within the framework of nonlinear realizations.

2.2. Two-point structures

Given two points z_1 and z_2 in superspace, it is useful to introduce (anti-)chiral combinations

$$\begin{aligned} x^a_{\bar{1}2} &= -x^a_{2\bar{1}} = x^a_{1-} - x^a_{2+} + 2\mathrm{i}\theta_{2i}\sigma^a\bar{\theta}^i_1 \\ \theta_{12} &= \theta_1 - \theta_2 \qquad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 \end{aligned} \tag{2.21}$$

which are invariants of the Q-supersymmetry transformations (the notation '$x_{\bar{1}2}$' indicates that $x_{\bar{1}2}$ is antichiral with respect to z_1 and chiral with respect to z_2). As a consequence of (2.19), they transform semi-covariantly with respect to the superconformal group

$$\begin{aligned} \delta x^{\dot{\alpha}\alpha}_{\bar{1}2} &= -\left(\hat{\bar{\omega}}^{\dot{\alpha}}{}_{\dot{\beta}}(z_1) - \delta^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\sigma}(z_1)\right) x^{\dot{\beta}\alpha}_{\bar{1}2} - x^{\dot{\alpha}\beta}_{\bar{1}2}\left(\hat{\omega}_\beta{}^\alpha(z_2) - \delta_\beta{}^\alpha\sigma(z_2)\right) \\ \delta\theta^\alpha_{12i} &= \mathrm{i}\big(\hat{\Lambda}_i{}^j(z_1) + \frac{2}{\mathcal{N}}\left(\bar{\sigma}(z_1) - \sigma(z_1)\right)\delta_i{}^j\big)\theta^\alpha_{12j} - \mathrm{i}\hat{\bar{\eta}}_{\dot{\beta}i}(z_1)x^{\dot{\beta}\alpha}_{\bar{1}2} \\ &\quad -\theta^\beta_{12i}\left(\hat{\omega}_\beta{}^\alpha(z_2) - \delta_\beta{}^\alpha\sigma(z_2)\right) \end{aligned} \tag{2.22}$$

Following [11], it is useful to introduce a conformally covariant $\mathcal{N}\times\mathcal{N}$ matrix†

$$u_i{}^j(z_{12}) = \delta_i{}^j - 4\mathrm{i}\frac{\theta_{12i}x_{\bar{1}2}\bar{\theta}^j_{12}}{x_{\bar{1}2}{}^2} = \delta_i{}^j + 4\mathrm{i}\theta_{12i}\tilde{x}_{\bar{1}2}{}^{-1}\bar{\theta}^j_{12} \tag{2.23}$$

with the basic properties

$$u^\dagger(z_{12})u(z_{12}) = 1, \qquad u^{-1}(z_{12}) = u(z_{21}), \qquad \det u(z_{12}) = \frac{x_{\bar{1}2}{}^2}{x_{\bar{2}1}{}^2}. \tag{2.24}$$

In accordance with (2.22), the unimodular unitary matrix

$$\hat{u}_i{}^j(z_{12}) = \left(\frac{x_{\bar{2}1}{}^2}{x_{\bar{1}2}{}^2}\right)^{1/\mathcal{N}} u_i{}^j(z_{12}) \tag{2.25}$$

transforms as

$$\delta\hat{u}_i{}^j(z_{12}) = \mathrm{i}\hat{\Lambda}_i{}^k(z_1)\hat{u}_k{}^j(z_{12}) - \mathrm{i}\hat{u}_i{}^k(z_{12})\hat{\Lambda}_k{}^j(z_2). \tag{2.26}$$

2.3. Three-point structures

Given three superspace points z_1, z_2 and z_3, one can define superconformally covariant bosonic and fermionic variables $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$, where $\boldsymbol{Z}_1 = (\boldsymbol{X}^a_1, \Theta^{\alpha i}_1, \bar{\Theta}^{\dot{\alpha}}_{1\,i})$ are [9, 11]

$$\begin{aligned} \boldsymbol{X}_1 &\equiv \tilde{x}_{1\bar{2}}{}^{-1}\tilde{x}_{\bar{2}3}\tilde{x}_{3\bar{1}}{}^{-1}, \qquad \bar{\boldsymbol{X}}_1 = \boldsymbol{X}^\dagger_1 = -\tilde{x}_{1\bar{3}}{}^{-1}\tilde{x}_{\bar{3}2}\tilde{x}_{2\bar{1}}{}^{-1} \\ \tilde{\Theta}^i_1 &\equiv \mathrm{i}\left(\tilde{x}_{\bar{2}1}{}^{-1}\bar{\theta}^i_{12} - \tilde{x}_{\bar{3}1}{}^{-1}\bar{\theta}^i_{13}\right) = \tfrac{1}{4}\tilde{D}^i_1 \ln\frac{x_{\bar{2}1}{}^2}{x_{\bar{3}1}{}^2} \\ \tilde{\bar{\Theta}}_{1\,i} &\equiv \mathrm{i}\left(\theta_{12i}\tilde{x}_{\bar{1}2}{}^{-1} - \theta_{13i}\tilde{x}_{\bar{1}3}{}^{-1}\right) = \tfrac{1}{4}\tilde{\bar{D}}_{1\,i}\ln\frac{x_{\bar{1}2}{}^2}{x_{\bar{1}3}{}^2} \end{aligned} \tag{2.27}$$

and $\boldsymbol{Z}_2$, $\boldsymbol{Z}_3$ are obtained from here by cyclically permuting indices. These structures possess remarkably simple transformation rules under superconformal transformations:

$$\begin{aligned} \delta\boldsymbol{X}_{1\alpha\dot{\alpha}} &= \left(\hat{\omega}_\alpha{}^\beta(z_1) - \delta_\alpha{}^\beta\sigma(z_1)\right)\boldsymbol{X}_{1\beta\dot{\alpha}} + \boldsymbol{X}_{1\alpha\dot{\beta}}\left(\hat{\bar{\omega}}^{\dot{\beta}}{}_{\dot{\alpha}}(z_1) - \delta^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\sigma}(z_1)\right) \\ \delta\Theta^i_{1\,\alpha} &= \hat{\omega}_\alpha{}^\beta(z_1)\Theta^i_{1\,\beta} - \mathrm{i}\Theta^j_{1\,\alpha}\hat{\Lambda}_j{}^i(z_1) - \frac{1}{\mathcal{N}}\big((\mathcal{N}-2)\sigma(z_1) + 2\bar{\sigma}(z_1)\big)\Theta^i_{1\,\alpha} \end{aligned} \tag{2.28}$$

† We use the notation adopted in [35, 38]. When the spinor indices are not indicated explicitly, the following matrix-like conventions are assumed [9]: $\psi = (\psi^\alpha)$, $\tilde{\psi} = (\psi_\alpha)$, $\bar{\psi} = (\bar{\psi}^{\dot{\alpha}})$, $\tilde{\bar{\psi}} = (\bar{\psi}_{\dot{\alpha}})$, $x = (x_{\alpha\dot{\alpha}})$, $\tilde{x} = (x^{\dot{\alpha}\alpha})$; but $x^2 \equiv x^a x_a = -\frac{1}{2}\,\mathrm{tr}(\tilde{x}x)$, and hence $\tilde{x}^{-1} = -x/x^2$.

and turn out to be essential building blocks for correlations functions of quasi-primary superfields.

Among important properties of $\boldsymbol{Z}$'s are the following:

$$\begin{aligned} \boldsymbol{X}_1{}^2 &= \frac{x_{\bar{2}3}{}^2}{x_{\bar{2}1}{}^2 x_{\bar{1}3}{}^2}, \qquad \bar{\boldsymbol{X}}_1{}^2 = \frac{x_{\bar{3}2}{}^2}{x_{\bar{3}1}{}^2 x_{\bar{1}2}{}^2} \\ \boldsymbol{X}_{1\alpha\dot{\alpha}} - \bar{\boldsymbol{X}}_{1\alpha\dot{\alpha}} &= 4\mathrm{i}\Theta^i_{1\,\alpha}\bar{\Theta}_{1\,\dot{\alpha}i} \end{aligned} \tag{2.29}$$

and further relations obtained by cyclic permutation of labels. The variables $\boldsymbol{Z}$ with different labels are related to each other:

$$\begin{aligned} &\tilde{x}_{\bar{1}3}\boldsymbol{X}_3\tilde{x}_{\bar{3}1} = -\bar{\boldsymbol{X}}_1{}^{-1}, && \tilde{x}_{\bar{1}3}\bar{\boldsymbol{X}}_3\tilde{x}_{\bar{3}1} = -\boldsymbol{X}_1{}^{-1} \\ &\tilde{x}_{\bar{1}3}\tilde{\Theta}^i_3 u_i{}^j(z_{31}) = -\boldsymbol{X}_1{}^{-1}\tilde{\Theta}^j_1, && u_i{}^j(z_{13})\tilde{\bar{\Theta}}_{3\,j}\tilde{x}_{\bar{3}1} = \tilde{\bar{\Theta}}_{1\,i}\bar{\boldsymbol{X}}_1{}^{-1}. \end{aligned} \tag{2.30}$$

With the aid of the matrices $u(z_{rs})$, $r, s = 1, 2, 3$, defined in (2.24), one can construct unitary matrices [11]

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{Z}_3) = u(z_{31})u(z_{12})u(z_{23}), && \boldsymbol{u}_i{}^j(\boldsymbol{Z}_3) = \delta_i{}^j - 4\mathrm{i}\tilde{\bar{\Theta}}_{3\,i}\boldsymbol{X}_3{}^{-1}\tilde{\Theta}^j_3 \\ &\boldsymbol{u}^\dagger(\boldsymbol{Z}_3) = u(z_{32})u(z_{21})u(z_{13}), && \boldsymbol{u}^{\dagger\,j}_i(\boldsymbol{Z}_3) = \delta_i{}^j + 4\mathrm{i}\tilde{\bar{\Theta}}_{3\,i}\bar{\boldsymbol{X}}_3{}^{-1}\tilde{\Theta}^j_3 \end{aligned} \tag{2.31}$$

transforming at z_3 only. Their properties are

$$\boldsymbol{u}^\dagger(\boldsymbol{Z}_3) = \boldsymbol{u}^{-1}(\boldsymbol{Z}_3), \qquad \det \boldsymbol{u}(\boldsymbol{Z}_3) = \frac{\boldsymbol{X}_3{}^2}{\bar{\boldsymbol{X}}_3{}^2}. \tag{2.32}$$

It is worth noting that $\det \boldsymbol{u}(\boldsymbol{Z}_3)$ is a superconformal invariant [11] and from (2.29) one immediately obtains

$$\frac{\boldsymbol{X}_1{}^2}{\bar{\boldsymbol{X}}_1{}^2} = \frac{\boldsymbol{X}_2{}^2}{\bar{\boldsymbol{X}}_2{}^2} = \frac{\boldsymbol{X}_3{}^2}{\bar{\boldsymbol{X}}_3{}^2}. \tag{2.33}$$

2.4. Specific features of $\mathcal{N} = 2$ theory

In the case of $\mathcal{N} = 2$, we have at our disposal the $SU(2)$-invariant tensors $\varepsilon_{ij} = -\varepsilon_{ji}$ and $\varepsilon^{ij} = -\varepsilon^{ji}$, normalized to $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$. They can be used to raise and lower isoindices

$$C^i = \varepsilon^{ij}C_j, \qquad C_i = \varepsilon_{ij}C^j. \tag{2.34}$$

Now, the condition of unimodularity of the matrix defined in (2.25)

$$\hat{u}_i{}^j(z_{12}) = \left(\frac{x_{\bar{2}1}{}^2}{x_{\bar{1}2}{}^2}\right)^{1/2} u_i{}^j(z_{12}) \tag{2.35}$$

takes the form

$$\left(\hat{u}^{-1}(z_{12})\right)_i{}^j = \hat{u}_i{}^j(z_{21}) = \varepsilon^{jk}\hat{u}_k{}^l(z_{12})\varepsilon_{li} \tag{2.36}$$

which can be written as

$$\hat{u}_{ji}(z_{21}) = -\hat{u}_{ij}(z_{12}). \tag{2.37}$$

The importance of this relation is that it implies that the two-point function

$$\begin{aligned} A_{i_1i_2}(z_1, z_2) &\equiv \frac{\hat{u}_{i_1i_2}(z_{12})}{\left(x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2\right)^{1/2}} = -\frac{\hat{u}_{i_2i_1}(z_{21})}{\left(x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2\right)^{1/2}} \\ &= \frac{u_{i_1i_2}(z_{12})}{x_{\bar{1}2}{}^2} = -\frac{u_{i_2i_1}(z_{21})}{x_{\bar{2}1}{}^2} \end{aligned} \tag{2.38}$$

is analytic in z_1 and z_2 for $z_1 \neq z_2$,

$$D_{1\alpha(j_1} A_{i_1)i_2}(z_1, z_2) = \bar{D}_{1\dot{\alpha}(j_1} A_{i_1)i_2}(z_1, z_2) = 0. \tag{2.39}$$

As we will see later, $A_{i_1 i_2}(z_1, z_2)$ is a building block of correlation functions of analytic quasi-primary superfields like the $\mathcal{N} = 2$ flavour current superfields. It is worth noting that unitarity of $\hat{u}(z_{12})$ now implies

$$\overline{\hat{u}_{ij}(z_{12})} = \hat{u}^{ij}(z_{12}). \tag{2.40}$$

The above properties of the matrices $\hat{u}_{rs}$, where $r, s = 1, 2, 3$, have natural counterparts for $\boldsymbol{u}(\boldsymbol{Z}_s)$, with $\boldsymbol{u}(\boldsymbol{Z}_3)$ defined in (2.31). We introduce the unitary unimodular 2×2 matrix

$$\hat{\boldsymbol{u}}(\boldsymbol{Z}_3) = \left(\frac{\bar{\boldsymbol{X}}_3{}^2}{\boldsymbol{X}_3{}^2}\right)^{1/2} \boldsymbol{u}(\boldsymbol{Z}_3), \qquad \det \hat{\boldsymbol{u}}(\boldsymbol{Z}_3) = 1, \qquad \hat{\boldsymbol{u}}^\dagger(\boldsymbol{Z}_3)\hat{\boldsymbol{u}}(\boldsymbol{Z}_3) = 1 \tag{2.41}$$

with the superconformal transformation law

$$\delta \hat{\boldsymbol{u}}_i{}^j(\boldsymbol{Z}_3) = \hat{\Lambda}_i{}^k(z_3)\hat{\boldsymbol{u}}_k{}^j(\boldsymbol{Z}_3) - \hat{\boldsymbol{u}}_i{}^k(\boldsymbol{Z}_3)\hat{\Lambda}_k{}^j(z_3). \tag{2.42}$$

Since $\hat{\boldsymbol{u}}(\boldsymbol{Z}_3)$ is unimodular and unitary, we have

$$\begin{aligned} &\mathrm{tr}\, \hat{\boldsymbol{u}}^\dagger(\boldsymbol{Z}_3) = \mathrm{tr}\, \hat{\boldsymbol{u}}(\boldsymbol{Z}_3) \\ &\hat{\boldsymbol{u}}^\dagger_{ji}(\boldsymbol{Z}_3) = -\hat{\boldsymbol{u}}_{ij}(\boldsymbol{Z}_3) \end{aligned} \tag{2.43}$$

and from here one can readily deduce the useful identities

$$\begin{aligned} &2\left(\frac{1}{\bar{\boldsymbol{X}}_3{}^2} - \frac{1}{\boldsymbol{X}_3{}^2}\right) = (\boldsymbol{X}_3 \cdot \bar{\boldsymbol{X}}_3)\left(\frac{1}{(\bar{\boldsymbol{X}}_3{}^2)^2} - \frac{1}{(\boldsymbol{X}_3{}^2)^2}\right) \\ &\Theta_{3(i} \frac{\boldsymbol{X}_3}{(\boldsymbol{X}_3{}^2)^2} \bar{\Theta}_{3\,j)} = \Theta_{3(i} \frac{\bar{\boldsymbol{X}}_3}{(\bar{\boldsymbol{X}}_3{}^2)^2} \bar{\Theta}_{3\,j)}. \end{aligned} \tag{2.44}$$

3. Correlators of $\mathcal{N} = 2$ quasi-primary superfields

3.1. Quasi-primary superfields

In $\mathcal{N}$-extended superconformal field theory, a quasi-primary superfield $\mathcal{O}^{\mathcal{A}}_{\mathcal{I}}(z)$, carrying some number of undotted and dotted spinor indices, denoted collectively by the superscript '$\mathcal{A}$', and transforming in a representation T of the R-symmetry $SU(\mathcal{N})$ with respect to the subscript '$\mathcal{I}$', is defined by the following infinitesimal transformation law under the superconformal group:

$$\begin{aligned} \delta\mathcal{O}^{\mathcal{A}}_{\mathcal{I}}(z) = -\xi\,\mathcal{O}^{\mathcal{A}}_{\mathcal{I}}(z) &+ (\hat{\omega}^{\gamma\delta}(z) M_{\gamma\delta} + \hat{\bar{\omega}}^{\dot{\gamma}\dot{\delta}}(z)\bar{M}_{\dot{\gamma}\dot{\delta}})^{\mathcal{A}}{}_{\mathcal{B}}\mathcal{O}^{\mathcal{B}}_{\mathcal{I}}(z) \\ &+ \mathrm{i}\hat{\Lambda}^k{}_l(z)(R^l{}_k)_{\mathcal{I}}{}^{\mathcal{J}}\mathcal{O}^{\mathcal{A}}_{\mathcal{J}}(z) - 2\,(q\sigma(z) + \bar{q}\bar{\sigma}(z))\,\mathcal{O}^{\mathcal{A}}_{\mathcal{I}}(z). \end{aligned} \tag{3.1}$$

Here $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$ are the Lorentz generators which act on the undotted and dotted spinor indices, respectively, while $R^i{}_j$ are the generators of $SU(\mathcal{N})$. The parameters q and $\bar{q}$ determine the dimension $(q + \bar{q})$ and $U(1)$ R-symmetry charge $(q - \bar{q})$ of the superfield, since for a combined scale and $U(1)$ chiral transformation

$$\delta x^m = \lambda x^m, \qquad \delta\theta^\alpha_i = \tfrac{1}{2}(\lambda + \mathrm{i}\Omega)\theta^\alpha_i, \qquad \delta\bar{\theta}^i_{\dot{\alpha}} = \tfrac{1}{2}(\lambda - \mathrm{i}\Omega)\bar{\theta}^i_{\dot{\alpha}} \tag{3.2}$$

we have

$$\sigma(z) = \frac{1}{2}\left(\lambda + \mathrm{i}\frac{\mathcal{N}}{\mathcal{N} - 4}\Omega\right). \tag{3.3}$$

In this paper we are mainly interested in two- and three-point correlation functions of the supercurrent $\mathcal{J}(z)$ and a flavour current superfield $\mathcal{L}_{(ij)}(z)$ in $\mathcal{N}=2$ superconformal theory. The reality condition $\bar{\mathcal{J}}=\mathcal{J}$ and the supercurrent conservation equation

$$D^{ij}\mathcal{J}=\bar{D}^{ij}\mathcal{J}=0 \tag{3.4}$$

uniquely fix the superconformal transformation law of $\mathcal{J}$

$$\delta\mathcal{J}(z)=-\xi\,\mathcal{J}(z)-2\,(\sigma(z)+\bar{\sigma}(z))\,\mathcal{J}(z). \tag{3.5}$$

As for the flavour current superfield, the reality condition $\overline{\mathcal{L}_{ij}}=\mathcal{L}^{ij}$ and the conservation (analyticity) equation

$$D^{(i}_{\alpha}\mathcal{L}^{jk)}=\bar{D}^{(i}_{\dot{\alpha}}\mathcal{L}^{jk)}=0 \tag{3.6}$$

fix its transformation law to

$$\delta\mathcal{L}_{ij}(z)=-\xi\,\mathcal{L}_{ij}(z)+2\mathrm{i}\hat{\Lambda}_{(i}{}^{k}(z)\mathcal{L}_{j)k}(z)-2\,(\sigma(z)+\bar{\sigma}(z))\,\mathcal{L}_{ij}(z). \tag{3.7}$$

Similar to the $\mathcal{N}=1$ consideration of [44], the transformations (3.5) and (3.7) can also be obtained as invariance conditions with respect to combined diffeomorphisms and Weyl transformations in the superconformal theory coupled to a $\mathcal{N}=2$ supergravity background.

3.2. Two-point functions

According to the general prescription of [9, 11], the two-point function of a quasi-primary superfield $\mathcal{O}_{\mathcal{I}}$ (carrying no Lorentz indices) with its conjugate $\bar{\mathcal{O}}^{\mathcal{J}}$ reads

$$\langle\mathcal{O}_{\mathcal{I}}(z_1)\bar{\mathcal{O}}^{\mathcal{J}}(z_2)\rangle=C_{\mathcal{O}}\frac{T_{\mathcal{I}}{}^{\mathcal{J}}\left(\hat{u}(z_{12})\right)}{(x_{\bar{1}2}{}^2)^{\bar{q}}(x_{\bar{2}1}{}^2)^{q}} \tag{3.8}$$

with $C_{\mathcal{O}}$ a normalization constant. Here T denotes the representation of $SU(\mathcal{N})$ to which $\mathcal{O}_{\mathcal{I}}$ belongs.

For the two-point function of the $\mathcal{N}=2$ supercurrent, the above prescription leads to

$$\langle\mathcal{J}(z_1)\mathcal{J}(z_2)\rangle=c_{\mathcal{J}}\frac{1}{x_{\bar{1}2}{}^2x_{\bar{2}1}{}^2}. \tag{3.9}$$

Using the identity

$$\bar{D}_1{}^{ij}\frac{1}{x_{\bar{1}2}{}^2}=4\mathrm{i}D_1{}^{ij}\delta^8_+(z_1,z_2), \tag{3.10}$$

where $\delta^8_+(z_1,z_2)$ denotes the $\mathcal{N}=2$ chiral delta function,

$$\delta^8_+(z_1,z_2)=\tfrac{1}{16}\bar{D}^4\delta^{12}(z_1-z_2),\qquad \bar{D}^4=\tfrac{1}{3}\bar{D}^{ij}\bar{D}_{ij}, \tag{3.11}$$

we immediately see that the supercurrent conservation equation is satisfied at non-coincident points

$$D_1{}^{ij}\langle\mathcal{J}(z_1)\mathcal{J}(z_2)\rangle=\bar{D}_1{}^{ij}\langle\mathcal{J}(z_1)\mathcal{J}(z_2)\rangle=0,\qquad z_1\neq z_2. \tag{3.12}$$

In this paper we leave aside the analysis of singular behaviour at coincident points, see [9] for details.

In the case of two-point function of the $\mathcal{N}=2$ flavour current superfield $\mathcal{L}_{ij}$, the above prescription gives

$$\langle \mathcal{L}_{i_1 j_1}(z_1)\mathcal{L}^{i_2 j_2}(z_2)\rangle = c_{\mathcal{L}} \frac{\hat{u}_{i_1}{}^{i_2}(z_{12})\hat{u}_{j_1}{}^{j_2}(z_{12}) + \hat{u}_{i_1}{}^{j_2}(z_{12})\hat{u}_{j_1}{}^{i_2}(z_{12})}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}. \tag{3.13}$$

Because of equation (2.39), the relevant conservation equation is satisfied

$$D_{1\alpha(k_1}\langle \mathcal{L}_{i_1 j_1)}(z_1)\mathcal{L}^{i_2 j_2}(z_2)\rangle = \bar{D}_{1\dot{\alpha}(k_1}\langle \mathcal{L}_{i_1 j_1)}(z_1)\mathcal{L}^{i_2 j_2}(z_2)\rangle = 0 \tag{3.14}$$

for $z_1 \neq z_2$.

3.3. *Three-point functions*

According to the general prescription of [9, 11], the three-point function of quasi-primary superfields $\mathcal{O}^{(1)}_{\mathcal{I}_1}$, $\mathcal{O}^{(2)}_{\mathcal{I}_2}$ and $\mathcal{O}^{(3)}_{\mathcal{I}_3}$ reads

$$\langle \mathcal{O}^{(1)}_{\mathcal{I}_1}(z_1)\mathcal{O}^{(2)}_{\mathcal{I}_2}(z_2)\mathcal{O}^{(3)}_{\mathcal{I}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{I}_1}{}^{\mathcal{J}_1}\left(\hat{u}(z_{13})\right) T^{(2)}{}_{\mathcal{I}_2}{}^{\mathcal{J}_2}\left(\hat{u}(z_{23})\right)}{(x_{\bar{1}3}{}^2)^{\bar{q}_1}(x_{\bar{3}1}{}^2)^{q_1}(x_{\bar{2}3}{}^2)^{\bar{q}_2}(x_{\bar{3}2}{}^2)^{q_2}} H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\boldsymbol{Z}_3). \tag{3.15}$$

Here $H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\boldsymbol{Z}_3)$ transforms as an isotensor at z_3 in the representations $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ with respect to the indices $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{I}_3$, respectively, and possesses the homogeneity property

$$\begin{aligned} &H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\Delta\bar{\Delta}\boldsymbol{X}, \Delta\Theta, \bar{\Delta}\bar{\Theta}) = \Delta^{2a}\bar{\Delta}^{2\bar{a}} H_{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}(\boldsymbol{X}, \Theta, \bar{\Theta}) \\ &a - 2\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3, \qquad \bar{a} - 2a = q_1 + q_2 - \bar{q}_3. \end{aligned} \tag{3.16}$$

In general, the latter equation admits a finite number of linearly independent solutions, and this can be considerably reduced by taking into account the symmetry properties, superfield conservation equations and, of course, the superfield constraints (chirality or analyticity).

3.3.1. *The $\mathcal{N}=2$ supercurrent.*

We are going to analyse the three-point function of the $\mathcal{N}=2$ supercurrent for which we should have

$$\begin{aligned} &\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{J}(z_3)\rangle = \frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2} H(\boldsymbol{Z}_3), \\ &H(\Delta\bar{\Delta}\boldsymbol{X}, \Delta\Theta, \bar{\Delta}\bar{\Theta}) = (\Delta\bar{\Delta})^{-2} H(\boldsymbol{X}, \Theta, \bar{\Theta}) \end{aligned} \tag{3.17}$$

where the real function $H(\boldsymbol{Z}_3)$ has to be compatible with the supercurrent conservation equation and the symmetry properties with respect to transposition of indices. Since $H(\boldsymbol{Z}_3)$ is invariant under $U(1)\times SU(2)$ R-transformations, we have $H(\boldsymbol{X}_3, \Theta_3, \bar{\Theta}_3) = H'(\boldsymbol{X}_3, \bar{\boldsymbol{X}}_3)$, as a consequence of (2.29).

When analysing the restrictions imposed by the $\mathcal{N}=2$ conservation equations, it proves advantageous, following similar $\mathcal{N}=1$ considerations in [9], to make use of conformally covariant operators $\mathcal{D}_{\bar{A}} = (\partial/\partial \boldsymbol{X}^a_3, \mathcal{D}_{\alpha i}, \bar{\mathcal{D}}^{\dot{\alpha} i})$ and $\mathcal{Q}_{\bar{A}} = (\partial/\partial \boldsymbol{X}^a_3, \mathcal{Q}_{\alpha i}, \bar{\mathcal{Q}}^{\dot{\alpha} i})$ defined by

$$\begin{aligned} &\mathcal{D}_{\alpha i} = \frac{\partial}{\partial \Theta_3^{\alpha i}} - 2\mathrm{i}(\sigma^a)_{\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}_{3i}\frac{\partial}{\partial \boldsymbol{X}^a_3}, \qquad \bar{\mathcal{D}}^{\dot{\alpha} i} = \frac{\partial}{\partial \bar{\Theta}_{3\dot{\alpha} i}} \\ &\mathcal{Q}_{\alpha i} = \frac{\partial}{\partial \Theta_3^{\alpha i}}, \qquad \bar{\mathcal{Q}}^{\dot{\alpha} i}\frac{\partial}{\partial \bar{\Theta}_{3\dot{\alpha} i}} + 2\mathrm{i}\Theta^i_{3\alpha}(\tilde{\sigma}^a)^{\dot{\alpha}\alpha}\frac{\partial}{\partial \boldsymbol{X}^a_3} \\ &[\mathcal{D}_{\bar{A}}, \mathcal{Q}_{\bar{B}}\} = 0. \end{aligned} \tag{3.18}$$

These operators emerge via the relations

$$\begin{aligned} D_{1\alpha}{}^{i} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) &= -\mathrm{i}(\tilde{x}_{\bar{3}1}{}^{-1})_{\alpha\dot\beta} u_j{}^{i}(z_{31}) \bar{\mathcal{D}}^{\dot\beta j} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \\ \bar{D}_{1\dot\alpha i} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) &= -\mathrm{i}(\tilde{x}_{\bar{1}3}{}^{-1})_{\beta\dot\alpha} u_i{}^{j}(z_{13}) \mathcal{D}^{b}_{j} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \\ D_{2\alpha}{}^{i} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) &= \mathrm{i}(\tilde{x}_{\bar{3}2}{}^{-1})_{\alpha\dot\beta} u_j{}^{i}(z_{32}) \bar{\mathcal{Q}}^{\dot\beta j} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \\ \bar{D}_{2\dot\alpha i} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) &= \mathrm{i}(\tilde{x}_{\bar{2}3}{}^{-1})_{\beta\dot\alpha} u_i{}^{j}(z_{23}) \mathcal{Q}^{b}_{j} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \end{aligned} \tag{3.19}$$

where $t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3)$ is an arbitrary function.

With the aid of these operators, one can prove the identity

$$D_1{}^{\alpha i} D_{1\alpha}{}^{j} \left(\frac{1}{x_{\bar{3}1}{}^2} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \right) = -\frac{u^i{}_k(z_{13}) u^j{}_l(z_{13})}{(x_{\bar{1}3}{}^2)^2} \bar{\mathcal{D}}^{k}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha l} t(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) \tag{3.20}$$

and similar ones involving the operators $\bar{D}_{1ij}$, $D_2{}^{ij}$ and $\bar{D}_{2ij}$.

Now, the supercurrent conservation equation (3.4) leads to the requirements

$$\mathcal{D}_{ij} H(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) = \bar{\mathcal{D}}^{ij} H(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) = 0 \tag{3.21}$$

and to similar ones with $\mathcal{D}$'s $\longrightarrow$ $\mathcal{Q}$'s. Since $\bar{\mathcal{D}}^{\dot\alpha i}$ and $\mathcal{Q}_{\alpha i}$ coincide with partial fermionic derivatives the above equations imply

$$\frac{\partial^2}{\partial\Theta^i_\alpha \partial\Theta^{\alpha j}} H(\boldsymbol{X}, \Theta, \bar\Theta) = \frac{\partial^2}{\partial\bar\Theta^{\dot\alpha}_i \partial\bar\Theta_{\dot\alpha j}} H(\boldsymbol{X}, \Theta, \bar\Theta) = 0, \tag{3.22}$$

and therefore the power series of $H(\boldsymbol{X}, \Theta, \bar\Theta)$ in the Grassmann variables Θ contains only a few terms.

The general solution for $H(\boldsymbol{Z}_3)$ compatible with all the physical requirements on the three-point function of the $\mathcal{N} = 2$ supercurrent reads

$$H(\boldsymbol{X}_3, \Theta_3, \bar\Theta_3) = A \left(\frac{1}{\boldsymbol{X}_3{}^2} + \frac{1}{\bar{\boldsymbol{X}}_3{}^2} \right) + B \frac{\Theta_3^{\alpha\beta} \boldsymbol{X}_{3\alpha\dot\alpha} \boldsymbol{X}_{3\beta\dot\beta} \bar\Theta_3^{\dot\alpha\dot\beta}}{(\boldsymbol{X}_3{}^2)^2} \tag{3.23}$$

where

$$\Theta_3^{\alpha\beta} = \Theta_3^{(\alpha\beta)} = \Theta_3^{\alpha i} \Theta^{\beta}_{3\,i}, \qquad \bar\Theta_3^{\dot\alpha\dot\beta} = \bar\Theta_3^{(\dot\alpha\dot\beta)} = \bar\Theta^{\dot\alpha}_{3\,i} \bar\Theta_3^{\dot\alpha i} \tag{3.24}$$

and A, B are real parameters. Note that the second structure is nilpotent.

Let us comment on the derivation of this solution. First, it is straightforward to check that the functions $\boldsymbol{X}_3{}^{-2}$ and $\bar{\boldsymbol{X}}_3{}^{-2}$ satisfy equation (3.21). They enter $H(\boldsymbol{Z}_3)$ with the same real coefficient, since H must be real and invariant under the replacement $z_1 \leftrightarrow z_2$ that acts on $\boldsymbol{X}_3$ and $\bar{\boldsymbol{X}}_3$ by $\boldsymbol{X}_3 \leftrightarrow -\bar{\boldsymbol{X}}_3$. The second term in (3.23) is a solution to (3.21) due to the special $\mathcal{N} = 2$ identity

$$D^{ij} \theta^{\alpha\beta} = \bar{D}^{ij} \bar\theta^{\dot\alpha\dot\beta} = 0. \tag{3.25}$$

It is important to demonstrate that the second term in (3.23) is real, i.e. that

$$\frac{\Theta_3^{\alpha\beta} \boldsymbol{X}_{3\alpha\dot\alpha} \boldsymbol{X}_{3\beta\dot\beta} \bar\Theta_3^{\dot\alpha\dot\beta}}{(\boldsymbol{X}_3{}^2)^2} = \frac{\Theta_3^{\alpha\beta} \bar{\boldsymbol{X}}_{3\alpha\dot\alpha} \bar{\boldsymbol{X}}_{3\beta\dot\beta} \bar\Theta_3^{\dot\alpha\dot\beta}}{(\bar{\boldsymbol{X}}_3{}^2)^2}. \tag{3.26}$$

Using the identity $\bar{\boldsymbol{X}}_3^a = \boldsymbol{X}_3^a + 2\mathrm{i}\Theta_3^i \sigma^a \bar\Theta_{3\,i}$, we first represent $\bar{\boldsymbol{X}}_3{}^{-2}$ as a function of $\boldsymbol{X}_3$, Θ_3, $\bar\Theta_3$:

$$\frac{1}{\bar{\boldsymbol{X}}_3{}^2} = \frac{1}{\boldsymbol{X}_3{}^2} - 4\mathrm{i} \frac{\Theta_3^i \boldsymbol{X}_3 \bar\Theta_{3\,i}}{(\boldsymbol{X}_3{}^2)^2} - 8 \frac{\Theta_3^{\alpha\beta} \boldsymbol{X}_{3\alpha\dot\alpha} \boldsymbol{X}_{3\beta\dot\beta} \bar\Theta_3^{\dot\alpha\dot\beta}}{(\boldsymbol{X}_3{}^2)^3}. \tag{3.27}$$

We then apply the same identity to express $\Theta_3^i \boldsymbol{X}_3 \bar\Theta_{3\,i}$ in the second term via $\boldsymbol{X}_3$ and $\bar{\boldsymbol{X}}_3$. Now, equation (3.26) follows from (3.27) and the first identity in (2.44).

Using (2.30), one can check that the three-point function (3.17) and (3.23) is completely symmetric in its arguments.

3.3.2. Flavour current superfields. Let us turn to the three-point function of flavour current superfields $\mathcal{L}^{\bar{a}}_{ij}$

$$\langle \mathcal{L}^{\bar{a}}_{i_1 j_1}(z_1)\mathcal{L}^{\bar{b}}_{i_2 j_2}(z_2)\mathcal{L}^{\bar{c}}_{i_3 j_3}(z_3)\rangle = \frac{\hat{u}_{i_1}{}^{k_1}(z_{13})\hat{u}_{j_1}{}^{l_1}(z_{13})\hat{u}_{i_2}{}^{k_2}(z_{23})\hat{u}_{j_2}{}^{l_2}(z_{23})}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \cdot H^{\bar{a}\bar{b}\bar{c}}_{(k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{Z}_3), \tag{3.28}$$

with

$$H^{\bar{a}\bar{b}\bar{c}}_{(k_1 l_1)(k_2 l_2)(i_3 j_3)}(\Delta\bar{\Delta}\boldsymbol{X}, \Delta\Theta, \bar{\Delta}\bar{\Theta}) = (\Delta\bar{\Delta})^{-2} H^{\bar{a}\bar{b}\bar{c}}_{(k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}). \tag{3.29}$$

Using relations (2.39) and (3.19), the flavour current conservation equations (3.6) are equivalent to

$$\begin{aligned} &\mathcal{D}_{\alpha(i_1} H^{\bar{a}\bar{b}\bar{c}}_{k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = \bar{\mathcal{D}}_{\dot{\alpha}(i_1} H^{\bar{a}\bar{b}\bar{c}}_{k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = 0, \\ &\mathcal{Q}_{\alpha(i_2} H^{\bar{a}\bar{b}\bar{c}}_{|k_1 l_1|k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = \bar{\mathcal{Q}}_{\dot{\alpha}(i_2} H^{\bar{a}\bar{b}\bar{c}}_{|k_1 l_1|k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = 0. \end{aligned} \tag{3.30}$$

In particular, since $\bar{\mathcal{D}}^{\dot{\alpha}i}$ and $\mathcal{Q}_{\alpha i}$ are just partial Grassmann derivatives, we should have

$$\frac{\partial}{\partial\bar{\Theta}_{3\,\dot{\alpha}}{}^{(i_1}} H^{\bar{a}\bar{b}\bar{c}}_{k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = \frac{\partial}{\partial\Theta_{3\,\alpha}{}^{(i_2}} H^{\bar{a}\bar{b}\bar{c}}_{|k_1 l_1|k_2 l_2)(i_3 j_3)}(\boldsymbol{X}, \Theta, \bar{\Theta}) = 0. \tag{3.31}$$

The most general form for the correlation function in question is of the form (3.28) with

$$H^{\bar{a}\bar{b}\bar{c}}_{(k_1 l_1)(k_2 l_2)(i_3 j_3)}(\boldsymbol{Z}_3) = f^{\bar{a}\bar{b}\bar{c}} \frac{\varepsilon_{i_3(k_1}\hat{\boldsymbol{u}}_{l_1)(l_2}(\boldsymbol{Z}_3)\varepsilon_{k_2)j_3}}{(\boldsymbol{X}_3{}^2\bar{\boldsymbol{X}}_3{}^2)^{1/2}} + (i_3 \longleftrightarrow j_3) \tag{3.32}$$

with $f^{\bar{a}\bar{b}\bar{c}} = f^{[\bar{a}\bar{b}\bar{c}]}$ a completely antisymmetric tensor, proportional to the structure constants of the flavour group. In contrast to $\mathcal{N} = 1$ supersymmetry [9], the three-point correlation function of flavour currents does not admit an anomalous term proportional to an overall completely symmetric group tensor, $d^{\bar{a}\bar{b}\bar{c}} = d^{(\bar{a}\bar{b}\bar{c})}$. This is a consequence of the fact that the $\mathcal{N} = 2$ conservation equations (3.6) do not admit non-trivial deformations; see also below.

3.3.3. Mixed correlators. The three-point function involving two $\mathcal{N} = 2$ supercurrent insertions and a flavour $\mathcal{N} = 2$ current superfield, turns out to vanish

$$\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{L}_{ij}(z_3)\rangle = 0. \tag{3.33}$$

On general grounds, the only possible expression for such a correlation function compatible with the conservation equations and reality properties should read

$$\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{L}_{ij}(z_3)\rangle = P \frac{1}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \frac{\hat{\boldsymbol{u}}_{(ij)}(\boldsymbol{Z}_3)}{(\boldsymbol{X}_3{}^2\bar{\boldsymbol{X}}_3{}^2)^{1/2}} \tag{3.34}$$

with P a real constant. However, the right-hand side is easily seen to be antisymmetric with respect to the transposition $z_1 \leftrightarrow z_2$ acting as $\boldsymbol{X}_3 \leftrightarrow -\bar{\boldsymbol{X}}_3$, and hence $\hat{\boldsymbol{u}}_{(ij)}(\boldsymbol{Z}_3) \leftrightarrow \hat{\boldsymbol{u}}^{\dagger}_{(ij)}(\boldsymbol{Z}_3) = -\hat{\boldsymbol{u}}_{(ij)}(\boldsymbol{Z}_3)$. Therefore, we must set $P = 0$.

For the three-point function with two flavour currents and one supercurrent insertion one finds

$$\begin{aligned} \langle \mathcal{L}^{\bar{a}}_{i_1 j_1}(z_1)\mathcal{L}^{\bar{b}}_{i_2 j_2}(z_2)\mathcal{J}(z_3)\rangle &= d\delta^{\bar{a}\bar{b}} \frac{\hat{u}_{i_1}{}^{k_1}(z_{13})\hat{u}_{j_1}{}^{l_1}(z_{13})\hat{u}_{i_2}{}^{k_2}(z_{23})\hat{u}_{j_2}{}^{l_2}(z_{23})}{x_{\bar{3}1}{}^2 x_{\bar{1}3}{}^2 x_{\bar{3}2}{}^2 x_{\bar{2}3}{}^2} \\ &\quad \times \frac{\varepsilon_{k_2(k_1}\hat{\boldsymbol{u}}_{l_1)l_2}(\boldsymbol{Z}_3) + \varepsilon_{l_2(k_1}\hat{\boldsymbol{u}}_{l_1)k_2}(\boldsymbol{Z}_3)}{(\boldsymbol{X}_3{}^2\bar{\boldsymbol{X}}_3{}^2)^{1/2}}, \end{aligned} \tag{3.35}$$

where d is a real parameter which can be related, via supersymmetric Ward identities, to the parameter $c_{\mathcal{L}}$ in the two-point function (3.13).

3.4. Example: $\mathcal{N} = 4$ super-Yang–Mills

Let us consider the harmonic superspace formulation for $\mathcal{N} = 4$ super-Yang–Mills theory

$$S[V^{++}, q^+, \breve{q}^+] = \frac{1}{2g^2} \operatorname{tr} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, W^2 - \frac{1}{g^2} \operatorname{tr} \int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)}\, \breve{q}^+ \left(D^{++} + \mathrm{i}V^{++}\right) q^+. \qquad (3.36)$$

Since the hypermultiplet q^+ belongs to the adjoint representation of the gauge group, we can unify q^+ and $\breve{q}^+$ in an isospinor

$$q^+_{\bar{\imath}} = (q^+, \breve{q}^+), \qquad q^{+\bar{\imath}} = \varepsilon^{\bar{\imath}\bar{\jmath}} q^+_{\bar{\jmath}} = (\breve{q}^+, -q^+), \qquad (q^+_{\bar{\imath}})\breve{} = q^{+\bar{\imath}} \qquad (3.37)$$

such that the action takes the form (with $\nabla = D^{++} + \mathrm{i}V^{++}$)

$$S = \frac{1}{2g^2} \operatorname{tr} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, W^2 - \frac{1}{2g^2} \operatorname{tr} \int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)}\, \breve{q}^{+\bar{\imath}} \overset{\longleftrightarrow}{\nabla}{}^{--} q^+_{\bar{\imath}}. \qquad (3.38)$$

This form makes it explicit that the theory manifestly possesses the flavour symmetry $SU_F(2)$, in addition to the $\mathcal{N} = 2$ automorphism group $SU_R(2) \times U_R(1)$. The full group $SU_R(2) \times U_R(1) \times SU_F(2)$ is the maximal subgroup of $SU_R(4)$—the R-symmetry group of the $\mathcal{N} = 4$ SYM—which can be made manifest in the framework of the $\mathcal{N} = 2$ superspace formulation. While the conserved currents for $SU_R(2) \times U_R(1)$ belong to the supercurrent

$$\mathcal{J} = \frac{1}{g^2} \operatorname{tr} \left(\bar{W} W - \tfrac{1}{4} q^{+\bar{\imath}} \overset{\longleftrightarrow}{\nabla}{}^{--} q^+_{\bar{\imath}} \right), \qquad (3.39)$$

the currents for $SU_F(2)$ belong to the flavour current supermultiplet

$$\mathcal{L}^{++\bar{a}}(z, u) \propto \mathrm{i} q^{+\bar{\imath}} (\tau^{\bar{a}})_{\bar{\imath}}{}^{\bar{\jmath}}\, q^+_{\bar{\jmath}} = u^{+i} u^{+i} \mathcal{L}^{\bar{a}}_{ij}(z), \qquad (3.40)$$

with $\tau^{\bar{a}}$ the Pauli matrices; here the latter equality is valid on-shell. The fact that $\langle \mathcal{J}\mathcal{J}\mathcal{L} \rangle$ vanishes identically, whereas $\langle \mathcal{L}\mathcal{L}\mathcal{J} \rangle$ is generically non-zero is now a simple consequence of group theory. In fact, group theory restricts the structure of the correlation function of three $\mathcal{N} = 4$ $SU_R(4)$ currents to be proportional to $\operatorname{tr}(t^I t^J t^K)$ where t^I is a $SU_R(4)$ generator. By considering the action of the $\mathcal{N} = 2$ $U_R(1)$ symmetry, one finds that the correct embedding $u_R(1) \subset su_R(4)$ is diag(+1, +1, −1, −1). Also $su_R(2) \bigoplus su_F(2) \subset su_R(4)$ is embedded as $\operatorname{diag}(su_R(2), su_F(2))$. The result stated above now follows immediately. Three- and four-point functions of the flavour currents (3.40) have been computed at two loops in [45].

4. Reduction to $\mathcal{N} = 1$ superfields

From the point of view of $\mathcal{N} = 1$ superconformal symmetry, any $\mathcal{N} = 2$ quasi-primary superfield consists of several $\mathcal{N} = 1$ quasi-primary superfields. Having computed the correlation functions of $\mathcal{N} = 2$ quasi-primary superfields, one can read off all correlators of their $\mathcal{N} = 1$ superconformal components.

When restricting ourselves to the subgroup $SU(2, 2|1) \in SU(2, 2|2)$, all matrix elements of $\boldsymbol{h}(z)$ (2.19) with $i, j = \underline{2}$ should vanish, and hence we have to set

$$\hat{\Lambda}_{\underline{1}}{}^{\underline{2}} = \hat{\Lambda}_{\underline{2}}{}^{\underline{1}} = 0, \qquad \mathrm{i}\hat{\Lambda}_{\underline{1}}{}^{\underline{1}} = -\mathrm{i}\hat{\Lambda}_{\underline{2}}{}^{\underline{2}} = \bar{\sigma} - \sigma. \qquad (4.1)$$

Therefore, the $\mathcal{N} = 1$ $U(1)$ R-transformation is a combination of $\mathcal{N} = 2$ $U(1)$ and special $SU(2)$ R-transformations.

Keeping equation (4.1) in mind, from the $\mathcal{N} = 2$ supercurrent transformation law (3.5) one deduces the transformation of the $\mathcal{N} = 1$ currents (1.10)

$$\begin{aligned} \delta J &= -\xi J - 2(\sigma + \bar{\sigma}) J \\ \delta J_\alpha &= -\xi J_\alpha + \hat{\omega}_\alpha{}^\beta J_\beta - (3\sigma + 2\bar{\sigma}) J_\alpha \\ \delta J_{\alpha\dot{\alpha}} &= -\xi J_{\alpha\dot{\alpha}} + (\hat{\omega}_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}} + \bar{\hat{\omega}}_{\dot{\alpha}}{}^{\dot{\beta}} \delta_\alpha{}^\beta) J_{\beta\dot{\beta}} - 3(\sigma + \bar{\sigma}) J_{\alpha\dot{\alpha}}. \end{aligned} \tag{4.2}$$

These superconformal transformations are uniquely singled out by the relevant conservation equations

$$\begin{aligned} D^2 J &= \bar{D}^2 J = 0 \\ D^\alpha J_\alpha &= \bar{D}^2 J_\alpha = 0 \\ D^\alpha J_{\alpha\dot{\alpha}} &= \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0. \end{aligned} \tag{4.3}$$

In the case of $\mathcal{N} = 2$ flavour current superfield $\mathcal{L}^{ij}$, its most interesting $\mathcal{N} = 1$ component containing the conserved current,

$$L \equiv \mathrm{i}\mathcal{L}^{\underline{12}}| = \bar{L} \tag{4.4}$$

satisfies the standard $\mathcal{N} = 1$ conservation equation

$$D^2 L = \bar{D}^2 L = 0, \tag{4.5}$$

and, therefore, its superconformal transformation rule is similar to J,

$$\delta L = -\xi L - 2(\sigma + \bar{\sigma}) L. \tag{4.6}$$

The same $\mathcal{N} = 1$ transformation follows from (3.7).

4.1. Two-point functions

Using the explicit form (3.9) of the $\mathcal{N} = 2$ supercurrent two-point function, one can read off the two-point functions of the $\mathcal{N} = 1$ quasi-primary superfields contained in $\mathcal{J}$†

$$\begin{aligned} \langle J(z_1) J(z_2) \rangle &= c_{\mathcal{J}} \frac{1}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}, \\ \langle J_\alpha(z_1) \bar{J}_{\dot{\beta}}(z_2) \rangle &= 4\mathrm{i} c_{\mathcal{J}} \frac{(x_{1\bar{2}})_{\alpha\dot{\beta}}}{x_{\bar{1}2}{}^2 (x_{\bar{2}1}{}^2)^2}, \\ \langle J_{\alpha\dot{\alpha}}(z_1) J_{\beta\dot{\beta}}(z_2) \rangle &= \frac{64}{3} c_{\mathcal{J}} \frac{(x_{1\bar{2}})_{\alpha\dot{\beta}} (x_{2\bar{1}})_{\beta\dot{\alpha}}}{(x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2)^2}. \end{aligned} \tag{4.7}$$

These results are in agreement with $\mathcal{N} = 1$ superconformal considerations [9]. Similarly, the two-point function of the $\mathcal{N} = 1$ flavour current superfield (4.4) follows from (3.13)

$$\langle L(z_1) L(z_2) \rangle = c_{\mathcal{L}} \frac{1}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2}. \tag{4.8}$$

† Here and below, all building blocks are expressed in $\mathcal{N} = 1$ superspace.

4.2. Three-point functions

We now present several $\mathcal{N}=1$ three-point functions which are encoded in that of the $\mathcal{N}=2$ supercurrent, given by equations (3.17) and (3.23). First of all, for the leading $\mathcal{N}=1$ component of $\mathcal{J}$ one immediately obtains

$$\langle J(z_1)J(z_2)J(z_3)\rangle = \frac{A}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\left(\frac{1}{\boldsymbol{X}_3{}^2}+\frac{1}{\bar{\boldsymbol{X}}_3{}^2}\right) \tag{4.9}$$

The second term in (3.23) does not contribute to this three-point function, since $\Theta^{\alpha\beta}$ is equal to zero for $\theta_{\underline{2}}=\bar{\theta}^{\underline{2}}=0$.

The derivation of three-point functions involving the $\mathcal{N}=1$ supercurrent is technically more complicated. In accordance with equation (1.10), $J_{\alpha\dot\alpha}$ is obtained from $\mathcal{J}$ by applying the operator

$$\Delta_{\alpha\dot\alpha} = \tfrac{1}{2}[D^{\underline{2}}_\alpha, \bar{D}_{\dot\alpha\underline{2}}] - \tfrac{1}{6}[D^{\underline{1}}_\alpha, \bar{D}_{\dot\alpha\underline{1}}] \tag{4.10}$$

and, then, the Grassmann variables $\theta_{\underline{2}}$ and $\bar\theta^{\underline{2}}$ have to be switched off. One can prove the following useful relations:

$$\begin{aligned}
\langle J_{\alpha\dot\alpha}(z_1)J(z_2)J(z_3)\rangle &= \frac{(\tilde{x}_{1\bar{3}}{}^{-1})_{\alpha\dot\gamma}(\tilde{x}_{3\bar{1}}{}^{-1})_{\gamma\dot\alpha}}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\Delta_{(\mathcal{D})}{}^{\gamma\dot\gamma} H(\boldsymbol{Z}_3)|,\\
\langle J_{\alpha\dot\alpha}(z_1)J_{\beta\dot\beta}(z_2)J(z_3)\rangle &= \frac{(\tilde{x}_{1\bar{3}}{}^{-1})_{\alpha\dot\gamma}(\tilde{x}_{3\bar{1}}{}^{-1})_{\gamma\dot\alpha}(\tilde{x}_{2\bar{3}}{}^{-1})_{\beta\dot\delta}(\tilde{x}_{3\bar{2}}{}^{-1})_{\delta\dot\beta}}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\\
&\quad\times\Delta_{(\mathcal{D})}{}^{\gamma\dot\gamma}\,\Delta_{(\mathcal{Q})}{}^{\delta\dot\delta} H(\boldsymbol{Z}_3)|
\end{aligned} \tag{4.11}$$

where $H(\boldsymbol{Z}_3)$ is given by equation (3.23) and the operators $\Delta_{(\mathcal{D})}$ and $\Delta_{(\mathcal{Q})}$ are constructed in terms of the conformally covariant derivatives (3.18)

$$\Delta_{(\mathcal{D})}{}^{\alpha\dot\alpha} = \tfrac{1}{2}[\mathcal{D}^\alpha_{\underline{2}}, \bar{\mathcal{D}}^{\dot\alpha\underline{2}}] - \tfrac{1}{6}[\mathcal{D}^\alpha_{\underline{1}}, \bar{\mathcal{D}}^{\dot\alpha\underline{1}}], \qquad \Delta_{(\mathcal{Q})}{}^{\alpha\dot\alpha} = \tfrac{1}{2}[\mathcal{Q}^\alpha_{\underline{2}}, \bar{\mathcal{Q}}^{\dot\alpha\underline{2}}] - \tfrac{1}{6}[\mathcal{Q}^\alpha_{\underline{1}}, \bar{\mathcal{Q}}^{\dot\alpha\underline{1}}]. \tag{4.12}$$

Direct calculations lead to

$$\begin{aligned}
\langle J(z_1)J(z_2)J_{\alpha\dot\alpha}(z_3)\rangle &= -\tfrac{1}{12}(8A-3B)\frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\\
&\quad\times\left(\frac{2(\boldsymbol{P}_3\cdot\boldsymbol{X}_3)\boldsymbol{X}_{3\alpha\dot\alpha}+\boldsymbol{X}_3{}^2\boldsymbol{P}_{3\alpha\dot\alpha}}{(\boldsymbol{X}_3{}^2)^2}+(\boldsymbol{X}_3\leftrightarrow-\bar{\boldsymbol{X}}_3)\right),
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
\langle J_{\alpha\dot\alpha}(z_1)J_{\beta\dot\beta}(z_2)J(z_3)\rangle &= -\frac{4}{9}(8A+3B)\frac{(x_{1\bar{3}})_{\alpha\dot\gamma}(x_{3\bar{1}})_{\gamma\dot\alpha}(x_{2\bar{3}})_{\beta\dot\delta}(x_{3\bar{2}})_{\delta\dot\beta}}{(x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2)^2}\\
&\quad\times\left(\frac{\boldsymbol{X}_3{}^{\gamma\dot\gamma}\boldsymbol{X}_3{}^{\delta\dot\delta}}{(\boldsymbol{X}_3{}^2)^3}+\frac{1}{2}\frac{\varepsilon^{\gamma\delta}\varepsilon^{\dot\gamma\dot\delta}}{(\boldsymbol{X}_3{}^2)^2}+(\boldsymbol{X}_3\leftrightarrow-\bar{\boldsymbol{X}}_3)\right),
\end{aligned} \tag{4.14}$$

with $\boldsymbol{P}_a$ defined by [9]

$$\bar{\boldsymbol{X}}_a-\boldsymbol{X}_a=\mathrm{i}\boldsymbol{P}_a, \qquad \boldsymbol{P}_a=2\Theta\sigma_a\bar\Theta. \tag{4.15}$$

Equation (4.14) presents itself a nice consistency check. In $\mathcal{N}=1$ superconformal field theory, the three-point function $\langle J_{\alpha\dot\alpha}J_{\beta\dot\beta}L\rangle$ of two supercurrents with one flavour current superfield L is uniquely determined up to an overall constant [9]. Any $\mathcal{N}=2$ superconformal field theory, considered as a particular $\mathcal{N}=1$ superconformal model, possesses a special flavour current superfield, $L=J$. Therefore, the only possible arbitrariness in the structure of the correlation function $\langle J_{\alpha\dot\alpha}J_{\beta\dot\beta}J\rangle$ is an overall constant. However, J and $J_{\alpha\dot\alpha}$ are parts of the

$\mathcal{N}=2$ supercurrent $\mathcal{J}$, and hence the three-point function $\langle J_{\alpha\dot\alpha} J_{\beta\dot\beta} J\rangle$ follows from $\langle \mathcal{J}\mathcal{J}\mathcal{J}\rangle$. Since the latter contains two linearly independent forms, given in equation (3.23), there are two possibilities: (a) either the A- or B-term in (3.23) does not contribute to $\langle J_{\alpha\dot\alpha} J_{\beta\dot\beta} J\rangle$; (b) both the A- and B-term produce the same functional contribution to $\langle J_{\alpha\dot\alpha} J_{\beta\dot\beta} J\rangle$ modulo overall constants. Equation (4.14) tells us that option (b) is realized.

The calculation of $\langle J_{\alpha\dot\alpha} J_{\beta\dot\beta} J_{\gamma\dot\gamma}\rangle$ is much more tedious. To derive this correlation function, one has to act with the operator (4.10) on each argument of $\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{J}(z_3)\rangle$. However, since by construction H in (3.17) is a function of $\boldsymbol{Z}_3$, it turns out to be quite difficult to control superconformal covariance at intermediate stages of the calculation when acting with $\Delta_{\gamma\dot\gamma}$ on the third argument of $\langle \mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{J}(z_3)\rangle$. A way out is as follows. One first computes

$$\langle \Delta_{\alpha\dot\alpha}\mathcal{J}(z_1)\mathcal{J}(z_2)\mathcal{J}(z_3)\rangle = \frac{(\tilde{x}_{1\bar{3}}{}^{-1})_{\alpha\dot\sigma}(\tilde{x}_{3\bar{1}}{}^{-1})_{\sigma\dot\alpha}}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}$$
$$\times\Big(\tfrac{64}{3}\theta^{\sigma}_{13\underline{2}}\bar\theta^{\dot\sigma 2}_{13} H(\boldsymbol{Z}_3) + \tfrac{16}{3}\theta^{\sigma}_{13\underline{2}}\, u_k{}^{\underline{2}}(z_{31})\bar{\mathcal{D}}^{\dot\sigma k} H(\boldsymbol{Z}_3) - \tfrac{16}{3}\bar\theta^{\dot\sigma 2}_{13} u_{\underline{2}}{}^{k}(z_{13})\mathcal{D}^{\sigma}_{k} H(\boldsymbol{Z}_3)$$
$$+\tfrac{1}{2}\big\{u_{\underline{2}}{}^{k}(z_{13})u_l{}^{\underline{2}}(z_{31}) - \tfrac{1}{3}u_{\underline{1}}{}^{k}(z_{13})u_l{}^{\underline{1}}(z_{31})\big\}[\mathcal{D}^{\sigma}_{k}, \bar{\mathcal{D}}^{\dot\sigma l}]H(\boldsymbol{Z}_3)\Big)$$

and next expresses $H(\boldsymbol{Z}_3)$, $\mathcal{D}^{\sigma}_{k} H(\boldsymbol{Z}_3)$ and $[\mathcal{D}^{\sigma}_{k}, \bar{\mathcal{D}}^{\dot\sigma l}]H(\boldsymbol{Z}_3)$ as functions of $\boldsymbol{Z}_1$ with the help of identities (2.30). After that it is a simple, but time-consuming procedure, to complete the computation of $\langle J_{\alpha\dot\alpha} J_{\beta\dot\beta} J_{\gamma\dot\gamma}\rangle$. The result reads

$$\langle J_{\alpha\dot\alpha}(z_1) J_{\beta\dot\beta}(z_2) J_{\gamma\dot\gamma}(z_3)\rangle = \frac{(x_{1\bar{3}})_{\alpha\dot\sigma}(x_{3\bar{1}})_{\sigma\dot\alpha}(x_{2\bar{3}})_{\beta\dot\delta}(x_{3\bar{2}})_{\delta\dot\beta}}{(x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2)^2} H^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\boldsymbol{X}_3, \bar{\boldsymbol{X}}_3),$$
$$H^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\boldsymbol{X}_3, \bar{\boldsymbol{X}}_3) = h^{\dot\sigma\sigma,\dot\delta\delta}{}_{\gamma\dot\gamma}(\boldsymbol{X}_3, \bar{\boldsymbol{X}}_3) + h^{\dot\delta\delta,\dot\sigma\sigma}{}_{\gamma\dot\gamma}(-\bar{\boldsymbol{X}}_3, -\boldsymbol{X}_3), \tag{4.16}$$

where

$$h^{\dot\alpha\alpha,\dot\beta\beta}{}_{\gamma\dot\gamma}(\boldsymbol{X}, \bar{\boldsymbol{X}}) = \frac{64}{27}(26A - \tfrac{9}{4}B)\frac{\mathrm{i}}{(\boldsymbol{X}^2)^2}\boldsymbol{X}^{\dot\beta\alpha}\delta^{\dot\alpha}_{\gamma}\delta^{\beta}_{\dot\gamma}$$
$$-\frac{8}{27}(8A - 9B)\frac{1}{(\boldsymbol{X}^2)^3}\Bigg(2\big(\boldsymbol{X}^{\dot\alpha\alpha}\boldsymbol{P}^{\dot\beta\beta} + \boldsymbol{X}^{\dot\beta\beta}\boldsymbol{P}^{\dot\alpha\alpha}\big)\boldsymbol{X}_{\gamma\dot\gamma}$$
$$-3\boldsymbol{X}^{\dot\alpha\alpha}\boldsymbol{X}^{\dot\beta\beta}\bigg(\boldsymbol{P}_{\gamma\dot\gamma} + 2\frac{(\boldsymbol{P}\cdot\boldsymbol{X})}{\boldsymbol{X}^2}\boldsymbol{X}_{\gamma\dot\gamma}\bigg)$$
$$+2\big((\boldsymbol{P}\cdot\boldsymbol{X})\boldsymbol{X}^{\alpha\dot\alpha} - \boldsymbol{X}^2\boldsymbol{P}^{\alpha\dot\alpha}\big)\delta^{\beta}_{\gamma}\delta^{\dot\beta}_{\dot\gamma} + 2\big((\boldsymbol{P}\cdot\boldsymbol{X})\boldsymbol{X}^{\beta\dot\beta} - \boldsymbol{X}^2\boldsymbol{P}^{\beta\dot\beta}\big)\delta^{\alpha}_{\gamma}\delta^{\dot\alpha}_{\dot\gamma}$$
$$+\big(4(\boldsymbol{P}\cdot\boldsymbol{X})\boldsymbol{X}^{\alpha\dot\beta} + \boldsymbol{X}^2\boldsymbol{P}^{\alpha\dot\beta}\big)\delta^{\beta}_{\gamma}\delta^{\dot\alpha}_{\dot\gamma} + \big(4(\boldsymbol{P}\cdot\boldsymbol{X})\boldsymbol{X}^{\beta\dot\alpha} + \boldsymbol{X}^2\boldsymbol{P}^{\beta\dot\alpha}\big)\delta^{\alpha}_{\gamma}\delta^{\dot\beta}_{\dot\gamma}\Bigg). \tag{4.17}$$

It is convenient to rewrite this result in vector notation

$$h^{abc}(\boldsymbol{X}, \bar{\boldsymbol{X}}) \equiv -\tfrac{1}{8}(\sigma^a)_{\alpha\dot\alpha}(\sigma^a)_{\beta\dot\beta}(\tilde\sigma^c)^{\dot\gamma\gamma} h^{\dot\alpha\alpha,\dot\beta\beta}{}_{\gamma\dot\gamma}(\boldsymbol{X}, \bar{\boldsymbol{X}})$$
$$= -\frac{16}{27}(26A - \tfrac{9}{4}B)\frac{\mathrm{i}}{(\boldsymbol{X}^2)^2}\big(\boldsymbol{X}^a\eta^{bc} + \boldsymbol{X}^b\eta^{ac} - \boldsymbol{X}^c\eta^{ab} + \mathrm{i}\varepsilon^{abcd}\boldsymbol{X}_d\big)$$
$$-\frac{8}{27}(8A - 9B)\frac{1}{(\boldsymbol{X}^2)^3}\Bigg(2\big(\boldsymbol{X}^a\boldsymbol{P}^b + \boldsymbol{X}^b\boldsymbol{P}^a\big)\boldsymbol{X}^c$$
$$-3\boldsymbol{X}^a\boldsymbol{X}^b\bigg(\boldsymbol{P}^c + 2\frac{(\boldsymbol{P}\cdot\boldsymbol{X})}{\boldsymbol{X}^2}\boldsymbol{X}^c\bigg) - (\boldsymbol{P}\cdot\boldsymbol{X})\big(3(\boldsymbol{X}^a\eta^{bc} + \boldsymbol{X}^b\eta^{ac}) - 2\boldsymbol{X}^c\eta^{ab}\big)$$
$$+\tfrac{1}{2}\boldsymbol{X}^2\big(\boldsymbol{P}^a\eta^{bc} + \boldsymbol{P}^b\eta^{ac} + \boldsymbol{P}^c\eta^{ab}\big)\Bigg). \tag{4.18}$$

Our final relations (4.16) and (4.18) agree perfectly with the general structure of the three-point function of the supercurrent in $\mathcal{N} = 1$ superconformal field theory [9].

Using the results of [9], it is easy to express A and B in terms of the anomaly coefficients [12]

$$a = \tfrac{1}{24}(5n_V + n_H), \qquad c = \tfrac{1}{12}(2n_V + n_H), \tag{4.19}$$

where n_V and n_H denote the number of free $\mathcal{N} = 2$ vector multiplets and hypermultiplets, respectively. We obtain†

$$A = \frac{3}{64\pi^6}(4a - 3c), \qquad B = \frac{1}{8\pi^6}(4a - 5c). \tag{4.20}$$

In $\mathcal{N} = 1$ supersymmetry, a superconformal Ward identity relates the coefficient in the two-point function of the supercurrent (4.7) to the anomaly coefficient c as follows [9]:

$$c_{\mathcal{J}} = \frac{3}{8\pi^4}c. \tag{4.21}$$

In terms of the coefficients A and B this relation reads

$$\frac{2}{\pi^2}c_{\mathcal{J}} = 8A - 3B. \tag{4.22}$$

In $\mathcal{N} = 1$ supersymmetry, there also exists a superconformal Ward identity which relates the coefficients in the following correlation functions:

$$\langle L(z_1)L(z_2)\rangle = \frac{c_L}{x_{\bar{1}2}{}^2 x_{\bar{2}1}{}^2},$$

$$\langle L(z_1)L(z_2)J_{\alpha\dot{\alpha}}(z_3)\rangle = \frac{D}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\left(\frac{2(\boldsymbol{P}_3\cdot\boldsymbol{X}_3)\boldsymbol{X}_{3\alpha\dot{\alpha}} + \boldsymbol{X}_3{}^2\boldsymbol{P}_{3\alpha\dot{\alpha}}}{(\boldsymbol{X}_3{}^2)^2} + \text{c.c.}\right)$$

of a current superfield L. A nice consequence of our consideration is that $\mathcal{N} = 2$ supersymmetry allows us to fix up this Ward identity without working it out explicitly. The point is that the $\mathcal{N} = 2$ supercurrent contains a special current superfield, that is J. Therefore, from the first relation in (4.7) and equation (4.13) we deduce

$$D = -\frac{1}{6\pi^2}c_L. \tag{4.23}$$

Let us turn to the three-point function of the $\mathcal{N} = 2$ flavour current superfield given by equations (3.28) and (3.32). From these relations one reads off the three-point function of the $\mathcal{N} = 1$ component (4.4)

$$\langle L^{\bar{a}}(z_1)L^{\bar{b}}(z_2)L^{\bar{c}}(z_3)\rangle = \frac{1}{4}f^{\bar{a}\bar{b}\bar{c}}\frac{\mathrm{i}}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\left(\frac{1}{\bar{\boldsymbol{X}}_3{}^2} - \frac{1}{\boldsymbol{X}_3{}^2}\right). \tag{4.24}$$

Here we have used the identities

$$\boldsymbol{u}_{\underline{1}}{}^{\underline{1}}(\boldsymbol{Z}_3)| = \det\boldsymbol{u}(\boldsymbol{Z}_3)|, \qquad \boldsymbol{u}_{\underline{1}}{}^{\underline{2}}(\boldsymbol{Z}_3)| = \boldsymbol{u}_{\underline{2}}{}^{\underline{1}}(\boldsymbol{Z}_3)| = 0, \qquad \boldsymbol{u}_{\underline{2}}{}^{\underline{2}}(\boldsymbol{Z}_3)| = 1. \tag{4.25}$$

It is worth noting that Ward identities allow us to represent $f^{\bar{a}\bar{b}\bar{c}}$ as a product of $c_{\mathcal{L}}$ and the structure constants of the flavour symmetry group, see [9] for more details.

† Our definition of the $\mathcal{N} = 1$ supercurrent corresponds to that adopted in [35] and differs in sign from Osborn's convention [9].

In $\mathcal{N} = 1$ superconformal field theory, the three-point function of flavour current superfields L contains, in general, two linearly independent forms [9]:

$$\langle L^{\bar{a}}(z_1) L^{\bar{b}}(z_2) L^{\bar{c}}(z_3)\rangle = \frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\left\{ \mathrm{i} f^{[\bar{a}\bar{b}\bar{c}]}\left(\frac{1}{\boldsymbol{X}_3{}^2} - \frac{1}{\bar{\boldsymbol{X}}_3{}^2}\right) + d^{(\bar{a}\bar{b}\bar{c})}\left(\frac{1}{\boldsymbol{X}_3{}^2} + \frac{1}{\bar{\boldsymbol{X}}_3{}^2}\right)\right\}.$$

The second term, involving a completely symmetric group tensor $d^{\bar{a}\bar{b}\bar{c}}$, reflects the presence of chiral anomalies in the theory. The field-theoretic origin of this term is due to the fact that the $\mathcal{N} = 1$ conservation equation $\bar{D}^2 L = D^2 L = 0$ admits a non-trivial deformation

$$\bar{D}^2 \langle L^{\bar{a}}\rangle \propto d^{\bar{a}\bar{b}\bar{c}} W^{\bar{b}\alpha} W^{\bar{c}}_{\alpha}$$

when the chiral flavour current is coupled to a background vector multiplet. Equation (4.24) tells us that the flavour currents are anomaly free in $\mathcal{N} = 2$ superconformal theory. This agrees with the facts that: (a) $\mathcal{N} = 2$ super-Yang–Mills models are non-chiral; (b) the $\mathcal{N} = 2$ conservation equation (3.6) does not possess non-trivial deformations.

Finally, from the three-point function (3.35) we immediately deduce

$$\begin{aligned}\langle L^{\bar{a}}(z_1) L^{\bar{b}}(z_2) J(z_3)\rangle &= \frac{d}{2}\delta^{\bar{a}\bar{b}} \frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\left(\frac{1}{\boldsymbol{X}_3{}^2} + \frac{1}{\bar{\boldsymbol{X}}_3{}^2}\right),\\ \langle L^{\bar{a}}(z_1) L^{\bar{b}}(z_2) J_{\alpha\dot{\alpha}}(z_3)\rangle &= -\frac{2d}{3}\delta^{\bar{a}\bar{b}} \frac{1}{x_{\bar{1}3}{}^2 x_{\bar{3}1}{}^2 x_{\bar{2}3}{}^2 x_{\bar{3}2}{}^2}\\ &\quad\times\left(\frac{2(\boldsymbol{P}_3\cdot\boldsymbol{X}_3)\boldsymbol{X}_{3\alpha\dot{\alpha}} + \boldsymbol{X}_3{}^2 \boldsymbol{P}_{3\alpha\dot{\alpha}}}{(\boldsymbol{X}_3{}^2)^2} + (\boldsymbol{X}_3 \leftrightarrow -\bar{\boldsymbol{X}}_3)\right).\end{aligned} \tag{4.26}$$

Now, the Ward identity (4.23) implies

$$d = \frac{1}{4\pi^2} c_{\mathcal{L}}. \tag{4.27}$$

5. Discussion

Our main objective in this paper was to determine the restrictions of the general structure of two- and three-point functions of conserved currents imposed by $\mathcal{N} = 2$ superconformal symmetry. This was done in a manifestly supersymmetric formalism. The results are contained in sections 3.2 and 3.3. In particular, we have shown that the three-point function of the supercurrent allows for two independent structures. In the appendices we show that the minimal supergravity multiplet can be described in harmonic superspace by two real unconstrained prepotentials: harmonic G and analytic v_5^{++}. This is the superfield parametrization which allows us to derive the supercurrent and multiplet of anomalies as the response of the matter action to small disturbances of the supergravity prepotentials.

In this paper, the results about the structure of the correlation functions were completely determined by $\mathcal{N} = 2$ superconformal symmetry. The results for specific models only differ in the value of the numerical coefficients. They can be determined in perturbation theory using supergraph techniques.

An interesting open problem is the issue of non-renormalization theorems for the correlation functions of conserved currents. For a recent discussion for $\mathcal{N} = 4$, see [46].

There exists an off-shell formulation of $\mathcal{N} = 3$ SYM theory, [47]. Since $\mathcal{N} = 3$ and $\mathcal{N} = 4$ SYM are dynamically equivalent, it can be used to find further restrictions and possible non-renormalization theorems on the $\mathcal{N} = 4$ correlation functions.

Another interesting problem is the structure of superconformal anomalies of $\mathcal{N} = 2$ matter systems in a supergravity background. Such anomalies are responsible for the three-point function of the $\mathcal{N} = 2$ supercurrent studied in section 3. The results of appendices A and B provide the natural prerequisites for the analysis of the $\mathcal{N} = 2$ superconformal anomalies.

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Appendix A. Supergravity multiplets

In this appendix we briefly review harmonic superspace and discuss the Weyl multiplet and the minimal supergravity multiplet in some detail.

In rigid supersymmetry, all known $\mathcal{N} = 2$ supersymmetric theories in four spacetime dimensions can be described in terms of fields living in $\mathcal{N} = 2$ harmonic superspace $\mathbb{R}^{4|8} \times SU(2)/U(1)$ introduced by GIKOS [39]. Along with the standard coordinates $z = (x^m, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})$ of $\mathbb{R}^{4|8}$ ($\bar\theta^{\dot\alpha i} = \overline{\theta^\alpha_i}$), this superspace involves the internal harmonic variables $u^\pm_i$ which are constrained by $u^{+i}u^-_i = 1$ and defined modulo phase rotations with charge ±1. Harmonic superspace possesses a supersymmetric subspace, with half the fermionic coordinates of the full superspace, defined to be spanned by the variables

$$\left\{\zeta^M, u^\pm_i\right\}, \qquad \zeta^M = (x^m_A, \theta^{+\hat\alpha}) = (x^m_A, \theta^{+\alpha}, \bar\theta^{+\dot\alpha}) \tag{A.1}$$

where†

$$x^m_A = x^m - 2\mathrm{i}\theta^{(i}\sigma^m\bar\theta^{j)}u^+_i u^+_j, \qquad \theta^{\pm\hat\alpha} = \theta^{\hat\alpha i}u^\pm_i. \tag{A.2}$$

The analytic subspace (A.1) is closed under $\mathcal{N} = 2$ super-Poincaré and superconformal transformations [23, 39]. In addition, it is invariant under the generalized conjugation '$\breve{\ }$' defined as [39]

$$\breve{}: x^m_A \to x^m_A, \qquad \theta^+_\alpha \to \bar\theta^+_{\dot\alpha}, \qquad \bar\theta^+_{\dot\alpha} \to -\theta^+_\alpha, \qquad u^{\pm i} \to -u^\pm_i, \qquad u^\pm_i \to u^{\pm i}.$$

The fundamental importance of analytic subspace (A.1) lies in the fact that the $\mathcal{N} = 2$ matter multiplets (hypermultiplets and vector multiplets) can be described in terms of unconstrained analytic superfields living in the analytic subspace (A.1).

In harmonic superspace, there is a universal gauge principle to introduce couplings to Yang–Mills and supergravity [39]. Consider the rigid supersymmetric operators D^{++} and D^{--} defined as

$$D^{\pm\pm} = \partial^{\pm\pm} - 2\mathrm{i}\theta^\pm\sigma^m\bar\theta^\pm\partial_m + \theta^{\pm\hat\alpha}\partial^\pm_{\hat\alpha}, \tag{A.3}$$

where $\partial^{\pm\pm} = u^{\pm i}\partial/\partial u^{\mp i}$, $\partial_m = \partial/\partial x^m_A$, $\partial^\pm_{\hat\alpha} = \partial/\partial\theta^{\mp\hat\alpha}$. The fundamental property of D^{++} is that if ϕ is analytic, i.e. if $\partial^+_{\hat\alpha}\phi = 0$, then so is $D^{++}\phi$. It turns our that switching on the Yang–Mills or supergravity couplings is equivalent to the requirement that D^{++} must be deformed to acquire a connection or non-trivial vielbeins, in such a way that the deformed operator still preserves analyticity.

† Equation (A.2) defines the so-called analytic basis of harmonic superspace, while the original basis $\{z, u^\pm_i\}$ is called central. In what follows, we mainly use the analytic basis and do not indicate the subscript 'A' explicitly.

A.1. Weyl multiplet

In this subsection we start by reviewing the harmonic superspace realization [23, 24] of the Weyl multiplet [26] describing $N = 2$ conformal supergravity and comprising 24 + 24 off-shell degrees of freedom. Then, we will present a new parametrization for the conformal supergravity prepotentials and describe several gauge fixings.

According to [23, 24], the conformal supergravity gauge fields are identified with the vielbein components of a real covariant derivative

$$\mathcal{D}^{++} = \partial^{++} + \mathcal{H}^{++M}\partial_M + \mathcal{H}^{(+4)}\partial^{--} + \mathcal{H}^{+\hat{\alpha}}\partial^+_{\hat{\alpha}} \tag{A.4}$$

that is required to move every analytic superfield into an analytic one. Hence $\mathcal{H}^{++M} \equiv (\mathcal{H}^{++m}, \mathcal{H}^{+++\alpha}, \breve{\mathcal{H}}^{+++\dot{\alpha}})$ and $\mathcal{H}^{(+4)}$ are analytic, while $\mathcal{H}^{+\hat{\alpha}} \equiv (\mathcal{H}^{+\alpha}, \breve{\mathcal{H}}^{+\dot{\alpha}})$ are unconstrained superfields. The supergravity gauge transformations act on $\mathcal{D}^{++}$ and a scalar superfield U via the rule ($D^0 = u^{+i}\partial/\partial u^{+i} - u^{-i}\partial/\partial u^{-i}$)

$$\delta\mathcal{D}^{++} = [\lambda + \rho, \mathcal{D}^{++}] + \lambda^{++}D^0, \qquad \delta U = (\lambda + \rho)U \tag{A.5}$$

where

$$\lambda = \lambda^M\partial_M + \lambda^{++}\partial^{--}, \qquad \rho = \rho^{-\hat{\alpha}}\partial^+_{\hat{\alpha}} \tag{A.6}$$

such that every analytic superfield of $U(1)$ charge p, $\Phi^{(p)}$, remains analytic

$$\delta\Phi^{(p)} = \lambda\Phi^{(p)}, \qquad \partial^+_{\hat{\alpha}}\Phi^{(p)} = \partial^+_{\hat{\alpha}}\delta\Phi^{(p)} = 0. \tag{A.7}$$

Therefore, the parameters $\lambda^M = (\lambda^m, \lambda^{+\alpha}, \breve{\lambda}^{+\dot{\alpha}})$ and λ^{++} are analytic, while $\rho^{-\hat{\alpha}} = (\rho^{-\alpha}, \breve{\rho}^{-\dot{\alpha}})$ are unconstrained superfields.

The supergravity gauge transformations are induced by special reparametrizations of harmonic superspace

$$\begin{aligned}
&\delta\zeta^M = -\lambda^M(\zeta, u), \\
&\delta u^{+i} = -\lambda^{++}(\zeta, u)u^{-i}, \qquad \delta u^{-i} = 0, \\
&\delta\theta^{-\hat{\alpha}} = -\rho^{-\hat{\alpha}}(\zeta, \theta^-, u)
\end{aligned} \tag{A.8}$$

which leave the analytic subspace invariant.

Since $\mathcal{D}^{++}$ contains a number of independent vielbeins, it is far from obvious in the above picture how to generate a single scalar supercurrent from the host of harmonic vielbeins. In addition, there is a technical complication—some vielbeins possess non-vanishing values in the flat superspace limit (A.3). To find a way out, it is sufficient to recall the standard wisdom of superfield $\mathcal{N} = 1$ supergravity [40]. In equations (A.4) and (A.6) the covariant derivative and gauge parameters are decomposed with respect to the superspace partial derivatives. To have a simple flat superspace limit (which would correspond to vanishing values for all the supergravity prepotential), it is convenient to decompose $\mathcal{D}^{++}$ and λ, ρ with respect to flat covariant derivatives $D^{\pm\pm}$, $D_M = (\partial_m, D^-_\alpha, \bar{D}^-_{\dot{\alpha}})$ and $D^+_{\hat{\alpha}} = \partial/\partial\theta^{-\hat{\alpha}}$; i.e.

$$\mathcal{D}^{++} = D^{++} + H^{++M}D_M + H^{(+4)}D^{--} + H^{+\hat{\alpha}}D^+_{\hat{\alpha}} \tag{A.9}$$

$$\lambda = \Lambda^M D_M + \Lambda^{++}D^{--} \qquad \rho = \rho^{-\hat{\alpha}}D^+_{\hat{\alpha}} \tag{A.10}$$

where $H^{(+4)} = \mathcal{H}^{(+4)}$, $\Lambda^{++} = \lambda^{++}$. In such a parametrization, the vielbeins H^{++M} and $H^{(+4)}$ are no longer independent, but they are instead expressed via a single unconstrained superfield. Really, since we must have

$$D^+_\alpha\mathcal{D}^{++}\Phi^{(p)} = 0,$$

for any analytic superfield $\Phi^{(p)}$, using the algebra of flat covariant derivatives leads to

$$D^+_\alpha H^{++\beta\dot\beta} - 2\mathrm{i}\delta^\beta_\alpha \breve{H}^{+++\dot\beta} = 0, \qquad D^+_\alpha H^{(+4)} = 0,$$
$$D^+_\alpha H^{+++\beta} - \delta^\beta_\alpha H^{(+4)} = 0, \qquad D^+_\alpha \breve{H}^{+++\dot\beta} = 0. \tag{A.11}$$

The general solution to these equations (and their conjugates) reads

$$H^{++\alpha\dot\alpha} = -\mathrm{i}D^{+\alpha}\bar{D}^{+\dot\alpha}G,$$
$$H^{+++\alpha} = -\tfrac{1}{8}D^{+\alpha}(\bar{D}^+)^2G, \qquad \breve{H}^{+++\dot\alpha} = \tfrac{1}{8}\bar{D}^{+\dot\alpha}(D^+)^2G, \tag{A.12}$$
$$H^{(+4)} = \tfrac{1}{16}(D^+)^2(\bar{D}^+)^2G \equiv (D^+)^4G$$

with $G(\zeta, \theta^-, u)$ a real unconstrained harmonic superfield, $\breve{G} = G$. The prepotential introduced is defined modulo pre-gauge transformations

$$\delta G = \tfrac{1}{4}(D^+)^2\Omega^{--} + \tfrac{1}{4}(\bar{D}^+)^2\breve{\Omega}^{--} \tag{A.13}$$

where Ω^{--} is a complex unconstrained parameter.

Similar to H^{++M} and $H^{(+4)}$, the gauge parameters Λ^M and Λ^{++} in equation (A.10) are expressed via a single real unconstrained superfield $l^{--}(\zeta, \theta^-, u)$, $\breve{l}^{--} = l^{--}$, as

$$\Lambda^{\alpha\dot\alpha} = -\mathrm{i}D^{+\alpha}\bar{D}^{+\dot\alpha}l^{--}, \qquad \Lambda^{++} = (D^+)^4 l^{--},$$
$$\Lambda^{+\alpha} = -\tfrac{1}{8}D^{+\alpha}(\bar{D}^+)^2 l^{--}, \qquad \breve{\Lambda}^{+\dot\alpha} = \tfrac{1}{8}\bar{D}^{+\dot\alpha}(D^+)^2 l^{--}. \tag{A.14}$$

From equation (A.5) one can read off the transformations of H^{++M}, $H^{(+4)}$ and $H^{+\hat\alpha}$:

$$\delta H^{++M} = \lambda H^{++M} - \tilde{\mathcal{D}}^{++}\Lambda^M, \qquad \delta H^{(+4)} = \lambda H^{(+4)} - \mathcal{D}^{++}\Lambda^{++},$$
$$\delta H^{+\hat\alpha} = (\lambda + \rho)H^{+\hat\alpha} - \Lambda^{+\hat\alpha} - \mathcal{D}^{++}\rho^{-\hat\alpha}, \tag{A.15}$$

where

$$\tilde{\mathcal{D}}^{++} = \mathcal{D}^{++} - H^{+\hat\alpha}D^+_{\hat\alpha}.$$

Since the parameters $\rho^{-\hat\alpha}$ are unconstrained, $H^{+\hat\alpha}$ can be gauged away

$$H^{+\hat\alpha} = 0. \tag{A.16}$$

Then, the residual gauge freedom is constrained by

$$\mathcal{D}^{++}\rho^{-\hat\alpha} = -\Lambda^{+\hat\alpha}. \tag{A.17}$$

In what follows, we will assume equation (A.16), hence $\mathcal{D}^{++}$ and $\tilde{\mathcal{D}}^{++}$ coincide.

From (A.15) it is easy to read off the transformation law of G. It is sufficient to note the identities

$$[D^+_{\hat\alpha}, \lambda] = 0, \qquad [D^+_{\hat\alpha}, \mathcal{D}^{++}] = 0 \tag{A.18}$$

where the latter holds for (A.16) only. Therefore, from equations (A.12), (A.14) and (A.15) we deduce

$$\delta G = \lambda G - \mathcal{D}^{++}l^{--}. \tag{A.19}$$

Now, equations (A.13) and (A.19) determine the full supergravity gauge group.

It is instructive to examine (A.19) in linearized theory

$$\delta G = -D^{++}l^{--}. \tag{A.20}$$

In the central basis D^{++} coincides with ∂^{++}, and $G(z,u)$ and $l^{--}(z,u)$ are

$$
\begin{aligned}
G(z,u) &= \boldsymbol{G}(z) + \sum_{n=1}^{\infty} G^{(i_1\cdots i_n j_1\cdots j_n)}(z) u^+_{i_1}\cdots u^+_{i_n} u^-_{j_1}\cdots u^-_{j_n}, \\
l^{--}(z,u) &= \sum_{n=1}^{\infty} l^{(i_1\cdots i_{n-1} j_1\cdots j_{n+1})}(z) u^+_{i_1}\cdots u^+_{i_{n-1}} u^-_{j_1}\cdots u^-_{j_{n+1}}
\end{aligned}
\tag{A.21}
$$

where $\boldsymbol{G}(z)$, $G^{(i_1\cdots i_{2n})}(z)$ and $l^{(i_1\cdots i_{2n})}(z)$ are unconstrained superfields. Since $D^{++}u^+_i = 0$ and $D^{++}u^-_i = u^+_i$, equation (A.20) tells us that all the components $G^{(i_1\cdots i_{2n})}$, $n = 1, 2, \ldots$, can be gauged away to arrive at the gauge condition

$$D^{++}G = 0. \tag{A.22}$$

The surviving gauge freedom consists of those combined transformations (A.13) and (A.19) which preserve the above gauge condition, that is

$$\delta\boldsymbol{G}(z) = \tfrac{1}{12} D_{ij}\Omega^{ij}(z) + \tfrac{1}{12}\bar{D}_{ij}\bar{\Omega}^{ij}(z) \tag{A.23}$$

where $\Omega^{ij}(z)$ is the leading component in the harmonic expansion of $\Omega^{--}(z,u)$ (A.13). The linearized prepotential of conformal supergravity $\boldsymbol{G}(z)$ and its gauge freedom (A.23) is precisely what follows from the structure of the $\mathcal{N} = 2$ supercurrent discussed in the introduction.

Instead of imposing the gauge condition (A.22), one can take a different course. Since $H^{(+4)}$ is analytic, it follows from (A.15) that we can achieve the gauge [23, 24]

$$H^{(+4)} = 0 \tag{A.24}$$

which restricts the residual gauge freedom to

$$\mathcal{D}^{++}\Lambda^{++} = 0. \tag{A.25}$$

Now, from (A.12) and (A.24) we obtain

$$G = D^{+\alpha}\Psi^-_\alpha + \bar{D}^{+\dot\alpha}\breve{\Psi}^-_{\dot\alpha} \tag{A.26}$$

where $\Psi^-_\alpha(z,u)$ is an unconstrained harmonic spinor superfield of $U(1)$ charge -1. Equation (A.25) defines a linear analytic superfield in conformal supergravity background. In *linearized* theory, the general solution of equation (A.25) is well known:

$$\Lambda^{++} = (D^+)^4\left\{u^-_i u^-_j D^{ij}V(z) + u^-_i u^-_j \bar{D}^{ij}\bar{V}(z)\right\}. \tag{A.27}$$

Therefore, from here and (A.14) we can completely specify the residual gauge freedom:

$$l^{--}(z,u) = D^{+\alpha}\Upsilon^{(-3)}_\alpha(z,u) + \bar{D}^{+\dot\alpha}\breve{\Upsilon}^{(-3)}_{\dot\alpha}(z,u) + u^-_i u^-_j\left(D^{ij}V(z) + \bar{D}^{ij}\bar{V}(z)\right) \tag{A.28}$$

with an unconstrained harmonic parameter $\Upsilon^{(-3)}_\alpha(z,u)$. Using $\Upsilon^{(-3)}_\alpha$ transformations, we can gauge away all Ψ^-_α but the leading component in its harmonic expansion

$$\Psi^-_\alpha(z,u) = \Psi^i_\alpha(z)u^-_i. \tag{A.29}$$

$\Psi^i_\alpha(z)$ is nothing but the Gates–Siegel prepotential [25].

A.2. Minimal multiplet

The so-called minimal supergravity multiplet [26] is obtained by coupling the Weyl multiplet to an Abelian vector multiplet which is a real analytic superfield $\mathcal{V}_5^{++}(\zeta, u)$. $\mathcal{V}_5^{++}(\zeta, u)$ transforms as a scalar under (A.5) and possesses its own gauge freedom [23, 24, 39]

$$\delta\mathcal{V}_5^{++} = -\mathcal{D}^{++}\lambda_5 \tag{A.30}$$

with $\lambda_5(\zeta, u)$ an arbitrary real analytic parameter. This vector multiplet is a gauge field for the central charge Δ that can be understood as the derivative in an extra bosonic coordinate x^5, $\Delta = \partial/\partial x^5$, on which matter supermultiplets may depend. For matter supermultiplets with central charge, the definition (A.4) should be replaced by

$$\mathcal{D}_\Delta^{++} = \mathcal{D}^{++} + \mathcal{V}_5^{++}\Delta \tag{A.31}$$

and the transformations (A.5) extend to

$$\delta\mathcal{D}_\Delta^{++} = [\lambda + \rho + \lambda_5\Delta, \mathcal{D}_\Delta^{++}] + \lambda^{++}D^0, \qquad \delta U = (\lambda + \rho + \lambda_5\Delta)U. \tag{A.32}$$

The limit of rigid supersymmetry corresponds to the choice when all H-vielbeins in (A.9) vanish and $\mathcal{V}_5^{++}$ can be brought to the form

$$\mathcal{V}_{5,\mathrm{flat}}^{++} = \mathrm{i}\left((\theta^+)^2 - (\bar\theta^+)^2\right). \tag{A.33}$$

That is why $\mathcal{V}_5^{++}$ must in general satisfy a global restriction that its scalar component field $\mathcal{Z}(x)$ defined by

$$\begin{aligned}\mathcal{V}_5^{++}(\zeta, u) &\sim (\theta^+)^2\breve{\mathcal{Z}}(x, u) + (\bar\theta^+)^2\mathcal{Z}(x, u),\\ \mathcal{Z}(x, u) &= \mathcal{Z}(x) + \sum_{n=1}^{\infty}\mathcal{Z}^{(i_1\cdots i_n j_1\cdots j_n)}(x)u_{i_1}^+\cdots u_{i_n}^+u_{j_1}^-\cdots u_{j_n}^-\end{aligned} \tag{A.34}$$

be non-vanishing over the spacetime, $\mathcal{Z}(x) \neq 0$. Then, ordinary local scale and chiral transformations (contained in (A.10)) can be used to bring $\mathcal{Z}(x)$ to its flat form (A.33) (all remaining components in (A.34) turn out to be gauge degrees of freedom). Let $\boldsymbol{D}_\alpha^i$, $\bar{\boldsymbol{D}}_{\dot\alpha i}$ and $\boldsymbol{D}^{\pm\pm}$ be the flat covariant derivatives with central charge. In the central basis, $\boldsymbol{D}^{\pm\pm}$ coincide with $\partial^{\pm\pm}$, while $\boldsymbol{D}_\alpha^i$ and $\bar{\boldsymbol{D}}_{\dot\alpha i}$ are

$$\boldsymbol{D}_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} + \mathrm{i}(\sigma^m\bar\theta^i)\partial_m - \mathrm{i}\theta_\alpha^i\Delta, \qquad \bar{\boldsymbol{D}}_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - \mathrm{i}(\theta_i\sigma^m)_{\dot\alpha}\partial_m - \mathrm{i}\bar\theta_{\dot\alpha i}\Delta. \tag{A.35}$$

In the analytic basis which we mainly use, $\boldsymbol{D}_{\hat\alpha}^+$ coincide with $D_{\hat\alpha}^+$ and the other derivatives are [39]: $\boldsymbol{D}_{\hat\alpha}^- = D_{\hat\alpha}^- - 2\mathrm{i}\theta_{\hat\alpha}^-\Delta$, $\boldsymbol{D}^{\pm\pm} = D^{\pm\pm} + \mathrm{i}\left((\theta^\pm)^2 - (\bar\theta^\pm)^2\right)\Delta$.

The above global restriction on $\mathcal{V}_5^{++}$ gets automatically accounted for if, instead of using the representations (A.9) and (A.10), we start decomposing the harmonic covariant derivative and gauge parameters with respect to the flat covariant derivatives with central charge

$$\mathcal{D}_\Delta^{++} = \boldsymbol{D}^{++} + H^{++M}\boldsymbol{D}_M + H^{(+4)}\boldsymbol{D}^{--} + H^{+\hat\alpha}D_{\hat\alpha}^+ + V_5^{++}\Delta \tag{A.36}$$

$$\lambda + \lambda_5\Delta = \Lambda^M\boldsymbol{D}_M + \Lambda^{++}\boldsymbol{D}^{--} + \Lambda_5\Delta. \tag{A.37}$$

Then, the flat superspace limit would correspond to $V_5^{++} = 0$. However, such a representation is sensible only if the matter multiplets U under consideration are characterized by a constant central charge, $\Delta U^I = \mathrm{i}M^I{}_J U^J$, with $M = (M^I{}_J)$ a constant mass matrix independent of the supergravity prepotentials. Such a situation appears, for example, for hypermultiplets described by unconstrained analytic superfields. However, it is well known that there exist $\mathcal{N} = 2$ supermultiplets which contain finitely many auxiliary fields and possess an intrinsic central charge. This means that setting the central charge to be constant is equivalent to putting the theory on-shell (for example, this applies to the hypermultiplet with $8+8$ off-shell degrees of freedom). To have a finite number of component fields in such theories, one has to impose special constraints on 'primary' superfields U in order that the series $\{U, \Delta U, \Delta\Delta U, \ldots\}$ contains only a few functionally independent representatives. The constraints imposed must determine not only the field content but also specify the off-shell central charge as a non-trivial functional of the supergravity prepotentials. For such theories, the representation (A.36) is useless, because the flat derivatives $\boldsymbol{D}^{\pm\pm}$ and $\boldsymbol{D}_M$ involve the 'curved' central charge.

In the representation (A.36) the requirement

$$D^+_\alpha \mathcal{D}^{++}_\Delta \Phi^{(p)} = 0$$

for any analytic superfield $\Phi^{(p)}$, implies that the set of equations (A.11) should be extended to include one more relation

$$D^+_\alpha V^{++}_5 - 2\mathrm{i} H^{+++}_\alpha = 0. \tag{A.38}$$

Now, the general solution of the constraints (A.11) and (A.38) is given by equation (A.12) along with

$$V^{++}_5 = \tfrac{1}{4}\mathrm{i}(D^+)^2 G - \tfrac{1}{4}\mathrm{i}(\bar{D}^+)^2 G + v^{++}_5, \qquad D^+_{\hat{\alpha}} v^{++}_5 = 0. \tag{A.39}$$

The pre-gauge invariance (A.13) turns into

$$\begin{aligned} \delta G &= \tfrac{1}{4}(D^+)^2\Omega^{--} + \tfrac{1}{4}(\bar{D}^+)^2\breve{\Omega}^{--}, \\ \delta v^{++}_5 &= \mathrm{i}(D^+)^4\Omega^{--} - \mathrm{i}(D^+)^4\breve{\Omega}^{--}. \end{aligned} \tag{A.40}$$

We see that the minimal multiplet is described by the two prepotentials G and v^{++}_5, the latter being a real analytic superfield.

The operator (A.37) must move every analytic superfield into an analytic one. This restricts the parameters Λ^M and Λ^{++} to have the form (A.14), while Λ_5 reads

$$\Lambda_5 = \tfrac{1}{4}\mathrm{i}(D^+)^2 l^{--} - \tfrac{1}{4}\mathrm{i}(\bar{D}^+)^2 l^{--} + \hat{\lambda}_5, \qquad D^+_{\hat{\alpha}}\hat{\lambda}_5 = 0 \tag{A.41}$$

where $\hat{\lambda}_5$ is an arbitrary real analytic superfield.

In the gauge (A.16), H^{++M}, $H^{(+4)}$ and V^{++}_5 transform as follows

$$\begin{aligned} &\delta H^{++M} = \lambda H^{++M} - \mathcal{D}^{++}\Lambda^M, \qquad \delta H^{(+4)} = \lambda H^{(+4)} - \mathcal{D}^{++}\Lambda^{++} \\ &\delta V^{++_5} = \lambda V^{++}_5 - \mathcal{D}^{++}\Lambda_5 \end{aligned} \tag{A.42}$$

and hence

$$\delta v^{++}_5 = \lambda v^{++}_5 - \mathcal{D}^{++}\hat{\lambda}_5. \tag{A.43}$$

As concerns the prepotential G, from (A.42) we again deduce its transformation (A.19).

Appendix B. Supercurrent and multiplet of anomalies

Given a matter system coupled to the minimal supergravity multiplet, we define the supercurrent and multiplet of anomalies

$$\mathcal{J} = \frac{\delta S}{\delta G}, \qquad \mathcal{T}^{++} = \frac{\delta S}{\delta v^{++}_5} \tag{B.1}$$

where S is the matter action. Here the variational derivatives with respect to the supergravity prepotentials are defined as follows:

$$\delta S = \int \mathrm{d}^{12}z\,\mathrm{d}u\,\delta G\,\frac{\delta S}{\delta G} + \int \mathrm{d}u\,\mathrm{d}\zeta^{(-4)}\,\delta v_5^{++}\,\frac{\delta S}{\delta v_5^{++}}. \tag{B.2}$$

As is seen, the supercurrent $\mathcal{J}$ is a real harmonic superfield, $\breve{\mathcal{J}} = \mathcal{J}$, while the multiplet of anomalies $\mathcal{T}^{++}$ is a real analytic superfield, $\breve{\mathcal{T}}^{++} = \mathcal{T}^{++}$, $D^+_{\hat{\alpha}}\mathcal{T}^{++} = 0$. By construction, both $\mathcal{J}$ and $\mathcal{T}^{++}$ are inert with respect to the central charge transformations.

The action is required to be invariant under pre-gauge transformations (A.40). This means

$$\delta S = \tfrac{1}{4}\int \mathrm{d}^{12}z\,\mathrm{d}u\,\mathcal{J}(D^+)^2\Omega^{--} + \mathrm{i}\int \mathrm{d}u\,\mathrm{d}\zeta^{(-4)}\,\mathcal{T}^{++}(D^+)^4\Omega^{--} + \text{c.c.}$$
$$= \int \mathrm{d}^{12}z\,\mathrm{d}u\,\Omega^{--}\left\{\tfrac{1}{4}(D^+)^2\mathcal{J} + \mathrm{i}\mathcal{T}^{++}\right\} + \text{c.c.} = 0$$

for arbitrary Ω^{--}. As a consequence, we obtain

$$\tfrac{1}{4}(D^+)^2\mathcal{J} + \mathrm{i}\mathcal{T}^{++} = 0, \qquad \tfrac{1}{4}(\bar{D}^+)^2\mathcal{J} - \mathrm{i}\mathcal{T}^{++} = 0. \tag{B.3}$$

The action must also be invariant under the superspace general coordinate transformation group. The group acts on the prepotentials G and v_5^{++} according to equations (A.19) and (A.43), respectively. These transformations should be supplemented by those of the matter superfields. On-shell, the invariance of S with respect to (A.19) and (A.43) turns out to imply very strong restrictions on $\mathcal{J}$ and $\mathcal{T}^{++}$, in addition to the conservation law (B.3). Let us describe here the implications of general coordinate invariance for the simplest and most interesting case of a flat superspace when $G = v_5^{++} = 0$ (in general, the analysis is basically the same but requires more involved technical tools). For such a background equations (A.19) and (A.43) reduce to the linearized transformations

$$\delta G = -D^{++}l^{--}, \qquad \delta v_5^{++} = -D^{++}\hat{\lambda}_5. \tag{B.4}$$

Now, the invariance of S with respect to the l^{--} transformations means

$$\delta S = -\int \mathrm{d}^{12}z\,\mathrm{d}u\,(D^{++}l^{--})\mathcal{J} = \int \mathrm{d}^{12}z\,\mathrm{d}u\,l^{--}D^{++}\mathcal{J} = 0 \tag{B.5}$$

for arbitrary l^{--}, and hence

$$D^{++}\mathcal{J} = 0. \tag{B.6}$$

We see that the matter supercurrent in Minkowski superspace is u-independent, $J = J(z)$. On the same grounds, the invariance of S with respect to the central charge $\hat{\lambda}_5$-transformations implies

$$D^{++}\mathcal{T}^{++} = 0. \tag{B.7}$$

The general solution of this equation in the central frame reads

$$\mathcal{T}^{++}(z,u) = \mathcal{T}^{(ij)}(z)u_i^+u_j^+. \tag{B.8}$$

Since $\mathcal{T}^{++}(z,u)$ has to be analytic, the multiplet of anomalies $\mathcal{T}^{(ij)}(z)$ satisfies equation (1.1).

Appendix C. Matter models in the supergravity background

In this section we will describe $\mathcal{N} = 2$ supersymmetric models, both with an intrinsic central charge and models with a constant central charge.

C.1. Models with intrinsic central charge

We will use analytic densities $\Phi^{(p)}_{\{w\}}$ transforming as

$$\delta\Phi^{(p)}_{\{w\}} = (\lambda + \lambda_5\Delta)\Phi^{(p)}_{\{w\}} + w\Lambda\Phi^{(p)}_{\{w\}}, \qquad D^+_{\hat\alpha}\Phi^{(p)}_{\{w\}} = D^+_{\hat\alpha}\delta\Phi^{(p)}_{\{w\}} = 0 \qquad \text{(C.1)}$$

where the variation of the analytic subspace measure $\mathrm{d}u\,\mathrm{d}\zeta^{(-4)}$ with respect to general coordinate transformations (A.8) is given by the analytic superfield

$$\Lambda \equiv (-1)^M D_M\Lambda^M + D^{--}\Lambda^{++}, \qquad D^+_{\hat\alpha}\Lambda = 0. \qquad \text{(C.2)}$$

We will be mainly interested in analytic densities $\Psi^{(p)}_{\{p/2\}}$ on which we can consistently impose the constraint

$$\left(\mathcal{D}^{++} + \mathcal{V}^{++}_5\Delta + \tfrac{1}{2}p\Gamma^{++}\right)\Psi^{(p)}_{\{p/2\}} = 0 \qquad \text{(C.3)}$$

where the analytic connection Γ^{++} is defined by [33]

$$\Gamma^{++} = (-1)^M D_M H^{++M} + D^{--}H^{(+4)}, \qquad D^+_{\hat\alpha}\Gamma^{++} = 0 \qquad \text{(C.4)}$$

and transforms as

$$\delta\Gamma^{++} = \lambda\Gamma^{++} - \mathcal{D}^{++}\Lambda - 2\Lambda^{++}. \qquad \text{(C.5)}$$

The above constraint turns out to be gauge covariant only if $p = 2w$.

To construct a supersymmetric action, let us specify an analytic density $\Psi^{(2)}_{\{1\}} \equiv \mathcal{L}^{++}$ subject to the constraint (C.3). Then, the integral

$$S = \int \mathrm{d}u\,\mathrm{d}\zeta^{(-4)}\,\mathcal{V}^{++}_5\mathcal{L}^{++} \qquad \text{(C.6)}$$

proves to be invariant under the supergravity gauge transformations. Indeed, since $\mathcal{L}^{++}$ is an analytic density of weight 1, and $\mathcal{V}^{++}_5$ is a scalar superfield, their product transforms into a total derivative

$$\begin{aligned}\delta\left(\mathcal{V}^{++}_5\mathcal{L}^{++}\right) &= (-1)^M D_M\left(\Lambda^M\mathcal{V}^{++}_5\mathcal{L}^{++}\right) + D^{--}\left(\Lambda^{++}\mathcal{V}^{++}_5\mathcal{L}^{++}\right),\\ &= (D^+)^4 D^{--}\left(l^{--}\mathcal{V}^{++}_5\mathcal{L}^{++}\right)\end{aligned} \qquad \text{(C.7)}$$

under (A.8), and the action (C.6) remains invariant. Here we have used equation (A.14). As concerns the central charge transformations, we have

$$\delta\mathcal{V}^{++}_5 = -\mathcal{D}^{++}\lambda_5 \qquad \delta\mathcal{L}^{++} = \lambda_5\Delta\mathcal{L}^{++} \qquad \text{(C.8)}$$

and, modulo total derivatives, the variation of S vanishes

$$\delta S = \int \mathrm{d}u\,\mathrm{d}\zeta^{(-4)}\,\lambda_5\left(\mathcal{D}^{++} + \mathcal{V}^{++}_5\Delta + \Gamma^{++}\right)\mathcal{L}^{++} = 0 \qquad \text{(C.9)}$$

as a consequence of (C.3). The above prescription to construct supersymmetric invariants is a natural generalization of the action rule given in [41] for $\mathcal{N} = 2$ rigid supersymmetric theories with a gauged central charge.

Now, let us turn to a hypermultiplet with intrinsic central charge in a conformal supergravity background. The hypermultiplet is described by a constrained analytic superfield $\boldsymbol{q}^+ \equiv \Psi^{(1)}_{\{1/2\}}$ and its conjugate $\breve{\boldsymbol{q}}^+$. It can be shown that the analyticity of $\boldsymbol{q}^+$ and the basic constraint

$$\left(\mathcal{D}^{++} + \mathcal{V}^{++}_5\Delta + \tfrac{1}{2}\Gamma^{++}\right)\boldsymbol{q}^+ = 0 \qquad \text{(C.10)}$$

determine the central charge Δ as a non-trivial operator depending on the supergravity prepotentials. The hypermultiplet dynamics is described by the Lagrangian

$$\mathcal{L}^{++} = \frac{1}{2}\breve{\boldsymbol{q}}^{+} \overleftrightarrow{\boldsymbol{\Delta}} \boldsymbol{q}^{+} - \mathrm{i}m\breve{\boldsymbol{q}}^{+}\boldsymbol{q}^{+}. \tag{C.11}$$

The corresponding equation of motion enforces the central charge to be constant [42]

$$\frac{\delta S}{\delta q^{+}} = 0 \quad \Longrightarrow \quad \Delta \boldsymbol{q}^{+} = \mathrm{i}m\boldsymbol{q}^{+}. \tag{C.12}$$

C.2. Models with constant central charge

Let us consider a dual, for applications a more useful description of the hypermultiplet in terms of an *unconstrained* analytic superfield $q^{+}(\zeta, u)$ and its conjugate $\breve{q}^{+}(\zeta, u)$. The dynamical superfield is defined to transform as a density of weight $\frac{1}{2}$,

$$\delta q^{+} = (\lambda + \lambda_5\Delta)\, q^{+} + \tfrac{1}{2}\Lambda q^{+} \tag{C.13}$$

and its central charge is chosen to be constant

$$\Delta q^{+} = \mathrm{i}mq^{+} \tag{C.14}$$

off-shell. The dynamics is described in curved superspace by the action

$$S = -\int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)} \left\{ \tfrac{1}{2}\breve{q}^{+} \overleftrightarrow{\mathcal{D}}^{++} q^{+} + \mathrm{i}m\mathcal{V}_5^{++}\breve{q}^{+}q^{+} \right\} \tag{C.15}$$

which reduces to that given in [24] for $m = 0$. The action is invariant under all local symmetries. The corresponding equation of motion reads

$$\frac{\delta S}{\delta q^{+}} = 0 \quad \Longrightarrow \quad \left(\mathcal{D}^{++} + \mathcal{V}_5^{++}\Delta + \tfrac{1}{2}\Gamma^{++}\right) q^{+} = 0. \tag{C.16}$$

Comparing equations (C.10) and (C.12) with (C.16) and (C.14), we see that the two hypermultiplet models are equivalent. However, the equation of motion in one model turns into the off-shell constraint in the other and vice versa.

The basic advantage of this model is that off the mass shell the dynamical variable q^{+} is an unconstrained superfield independent of the supergravity prepotentials. That is why one can readily vary the action with respect to these prepotentials. Using equations (A.12) and (A.39) gives

$$\breve{q}^{+}\mathcal{D}^{++}q^{+} + \mathrm{i}m\mathcal{V}_5^{++}\breve{q}^{+}q^{+} = \breve{q}^{+}\boldsymbol{D}^{++}q^{+} + (D^{+})^4\left\{\breve{q}^{+}G\boldsymbol{D}^{--}q^{+}\right\} + \mathrm{i}mv_5^{++}\breve{q}^{+}q^{+}. \tag{C.17}$$

We therefore obtain

$$\mathcal{J} = -\tfrac{1}{2}\breve{q}^{+} \overleftrightarrow{\boldsymbol{D}}^{--} q^{+}, \qquad \mathcal{T}^{++} = -\mathrm{i}m\breve{q}^{+}q^{+}. \tag{C.18}$$

Let us compute the supercurrent and multiplet of anomalies (C.18) in flat superspace where the equation of motion (C.16) becomes

$$\boldsymbol{D}^{++}q^{+} = 0. \tag{C.19}$$

In the central basis, $\boldsymbol{D}^{\pm\pm}$ coincide with $\partial^{\pm\pm}$, and the on-shell superfields read

$$q^{+} = q^{i}(z)u_i^{+}, \qquad \breve{q}^{+} = \bar{q}_i(z)u^{+i}, \qquad \bar{q}_i = \overline{q^{i}}. \tag{C.20}$$

Now, equation (C.18) leads to

$$\mathcal{J} = -\tfrac{1}{2}\bar{q}_i q^{i}, \qquad \mathcal{T}^{++} = \mathcal{T}^{ij}(z)u_i^{+}u_j^{+}, \qquad \mathcal{T}^{ij} = \mathrm{i}m\bar{q}^{(i}q^{j)}. \tag{C.21}$$

What we have derived is exactly the $\mathcal{N} = 2$ supercurrent and multiplet of anomalies found by Sohnius [14].

The above consideration can be generalized to the case of a general renormalizable super-Yang–Mills system in curved superspace with action

$$S = \frac{1}{2g^2} \,\mathrm{tr} \int \mathrm{d}^4x\, \mathrm{d}^4\Theta\, \mathcal{E}\mathcal{W}^2 - \int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)}\, \breve{q}^+ \left\{ \tfrac{1}{2} \overleftrightarrow{\mathcal{D}}^{++} + \mathrm{i}\mathcal{V}^{++} + \mathrm{i}\mathcal{V}_5^{++} M \right\} q^+. \tag{C.22}$$

Here $\mathcal{V}^{++} = \mathcal{V}_I^{++}(\zeta, u) R_I$ is the Yang–Mills gauge superfield, and $\mathcal{W}$ is the corresponding covariantly chiral strength; $\mathcal{E}$ denotes the $\mathcal{N} = 2$ chiral density [32, 43]. The constant mass matrix M is required to be Hermitian and to commute with the gauge group, $[\mathcal{V}^{++}, M] = 0$. In flat superspace, the corresponding supercurrent and multiplet of anomalies read

$$\mathcal{J} = \frac{1}{g^2} \,\mathrm{tr}\left(\bar{\mathcal{W}}\mathcal{W}\right) - \frac{1}{2} \breve{q}^+ \overleftrightarrow{\nabla}^{--} q^+, \qquad \mathcal{T}^{++} = -\mathrm{i}\breve{q}^+ M q^+ \tag{C.23}$$

where ∇^{--} denotes the proper gauge covariant harmonic derivative. In the central basis, ∇^{--} coincides with ∂^{--}, and on-shell

$$q^+ = q^i(z) u_i^+, \qquad \nabla_\alpha^{(i} q^{j)} = \bar{\nabla}_{\dot{\alpha}}^{(i} q^{j)} = 0 \tag{C.24}$$

where ∇_α^i and $\bar{\nabla}_i^{\dot{\alpha}}$ denote ordinary u-independent gauge covariant derivatives. Therefore, from equation (C.23) we obtain

$$\mathcal{J} = \frac{1}{g^2} \,\mathrm{tr}\left(\bar{\mathcal{W}}\mathcal{W}\right) - \frac{1}{2} \bar{q}_i q^i, \qquad \mathcal{T}^{ij} = \mathrm{i}\bar{q}^{(i} M q^{j)}. \tag{C.25}$$

It is worth noting that (C.22) describes a curved superspace extension of the $\mathcal{N} = 4$ super-Yang–Mills theory if $M = 0$ and if q^+ transforms in the adjoint representation of the gauge group.

It is well known that $\mathcal{N} = 2$ Poincaré or de Sitter supergravity cannot be formulated solely in terms of the minimal multiplet [26, 42]. To find a consistent action for Poincaré supergravity, one has to couple the minimal multiplet to an auxiliary multiplet whose role is to compensate some local transformations. Such a compensator may contain finitely many [26] or an infinite number [24] of off-shell component fields. The three known minimal formulations [26] comprising 40 + 40 off-shell degrees of freedom and their compensators are: (I) nonlinear multiplet; (II) hypermultiplet with intrinsic central charge (C.10); (III) improved tensor multiplet. In principle, one can elaborate on non-minimal supergravity formulations with $n + n$ off-shell degrees of freedom, $40 < n < \infty$. Finally, there exists the maximal formulation [24] whose compensator is a single q^+ hypermultiplet considered in this subsection. In all cases, the supergravity action is a sum of the action of the minimal multiplet and that for the compensator [24, 26].

No matter what compensator we choose, it does not enter the minimal classical action (C.22) corresponding to general $\mathcal{N} = 2$ renormalizable SYM models. Therefore, the choice of compensator has no impact on the structure of the supercurrent at the classical level. The main effect of the compensator is to ensure self-consistency of the dynamics of the full supergravity–matter system.

If we give up the requirement of renormalizability, the compensator can tangle with $\mathcal{N} = 2$ matter. This is the case for general quaternionic off-shell sigma models in curved harmonic superspace [34]. However, then we deal with effective field theories (e.g. low-energy string actions) and can treat the compensator as part of the matter sector coupled to $\mathcal{N} = 2$ conformal supergravity.

As an example of more general dynamics, let us consider the $\mathcal{N}=2$ rigid supersymmetric sigma model

$$S=-\frac{1}{2}\int \mathrm{d}u\,\mathrm{d}\zeta^{(-4)}\left\{\breve{q}^{+}\overleftrightarrow{\boldsymbol{D}}^{++}q^{+}+\frac{1}{2}\lambda(\breve{q}^{+}q^{+})^{2}\right\} \tag{C.26}$$

with λ the coupling constant. The bosonic sector of this model describes the Taub–NUT gravitational instanton with a scalar potential generated by the central charge. To lift the model to curved superspace, one has to couple the dynamical superfields not only to the minimal supergravity multiplet, but also to an unconstrained analytic density ω [23, 24]. As a result, the coupling to supergravity is characterized by $\mathcal{J}$ and $\mathcal{T}^{++}$ given, in the flat superspace limit, by (C.18) along with the analytic superfield $\mathcal{T}^{(+4)}=\delta S/\delta\omega=-\frac{1}{2}\lambda(\breve{q}^{+}q^{+})^{2}$. The conservation equations (B.3) and (B.7) remain unchanged, but equation (B.6) gets modified to

$$D^{++}\mathcal{J}+D^{--}\mathcal{T}^{(+4)}=0 \tag{C.27}$$

and therefore $\mathcal{J}$ becomes u-dependent (note that $(D^{++})^{2}\mathcal{J}=0$, since $\boldsymbol{D}^{++}(\breve{q}^{+}q^{+})=D^{++}(\breve{q}^{+}q^{+})=0$ on-shell).

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