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Refined Algebraic Quantization in the oscillator representation of $\mathrm{SL}(2,\mathbb{R})$

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Abstract

We investigate Refined Algebraic Quantization (RAQ) with group averaging in a constrained Hamiltonian system with unreduced phase space $T^*\mathbb{R}^4$ and gauge group $\mathrm{SL}(2,\mathbb{R})$. The reduced phase space $\mathcal{M}$ is connected and contains four mutually disconnected 'regular' sectors with topology $\mathbb{R}\times S^1$, but these sectors are connected to each other through an exceptional set where $\mathcal{M}$ is not a manifold and where $\mathcal{M}$ has non-Hausdorff topology. The RAQ physical Hilbert space $\mathcal{H}_{\rm phys}$ decomposes as $\mathcal{H}_{\rm phys} \simeq \oplus\mathcal{H}_i$, where the four subspaces $\mathcal{H}_i$ naturally correspond to the four regular sectors of $\mathcal{M}$. The RAQ observable algebra $\mathcal{A}_{\rm obs}$, represented on $\mathcal{H}_{\rm phys}$, contains natural subalgebras represented on each $\mathcal{H}_i$. The group averaging takes place in the oscillator representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R}^{2,2})$, and ensuring convergence requires a subtle choice for the test state space: the classical analogue of this choice is to excise from $\mathcal{M}$ the exceptional set while nevertheless retaining information about the connections between the regular sectors. A quantum theory with the Hilbert space $\mathcal{H}_{\rm phys}$ and a finitely-generated observable subalgebra of $\mathcal{A}_{\rm obs}$ is recovered through both Ashtekar's Algebraic Quantization and Isham's group theoretic

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quantization.

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I. INTRODUCTION

In the quantization of constrained systems, one proposal for defining an inner product on the physical Hilbert space is to induce this inner product from an auxiliary Hilbert space $\mathcal{H}_{\rm aux}$ via averaging over the gauge group. The construction of $\mathcal{H}_{\rm aux}$ draws input from the kinematical structure of the theory before imposing the constraints, and the constraints enter through an operator representation of the gauge group on $\mathcal{H}_{\rm aux}$. The method has emerged and been applied in various contexts; see [1–9] and the references therein.

A major open question with group averaging is the sense in which the averaging can be made to converge. One may encounter situations where the group averaging diverges merely because of some ill-chosen piece of technical input, and modifying the input leads to a well-defined theory. On the other hand, one may also encounter situations where convergence of the group averaging is precluded by some physically interesting property of the system. For example, within the Refined Algebraic Quantization framework of [8], a convergent group averaging cannot yield a theory with superselection sectors, while a well-defined theory with superselection sectors may nevertheless be recovered through a suitable renormalization of the averaging [9].

In this paper we study group averaging in a quantum mechanical system whose constraints generate the gauge group $\mathrm{SL}(2,\mathbb{R})$. The classical phase space is $\Gamma = T^*\mathbb{R}^4$, and the three classical constraints on Γ are homogeneous quadratic functions of the global canonical phase space coordinates. The system was introduced by Montesinos, Rovelli, and Thiemann [10] as an analogue of general relativity with two "Hamiltonian"-type constraints, quadratic in the momenta, and one "momentum"-type constraint, linear in the momenta. The reduced phase space $\mathcal{M}$ is connected, and it contains four mutually disconnected 'regular' sectors with topology $\mathbb{R} \times S^1$, but connecting these sectors there is an exceptional set where $\mathcal{M}$ is not a manifold and the topology of $\mathcal{M}$ is non-Hausdorff. One thus anticipates quantization to produce a theory with four 'regular' sectors, with subtleties in those aspects of quantization that try in some sense to connect these four sectors. We shall see that this is indeed the case, and when group averaging is used in the quantization, the subtleties emerge precisely in the convergence of the group averaging.

We consider two quantization approaches. First, we recall that Γ admits an explicitly-known $\mathfrak{o}(2,2)$ Poisson bracket algebra $\mathcal{A}_{\rm class}$ of constants of motion ("observables") that separates the regular sectors of $\mathcal{M}$ [10]. We therefore carry through Ashtekar's Algebraic Quantization program [11,12] with $\mathcal{A}_{\rm class}$ promoted into a quantum observable star-algebra $\mathcal{A}^{(\star)}_{\rm phy}$. In agreement with the results of [10], we find four distinct Hilbert spaces, each corresponding to one of the regular sectors of $\mathcal{M}$. We then add to $\mathcal{A}^{(\star)}_{\rm phy}$ four new generators whose classical counterparts act on the four sectors of $\mathcal{M}$ as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ permutation subgroup, and we carry through Algebraic Quantization with the resulting larger observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$. Expectedly, the emerging Hilbert space $\mathcal{H}_+$ is the direct sum of the previous four individual Hilbert spaces. We also show that $\mathcal{H}_+$ with the observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$ can be recovered by applying Isham's group theoretic quantization [13] to an $\mathrm{O}(2,2)$ action on Γ: the infinitesimal generators of the action of the connected subgroup $\mathrm{O_c}(2,2)$ are precisely the classical observables in $\mathcal{A}_{\rm class}$.

We then consider a group averaging approach. For concreteness, and to a considerable degree without loss of generality [14], we adopt the formalism of Refined Algebraic Quantization (RAQ) [4,8,14]. The structure of Γ and the classical constraints suggests a natural choice for $\mathcal{H}_{\rm aux}$ and for the representation of the gauge group $\mathrm{SL}(2,\mathbb{R})$: this representation is isomorphic to the oscillator representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R}^{2,2})$ [15]. $\mathcal{H}_{\rm aux}$ also carries a representation of the Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$, and this representation commutes with the $\mathrm{SL}(2,\mathbb{R})$-action. With a suitable choice for the RAQ linear space $\Phi \subset \mathcal{H}_{\rm aux}$ of test states, we find that the group averaging converges in absolute value and produces a nontrivial physical Hilbert space $\mathcal{H}_{\rm phys}$. $\mathcal{H}_{\rm phys}$ is isomorphic to $\mathcal{H}_{+}$, and the representation of the RAQ observable algebra $\mathcal{A}_{\rm obs}$ on $\mathcal{H}_{\rm phys}$ contains a subrepresentation isomorphic to the representation of $\mathcal{A}^{(\star)}_{\rm phy+}$ on $\mathcal{H}_{+}$. (For technical reasons, these isomorphisms are antilinear.) In this sense, the RAQ quantum theory contains the Algebraic Quantization quantum theory. Further, the uniqueness theorem of [8] shows that our choices for $\mathcal{H}_{\rm aux}$, the $\mathrm{SL}(2,\mathbb{R})$-action, and Φ completely determine the RAQ quantum theory even without group averaging: with our choices, the only freedom in the RAQ rigging map is a multiplicative constant.

Now to the promised subtleties. In the Algebraic Quantization approach, the subtlety occurs with the choice of the linear space on which the constraints are solved. The 'natural' first candidate $\tilde{V}$ for this linear space contains a one-dimensional subspace that, by the spectral properties of $\mathcal{A}^{(\star)}_{\rm phy}$, corresponds classically to the exceptional set in $\mathcal{M}$. This subspace turns however out to have zero norm, and one does not recover a Hilbert space. The remedy is simply to drop the troublesome one-dimensional subspace from $\tilde{V}$, with the results mentioned above.

In the RAQ approach, the subtlety occurs with the choice of the test state space. The structure of the quantum constraint operators and the $\mathrm{SL}(2,\mathbb{R})$-action suggests a natural choice $\tilde{\Phi}$, but it turns out that the group averaging fails to converge precisely on the subspace of $\tilde{\Phi}$ where it attempts to produce the "zero norm" vectors encountered in the Algebraic Quantization. The remedy is again to ensure that the troublesome subspace does not appear in the physical Hilbert space, but now this has to be done by modifying the test state space, and as the definition of observables in RAQ is intimately related to the test state space, care must be taken in order that the RAQ observable algebra remain large enough to allow a comparison with the Algebraic Quantization observable algebra. Our choice, Φ, was found by scrutinizing the explicitly-known $\mathcal{A}^{(\star)}_{\rm phy+}$-action on $\tilde{\Phi}$.

The rest of the paper is as follows. In section II we review and analyze the classical system [10], paying special attention to the classical observable algebra $\mathcal{A}_{\rm class}$, its pull-backs to the various parts of the reduced phase space, and the associated $\mathrm{O}(2,2)$ action. The Algebraic Quantization and the group theoretic quantization are carried out in section III. Section IV presents a concise outline of RAQ with group averaging, in the specific formulation of [8], and section V carries out RAQ in our system. Section VI contains a brief discussion. Appendices A and B collect some relevant facts about $\mathrm{SL}(2,\mathbb{R})$, its covering groups, and their oscillator representations [15]. Certain technical calculations concerning the group averaging are given in appendices C and D.

II. CLASSICAL DYNAMICS

In this section we review and analyze the classical system introduced in [10]. Some relevant facts about the group $\mathrm{SL}(2,\mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ are collected in appendix A.

The phase space is $\Gamma := T^*\mathbb{R}^4 \simeq \mathbb{R}^8$, with the global coordinate functions (u^1, u^2, v^1, v^2) for the base and (p^1, p^2, π^1, π^2) for the cotangent fibers. The symplectic structure is $\Omega = \sum_i (dp^i \wedge du^i + d\pi^i \wedge dv^i)$. We adopt the vector notation $(u^1, u^2) := \vec{u}$, $(v^1, v^2) := \vec{v}$, $(p^1, p^2) := \vec{p}$, $(\pi^1, \pi^2) := \vec{\pi}$, and we indicate a contraction in the suppressed two-dimensional indices by a dot product.

The action reads

$$S = \int dt \left(\vec{p} \cdot \dot{\vec{u}} + \vec{\pi} \cdot \dot{\vec{v}} - N^1 H_1 - N^2 H_2 - \lambda D \right) \quad , \tag{2.1}$$

where N^1, N^2, and λ are Lagrange multipliers, and the three constraints are

$$H_1 := \tfrac{1}{2} \left(\vec{p}^{\,2} - \vec{v}^{\,2} \right) \quad , \tag{2.2a}$$

$$H_2 := \tfrac{1}{2} \left(\vec{\pi}^{\,2} - \vec{u}^{\,2} \right) \quad , \tag{2.2b}$$

$$D := \vec{u} \cdot \vec{p} - \vec{v} \cdot \vec{\pi} \quad . \tag{2.2c}$$

The Poisson bracket algebra of the constraints is

$$\{H_1 \, , \, H_2\} = D \quad , \tag{2.3a}$$

$$\{D \, , \, H_1\} = 2H_1 \quad , \tag{2.3b}$$

$$\{D \, , \, H_2\} = -2H_2 \quad , \tag{2.3c}$$

which is isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ in the basis (A3) of appendix A. The system is therefore a first class constrained system [16]. The gauge group generated by the constraints is $\mathrm{SL}(2,\mathbb{R})$, and its action on Γ is [10]

$$\begin{aligned} \begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix} &\mapsto g \begin{pmatrix} \vec{u} \\ \vec{p} \end{pmatrix} \quad , \\ \begin{pmatrix} \vec{\pi} \\ \vec{v} \end{pmatrix} &\mapsto g \begin{pmatrix} \vec{\pi} \\ \vec{v} \end{pmatrix} \quad , \end{aligned} \tag{2.4}$$

where g is an 2×2 matrix in $\mathrm{SL}(2,\mathbb{R})$.

The reduced phase space $\mathcal{M}$ is, by definition, the quotient of the constraint hypersurface under the $\mathrm{SL}(2,\mathbb{R})$ action (2.4). The topology of $\mathcal{M}$ is induced from Γ, and wherever the geometry of $\mathcal{M}$ is sufficiently regular, $\mathcal{M}$ inherits from Γ also a differentiable structure and a real analytic structure.

$\mathcal{M}_0$ decomposes naturally into six subsets, which we denote respectively by $\mathcal{M}_0$, $\mathcal{M}_{\mathrm{ex}}$, and $\mathcal{M}_{\epsilon_1,\epsilon_2}$, where $\epsilon_i \in \{1, -1\}$. For the points in $\mathcal{M}_{\epsilon_1,\epsilon_2}$, unique representatives in Γ are

$$\begin{aligned} \vec{u} &= \sqrt{r}\,(1, 0) \quad , \\ \vec{p} &= \sqrt{r}\,(0, \epsilon_1) \quad , \\ \vec{v} &= \sqrt{r}\,(\cos\varphi, -\epsilon_1\epsilon_2 \sin\varphi) \quad , \\ \vec{\pi} &= \sqrt{r}\,(\sin\varphi, +\epsilon_1\epsilon_2 \cos\varphi) \quad , \end{aligned} \tag{2.5}$$

where $r > 0$ and $0 \le \varphi < 2\pi$. For the points in $\mathcal{M}_{\rm ex}$, unique representatives in Γ are

$$\begin{aligned} &\vec{u} = (\cos\theta, \sin\theta) \quad , \\ &\vec{\pi} = (\cos\varphi, \sin\varphi) \quad , \\ &\vec{v} = \vec{p} = 0 \quad , \end{aligned} \tag{2.6}$$

where $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$. $\mathcal{M}_0$ contains a single point, whose unique representative in Γ is $\vec{u} = \vec{v} = \vec{p} = \vec{\pi} = 0$.

The four subsets $\mathcal{M}_{\epsilon_1,\epsilon_2}$ of $\mathcal{M}$ are disconnected. Each is open in $\mathcal{M}$ and has topology $\mathbb{R} \times S^1$, and each is coordinatized by the pair (r, φ) as shown in (2.5), with $r > 0$ and $(r, \varphi) \sim (r, \varphi + 2\pi)$. The pullback of Ω to each $\mathcal{M}_{\epsilon_1,\epsilon_2}$ is nondegenerate and equal to $-dr \wedge d\varphi$, thus making each $\mathcal{M}_{\epsilon_1,\epsilon_2}$ into a smooth symplectic manifold. We regard $\mathcal{M}_{\epsilon_1,\epsilon_2}$ as the four 'regular' sectors of $\mathcal{M}$, and we denote their union by $\mathcal{M}_{\rm reg}$.

$\mathcal{M}_{\rm ex}$ is a smooth two-dimensional manifold, and the pullback of Ω to $\mathcal{M}_{\rm ex}$ vanishes. The topology of $\mathcal{M}$ near $\mathcal{M}_{\rm ex}$ is severely non-Hausdorff: any neighborhood of any point in $\mathcal{M}_{\rm ex}$ contains $\mathcal{M}_0$, and there are pairs of points in $\mathcal{M}_{\rm ex}$ whose neighborhoods also overlap in every sector of $\mathcal{M}_{\rm reg}$. Finally, any neighborhood of $\mathcal{M}_0$ contains $\mathcal{M}_{\rm ex}$ and intersects all the sectors of $\mathcal{M}_{\rm reg}$.

We therefore see that $\mathcal{M}$ is connected: each of the disconnected sectors of $\mathcal{M}_{\rm reg}$ is attached to $\mathcal{M}_0$ and $\mathcal{M}_{\rm ex}$. It is clear from (2.5) that the subset $\mathcal{M}_{\rm reg} \cup \mathcal{M}_0$ can be visualized as four cones with a common tip, the tip consisting of the single point in $\mathcal{M}_0$ and being at $r \to 0_+$ in each $\mathcal{M}_{\epsilon_1,\epsilon_2}$ [10]. On the other hand, for fixed ϵ_2, the union of $\mathcal{M}_{1,\epsilon_2}$, $\mathcal{M}_{-1,\epsilon_2}$, and the $\theta = 0$ circle of $\mathcal{M}_{\rm ex}$ constitutes a smooth symplectic manifold with topology $\mathbb{R} \times S^1$: to see this, make in (2.5) a gauge transformation that multiplies $\vec{v}$ and $\vec{p}$ by $\sqrt{r}$ and divides $\vec{u}$ and $\vec{\pi}$ by $\sqrt{r}$, and allow r to take all real values. The union of $\mathcal{M}_{1,\epsilon_2}$, $\mathcal{M}_{-1,\epsilon_2}$, and the $\theta = \pi/2$ circle of $\mathcal{M}_{\rm ex}$ constitutes also a smooth symplectic manifold with topology $\mathbb{R} \times S^1$: to see this, make in (2.5) the analogous gauge transformation with $1/\sqrt{r}$ instead of $\sqrt{r}$. The union of $\mathcal{M}_{1,\epsilon_2}$, $\mathcal{M}_{-1,\epsilon_2}$, and both of these circles in $\mathcal{M}_{\rm ex}$ is a smooth symplectic non-Hausdorff manifold, with topology $\mathbb{R}' \times S^1$, where $\mathbb{R}'$ is the real line with doubled origin. The structure of $\mathcal{M}$ near $\mathcal{M}_{\rm ex}$ is therefore reminiscent of, but more involved than, the joining of the causal and noncausal sectors of Misner space [17], or the joining of the spacelike and timelike sectors in the solution space to Witten's 2+1 gravity on $\mathbb{R} \times T^2$ [18,19] or on $\mathbb{R} \times$(Klein bottle) [20].

We now turn to the observables. Consider on Γ the six functions [10]

$$\begin{aligned} O_{12} &:= u^1 p^2 - p^1 u^2, \qquad & O_{23} &:= u^2 v^1 - p^2 \pi^1, \\ O_{13} &:= u^1 v^1 - p^1 \pi^1, \qquad & O_{24} &:= u^2 v^2 - p^2 \pi^2, \\ O_{14} &:= u^1 v^2 - p^1 \pi^2, \qquad & O_{34} &:= \pi^1 v^2 - v^1 \pi^2 \ . \end{aligned} \tag{2.7}$$

The linear span of the O_{ij} is closed under Poisson brackets, and the Poisson bracket algebra is isomorphic to the Lie algebra $\mathfrak{o}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$. The basis (2.7) is adapted to the $\mathfrak{o}(2,2)$ form of the algebra, while a basis adapted to the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ form is

$$\begin{aligned} \tau_0^\eta &:= \tfrac{1}{2} (O_{12} - \eta O_{34}) \quad , \\ \tau_1^\eta &:= \tfrac{1}{2} (O_{13} - \eta O_{24}) \quad , \\ \tau_2^\eta &:= \tfrac{1}{2} (O_{23} + \eta O_{14}) \quad , \end{aligned} \tag{2.8}$$

where $\eta \in \{1, -1\}$: the Poisson brackets read

$$\begin{aligned}
\{\tau_1^\eta, \, \tau_2^{\eta'}\} &= -\delta^{\eta,\eta'} \tau_0^\eta \quad , \\
\{\tau_2^\eta, \, \tau_0^{\eta'}\} &= \delta^{\eta,\eta'} \tau_1^\eta \quad , \\
\{\tau_0^\eta, \, \tau_1^{\eta'}\} &= \delta^{\eta,\eta'} \tau_2^\eta \quad .
\end{aligned} \tag{2.9}$$

We record for future use that the τ_i^η satisfy for each η the identity

$$-\left(\tau_0^\eta\right)^2 + \left(\tau_1^\eta\right)^2 + \left(\tau_2^\eta\right)^2 = H_1 H_2 + \tfrac{1}{4} D^2 \quad . \tag{2.10}$$

Now, τ_i^η Poisson commute with the constraints and are thus by definition observables. We denote by $\mathcal{A}_{\text{class}}$ the classical observable algebra generated by $\{\tau_j^\eta\}$. The pullbacks of τ_i^η to $\mathcal{M}$ vanish on $\mathcal{M}_0$ and $\mathcal{M}_{\text{ex}}$, while on $\mathcal{M}_{\text{reg}}$ we have

$$\tau_0^\eta = \tfrac{1}{2}\epsilon_1(1+\eta\epsilon_2)\, r \quad , \tag{2.11a}$$

$$\tau_1^\eta = \tfrac{1}{2}(1+\eta\epsilon_2)\, r\cos\varphi \quad , \tag{2.11b}$$

$$\tau_2^\eta = -\tfrac{1}{2}\epsilon_1(1+\eta\epsilon_2)\, r\sin\varphi \quad . \tag{2.11c}$$

$\mathcal{A}_{\text{class}}$ therefore separates $\mathcal{M}_{\text{reg}}$. More precisely, for given η, the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra generated by $\{\tau_i^\eta\}$ vanishes on $\mathcal{M}_{1,-\eta}$ and $\mathcal{M}_{-1,-\eta}$ but separates $\mathcal{M}_{1,\eta} \cup \mathcal{M}_{-1,\eta}$, and on $\mathcal{M}_{\epsilon_1,\eta}$ τ_0^η has the definite sign ϵ_1.

We note in passing that τ_i^η are real analytic functions on Γ. For given ϵ_1 and ϵ_1', (2.11) therefore shows that $\mathcal{M}_{\epsilon_1,1}$ and $\mathcal{M}_{\epsilon_1',-1}$ cannot both belong to a connected real analytic manifold whose analytic structure would be induced from that of Γ.

By construction, exponentiating the Poisson bracket action of $\mathcal{A}_{\text{class}}$ on Γ yields on Γ the action of a connected group $\mathcal{G}$ that is locally $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, and this $\mathcal{G}$-action commutes with the gauge group action (2.4). Considering Γ in a polarization in which $(\vec{u}, \vec{\pi})$ are the 'coordinates' and $(\vec{p}, -\vec{v})$ are the 'momenta', it is immediate from (2.7) that this $\mathcal{G}$-action reads

$$\begin{pmatrix} \vec{u} \\ \vec{\pi} \end{pmatrix} \mapsto A \begin{pmatrix} \vec{u} \\ \vec{\pi} \end{pmatrix} \quad , \tag{2.12a}$$

$$\begin{pmatrix} \vec{p} \\ -\vec{v} \end{pmatrix} \mapsto \left(A^{-1}\right)^T \begin{pmatrix} \vec{p} \\ -\vec{v} \end{pmatrix} \quad , \tag{2.12b}$$

where A is a 4×4 matrix in the defining representation of $\mathrm{O}(2,2)$, and in the connected component $\mathrm{O_c}(2,2)$. Hence $\mathcal{G} = \mathrm{O_c}(2,2) \simeq [\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})]/\mathbb{Z}_2$. We use (2.12) to extend the $\mathcal{G}$-action to the action of $\mathcal{G}_+ := \mathrm{O}(2,2)$: the $\mathcal{G}_+$-action is generated by the $\mathcal{G}$-action and the four maps $P_{\epsilon_1,\epsilon_2}$, where $\epsilon_i \in \{1,-1\}$ and

$$P_{\epsilon_1,\epsilon_2} : (u^1, u^2, v^1, v^2, p^1, p^2, \pi^1, \pi^2) \mapsto (u^1, \epsilon_1 u^2, v^1, \epsilon_1\epsilon_2 v^2, p^1, \epsilon_1 p^2, \pi^1, \epsilon_1\epsilon_2 \pi^2) \quad . \tag{2.13}$$

It is clear that also the $\mathcal{G}_+$-action on Γ commutes with the gauge group action (2.4).

The induced $\mathcal{G}$-action on $\mathcal{M}$ is trivial on $\mathcal{M}_0$, maps $\mathcal{M}_{\text{ex}}$ transitively to itself, and maps each $\mathcal{M}_{\epsilon_1,\epsilon_2}$ transitively to itself. The induced $\mathcal{G}_+$-action on $\mathcal{M}$ is trivial on $\mathcal{M}_0$, maps $\mathcal{M}_{\text{ex}}$ transitively to itself, and maps $\mathcal{M}_{\text{reg}}$ transitively to itself, permuting the the four sectors of $\mathcal{M}_{\text{reg}}$ by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ permutation subgroup according to

$$P_{\epsilon_1',\epsilon_2'} : \mathcal{M}_{\epsilon_1,\epsilon_2} \to \mathcal{M}_{\epsilon_1'\epsilon_1,\epsilon_2'\epsilon_2} \quad . \tag{2.14}$$

III. ALGEBRAIC QUANTIZATION

In this section we quantize the system in the Algebraic Quantization framework of [11]. In this framework one first solves the quantum constraint equations on a linear space, without an inner product, and then seeks a Hermitian inner product such that the adjoint relations in the chosen quantum observable algebra reflect the reality relations in the corresponding classical observable algebra; we refer to [11,12] for overviews and more detail. Our analysis closely follows that in [10], the main difference being that we consider two possible choices for the classical observable algebra, arising respectively from the groups $\mathcal{G}$ and $\mathcal{G}_+$ introduced in section II. The connection to Isham's group theoretic quantization [13] is made at the end of the section.

We work in a "coordinate representation", starting with the linear space of smooth functions $\Psi(\vec{u},\vec{v})$ on $\mathbb{R}^4$. We shall frequently use the polar coordinates defined by $u^1+iu^2 = ue^{i\alpha}$, $v^1+iv^2 = ve^{i\beta}$, where $u\geq 0$, $v \geq 0$. Note that no inner product is introduced at this stage.

To begin, we promote the classical constraints (2.2) into quantum constraint operators. The momentum operators are

$$\widehat{\vec{p}} := -i\vec{\nabla}_u , \quad \widehat{\vec{\pi}} := -i\vec{\nabla}_v \quad , \tag{3.1}$$

and we order the quantum constraints as

$$\widehat{H}_1 := -\tfrac{1}{2}\left(\vec{\nabla}_u^2 + \vec{v}^2\right) \quad , \tag{3.2a}$$

$$\widehat{H}_2 := -\tfrac{1}{2}\left(\vec{\nabla}_v^2 + \vec{u}^2\right) \quad , \tag{3.2b}$$

$$\widehat{D} := -i\left(\vec{u}\cdot\vec{\nabla}_u - \vec{v}\cdot\vec{\nabla}_v\right) \quad , \tag{3.2c}$$

where $\vec{\nabla}_u^2 := \frac{\partial^2}{\partial(u^1)^2} + \frac{\partial^2}{\partial(u^2)^2}$, and similarly for $\vec{\nabla}_v^2$. The commutator algebra of the quantum constraints then closes as

$$\left[\widehat{H}_1\, , \, \widehat{H}_2\right] = i\widehat{D} \quad , \tag{3.3a}$$

$$\left[\widehat{D}\, , \, \widehat{H}_1\right] = 2i\widehat{H}_1 \quad , \tag{3.3b}$$

$$\left[\widehat{D}\, , \, \widehat{H}_2\right] = -2i\widehat{H}_2 \quad . \tag{3.3c}$$

Next, we define a set of quantum observables $\widehat{O}_{ij}$ by substituting the momentum operators (3.1) into the expressions (2.7) of the classical observables O_{ij}. As the resulting expressions contain no products of noncommuting operators, no issue of ordering arises. The operators $\widehat{O}_{ij}$ commute with the constraints (3.2), and their commutator algebra closes. As O_{ij} are real, we introduce on this algebra a star-operation by $\widehat{O}_{ij}^{\star} = \widehat{O}_{ij}$ and extending to the full algebra by antilinearity. We denote this star-algebra of physical observables by $\mathcal{A}^{(\star)}_{\rm phy}$.

We define in $\mathcal{A}^{(\star)}_{\rm phy}$ the operators $\widehat{\tau}^{\eta}_i$ by the hatted counterparts of (2.8), and we write

$$\widehat{\tau}^{\eta}_{\pm} := \widehat{\tau}^{\eta}_1 \pm i\widehat{\tau}^{\eta}_2 \quad . \tag{3.4}$$

The operators $\widehat{\tau}^\eta_0$ and $\widehat{\tau}^\eta_\pm$ generate $\mathcal{A}^{(\star)}_{\rm phy}$. The commutators are

$$[\widehat{\tau}^\eta_0, \widehat{\tau}^{\eta'}_\pm] = \pm\delta^{\eta,\eta'}\,\widehat{\tau}^\eta_\pm \quad , \tag{3.5a}$$
$$[\widehat{\tau}^\eta_+, \widehat{\tau}^{\eta'}_-] = -2\delta^{\eta,\eta'}\,\widehat{\tau}^\eta_0 \quad , \tag{3.5b}$$

and the star-operation reads

$$(\widehat{\tau}^\eta_0)^\star = \widehat{\tau}^\eta_0 \quad , \tag{3.6a}$$
$$(\widehat{\tau}^\eta_\pm)^\star = \widehat{\tau}^\eta_\mp \quad . \tag{3.6b}$$

The explicit expressions of the operators in the polar coordinates are

$$\widehat{\tau}^\eta_0 = -\tfrac{1}{2}i\,(\partial_\alpha + \eta\partial_\beta) \quad , \tag{3.7a}$$
$$\widehat{\tau}^\eta_\pm = \tfrac{1}{2}e^{\pm i(\alpha+\eta\beta)}\,\{uv + [\partial_u \pm (i/u)\partial_\alpha]\,[\partial_v \pm \eta(i/v)\partial_\beta]\} \quad . \tag{3.7b}$$

We now solve the quantum constraints by separation of variables. As shown in [10], solutions that are smooth functions of $(\vec{u},\vec{v})$ and separable in their angle dependence are multiples of the functions

$$\Psi_{m,\epsilon} := e^{im(\alpha+\epsilon\beta)}\;J_m(uv) \quad , \tag{3.8}$$

where $m\in\mathbb{Z}$, $\epsilon\in\{1,-1\}$, and J_m is the Bessel function of the first kind [21]. The functions $\Psi_{m,\epsilon}$ are linearly independent, with the exception that $\Psi_{0,+}=\Psi_{0,-}$. We denote the linear span of the $\Psi_{m,\epsilon}$ by $\tilde{V}$. As

$$\widehat{\tau}^\eta_0\Psi_{m,\epsilon} = \delta^{\eta,\epsilon}m\Psi_{m,\epsilon} \quad , \tag{3.9a}$$
$$\widehat{\tau}^\eta_\pm\Psi_{m,\epsilon} = \delta^{\eta,\epsilon}m\Psi_{m\pm1,\epsilon} \quad , \tag{3.9b}$$

$\tilde{V}$ carries a representation of $\mathcal{A}^{(\star)}_{\rm phy}$.

One could now find the subspaces of $\tilde{V}$ on which the representation of $\mathcal{A}^{(\star)}_{\rm phy}$ is (algebraically) irreducible, and look on each for an inner product in which the star-operation (3.6) becomes the adjoint operation,

$$(\widehat{\tau}^\eta_0)^\dagger = \widehat{\tau}^\eta_0 \quad , \tag{3.10a}$$
$$(\widehat{\tau}^\eta_\pm)^\dagger = \widehat{\tau}^\eta_\mp \quad . \tag{3.10b}$$

However, the only subspace on which such an inner product exists is the one-dimensional subspace generated by $\Psi_{0,+}$, and the resulting theory is physically uninteresting, as every operator in $\mathcal{A}^{(\star)}_{\rm phy}$ then annihilates the whole Hilbert space. There are four other subspaces carrying an irreducible representation of $\mathcal{A}^{(\star)}_{\rm phy}$, but each of these subspaces contains $\Psi_{0,+}$, and the adjoint relations (3.10) imply that $\Psi_{0,+}$ have a vanishing norm [cf. (3.12) and (3.13) below].

The way to remedy the situation is to note that the troublesome vector $\Psi_{0,+}$ is annihilated by every operator in $\mathcal{A}^{(\star)}_{\rm phy}$, and this vector can therefore be dropped at the outset. Let thus V be the linear span of $\{\Psi_{m,\epsilon}\mid m\neq 0\}$. V carries a representation of $\mathcal{A}^{(\star)}_{\rm phy}$, which reads as in (3.9) except that whenever $\Psi_{0,\epsilon}$ would occur on the right-hand side, it is replaced by the zero vector. V decomposes into the direct sum $V = \bigoplus V_{\epsilon_1,\epsilon_2}$, where $\epsilon_i \in \{1,-1\}$ and

$$V_{\epsilon_1,\epsilon_2} := \mathrm{span}\,\{\Psi_{m,\epsilon_2} \mid \epsilon_1 m > 0\} \quad . \tag{3.11}$$

Each $V_{\epsilon_1,\epsilon_2}$ carries an irreducible representation of $\mathcal{A}^{(\star)}_{\mathrm{phy}}$, and we therefore seek an inner product $(\,\cdot\,,\,\cdot\,)_{\epsilon_1,\epsilon_2}$ individually on each. Equations (3.9a) and (3.10a) imply that the $\Psi_{m,\epsilon}$ are orthogonal. Equations (3.9b) and (3.10b) yield the recurrence relation

$$\begin{aligned}(m\pm 1)^2\,(\Psi_m\,,\Psi_m) &= (\widehat{\tau}_\mp\Psi_{m\pm1}\,,\widehat{\tau}_\mp\Psi_{m\pm1})\\ &= (\Psi_{m\pm1}\,,\widehat{\tau}_\pm\widehat{\tau}_\mp\Psi_{m\pm1})\\ &= m(m\pm 1)\,(\Psi_{m\pm1}\,,\Psi_{m\pm1}) \quad ,\end{aligned} \tag{3.12}$$

where we have suppressed the index ϵ on the vectors, the index $\eta=\epsilon$ on $\widehat{\tau}_\pm$, and the index on the inner product. It follows, still suppressing the indices, that

$$(\Psi_m\,,\Psi_{m'}) = a|m|\delta_{m,m'} \quad , \tag{3.13}$$

where a is a positive constant, independent for each $V_{\epsilon_1,\epsilon_2}$.

It is clear that (3.13) defines on each $V_{\epsilon_1,\epsilon_2}$ an inner product satisfying the adjoint relations (3.10). Completion yields the four Hilbert spaces $\mathcal{H}_{\epsilon_1,\epsilon_2}$, and it follows from the asymptotic large order expansion of J_m [22] that every vector in these Hilbert spaces is represented by a function on the original configuration space $\mathbb{R}^4 = \{(\vec{u},\vec{v})\}$. Each $\mathcal{H}_{\epsilon_1,\epsilon_2}$ carries a representation of $\mathcal{A}^{(\star)}_{\mathrm{phy}}$ by densely-defined operators. For given η, the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra generated by $\{\widehat{\tau}^\eta_i\}$ is represented nontrivially on $\mathcal{H}_{\epsilon_1,\eta}$: the representation belongs to the discrete series [15,23–25] and, in the notation of [23], is known as $D^{\epsilon_1}_1$.

In each of these representations of $\mathcal{A}^{(\star)}_{\mathrm{phy}}$ on $\mathcal{H}_{\epsilon_1,\epsilon_2}$, the Casimir operators of both the trivial and nontrivial $\mathfrak{sl}(2,\mathbb{R})$ subalgebra take the value zero:

$$\left[-\left(\widehat{\tau}^\eta_0\right)^2 + \left(\widehat{\tau}^\eta_1\right)^2 + \left(\widehat{\tau}^\eta_2\right)^2\right]\mathcal{H}_{\epsilon_1,\epsilon_2} = 0 \quad . \tag{3.14}$$

In this sense, the quantum theory has preserved the identities (2.10) satisfied by the classical observables.

It is easy to extend the above analysis to the larger observable algebra $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$, generated by $\mathcal{A}^{(\star)}_{\mathrm{phy}}$ and the set $\left\{\widehat{P}_{\epsilon_1,\epsilon_2}\right\}$, where $\epsilon_i \in \{1,-1\}$ and

$$\left(\widehat{P}_{\epsilon_1,\epsilon_2}\Psi\right)(u^1,u^2,v^1,v^2) := \Psi(u^1,\epsilon_1 u^2,v^1,\epsilon_1\epsilon_2 v^2) \quad . \tag{3.15}$$

Note that $\widehat{P}_{\epsilon_1,\epsilon_2}$ is the operator analogue of the map $P_{\epsilon_1,\epsilon_2}$ (2.13) on Γ. The star-operation is extended to $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ by $\widehat{P}^\star_{\epsilon_1,\epsilon_2} = \widehat{P}_{\epsilon_1,\epsilon_2}$. As

$$\widehat{P}_{\epsilon_1,\epsilon_2}\Psi_{m,\epsilon} = \Psi_{\epsilon_1 m,\epsilon_2\epsilon} \quad , \tag{3.16}$$

the new operators permute the subspaces $V_{\epsilon_1,\epsilon_2}$ by an $\mathbb{Z}_2\times\mathbb{Z}_2$ permutation subgroup according to

$$\widehat{P}_{\epsilon'_1,\epsilon'_2} V_{\epsilon_1,\epsilon_2} = V_{\epsilon'_1\epsilon_1,\epsilon'_2\epsilon_2} \quad , \tag{3.17}$$

and the representation of $\mathcal{A}^{(\star)}_{\text{phy}+}$ on V is irreducible. Proceeding as above, we arrive at the Hilbert space $\mathcal{H}_+ := \bigoplus \mathcal{H}_{\epsilon_1,\epsilon_2}$, where the subspaces $\mathcal{H}_{\epsilon_1,\epsilon_2}$ are orthogonal and the inner product on each is given by (3.13), but now with the same a for all $\mathcal{H}_{\epsilon_1,\epsilon_2}$.

The quantum theories that we have obtained have a natural interpretation as quantizations of different subsets of the classical reduced phase space $\mathcal{M}$. For given ϵ_1 and ϵ_2, the representation of $\mathcal{A}^{(\star)}_{\text{phy}}$ on $\mathcal{H}_{\epsilon_1,\epsilon_2}$ is the quantum analogue of the pullback of the classical algebra $\mathcal{A}_{\text{class}}$ to $\mathcal{M}_{\epsilon_1,\epsilon_2}$, in that in each case the $\eta = -\epsilon_2$ $\mathfrak{sl}(2,\mathbb{R})$ subalgebra is trivial, and in the nontrivial $\mathfrak{sl}(2,\mathbb{R})$ subalgebra $\widehat{\tau}^{\epsilon_2}_0$ and and $\tau^{\epsilon_2}_0$ have the same definite sign. The Hilbert space $\mathcal{H}_{\epsilon_1,\epsilon_2}$ with the observable algebra $\mathcal{A}^{(\star)}_{\text{phy}}$ can therefore be thought of as a quantization of the sector $\mathcal{M}_{\epsilon_1,\epsilon_2}$. Similarly, the Hilbert space $\mathcal{H}_+$ with the observable algebra $\mathcal{A}^{(\star)}_{\text{phy}+}$ can be thought of as a quantization of all the four sectors of $\mathcal{M}_{\text{reg}}$.

One can also obtain our quantum theories via the group theoretic quantization of Isham [13]. As noted in section II, the $\mathcal{G}$-action (2.12) on Γ induces on each $\mathcal{M}_{\epsilon_1,\epsilon_2}$ a transitive $\mathcal{G}$-action, and also the transitive action of a subgroup $\text{SL}(2,\mathbb{R}) \subset \mathcal{G}$: this $\text{SL}(2,\mathbb{R})$-action is obtained by exponentiating the Poisson bracket action of the algebra (2.11). For group theoretic quantization on a given sector $\mathcal{M}_{\epsilon_1,\epsilon_2}$, we can therefore adopt this $\text{SL}(2,\mathbb{R})$ as the canonical group. In order to preserve the classical identities (2.10) in the quantum theory, we consider the irreducible unitary representations of $\text{SL}(2,\mathbb{R})$ in which the Casimir operator vanishes. The only such representations are the trivial representation and the discrete series representations $D^{\pm}_1$ [15,23–25]. $\widehat{\tau}^{\epsilon_2}_0$ vanishes in the trivial representation, whereas in each $D^{\pm}_1$ it as a definite sign, and it is in $D^{\epsilon_1}_1$ that this sign agrees with the sign of the classical function $\tau^{\epsilon_2}_0$ (2.11a) on $\mathcal{M}_{\epsilon_1,\epsilon_2}$. Thus, requiring the signs of $\widehat{\tau}^{\epsilon_2}_0$ and $\tau^{\epsilon_2}_0$ to agree picks the representation $D^{\epsilon_1}_1$: we arrive at the Hilbert space $\mathcal{H}_{\epsilon_1,\epsilon_2}$, and the observable algebra is the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of $\mathcal{A}^{(\star)}_{\text{phy}}$ with $\eta = \epsilon_2$. A similar argument can be made for group theoretic quantization on $\mathcal{M}_{\text{reg}}$ with the canonical group $\mathcal{G}_+ \simeq \text{O}(2,2) \simeq \text{O}_{\text{c}}(2,2) \times_s (\mathbb{Z}_2)^2$, arriving at $\mathcal{H}_+$ with the observable algebra $\mathcal{A}^{(\star)}_{\text{phy}+}$. As neither $\mathcal{M}_{\text{reg}}$ nor $\mathcal{G}_+$ is connected, it is perhaps not clear how unique the implementation of the group theoretic quantization in this case is, but $\mathcal{H}_+$ clearly does carry an irreducible unitary representation of $\mathcal{G}_+$. Further possibilities of implementing group theoretic quantization on $\mathcal{M}_{\text{reg}}$ and its four sectors are discussed in [26,27].

We end the section with two remarks:

1) One might have tried to include in the vector space of solutions to the constraints functions that are not smooth at $uv = 0$. In this case one can replace J_m in (3.8) by any linear combination of J_m and N_m, with m-independent coefficients, and the abstract construction of the Hilbert spaces goes through as above. However, when N_m is present, it is seen from the large order expansion of N_m [22] that the completion introduces in the Hilbert spaces vectors that cannot be represented by functions on the original configuration space.

2) One might have tried to include in the vector space of solutions to the constraints vectors that are not single-valued functions on the configuration space, thus allowing m in (3.8), or in the analogue of (3.8) with a linear combination of J_m and N_m, to be noninteger. The representation of $\mathcal{A}^{(\star)}_{\text{phy}}$ on this larger vector space takes again the form (3.9), and breaks

thus into irreducible representations classified by ϵ and the fractional part of m. However, in this case no inner product satisfying the adjoint relations (3.10) exists.

IV. FORMALISM OF REFINED ALGEBRAIC QUANTIZATION WITH GROUP AVERAGING

In this section we give a brief outline of Refined Algebraic Quantization (RAQ) with group averaging. The main purposes of the section are to fix the notation and to fix the particular version of RAQ: we follow the formulation of Giulini and Marolf [8]. We specialize throughout to the case where the gauge group is a connected unimodular Lie group.

A. Refined Algebraic Quantization

RAQ begins by implementing the quantum constraints as self-adjoint operators on an auxiliary Hilbert space $\mathcal{H}_{\rm aux}$. We assume that the commutator algebra of the constraints closes as a Lie algebra, so that the algebra exponentiates into a unitary representation $U(g)$ of a corresponding connected Lie group G on $\mathcal{H}_{\rm aux}$. We refer to G as the gauge group, and we assume that it is unimodular (that is, that the structure constants of the Lie algebra are traceless).

Next, RAQ solves the constraints in an enlargement of $\mathcal{H}_{\rm aux}$. To this end, one introduces a space of test states, a dense linear subspace $\Phi \subset \mathcal{H}_{\rm aux}$ such that the operators $U(g)$ map Φ to itself. The desired enlargement is the algebraic dual of Φ, denoted by Φ^* and topologized by the topology of pointwise convergence. For $f \in \Phi^*$ and $\phi \in \Phi$, we denote the dual action of f on ϕ by $f[\phi]$. Φ^* carries a representation $U^*(g)$ of G defined by the dual action: for $f \in \Phi^*$, $\big(U^*(g)f\big)[\phi] = f[U(g^{-1})\phi]$ for all $\phi \in \Phi$. Solutions to the quantum constraints are then by definition the elements $f \in \Phi^*$ for which $U^*(g)f = f$ for all $g \in G$.

The RAQ algebra of observables is completely determined by the structure specified above. An operator $\mathcal{O}$ on $\mathcal{H}_{\rm aux}$ is called gauge invariant if the domains of $\mathcal{O}$ and $\mathcal{O}^\dagger$ include Φ, $\mathcal{O}$ and $\mathcal{O}^\dagger$ map Φ to itself, and $\mathcal{O}$ commutes with the G-action on Φ: $\mathcal{O}U(g)\phi = U(g)\mathcal{O}\phi$ for all $g \in G$, $\phi \in \Phi$. Note that if $\mathcal{O}$ is gauge invariant, so is $\mathcal{O}^\dagger$. The observable algebra $\mathcal{A}_{\rm obs}$ is by definition the algebra of gauge invariant operators. $\mathcal{A}_{\rm obs}$ has on Φ^* an antilinear representation defined by the dual action [14]: for $f \in \Phi^*$, $(\mathcal{O}f)[\phi] = f[\mathcal{O}^\dagger\phi]$ for all $\phi \in \Phi$. Note that $\mathcal{A}_{\rm obs}$ does not need to be constructed or presented in any explicit sense.

The last ingredient in RAQ is a rigging map, which is by definition an antilinear map η from Φ to Φ^* satisfying four postulates:

(i) The image of η solves the constraints: Each vector in the image of η is invariant under the G-action on Φ^*.

(ii) η is real: $\eta(\phi_1)[\phi_2] = \overline{\eta(\phi_2)[\phi_1]}$ for all $\phi_1, \phi_2 \in \Phi$.

(iii) η is positive: $\eta(\phi)[\phi] \geq 0$ for all $\phi \in \Phi$.

(iv) η intertwines with the representations of the observable algebra on Φ and Φ^*: $\mathcal{O}(\eta\phi) = \eta(\mathcal{O}\phi)$ for all $\mathcal{O} \in \mathcal{A}_{\rm obs}$ and all $\phi \in \Phi$.

The input required in RAQ is now complete. As the final step, RAQ introduces on the image of η a Hermitian inner product by

$$\big(\eta(\phi_1), \eta(\phi_2)\big)_{\rm phys} := \eta(\phi_2)[\phi_1] \quad , \tag{4.1}$$

and completes the image of η in this inner product into a Hilbert space $\mathcal{H}_{\rm phys}$, which is by definition the physical Hilbert space of the theory. $\mathcal{H}_{\rm phys}$ carries an antilinear representation of $\mathcal{A}_{\rm obs}$, and the adjoint map in this representation (with respect to the inner product on $\mathcal{H}_{\rm phys}$) is by construction that induced by the adjoint map on $\mathcal{H}_{\rm aux}$. The representation of $\mathcal{A}_{\rm obs}$ on $\mathcal{H}_{\rm phys}$ is known to be nontrivial provided certain technical conditions hold [14].

B. Group averaging

The group averaging proposal in RAQ addresses the last ingredient above, the choice of the rigging map. The proposal seeks the rigging map as a suitable interpretation of the formal expression

$$\eta(|\phi\rangle) := \int_G dg\, \langle\phi| U(g) \quad , \tag{4.2}$$

where we have invoked the Dirac notation for the vector $|\phi\rangle \in \Phi$ and for its Hilbert dual vector $\langle\phi|$. The measure dg is the Haar measure on G (which is both left and right invariant by the unimodularity of G).

Consider now the formula

$$(\phi_2, \phi_1)_{\rm ga} := \int_G dg\, \big(\phi_2, U(g)\phi_1\big)_{\rm aux} \quad , \tag{4.3}$$

and suppose that the integral on the right-hand side converges in absolute value for all ϕ_1 and ϕ_2 in Φ. Formula (4.3) defines then on Φ the sesquilinear form $(\,\cdot\,,\,\cdot\,)_{\rm ga}$, and we interpret the group averaging proposal (4.2) as

$$\eta(\phi_1)[\phi_2] := (\phi_1, \phi_2)_{\rm ga} \quad . \tag{4.4}$$

The resulting map η clearly satisfies postulates (i), (ii), and (iv): (i) follows from the invariance of the Haar measure, and (ii) from the fact that $dg = d(g^{-1})$. If η further satisfies (iii), and if η is not identically zero, the group averaging proposal has then produced a rigging map.

Considerable control over the space of possible rigging maps is provided by the uniqueness theorem of Giulini and Marolf [8]. To state the theorem, we note [8] that if h is an L^1 function on G, the expression $\hat{h} := \int_G dg\, h(g) U(g)$ defines a bounded operator on $\mathcal{H}_{\rm aux}$, and the set of all such operators forms an algebra $\hat{\mathcal{A}}_G$. Suppose now that Φ is invariant under $\hat{\mathcal{A}}_G$, the integral in (4.3) converges in absolute value for all ϕ_1 and ϕ_2 in Φ, and the sesquilinear form $(\,\cdot\,,\,\cdot\,)_{\rm ga}$ on Φ is not identically zero. Then, if a rigging map exists, it is unique up to an overall multiple, and given by (4.4) [8].

V. REFINED ALGEBRAIC QUANTIZATION OF THE $\mathrm{SL}(2,\mathbb{R})$ SYSTEM

In this section we apply the RAQ formalism of in section IV to our system. To maintain a contact to the Algebraic Quantization of section III, we shall proceed so that the RAQ observable algebra $\mathcal{A}_{\rm obs}$ will turn out to contain the Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$.

A. Auxiliary Hilbert space and the gauge group

We take the auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$ to be $L^2(\mathbb{R}^4)$ of wave functions $\Psi(\vec{u}, \vec{v})$ in the inner product

$$(\Psi_1, \Psi_2)_{\mathrm{aux}} := \int d^2\vec{u}\, d^2\vec{v}\, \overline{\Psi}_1 \Psi_2 \quad . \tag{5.1}$$

We take the constraint operators to be given by (3.2).

The constraints are essentially self-adjoint on $\mathcal{H}_{\mathrm{aux}}$, and exponentiating $-i$ times their algebra yields on $\mathcal{H}_{\mathrm{aux}}$ a unitary representation U of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The group elements that appear in the Iwasawa decomposition (A7) are represented by

$$U\big(\exp(\beta e^-)\big) = \exp(-i\mu \widehat{H}_2) \quad , \tag{5.2a}$$

$$U\big(\exp(\lambda h)\big) = \exp\big(-i\lambda \widehat{D}\big) \quad , \tag{5.2b}$$

$$U\big(\exp[\theta(e^+ - e^-)]\big) = \exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big) \quad . \tag{5.2c}$$

$\exp(-i\mu\widehat{H}_2)$ and $\exp\big(-i\lambda\widehat{D}\big)$ act on the wave functions $\Psi(\vec{u}, \vec{v})$ respectively as

$$[\exp(-i\mu\widehat{H}_2)\,\Psi](\vec{u}, \vec{v}) = \int \frac{d^2\vec{v}'}{2\pi i\mu} \exp\left\{ \frac{i}{2}\left[\frac{(\vec{v}-\vec{v}')^2}{\mu} + \mu\vec{u}^2 \right] \right\} \Psi(\vec{u}, \vec{v}') \quad (\text{for } \mu \neq 0) \quad , \tag{5.3a}$$

$$[\exp(-i\lambda\widehat{D})\,\Psi](\vec{u}, \vec{v}) = \Psi(e^{-\lambda}\vec{u}, e^{\lambda}\vec{v}) \quad . \tag{5.3b}$$

Regarding $\exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big)$, it suffices to observe that

$$\widehat{H}_1 - \widehat{H}_2 = \widehat{H}^{\mathrm{sho}}_{\vec{u}} - \widehat{H}^{\mathrm{sho}}_{\vec{v}} \quad , \tag{5.4}$$

where $\widehat{H}^{\mathrm{sho}}_{\vec{u}}$ and $\widehat{H}^{\mathrm{sho}}_{\vec{v}}$ are the two-dimensional harmonic oscillator Hamiltonians in respectively $\vec{u}$ and $\vec{v}$,

$$\widehat{H}^{\mathrm{sho}}_{\vec{u}} := \tfrac{1}{2}\left(-\vec{\nabla}^2_u + \vec{u}^2\right) \quad , \tag{5.5a}$$

$$\widehat{H}^{\mathrm{sho}}_{\vec{v}} := \tfrac{1}{2}\left(-\vec{\nabla}^2_v + \vec{v}^2\right) \quad . \tag{5.5b}$$

It follows that $\exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big)$ is periodic in θ with period 2π. As discussed in appendix A, this shows that U is a representation of $\mathrm{SL}(2,\mathbb{R})$ [and not just a representation of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$]. In the terminology of RAQ, the gauge group G is thus $\mathrm{SL}(2,\mathbb{R})$.

The Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ is represented on $\mathcal{H}_{\mathrm{aux}}$ by densely-defined operators, and the star-operation of $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ is the adjoint map of $\mathcal{H}_{\mathrm{aux}}$. $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ clearly commutes both with the constraint operators (3.2) and with U on the respective common domains. $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ exponentiates into an $\mathrm{O}(2,2)$ action on $\mathcal{H}_{\mathrm{aux}}$: representing the states as functions of $(\vec{u}, \vec{\pi})$ via the Fourier-transform in $\vec{v}$, $\mathrm{O}(2,2)$ acts on the arguments of the functions by (2.12a). It is clear that this $\mathrm{O}(2,2)$ action commutes with U.

U is isomorphic to the oscillator representation of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R}^{2,2})$, and our $\mathrm{O}(2,2)$ action on $\mathcal{H}_{\mathrm{aux}}$ is isomorphic to the $\mathrm{O}(2,2)$ action on $L^2(\mathbb{R}^{2,2})$ known in this context [15]. We give a brief review of the oscillator representation in appendix B.

B. Test states

Next, we seek a suitable linear space of test states in $\mathcal{H}_{\rm aux}$. The decomposition (5.4) suggests that we make use of the eigenstates of the harmonic oscillator Hamiltonians (5.5). It is convenient to choose the eigenstates so that they are also eigenstates of the angular momentum operators $\hat{u}^1\hat{p}^2 - \hat{u}^2\hat{p}^1 = -i\partial_\alpha$ and $\hat{v}^1\hat{\pi}^2 - \hat{v}^2\hat{\pi}^1 = -i\partial_\beta$. These eigenstates are

$$\phi_{m,m';n,n'} := e^{i(m\alpha+m'\beta)}\, u^{|m|}\, v^{|m'|}\, L_n^{|m|}(\vec{u}^2)\, L_{n'}^{|m'|}(\vec{v}^2)\, \exp\left[-\tfrac{1}{2}(\vec{u}^2+\vec{v}^2)\right] \quad , \tag{5.6}$$

where the indices are integers with $n \geq 0$ and $n' \geq 0$, and the L's are the generalized Laguerre polynomials [28,29]. $\phi_{m,m';n,n'}$ is an eigenstate of $\widehat{H}^{\rm sho}_{\vec{u}}$ and $\widehat{H}^{\rm sho}_{\vec{v}}$ with the respective eigenvalues $|m| + 2n$ and $|m'| + 2n'$, and it is an eigenstate of $-i\partial_\alpha$ and $-i\partial_\beta$ with the respective eigenvalues m and m'. The states $\phi_{m,m';n,n'}$ form a linearly independent and orthogonal set in $\mathcal{H}_{\rm aux}$, satisfying

$$\big(\phi_{m,m';n,n'}, \phi_{\tilde{m},\tilde{m}';\tilde{n},\tilde{n}'}\big)_{\rm aux} = \frac{\pi^2(n+|m|)!\,(n'+|m'|)!}{n!\,(n')!}\,\delta_{m,\tilde{m}}\,\delta_{m',\tilde{m}'}\,\delta_{n,\tilde{n}}\,\delta_{n',\tilde{n}'} \quad , \tag{5.7}$$

and their linear span $\tilde{\Phi}$ is dense in $\mathcal{H}_{\rm aux}$. $\tilde{\Phi}$ consists of vectors of the form $P(\vec{u},\vec{v})\exp\left[-\frac{1}{2}(\vec{u}^2+\vec{v}^2)\right]$, where $P(\vec{u},\vec{v})$ is an arbitrary polynomial in the four coordinates (u^1,u^2,v^1,v^2): from this characterization it is clear that $\tilde{\Phi}$ is mapped to itself by the quantum constraint operators (3.2). Similarly, recalling that the Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$ is generated by (3.15) and the hatted counterparts of (2.7), it is clear that $\tilde{\Phi}$ is mapped to itself by $\mathcal{A}^{(\star)}_{\rm phy+}$.

$\tilde{\Phi}$ itself is not suitable for our RAQ the test state space. First, there is a technical issue in that $\tilde{\Phi}$ is not mapped to itself by the G-action U, as is immediate for example from (5.3b). The serious problem with $\tilde{\Phi}$ is, however, that the group averaging integral (4.3) is not convergent, as we show in appendix C: convergence fails when both angular momentum quantum numbers vanish. We now show how to modify $\tilde{\Phi}$ so that the group averaging integral becomes convergent, and we then use the group algebra technique of [8] to generate a test state space that is invariant under U and large enough for the uniqueness theorem of [8] to apply.

Let Φ_0 be the linear span of the set

$$B_0 := \left\{ \phi_{m,m';n,n'} \,\middle|\, |m| + |m'| > 0 \right\} \cup \left\{ (\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}) \right\} \quad . \tag{5.8}$$

What motivates this definition is that Φ_0 is mapped to itself by the Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\rm phy+}$. To see this, recall from above that $\tilde{\Phi}$ is mapped to itself by $\mathcal{A}^{(\star)}_{\rm phy+}$. It is therefore sufficient to consider the situation in which an element of $\mathcal{A}^{(\star)}_{\rm phy}$ acts on a vector in B_0 and produces a vector whose expansion in the basis $\{\phi_{m,m';n,n'}\}$ has components with $m = m' = 0$. From (3.15), (3.7), and the angle dependence in $\phi_{m,m';n,n'}$ (5.6), we see that the only nontrivial instance of how this can happen is the action of $\widehat{\tau}^\eta_\pm$ on $\phi_{\mp1,\mp\eta;n,n'}$, which reads by explicit computation [30]

$$\widehat{\tau}^\eta_\pm \phi_{\mp1,\mp\eta;n,n'} = (n+1)(n'+1)\,(\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}) \quad , \tag{5.9}$$

and this is in the linear span of B_0. Thus Φ_0 is mapped to itself by $\mathcal{A}^{(\star)}_{\rm phy+}$.

We claim that Φ_0 is dense in $\mathcal{H}_{\rm aux}$. To show this, recall from above that $\{\phi_{m,m';n,n'}\}$ is an orthogonal Hilbert space basis for $\mathcal{H}_{\rm aux}$. It is therefore sufficient to show that the linear subspace $W \subset \Phi$ spanned by $\{(\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1})\}$ is dense in the Hilbert subspace $\mathcal{H}_0 \subset \mathcal{H}_{\rm aux}$ spanned by $\{\phi_{0,0;n,n'}\}$. Suppose this is false. Then there exists a nonzero vector $v \in \mathcal{H}_0$ that is in the orthogonal complement of the closure of W. As $v \in \mathcal{H}_0$, we can write $v = \sum_{n,n'} b_{n,n'}\phi_{0,0;n,n'}$, where the coefficients satisfy $\sum_{n,n'} |b_{n,n'}|^2 < \infty$ by (5.7), and at least one coefficient is nonzero. However, the orthogonality of v with each $(\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}) \in W$ implies $b_{n,n'} = -b_{n+1,n'+1}$ for all n and n', and the sum $\sum_{n,n'} |b_{n,n'}|^2$ therefore diverges, which is a contradiction. Thus Φ_0 is dense in $\mathcal{H}_{\rm aux}$.

The crucial property of Φ_0 is that the group averaging integral (4.3) converges in absolute value for all ϕ_1 and ϕ_2 in Φ_0. This is shown in appendix C.

As Φ_0 is not mapped to itself by U, Φ_0 does not technically qualify as a test state space in our version of RAQ. A simple remedy would be to consider the space Φ_0', which is the closure of Φ_0 under the algebra generated by the operators $U(g)$ for $g \in G$. Φ_0' is clearly dense in $\mathcal{H}_{\rm aux}$ and invariant under U, and it thus satisfies the RAQ test state space conditions, and one could indeed successfully complete RAQ with Φ_0' as the test state space. However, we wish to work with a test state space to which the uniqueness theorem of Giulini and Marolf [8] applies. To this end, recall from section IV that an L^1 function h on G defines on $\mathcal{H}_{\rm aux}$ the bounded operator $\hat{h} := \int_G dg\, h(g) U(g)$, and the set of all such operators forms an algebra $\hat{\mathcal{A}}_G$. Let now Φ be the closure of Φ_0' under the action of $\hat{\mathcal{A}}_G$. It is clear that Φ is dense in $\mathcal{H}_{\rm aux}$ and invariant under U, and Φ thus satisfies the RAQ test state space conditions. It is also clear that Φ is mapped to itself by $\hat{\mathcal{A}}_G$, while Φ_0' is not.

We now adopt Φ as the RAQ test state space. As Φ_0 is mapped to itself by $\mathcal{A}^{(\star)}_{\rm phy+}$, so is Φ, and the RAQ observable algebra $\mathcal{A}_{\rm obs}$ therefore contains $\mathcal{A}^{(\star)}_{\rm phy+}$ as a subalgebra.

As a final remark, we note that Φ_0 is mapped to itself by the quantum constraint operators (3.2) [30], and therefore Φ_0' and Φ are also mapped to themselves by these operators. Φ_0, Φ_0', and Φ would therefore all qualify as test state spaces in formulations of RAQ that solve the constraints in terms of the constraint operators rather than in terms of the G-action U [4,14].

C. Group averaging and the physical Hilbert space

Consider now the group averaging. As mentioned above, the integral in (4.3) converges in absolute value for all ϕ_1 and ϕ_2 in Φ_0. It follows from Lemma 2 in [8] that the integral in (4.3) converges in absolute value for all ϕ_1 and ϕ_2 in Φ. The map η is therefore well defined by (4.3) and (4.4), and it satisfies the rigging map postulates with the possible exception of positivity.

To evaluate η, let $\phi_i \in \Phi$, and let h_i be L^1 functions on G. We then have from (4.3) and (4.4) [8]

$$\eta(\hat{h}_1\phi_1)[\phi_2] = \overline{\left(\int_G dg\, h_1(g)\right)}\, \eta(\phi_1)[\phi_2] \quad , \tag{5.10a}$$

$$\eta(\phi_1)[\hat{h}_2\phi_2] = \left(\int_G dg\, h_2(g) \right) \eta(\phi_1)[\phi_2] \quad . \tag{5.10b}$$

As further $\eta(\phi_1)[U(g_0)\phi_2] = \eta(U(g_0)\phi_1)[\phi_2] = \eta(\phi_1)[\phi_2]$, it suffices to evaluate $\eta(\phi_1)[\phi_2]$ for ϕ_1 and ϕ_2 in the set B_0 (5.8).

The explicit evaluation of η is done in appendix D. We can represent the vectors in the image of η as functions on $\mathbb{R}^4 = \{(\vec{u}, \vec{v})\}$, acting on the test states $\phi \in \Phi$ by

$$f[\phi] = \int d^2\vec{u}\, d^2\vec{v}\, f(\vec{u}, \vec{v})\phi(\vec{u}, \vec{v}) \quad . \tag{5.11}$$

We find

$$\eta(\phi_{m,m';n,n'}) = 2\pi^2(-1)^n[\mathrm{sgn}(m)]^m \delta_{|m|,|m'|}\, \delta_{n,n'}\, \frac{(n+|m|)!}{|m|\, n!}\, f_{m,(m'/m)} \quad , \quad |m| + |m'| > 0 \quad , \tag{5.12a}$$

$$\eta\left(\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}\right) = 0 \quad , \tag{5.12b}$$

where the functions $f_{m,\epsilon}$, with $m \in \mathbb{Z} \setminus \{0\}$ and $\epsilon \in \{1, -1\}$, are defined by

$$f_{m,\epsilon} := J_m(uv)\, e^{-im(\alpha+\epsilon\beta)} \quad . \tag{5.13}$$

The action (5.11) of $f_{m,\epsilon}$ on the vectors in B_0 reads [31]

$$f_{m,\epsilon}[\phi_{\tilde{m},\tilde{m}';n,n'}] = 2\pi^2(-1)^n[\mathrm{sgn}(m)]^m \delta_{m,\tilde{m}}\, \delta_{\epsilon m,\tilde{m}'}\, \delta_{n,n'}\, \frac{(n+|m|)!}{n!} \quad , \quad |\tilde{m}| + |\tilde{m}'| > 0 \quad , \tag{5.14a}$$

$$f_{m,\epsilon}[\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}] = 0 \quad . \tag{5.14b}$$

From this it is clear that the set $\{f_{m,\epsilon} \mid m \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1\}$ is linearly independent in Φ^* and a basis for the image of η.

What remains is to evaluate the (prospective) inner product on the image of η. From (4.1), (5.12a), and (5.14a), we find

$$(f_{m,\epsilon}, f_{m',\epsilon'})_{\mathrm{phys}} = |m|\, \delta_{m,m'} \delta_{\epsilon,\epsilon'} \quad . \tag{5.15}$$

As (5.15) is positive definite, all the rigging map postulates are satisfied, and (5.15) does define an inner product on the image of η. The physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ is obtained by completion. The asymptotic large order expansion of J_m [22] shows that every vector in $\mathcal{H}_{\mathrm{phys}}$ can be represented as a function on $\mathbb{R}^4 = \{(\vec{u}, \vec{v})\}$.

Finally, as Φ is invariant under $\hat{\mathcal{A}}_G$, the assumptions of the uniqueness theorem of Giulini and Marolf are satisfied. It follows that every rigging map for our triple $(\mathcal{H}_{\mathrm{aux}}, U, \Phi)$ is a multiple of the group averaging rigging map η.

D. Observables and the relation to Algebraic Quantization

As we have emphasized, the RAQ observable algebra $\mathcal{A}_{\mathrm{obs}}$ contains the Algebraic Quantization observable algebra $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ as a subalgebra, and the star-operation on $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ is the adjoint map of $\mathcal{H}_{\text{aux}}$. It follows that the antilinear representation of $\mathcal{A}_{\text{obs}}$ on $\mathcal{H}_{\text{phys}}$ contains an antilinear representation ρ_+ of $\mathcal{A}^{(\star)}_{\text{phy}+}$, and in ρ_+ the star-operation on $\mathcal{A}^{(\star)}_{\text{phy}+}$ is the adjoint map of $\mathcal{H}_{\text{phys}}$. ρ_+ acts on the basis $\{f_{m,\epsilon} \mid m \in \mathbb{Z} \setminus \{0\}, \epsilon = \pm 1\}$ of $\mathcal{H}_{\text{phys}}$ as

$$\rho_+(\widehat{\tau}^\eta_0) : f_{m,\epsilon} \mapsto \delta^{\eta,\epsilon} m f_{m,\epsilon} \quad , \tag{5.16a}$$

$$\rho_+(\widehat{\tau}^\eta_\pm) : f_{m,\epsilon} \mapsto \delta^{\eta,\epsilon} m f_{m\pm1,\epsilon} \quad , \tag{5.16b}$$

$$\rho_+(\widehat{P}_{\epsilon_1,\epsilon_2}) : f_{m,\epsilon} \mapsto f_{\epsilon_1 m,\epsilon_2\epsilon} \quad , \tag{5.16c}$$

where $f_{0,\epsilon}$, whenever it appears on the right-hand side, is understood to mean zero.

Comparing (5.16) to (3.9) and (3.16), and the RAQ inner product (5.15) to the Algebraic Quantization inner product (3.13), we see that ρ_+ is anti-isomorphic to the representation of $\mathcal{A}^{(\star)}_{\text{phy}+}$ on the Hilbert space $\mathcal{H}_+$ obtained in the Algebraic Quantization of section III, provided the inner products are normalized to agree. The $\mathrm{O}(2,2)$ action on $\mathcal{H}_+$ found in section III is anti-isomorphic to the $\mathrm{O}(2,2)$-action on $\mathcal{H}_{\text{phys}}$ induced by the $\mathrm{O}(2,2)$ action on $\mathcal{H}_{\text{aux}}$. In this sense, the RAQ quantum theory contains the Algebraic Quantization quantum theory.

VI. DISCUSSION

In this paper we have compared the Algebraic Quantization (AQ) framework and the Refined Algebraic Quantization (RAQ) framework in a constrained Hamiltonian system with unreduced phase space $\Gamma = T^*\mathbb{R}^4$ and gauge group $\mathrm{SL}(2,\mathbb{R})$. In both approaches we used input motivated by the structure of the classical constraints as quadratic functions on Γ. In AQ, we first solved the constraints on a suitable vector space, promoted an explicitly-known classical observable algebra into the quantum operator star-algebra $\mathcal{A}^{(\star)}_{\text{phy}+}$, and determined the inner product by requiring the star-operation on $\mathcal{A}^{(\star)}_{\text{phy}+}$ to coincide with the adjoint operation. In RAQ, we chose the auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ to be L^2 over the unreduced configuration space $\mathbb{R}^4$, and we promoted the classical $\mathrm{SL}(2,\mathbb{R})$ gauge transformations on Γ into a unitary $\mathrm{SL}(2,\mathbb{R})$-action on $\mathcal{H}_{\text{aux}}$. We took particular care to choose the RAQ test state space $\Phi \subset \mathcal{H}_{\text{aux}}$ so that the RAQ observable algebra $\mathcal{A}_{\text{obs}}$ contains $\mathcal{A}^{(\star)}_{\text{phy}+}$. Considering the similarity in these inputs, it is not surprising that the RAQ quantum theory turned out to contain the AQ quantum theory. We also investigated the $\mathrm{O}(2,2)$ group actions underlying the classical and quantum observable algebras, and we showed that the AQ quantum theory can be recovered through Isham's group theoretic quantization framework.

Both AQ and RAQ encountered with the zero angular momentum states a technical difficulty whose origin is in the structure of a certain pathological subset of the classical reduced phase space. The remedy was to ensure that such states do not appear in the physical Hilbert space. In AQ, the problem appeared in the guise of "zero norm" states in the prospective Hilbert space, and the cure was simply to drop the states already from the vector space on which the constraints are solved. In RAQ, on the other hand, the problem appeared as the divergence of the group averaging, and the cure now was to modify the space of test states. However, as the RAQ observable algebra is defined in terms of the test state space, the modification needed to be quite subtle in order that the RAQ observable algebra could still be meaningfully compared with the AQ observable algebra: here we took advantage of the explicit knowledge of the operators in $\mathcal{A}^{(\star)}_{\text{phy}+}$. This illustrates well how neither AQ nor RAQ is a *prescription* for quantization: they are schemes that need input at various steps, and making successful choices in the 'early' steps may require hindsight from the 'later' steps [12,14,32]. Also, this illustrates that although RAQ does not assume a single observable to be explicitly constructed, the knowledge of some observables of interest can be quite useful in making good choices at the various steps of RAQ.

As discussed in [10], the constraint algebra of our system is analogous to the constraint algebra of general relativity. Among the three constraints (2.2), H_1 and H_2 are "Hamiltonian"-type, quadratic in the momenta, while D is "momentum"-type, linear in the momenta, and the mixing of these two types of constraints in (2.3) is as in general relativity [33]. One consequence of this analogy is that one could introduce and investigate in our system also group averaging with Teitelboim's "causal" boundary condition [34–36]. In general relativity, this condition proposes that only positive lapses contribute to the path integral that defines the quantum mechanical propagation amplitude. If H_1 and H_2 are adopted as the analogue of the Hamiltonian constraint of general relativity at two spatial points [10], the causal boundary condition in our system yields an average over the semigroup of $\mathrm{SL}(2,\mathbb{R})$ matrices whose all entries are positive: integrating first over the lapses and then over the shift, the $\mathrm{SL}(2,\mathbb{R})$ elements emerge from the amplitude folding of [34,35] in the form

$$\exp(\nu h)\exp(\nu_+ e^+ + \nu_- e^-) \quad , \tag{6.1}$$

where $-\infty < \nu < \infty$ and $\nu_\pm > 0$, and we have explicitly verified that the measure emerging from the ghost integrations of [34,35] is the $\mathrm{SL}(2,\mathbb{R})$ Haar measure in the parametrization (6.1). It might be interesting to see whether a scattering theory of the type considered in [34–36] could be built on the causal boundary condition in our system.

We note in this context that allowing ν and $\nu_\pm$ to take all real values in (6.1) does not cover all of $\mathrm{SL}(2,\mathbb{R})$, and in particular it does not reach those matrices where the product of the diagonal elements is negative. In our system, the decomposition of the quantum propagation amplitude in the form given in [34,35], first integrating over the lapses and then over the shift, does thus not yield an average over the whole group when the lapses and the shift are allowed to take all real values. This phenomenon occurs also upon considering (6.1) in the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\!\!1\} \simeq \mathrm{O_c}(2,1)$. The phenomenon is therefore not a consequence of the fact that the exponential map from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$ is not onto, as the exponential map from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathrm{PSL}(2,\mathbb{R})$ is.

We saw in section II that the $\mathrm{O}(2,2)$ action on Γ looks simple in the polarization in which $(\vec{u},\vec{\pi})$ are the 'coordinates' and $(\vec{p},-\vec{v})$ are the 'momenta'. Similarly, we noted in section V that the $\mathrm{O}(2,2)$ action on $\mathcal{H}_{\text{aux}} = L^2\big(\{(\vec{u},\vec{v})\}\big)$ looks simple when Fourier-transformed to $L^2\big(\{\vec{u},\vec{\pi})\}\big)$. Attempting to quantize the system in a $(\vec{u},\vec{\pi})$-representation would however present difficulties. Adopting the $(\vec{u},\vec{\pi})$-representation in Algebraic Quantization and proceeding as in section III, one finds that the constraints cannot be solved in terms of smooth functions: the constraint $\widehat{H}_2\Psi = 0$ implies that the support of $\Psi(\vec{u},\vec{\pi})$ would need to be in some sense at $\vec{\pi}^2 - \vec{u}^2 = 0$. The reason underlying this difficulty is precisely that our solutions to the constraints in the $(\vec{u},\vec{v})$-representation are not square integrable, or even integrable, and Fourier-transforming them to a $(\vec{u},\vec{\pi})$-representation is a priori not defined. In RAQ, in contrast, the Fourier-transform to the $(\vec{u},\vec{\pi})$-representation is well-defined in $\mathcal{H}_{\text{aux}}$, there is no obstacle to constructing in this representation Φ, the G-action, or the group averaging sesquilinear form (4.3), and proving the absolute convergence of the integral in (4.3) is in fact technically simpler than in the $(\vec{u},\vec{v})$-representation. At the abstract level, one thus recovers isomorphic RAQ quantum theories in the $(\vec{u},\vec{v})$-representation and the $(\vec{u},\vec{\pi})$-representation. The difficulty of doing RAQ in the $(\vec{u},\vec{\pi})$-representation is a more practical one, namely, that the methods of appendix D now do not yield a representation of the image of η as functions on $\mathbb{R}^4 = \{(\vec{u},\vec{\pi})\}$, and one needs some other way to prove that η is positive and to evaluate η in some practical fashion.

The classical system admits a generalization in which $\vec{u}$ and $\vec{v}$ in the action functional (2.1)–(2.2) have respectively r and s components, for any nonnegative integers r and s. The phase space is $\Gamma_{r,s} := T^*\mathbb{R}^{r,s}$, the gauge group generated by the constraints is still $\mathrm{SL}(2,\mathbb{R})$, and $\Gamma_{r,s}$ has a natural $\mathrm{O}(r,s)$-action that commutes with the $\mathrm{SL}(2,\mathbb{R})$-action. One expects that this generalized system could be quantized with our methods, and that the quantum theory would reflect properties of the oscillator representation on $L^2(\mathbb{R}^{r,s})$ [15,37,38]. It is also possible to generalize the system to certain other gauge groups of interest by minor modification of the constraint structure in (2.1)–(2.2), such as to the $(1+1)$ Poincare group, or to the affine group on $\mathbb{R}$ (which is nonunimodular). We leave such generalizations subject to future work.

Note added: After this work was completed, a quantization of the system in the Algebraic Constraint Quantization framework of [39,40] was posted in [41]. As noted in [41], the quantum theory recovered therein is in essence identical to our Algebraic Quantization quantum theory.

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APPENDIX A: IWASAWA DECOMPOSITION OF $\mathrm{SL}(2,\mathbb{R})$

In this appendix we collect some well-known properties of $\mathrm{SL}(2,\mathbb{R})$. The notation follows [15].

$\mathrm{SL}(2,\mathbb{R})$ consists of real 2×2 matrices with unit determinant,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad ad - bc = 1 \quad . \tag{A1}$$

Each element of $\mathrm{SL}(2,\mathbb{R})$ admits a unique Iwasawa decomposition,

$$g = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad , \tag{A2}$$

where $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and $0 \leq \theta < 2\pi$. In terms of the parametrization (A2), the left and right invariant Haar measure reads $e^{2\lambda}\, d\lambda\, d\mu\, d\theta$.

A standard basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of the three matrices

$$\begin{aligned} h &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \\ e^{+} &:= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \\ e^{-} &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \end{aligned} \tag{A3}$$

whose commutators are

$$\begin{aligned} \left[h\,,\, e^{+}\right] &= 2e^{+} \quad , \\ \left[h\,,\, e^{-}\right] &= -2e^{-} \quad , \\ \left[e^{+}\,,\, e^{-}\right] &= h \quad . \end{aligned} \tag{A4}$$

A second standard basis is

$$\begin{aligned} \gamma_0 &:= \tfrac{1}{2}(e^{+} - e^{-}) \quad , \\ \gamma_1 &:= \tfrac{1}{2}(e^{+} + e^{-}) \quad , \\ \gamma_2 &:= \tfrac{1}{2}h \quad , \end{aligned} \tag{A5}$$

with the commutators

$$\begin{aligned} [\gamma_1\,,\, \gamma_2] &= -\gamma_0 \quad , \\ [\gamma_2\,,\, \gamma_0] &= \gamma_1 \quad , \\ [\gamma_0\,,\, \gamma_1] &= \gamma_2 \quad . \end{aligned} \tag{A6}$$

Each of the three matrices in (A2) is in the image of the exponential map from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{R})$. In terms of the exponential map, (A2) reads

$$g = \exp(\mu e^{-}) \exp(\lambda h) \exp[\theta(e^{+} - e^{-})] \quad . \tag{A7}$$

The decomposition (A2) encodes the first homotopy group $\mathbb{Z}$ of $\mathrm{SL}(2,\mathbb{R})$ entirely in the rightmost factor. The quotient map from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\!\!1\} \simeq \mathrm{O_c}(2,1)$ [the connected component of $\mathrm{O}(2,1)$] acts in the decomposition (A2) by the identification $(\mu,\lambda,\theta) \sim (\mu,\lambda,\theta+\pi)$. A unique Iwasawa decomposition of the form (A7) holds therefore also for covering groups of $\mathrm{O_c}(2,1)$: for the n-fold covering $0 \leq \theta < n\pi$, and for the universal covering $-\infty < \theta < \infty$.

APPENDIX B: OSCILLATOR REPRESENTATION OF THE DOUBLE COVER OF $\mathrm{SL}(2,\mathbb{R})$

In this appendix we recall some properties of the oscillator representation of the double cover of $\mathrm{SL}(2,\mathbb{R})$ [15]. We denote in this appendix the double cover of $\mathrm{SL}(2,\mathbb{R})$ by $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

1. Oscillator representation on $L^2(\mathbb{R})$

Consider on $L^2(\mathbb{R})$ the three essentially self-adjoint operators

$$\widehat{H}_1 := -\tfrac{1}{2}\partial_x^2 \quad , \tag{B1a}$$

$$\widehat{H}_2 := -\tfrac{1}{2}x^2 \quad , \tag{B1b}$$

$$\widehat{D} := -i\left(x\partial_x + \tfrac{1}{2}\right) \quad , \tag{B1c}$$

whose commutators form the $\mathfrak{sl}(2,\mathbb{R})$ algebra (3.3). Exponentiation yields a unitary representation ω of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R})$. The group elements that appear in the Iwasawa decomposition (A7) are represented by

$$\omega\big(\exp(\mu e^-)\big) = \exp(-i\mu\widehat{H}_2) \quad , \tag{B2a}$$

$$\omega\big(\exp(\lambda h)\big) = \exp\big(-i\lambda\widehat{D}\big) \quad , \tag{B2b}$$

$$\omega\big(\exp[\theta(e^+ - e^-)]\big) = \exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big) \quad . \tag{B2c}$$

The two first operators in (B2) act on functions $\psi(x)$ as

$$[\exp(-i\mu\widehat{H}_2)\,\psi](x) = e^{i\mu x^2/2}\,\psi(x) \quad , \tag{B3a}$$

$$[\exp(-i\lambda\widehat{D})\,\psi](x) = e^{-\lambda/2}\,\psi(e^{-\lambda}x) \quad , \tag{B3b}$$

while $\exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big)$ is the unit mass and frequency harmonic oscillator evolution operator. As $\exp\big(-i\theta(\widehat{H}_1 - \widehat{H}_2)\big)$ is periodic in θ with period 4π, ω is a representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ but not a representation of $\mathrm{SL}(2,\mathbb{R})$.

It is evident that ω decomposes into a sum of two unitary representations, one acting on even and the other on odd functions. It can be shown that these two representations are irreducible [15].

The oscillator representation can be formally written as

$$[\omega(g)\,\psi](x) = \int \frac{dy}{\sqrt{2\pi i b}} \exp\left[\frac{i\,(ay^2 + dx^2 - 2xy)}{2b}\right]\psi(y) \quad , \tag{B4}$$

where a, b and d are as shown in (A1) in the $\mathrm{SL}(2,\mathbb{R})$ representative of g [while g itself is in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$]. The singularities and branch cuts in the integral kernel in (B4) must however be interpreted consistently with the unambiguously-defined left-hand side. For example, when $g = \exp[\theta(e^+ - e^-)]$, $\omega(g)$ is the harmonic oscillator evolution operator, for which $a = d = \cos\theta$ and $b = \sin\theta$, and the integral kernel in (B4) is singular at $\theta = \pi n$.

The integral kernel in (B4) can be derived from the $\mathrm{SL}(2,\mathbb{R})$ action that the classical counterparts of the operators (B1) generate on $T^*\mathbb{R}$. Writing $g \in \mathrm{SL}(2,\mathbb{R})$ as in (A1), and denoting by (q,p) the usual canonical chart on $T^*\mathbb{R}$, this action reads

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \quad . \tag{B5}$$

(B5) preserves the symplectic structure $dp \wedge dq$ and is therefore a canonical transformation. For $b \neq 0$, one can express the old and new momenta as functions of the old and new coordinates, and the canonical transformation has then a generating function $S(q, q')$, satisfying

$$p'(q, q')\, dq' - p(q, q')\, dq = dS(q, q') \quad . \tag{B6}$$

Simple algebra yields

$$S(q, q') = \frac{aq^2 + dq'^2 - 2qq'}{2b} \quad . \tag{B7}$$

As $S(q, q')$ (B7) is quadratic in q and q', the integral kernel of the corresponding unitary transformation consists of the exponential $\exp[iS(q, q')]$ and a prefactor that does not depend on q or q'. Imposing unitarity yields the prefactor shown in (B4).

2. Oscillator representation on $L^2(\mathbb{R}^{r,s})$

Inverting the signs of both e^+ and e^- in the basis (A3) of $\mathfrak{sl}(2, \mathbb{R})$ is an automorphism of $\mathfrak{sl}(2, \mathbb{R})$. Inverting the signs of $\widehat{H}_1$ and $\widehat{H}_2$ in (B1) and proceeding as above yields therefore a representation ω^* of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on $L^2(\mathbb{R})$. The tensor product $\omega_{r,s}$ of r copies of ω and s copies of ω^* is naturally realized as a representation of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on $L^2(\mathbb{R}^{r,s})$, each ω acting on one of the first r coordinates and each ω^* acting on one of the last s coordinates. $\omega_{r,s}$ is a representation of $\mathrm{SL}(2, \mathbb{R})$ iff $r + s$ is even.

The group $\mathrm{O}(r, s)$ has a natural action on $L^2(\mathbb{R}^{r,s})$ by $\psi(x) \mapsto \psi(a^{-1}x)$, where a is in the defining matrix representation of $\mathrm{O}(r, s)$. This $\mathrm{O}(r, s)$ action commutes with $\omega_{r,s}$, and the spectral decomposition of one completely determines the spectral decomposition of the other [15,37].

The representation U of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}_{\mathrm{aux}}$ introduced in section V, and the representation of $\mathrm{O}(2, 2)$ on $\mathcal{H}_{\mathrm{aux}}$ generated by the observable algebra $\mathcal{A}^{(\star)}_{\mathrm{phy}+}$ therein, are isomorphic to the above structure with $r = s = 2$. The isomorphism is the Fourier transform in the last two coordinates in $\mathcal{H}_{\mathrm{aux}} \simeq L^2(\mathbb{R}^{2,2})$.

APPENDIX C: CONVERGENCE OF THE GROUP AVERAGING

In this appendix we show that the integral in (4.3),

$$\int_G dg\, \big(\phi_2, U(g)\phi_1\big)_{\mathrm{aux}} \quad , \tag{C1}$$

converges in absolute value for all ϕ_1 and ϕ_2 in the space Φ_0 defined in section V.

It suffices to consider ϕ_1 and ϕ_2 in the set B_0 (5.8). As the operators $-i\partial_\alpha$ and $-i\partial_\beta$ (which belong to $\mathcal{A}_{\mathrm{obs}}$) commute with $U(g)$, it suffices to consider the case where ϕ_1 and ϕ_2 have the same angular momentum quantum numbers, for otherwise the integrand in (C1) vanishes by the orthogonality (5.7).

We now consider separately the case where at least one angular momentum is nonzero and the case where both angular momenta are zero.

1. At least one angular momentum nonzero

We set $\phi_1 = \phi_{m,m';n,n'}$ and $\phi_2 = \phi_{m,m';\tilde{n},\tilde{n}'}$, where $|m| + |m'| > 0$.

We write g in the Iwasawa decomposition (A2). By (5.2), $U(g)$ is given by

$$U(g) = \exp(-i\mu\widehat{H}_2)\exp\bigl(-i\lambda\widehat{D}\bigr)\exp\bigl(-i\theta(\widehat{H}_1 - \widehat{H}_2)\bigr) \ . \tag{C2}$$

The Haar measure in (C1) reads $dg = e^{2\lambda}\, d\lambda\, d\mu\, d\theta$, and the integration is over all real values of λ and μ and over one 2π cycle in θ. As ϕ_1 is an eigenstate of the rightmost operator in (C2) with an eigenvalue of absolute value 1, it suffices to set $\theta = 0$ and consider the integral over λ and μ in the measure $e^{2\lambda}\, d\lambda\, d\mu$.

Let thus $U(g)$ be as in (C2) with $\theta = 0$ and $\mu \neq 0$. By (5.3), we have

$$\begin{aligned}\bigl(\phi_2, U(g)\phi_1\bigr)_{\text{aux}} &= \frac{4\pi^2[\operatorname{sgn}(m')]^{m'} z^{(k'-k)/2}}{i^{(m'+1)}\mu} \\ &\times \int du\, dv\, dv'\ u^{2k+1} v^{k'+1} (v')^{k'+1} J_{k'}(vv'/\mu) L_{\tilde{n}}^{k}(u^2) L_{\tilde{n}'}^{k'}(v^2) L_n^k(u^2/z) L_{n'}^{k'}\bigl(z(v')^2\bigr) \\ &\quad \times \exp\left\{-\tfrac{1}{2}\left[1 + (1/z) - i\mu\right]u^2 - \tfrac{1}{2}\left[1 - (i/\mu)\right]v^2 - \tfrac{1}{2}\left[z - (i/\mu)\right](v')^2\right\} \ ,\end{aligned} \tag{C3}$$

where $k := |m|$, $k' := |m'|$, $z := e^{2\lambda}$, and the integration is over positive values of u, v, and v'. The Bessel function $J_{k'}(vv'/\mu)$ has emerged from performing the angular part of the $d^2\vec{v}'$ integral in (5.3a). Here, and from now on, the individual components of $\vec{u}$ and $\vec{v}$ will not appear, and we always write $u = \sqrt{\vec{u}^2}$, $u^2 := \vec{u}^2$, and so on.

In (C3), we write out the generalized Laguerre polynomials as polynomials in their respective arguments. $L_n^k(u^2/z)$ yields a sum of numerical coefficients times $u^{2r}z^{-r}$, $L_{n'}^{k'}\bigl(z(v')^2\bigr)$ yields $(v')^{2r'} z^{r'}$, $L_{\tilde{n}}^k(u^2)$ yields u^{2s}, and $L_{\tilde{n}'}^{k'}(v^2)$ yields $v^{2s'}$, where r, r', s, and s' range over integers satisfying $0 \le r \le n$, $0 \le r' \le n'$, $0 \le s \le \tilde{n}$, and $0 \le s' \le \tilde{n}'$. (C3) equals therefore a sum over r, r', s, and s' of numerical coefficients times

$$\begin{aligned}&\frac{z^{r'-r+(k'-k)/2}}{\mu}\int du\, dv\, dv'\ u^{2r+2s+2k+1} v^{2s'+k'+1} (v')^{2r'+k'+1} J_{k'}(vv'/\mu) \\ &\qquad \times \exp\left\{-\tfrac{1}{2}\left[1 + (1/z) - i\mu\right]u^2 - \tfrac{1}{2}\left[1 - (i/\mu)\right]v^2 - \tfrac{1}{2}\left[z - (i/\mu)\right](v')^2\right\} \ .\end{aligned} \tag{C4}$$

In (C4), we perform first the elementary integral over u. We then perform the integral over v using (6.631.10) in [21]: the result involves the generalized Laguerre polynomial $L_{s'}^{k'}$ of argument $(v')^2/[2\mu(\mu - i)]$, and we expand this polynomial as a sum of numerical coefficients times $\bigl\{(v')^2/[\mu(\mu - i)]\bigr\}^{s''}$, where s'' ranges over integers satisfying $0 \le s'' \le s'$. The remaining integral over v' is elementary. Note that these integrals over u, v, and v' converge in absolute value. Collecting, we find that (C3) is a sum over r, r', s, s', and s'' of numerical coefficients times

$$\mu^{s'-s''}(1 + i\mu)^{r'-s'} z^{1+r'+s+(k'+k)/2}(1 + z + i\mu z)^{-r'-s''-k'-1}(1 + z - i\mu z)^{-r-s-k-1} \ , \tag{C5}$$

where $0 \le s'' \le s'$. An elementary analysis shows that (C5) is integrable in absolute value over $\{(z,\mu) \mid z > 0, \mu \in \mathbb{R}\}$ in the measure $\int dz\, d\mu$ provided $r + (k + k')/2 > 0$. As

$|m| + |m'| > 0$ by assumption, this condition is satisfied. Thus (C1) converges in absolute value.

We note that the assumption $|m| + |m'| > 0$ was only used in the final step, in showing the integrability of (C5). We also note that this assumption is necessary. Taking $\phi_1 = \phi_{0,0;0,0}$ and $\phi_2 = \phi_{0,0;n,n}$, (C2) and (C3) yield, using [31] and (6.631.4) in [21],

$$\big(\phi_2, U(g)\phi_1\big)_{\text{aux}} = 4\pi^2(-1)^n \times \frac{z[(1-z)^2 + \mu^2 z^2]^n}{[(1+z)^2 + \mu^2 z^2]^{n+1}} \ , \tag{C6}$$

and the integral of (C6) over the group in the Haar measure is unambiguously divergent.

2. Both angular momenta zero

We set $\phi_1 = \phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}$ and $\phi_2 = \phi_{0,0;\tilde{n},\tilde{n}'} + \phi_{0,0;\tilde{n}+1,\tilde{n}'+1}$. As above, it suffices to take $\theta = 0$ and consider the integral over λ and μ in the measure $e^{2\lambda}\, d\lambda\, d\mu$.

Let again $U(g)$ be as in (C2) with $\theta = 0$ and $\mu \neq 0$. By (5.3), we have

$$\begin{aligned}
\big(\phi_2, U(g)\phi_1\big)_{\text{aux}} = \frac{4\pi^2}{i\mu} \int & du\, dv\, dv'\; uvv' J_0(vv'/\mu) \left[L_{\tilde{n}}(u^2) L_{\tilde{n}'}(v^2) + L_{\tilde{n}+1}(u^2) L_{\tilde{n}'+1}(v^2)\right] \\
& \times \left[L_n(u^2/z) L_{n'}\big(z(v')^2\big) + L_{n+1}(u^2/z) L_{n'+1}\big(z(v')^2\big)\right] \\
& \times \exp\left\{-\tfrac{1}{2}\left[1 + (1/z) - i\mu\right] u^2 - \tfrac{1}{2}\left[1 - (i/\mu)\right] v^2 - \tfrac{1}{2}\left[z - (i/\mu)\right] (v')^2\right\} \ .
\end{aligned} \tag{C7}$$

In (C7), we write out $L_n(u^2/z)$ and $L_{n+1}(u^2/z)$ as polynomials in their arguments. The previous analysis [the integrability of (C5) for $k = k' = 0$ provided $r > 0$] shows that the nonconstant terms give integrable contributions. Consider therefore the expression where $L_n(u^2/z)$ and $L_{n+1}(u^2/z)$ in (C7) are each replaced by their constant term 1. We perform the integrals over u and v using (7.414.6) and (7.421.1) in [21], obtaining a sum of numerical constants times

$$\begin{aligned}
\frac{z(1 - z - i\mu z)^p (1 - i\mu)^{p'}}{(1 + z - i\mu z)^{p+1} (1 + i\mu)^{p'+1}} \int dv'\, v' L_{p'}\left(\frac{(v')^2}{1+\mu^2}\right) \exp\left\{-\frac{1}{2}\left[z + \frac{1}{(1+i\mu)}\right](v')^2\right\} \\
\times \left[L_{n'}\big(z(v')^2\big) + L_{n'+1}\big(z(v')^2\big)\right] \ ,
\end{aligned} \tag{C8}$$

where $(p, p') = (\tilde{n}, \tilde{n}')$ or $(p, p') = (\tilde{n}+1, \tilde{n}'+1)$. In (C8), we write out $L_{p'}$ as a sum of numerical coefficients times $(v')^{2s}(1+\mu^2)^{-s}$, where s ranges over integers satisfying $0 \leq s \leq p'$. We then perform the remaining integral by changing the integration variable from v' to $x := z(v')^2$ and using the formula

$$\int_0^\infty dx\, x^s \left[L_{n'}(x) + L_{n'+1}(x)\right] \exp\left[-\tfrac{1}{2}\left(1 + a^{-1}\right)x\right] = \frac{a^{s+1} P_{n',s}(a)}{(1+a)^{n'+s+2}} \ , \tag{C9}$$

where $P_{n',s}$ is a polynomial (whose precise numerical coefficients will not be needed) of order $n' + s$. The validity of (C9) for $s = 0$ follows from (7.414.7) in [21], and the validity for $s > 0$ follows by repeated differentiation with respect to a^{-1}. It then follows by elementary analysis that (C8) is integrable in absolute value over $\{(z, \mu) \mid z > 0, \mu \in \mathbb{R}\}$ in the measure $\int dz\, d\mu$. Thus (C1) converges in absolute value.

APPENDIX D: EVALUATION OF THE RIGGING MAP

In this appendix we evaluate the map η, given by (4.3) and (4.4), on the test function space Φ defined in section V.

It suffices to consider test states ϕ in the set B_0 (5.8). We consider separately the case where both angular momenta are nonzero and the case where at least one angular momentum is zero.

1. Both angular momenta nonzero

Suppose $m \neq 0 \neq m'$, and consider $U(g)\phi_{m,m';n,n'}$ as a function on $G \times \mathbb{R}^4$, where $G = \mathrm{SL}(2,\mathbb{R})$ is the gauge group and $\mathbb{R}^4 = \{(\vec{u},\vec{v})\}$ is the configuration space. By the methods of appendix C it is straightforward to show that $U(g)\phi_{m,m';n,n'}$ is integrable in absolute value over G pointwise in $(\vec{u},\vec{v})$, and that $\overline{\phi}U(g)\phi_{m,m';n,n'}$ is integrable in absolute value over $G \times \mathbb{R}^4$ for every $\phi \in \Phi_0$. It follows by Fubini's theorem that $\eta(\phi_{m,m';n,n'})$ can be represented by a function on $\mathbb{R}^4$, acting on test states $\phi \in \Phi$ by (5.11): we have

$$\eta(\phi_{m,m';n,n'}) = \overline{\chi_{m,m';n,n'}} \quad , \tag{D1}$$

where

$$\chi_{m,m';n,n'} := \int_G dg\, U(g)\phi_{m,m';n,n'} \quad , \tag{D2}$$

and the integral in (D2) is evaluated pointwise on $\mathbb{R}^4$. We shall now evaluate (D2).

We write $U(g)$ in the Iwasawa decomposition (C2) and write $z := e^{2\lambda}$. For $\mu \neq 0$, we obtain

$$\begin{aligned} U(g)\phi_{m,m';n,n'} = {} & \frac{[\mathrm{sgn}(m')]^{m'}\, e^{i(m\alpha+m'\beta)} z^{(k'-k)/2} e^{i\theta(k'-k+2n'-2n)}}{(i)^{m'+1}\mu} \\ & \times \int_0^\infty dv'\, u^k (v')^{k'+1} J_{k'}(vv'/\mu) L_n^k(u^2/z) L_{n'}^{k'}\big(z(v')^2\big) \\ & \qquad \times \exp\left[-\frac{1}{2}\left(\frac{u^2}{z} + z(v')^2\right) + \frac{i}{2}\left(\mu u^2 + \frac{v^2+(v')^2}{\mu}\right)\right] \quad , \end{aligned} \tag{D3}$$

where $k := |m|$ and $k' := |m'|$, and by assumption $k \geq 1$ and $k' \geq 1$. As in appendix C, the Bessel function $J_{k'}(vv'/\mu)$ has emerged from performing the angular part of the $d^2\vec{v}'$ integral in (5.3a). The integral in (D3) could be performed in terms of a generalized Laguerre polynomial using (7.421.4) in [21], but for us it will be more convenient to proceed directly with (D3).

We now integrate (D3) in the Haar measure $dg = e^{2\lambda}\, d\lambda\, d\mu\, d\theta = \frac{1}{2} dz\, d\mu\, d\theta$. By the above discussion, this integral converges in absolute value. We may assume $u > 0$ and $v > 0$. The integral over θ yields the factor $2\pi\delta_{k+2n,k'+2n'}$. In the remaining expression we first change the variable in the integral in (D3) from v' to $x := z(v')^2$, and we then change the variables in the outer integral $\int dz\, d\mu$ to $y := u^2/z$ and $p := u^2\mu$. We obtain

$$\chi_{m,m';n,n'} = \frac{\pi[\mathrm{sgn}(m')]^{m'}\delta_{k+2n,k'+2n'}\, e^{i(m\alpha+m'\beta)}}{2(i)^{m'+1}} \int_0^\infty dy\, y^{(k/2)-1} L_n^k(y) e^{-y/2}$$
$$\times \int_{-\infty}^{\infty} \frac{dp}{p} \int_0^\infty dx\, x^{k'/2} J_{k'}\left(\frac{uv\sqrt{xy}}{p}\right) L_{n'}^{k'}(x)\, \exp\left[-\frac{x}{2} + \frac{i}{2}\left(p + \frac{u^2v^2+xy}{p}\right)\right] \quad . \quad \text{(D4)}$$

We then interchange the order of the $\int dx$ and $\int dp$ integrals in (D4), justified by the absolute convergence of the double integral $\int dx\, dp$. Performing the $\int dp$ integral by (the absolutely convergent analytic continuation of) (6.635.3) in [21], we obtain

$$\begin{aligned}\chi_{m,m';n,n'} &= \pi^2 \delta_{k+2n,k'+2n'} e^{i(m\alpha+m'\beta)} J_{k'}(uv) \\ &\quad\times \int_0^\infty dy\, y^{(k/2)-1} L_n^k(y) e^{-y/2} \int_0^\infty dx\, x^{k'/2} J_{k'}\left(\sqrt{xy}\right) L_{n'}^{k'}(x) e^{-x/2} \\ &= 2\pi^2 (-1)^{n'} \delta_{k+2n,k'+2n'} e^{i(m\alpha+m'\beta)} J_{k'}(uv) \int_0^\infty dy\, y^{(k+k')/2-1} L_n^k(y) L_{n'}^{k'}(y) e^{-y} \quad , \quad \text{(D5)}\end{aligned}$$

where in the last step we have evaluated the $\int dx$ integral using [31].

Consider the remaining integral in (D5). Suppose $k' \geq k$. Because of the factor $\delta_{k+2n,k'+2n'}$, it suffices to consider $k' = k + 2s$ and $n = n' + s$ for some nonnegative integer s. We thus need to evaluate

$$\int_0^\infty dy\, y^{k+s-1} L_{n'+s}^k(y) L_{n'}^{k+2s}(y) e^{-y} \quad . \quad \text{(D6)}$$

Expanding $L_{n'}^{k+2s}(y)$ in (D6) as a polynomial in y yields integrals of the form

$$\int_0^\infty dy\, y^{k+q} L_{n'+s}^k(y) e^{-y} \quad , \quad \text{(D7)}$$

where $s - 1 \leq q \leq s + n' - 1$. The orthogonality of the generalized Laguerre polynomials [30] implies that (D7) vanishes for $0 \leq q < n' + s$. When $s > 0$, q is always in this range, and (D6) thus vanishes. When $s = 0$, the only value of q not in this range is $q = -1$, which comes from the constant term of the expanded $L_{n'}^k(y)$ in (D6): using [42], (7.414.7) in [21], and (15.1.40) in [22], we then find that (D6) for $s = 0$ is equal to $(n'+k)!/[k\,(n')!]$. Finally, the case $k' < k$ reduces to the case already considered by interchange of the primed and unprimed indices, and we find that (D5) vanishes.

Expressing the result in terms of the original indices, we have

$$\chi_{m,m';n,n'} = 2\pi^2(-1)^n [\mathrm{sgn}(m)]^m \delta_{|m|,|m'|}\, \delta_{n,n'}\, \frac{(n+|m|)!}{|m|\, n!}\, J_m(uv)\, e^{i(m\alpha+m'\beta)} \quad . \quad \text{(D8)}$$

The result (5.12a) then follows from (D1) and (5.13).

2. At least one angular momentum zero

What remains is to evaluate the map η for $\phi_{0,m';n,n'}$ with $m' \neq 0$, $\phi_{m,0;n,n'}$ with $m \neq 0$, and $\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}$. We shall show that η vanishes on these states.

A direct analysis along the above lines would run into a technical difficulty in that not all the analogous multiple integrals now converge in absolute value. It is however suggestive to note that the integrals are still conditionally convergent, and starting from the counterpart of (D2) and formally interchanging the integrations as above yields the result zero. For $\phi_{0,m';n,n'}$ and $\phi_{m,0;n,n'}$, (D2) yields the zero function and hence the zero vector in Φ^*. For $\phi_{0,0;n,n'}+\phi_{0,0;n+1,n'+1}$, the counterpart of (D2) yields a function proportional to $J_0(uv)$, which clearly solves the quantum constraints, but the dual action (5.11) of $J_0(uv)$ on every vector in Φ_0 vanishes [by the extension of (5.14a) to $m=0$], and as an element of Φ^* $J_0(uv)$ is thus identical to the zero vector. We now show that the result zero is indeed the correct one.

Consider first

$$\eta(\phi_{0,m';p,p'})[\phi] = (\phi_{0,m';p,p'}, \phi)_{\mathrm{ga}} \quad , \tag{D9}$$

where $m' \neq 0$ and $\phi \in B_0$ (5.8). As noted in appendix C, it suffices to consider $\phi = \phi_{0,m';n,n'}$. Using the Iwasawa decomposition (C2) in (4.3), the integral over θ shows that we can set $|m'| = 2(n-n')$, and a similar reasoning with $U(g)$ in (4.3) conjugated to act on the first argument shows that we can set $|m'| = 2(p-p')$. It therefore suffices to consider $(\phi_{0,\pm 2s;p+s,p}, \phi_{0,\pm 2s;n+s,n})_{\mathrm{ga}}$ with $s \geq 1$.

Consider thus $(\phi_{0,2s;p+s,p}, \phi_{0,2s;n+s,n})_{\mathrm{ga}}$ with $s \geq 1$. We recall that the operators $\widehat{\tau}^{\eta}_{\pm}$ (3.7) are in $\mathcal{A}_{\mathrm{obs}}$ and the adjoint of $\widehat{\tau}^{\eta}_{\pm}$ in $\mathcal{H}_{\mathrm{aux}}$ is $\widehat{\tau}^{\eta}_{\mp}$. Using properties of the generalized Laguerre polynomials [30] we find

$$\widehat{\tau}^{-}_{-}\phi_{1,2s-1;n+s-1,n} = -(n+s)\left(\phi_{0,2s;n+s-1,n-1} + \phi_{0,2s;n+s,n}\right) \quad , \tag{D10a}$$

$$\widehat{\tau}^{-}_{+}\phi_{0,2s;p+s,p} = -(p+1)\phi_{1,2s-1;p+s,p+1} + (p+2s)\phi_{1,2s-1;p+s-1,p} \quad , \tag{D10b}$$

where $\phi_{0,2s;n+s-1,n-1}$ for $n=0$ is understood as the zero vector. We therefore have

$$\begin{aligned}
&(n+s)\left[(\phi_{0,2s;p+s,p}, \phi_{0,2s;n+s-1,n-1})_{\mathrm{ga}} + (\phi_{0,2s;p+s,p}, \phi_{0,2s;n+s,n})_{\mathrm{ga}}\right] \\
&\quad = -(\phi_{0,2s;p+s,p}, \widehat{\tau}^{-}_{-}\phi_{1,2s-1;n+s-1,n})_{\mathrm{ga}} \\
&\quad = -(\widehat{\tau}^{-}_{+}\phi_{0,2s;p+s,p}, \phi_{1,2s-1;n+s-1,n})_{\mathrm{ga}} \\
&\quad = (p+1)(\phi_{1,2s-1;p+s,p+1}, \phi_{1,2s-1;n+s-1,n})_{\mathrm{ga}} + (p+2s)(\phi_{1,2s-1;p+s-1,p}, \phi_{1,2s-1;n+s-1,n})_{\mathrm{ga}} \\
&\quad = 0 \quad ,
\end{aligned} \tag{D11}$$

where the last equality follows from (5.12a) in the index range where (5.12a) has already been verified. By induction in n, (D11) implies $(\phi_{0,2s;p+s,p}, \phi_{0,2s;n+s,n})_{\mathrm{ga}} = 0$. An analogous argument shows $(\phi_{0,-2s;p+s,p}, \phi_{0,-2s;n+s,n})_{\mathrm{ga}} = 0$.

Thus $\eta(\phi_{0,m';p,p'}) = 0$ for $m' \neq 0$. A similar argument shows that $\eta(\phi_{m,0;p,p'}) = 0$ for $m \neq 0$. Finally, $\eta(\phi_{0,0;p,p'} + \phi_{0,0;p+1,p'+1}) = 0$ follows by applying an analogous reasoning to the relations [30]

$$\widehat{\tau}^{+}_{-}\phi_{1,1;n,n'} = (n+1)(n'+1)\left(\phi_{0,0;n,n'} + \phi_{0,0;n+1,n'+1}\right) \quad , \tag{D12a}$$

$$\widehat{\tau}^{+}_{+}\left(\phi_{0,0;p,p'} + \phi_{0,0;p+1,p'+1}\right) = \phi_{1,1;p-1,p'-1} + 2\phi_{1,1;p,p'} + \phi_{1,1;p+1,p'+1} \quad . \tag{D12b}$$

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