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An exceptional geometry for $d = 11$ supergravity?

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Abstract

We analyze the algebraic constraints of the generalized vielbein in $SO(1,2) \times SO(16)$ invariant $d = 11$ supergravity, and show that the bosonic degrees of freedom of $d = 11$ supergravity, which become the physical ones upon reduction to $d = 3$, can be assembled into an $E_{8(8)}$-valued vielbein already in eleven dimensions. A crucial role in the construction is played by the maximal nilpotent commuting subalgebra of $E_{8(8)}$, of dimension 36, suggesting a partial unification of general coordinate and tensor gauge transformations.

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1 Introduction.

One of the most remarkable properties of maximal ($d = 11$) supergravity [1] is the emergence of hidden symmetries of exceptional type in the reduction to lower dimensions [2]. Some time ago, it was shown [3, 4] that, already in eleven dimensions, this theory permits reformulations where the tangent space symmetry $SO(1, 10)$ is replaced by the local symmetries that would arise in the reduction to four and three dimensions, i.e. $SO(1, 3) \times SU(8)$ and $SO(1, 2) \times SO(16)$, respectively. The key point here is, of course, that this construction works without any assumptions restricting the dependence of theory on the coordinates, so these symmetries already exist in eleven dimensions. Besides its general interest, this result plays a pivotal role in establishing the consistency of the Kaluza Klein reduction of $d = 11$ supergravity in various non-trivial backgrounds [5, 6]. In this article, we will concentrate on the $SO(1, 2) \times SO(16)$ version of [4], which exhibits special features (especially "maximal unification" of symmetries), and which in the reduction to three dimensions yields directly maximal $N = 16$ supergravity[1].

The equivalence of the different versions of $d = 11$ supergravity is established at the level of the equations of motion by making special gauge choices, and does not extend off shell because the new versions mix equations of motion and Bianchi identities of the original theory. As shown in [3, 4], the bosonic fields, that would become scalar matter fields in the dimensional reduction, can be assigned to representations of the hidden global symmetry groups $E_{7(7)}$ and $E_{8(8)}$, respectively. For this purpose it was necessary to fuse the bosonic fields into new objects, christened "generalized vielbeine": these are soldering forms with upper world indices running over the internal dimensions, and lower indices belonging to the **56** and **248** representations of $E_{7(7)}$ and $E_{8(8)}$, respectively. The generalized vielbeine are subject to algebraic constraints which follow from their explicit expressions in terms of the vielbein of $d = 11$ supergravity. For the $SO(1, 2) \times SO(16)$ version of [4], there exists a differential constraint in addition, which is a $d = 11$ variant of the duality constraint by which vector fields are converted into scalars in three dimensions. However, it was not clear from previous work how to solve these constraints (apart from the torus reduction, explicit solutions are only known for the S^7 compactification of $d = 11$ supergravity [6]), and how to recover the correct counting of bosonic physical degrees of freedom.

An essential new element of the present work in comparison with previous results is our treatment of the 3-form potential A_{MNP}. Whereas in [3, 4] this field appeared only via its field strength $F_{MNPQ} = 24\partial_{[M}A_{NPQ]}$ inside the $E_{7(7)}$ and $E_{8(8)}$ connections given there, part of it here is merged

[1]The $E_{8(8)}$ invariance of maximal $N = 16$ supergravity in three dimensions was originally shown in [7], while its complete Lagrangian and supersymmetry transformations were derived in [8]. The latter follow directly by dimensional reduction of the variations presented in [4]. See also [9, 10] for a more recent treatment.

into an enlarged generalized vielbein. As a consequence, the latter now also transforms under tensor gauge transformations, suggesting a partial unification of coordinate and tensor gauge transformations. We show that the generalized vielbein is actually part of a full $E_{8(8)}$ matrix $\mathcal{V}$ (a "248-bein"), which now lives in eleven dimensions, and which incorporates the bosonic degrees of freedom of $d = 11$ supergravity, with the exception of the dreibein remaining from the 3+8 split (which does not propagate in the dimensionally reduced theory). Moreover, we present evidence for the existence of an "exceptional geometry" for $d = 11$ supergravity by displaying the action of the combined internal coordinate and tensor gauge transformations on this 248-bein. In deriving these results, we will make crucial use of certain special properties of the exceptional Lie algebra $E_{8(8)}$, in particular the existence of a maximal nilpotent abelian subalgebra of dimension 36, which is unique up to conjugation [11], and whose importance was recently emphasized in [10].

The present paper deals mainly with the algebraic relations obeyed by the generalized vielbein, and their solution. The corresponding differential relations will be discussed elsewhere. We believe that our results constitute further evidence for a hidden $E_{8(8)}$ structure of $d = 11$ supergravity, but there remain a number of open problems that must be dealt with; these include in particular the proper treatment of the tensor components $B_{\mu\nu m}$ and $B_{\mu\nu\rho}$, and the construction of an invariant action in terms of the 248-bein $\mathcal{V}$ in eleven dimensions.

2 $SO(1,2)\times SO(16)$ invariant $d = 11$ supergravity

We will first review $SO(1,2) \times SO(16)$ formulation of $d = 11$ supergravity, referring readers to [4] for further details. Our conventions concerning $E_{8(8)}$ as well as its $SO(16)$ and $SL(8,\mathbb{R})$ decompositions, which played an important role also in [10], are summarized in two appendices.

$SO(1,2) \times SO(16)$ invariant $d = 11$ supergravity [4] is derived from the original version of [1] by first splitting up the fields in a way that would be appropriate for the reduction to three dimensions, but without dropping the dependence on any coordinates, and then reassembling the pieces into new objects transforming under local $SO(1,2) \times SO(16)$. Hence we are still dealing with $d = 11$ supergravity, albeit in a very different guise. This is achieved by first breaking the original tangent space symmetry $SO(1,10)$ down to $SO(1,2) \times SO(8)$ by a partial gauge choice for the elfbein, and then reenlarging it to $SO(1,2) \times SO(16)$ by the introduction of new gauge degrees of freedom. The construction thus requires a 3+8 split of the $d = 11$ coordinates and tensor indices. The main task then is to identify the proper $SO(1,2) \times SO(16)$ covariant fields and to verify that all supersymmetry variations as well as the equations of motion can be entirely expressed in terms of the new fields.

In a first step one thus brings the elfbein into triangular form by (partial) use of local $SO(1,10)$ Lorentz invariance

$$E_M{}^A = \begin{pmatrix} \Delta^{-1} e_\mu{}^{\underline{\alpha}} & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \qquad \Delta := \det e_m{}^a, \tag{2.1}$$

Here curved $d = 11$ indices decompose as $M = (\mu, m), N = (\nu, n), \ldots$ with $\mu, \nu, \ldots = 0, 1, 2$ and $m, n, \ldots = 3, \ldots, 10$, and the associated flat indices are denoted by $\underline{\alpha}, \underline{\beta}, \ldots$ and $a, b, \ldots$, respectively[2] (as in [4] the $SO(1,2)$ indices $\underline{\alpha}, \underline{\beta}, \ldots$ are underlined to distinguish them from the $SO(8)$ spinor indices to be used below). The partially gauge fixed elfbein, whose form is preserved by the $SO(1,2) \times SO(8)$ subgroup of $SO(1,10)$, thus contains the Weyl rescaled dreibein $e_\mu{}^\alpha$, the Kaluza-Klein vector $B_\mu{}^m$ and the achtbein $e_m{}^a$ yielding the scalar degrees of freedom living in the coset $GL(8,\mathbb{R})/SO(8)$.

The remaining bosonic degrees of freedom reside in the 3-index field A_{MNP}, which gives rise to various scalar and tensor fields upon performing a 3+8 split of the indices. First of all, there are 56 scalars A_{mnp} and 28 vector fields

$$B_{\mu mn} := A_{\mu mn} - B_\mu{}^p A_{mnp} \tag{2.2}$$

If one were to reduce to three dimensions, the 8+28 vector fields $B_\mu{}^m$ and $B_{\mu mn}$ would be converted to 36 scalar degrees of freedom by means of a duality transformation. Here, they will be kept together with their dual scalars contained in the generalized vielbein, to which they are related by a nonlinear analog of the (linear) duality constraint of the reduced $d = 3$ supergravity.

In addition, A_{MNP} gives rise to (always in the 3+8 split)

$$\begin{aligned} B_{\mu\nu p} &:= A_{\mu\nu p} - 2B_{[\nu}{}^n A_{\mu]np} + B_\mu{}^m B_\nu{}^n A_{mnp} \\ B_{\mu\nu\rho} &:= A_{\mu\nu\rho} - 3B_{[\mu}{}^m A_{\nu\rho]m} + 3B_{[\mu}{}^m B_\nu{}^n A_{\rho]mn} \\ &\quad\; - B_\mu{}^m B_\nu{}^n B_\rho{}^p A_{mnp} \end{aligned} \tag{2.3}$$

These fields are subject to the tensor gauge transformations

$$\delta A_{MNP} = 3\partial_{[M} \xi_{NP]} \tag{2.4}$$

Under these, we have

$$\begin{aligned} \delta B_{\mu mn} &= \mathcal{D}_\mu \xi_{mn} + 2\partial_{[m} B_\mu{}^p \xi_{n]p} + 2\partial_{[m} \tilde{\xi}_{n]\mu} \\ \delta B_{\mu\nu m} &= \partial_m \tilde{\xi}_{\mu\nu} + 2\partial_m B_{[\mu}{}^n \tilde{\xi}_{n\nu]} + 2\mathcal{D}_{[\mu} \tilde{\xi}_{\nu]m} + \mathcal{B}_{\mu\nu}{}^n \xi_{nm} \\ \delta B_{\mu\nu\rho} &= 3\mathcal{D}_{[\mu} \tilde{\xi}_{\nu\rho]} - 3\mathcal{B}_{[\mu\nu}{}^m \tilde{\xi}_{\rho]m} \end{aligned} \tag{2.5}$$

[2]We hope that no confusion results from our double usage of $A, B, \ldots$ as both $SO(1,10)$ vector and $SO(16)$ spinor indices. It should always be clear from the context which is meant.

where

$$\mathcal{D}_\mu := \partial_\mu - B_\mu{}^m \partial_m \tag{2.6}$$

and

$$\mathcal{B}_{\mu\nu}{}^m := \mathcal{D}_\mu B_\nu{}^m - \mathcal{D}_\nu B_\mu{}^m \tag{2.7}$$

The parameters $\tilde{\xi}_{\mu\nu}$ and $\tilde{\xi}_{\mu m}$ are defined by

$$\begin{aligned} \tilde{\xi}_{\mu\nu} &:= \xi_{\mu\nu} + 2B_{[\mu}{}^m \xi_{\nu]m} + B_\mu{}^m B_\nu{}^n \xi_{mn} \\ \tilde{\xi}_{\mu m} &:= \xi_{\mu m} - B_\mu{}^n \xi_{nm} \end{aligned} \tag{2.8}$$

It is easy to see that in the dimensionally reduced theory, one can make use of the parameter components $\tilde{\xi}_{\mu m}$ and $\tilde{\xi}_{\mu\nu}$ to set $B_{\mu\nu m} = B_{\mu\nu\rho} = 0$. Since we have not been able so far to cast the supersymmetry variations of these components into a completely $SO(16)$ covariant form, this gauge would also be very convenient in the present setting. However, there appears to be an obstacle to this gauge choice if the full $d = 11$ coordinate dependence is retained.

To identify the proper $SO(16)$ covariant bosonic fields, we must first explain how to rewrite the fermion fields. The $d = 11$ gravitino $\Psi_A \equiv (\Psi_{\underline{\alpha}}, \Psi_a)$ has 32 spinor components, which split as $\mathbf{2} \otimes (\mathbf{8}_s \oplus \mathbf{8}_c)$ under the $SO(1,2) \times SO(8)$ subgroup of $SO(1,10)$. Suppressing $SO(1,2)$ spinor indices, we then assign the resulting fields to the $SO(1,2) \times SO(16)$ fields ψ_μ^I and $\chi^{\dot{A}}$ via the following prescription [4]

$$\psi_\mu^I := \begin{cases} \Delta^{-1/2} e_\mu{}^{\underline{\alpha}} \left(\Psi_{\underline{\alpha}\alpha} + \gamma_{\underline{\alpha}} \Gamma^a_{\alpha\dot{\beta}} \Psi_{a\dot{\beta}} \right) & \text{if } I = \alpha \\ \Delta^{-1/2} e_\mu{}^{\underline{\alpha}} \left(\Psi_{\underline{\alpha}\dot{\alpha}} - \gamma_{\underline{\alpha}} \Gamma^a_{\dot{\alpha}\beta} \Psi_{a\beta} \right) & \text{if } I = \dot{\alpha} \end{cases} \tag{2.9}$$

and

$$\chi^{\dot{A}} := \begin{cases} \Delta^{-1/2} \left(\Gamma^b \Gamma^a \right)_{\alpha\beta} \Psi_{b\beta} & \text{if } \dot{A} = (a\alpha) \\ -\Delta^{-1/2} \left(\Gamma^b \Gamma^a \right)_{\dot{\alpha}\dot{\beta}} \Psi_{b\dot{\beta}} & \text{if } \dot{A} = (\dot{\alpha}a) \end{cases} \tag{2.10}$$

where I and $\dot{A}$ are $SO(16)$ vector and (conjugate) spinor indices, respectively (see appendix B, and in particular (B.1) for the relevant $SO(8)$ decompositions).

The physical bosonic degrees of freedom, which correspond to the propagating 128 propagating scalar degrees of freedom of maximal $d = 3$ supergravity, are fused into an appropriate generalized vielbein. The relevant expressions are found by proceeding from the following $SO(16)$ invariant ansatz for the supersymmetry variations of the vector fields in terms of the fermions (2.9) and (2.10) (in a suitable normalization):

$$\delta B_\mu{}^m = \tfrac{1}{2} e^m_{IJ}\, \bar{\epsilon}^I \psi_\mu^J + e^m_A\, \Gamma^I_{A\dot{B}} \bar{\epsilon}^I \gamma_\mu \chi^{\dot{B}} \tag{2.11}$$

$$\delta B_{\mu mn} = \tfrac{1}{2} e_{mnIJ}\, \bar{\epsilon}^I \psi_\mu^J + e_{mnA} \Gamma^I_{A\dot{B}}\, \bar{\epsilon}^I \gamma_\mu \chi^{\dot{B}} \tag{2.12}$$

The explicit expressions in a special $SO(16)$ gauge for the new bosonic quantities appearing on the r.h.s. of this equation can be found by comparing the above expressions with the ones obtained directly from $d=11$ supergravity in the gauge (2.1). It is already known that [4]

$$(e^m_{IJ},\, e^m_A) := \begin{cases} \Delta^{-1} e_a{}^m \Gamma^a_{\alpha\dot\beta} & \text{if } [IJ] \text{ or } A=(\alpha\dot\beta) \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

again using the $SO(8)$ decompositions of appendix B. By contrast, the objects $(e_{mnIJ}\,,\, e_{mnA})$ related to the gauge fields $B_{\mu mn}$ – hence with antisymmetrized *lower* internal world indices – have not yet appeared in previous work. Matching the r.h.s. of (2.12) with the variations obtained directly from the $d=11$ supersymmetry variations of [1] we find

$$e_{mn\mathcal{A}} := \overset{\circ}{e}_{mn\mathcal{A}} + A_{mnp} e^p_{\mathcal{A}} \tag{2.14}$$

where

$$\overset{\circ}{e}_{mnIJ} := \begin{cases} \Delta^{-1} e_m{}^a e_n{}^b (\Gamma_{ab})_{\alpha\beta} & \text{if } [IJ]=[\alpha\beta] \\ -\Delta^{-1} e_m{}^a e_n{}^b (\Gamma_{ab})_{\dot\alpha\dot\beta} & \text{if } [IJ]=[\dot\alpha\dot\beta] \\ 0 & \text{if } [IJ]=[\alpha\dot\beta] \equiv -[\dot\beta\alpha] \end{cases} \tag{2.15}$$

$$\overset{\circ}{e}_{mnA} := \begin{cases} 4\Delta^{-1} e_{\underline{ma}} e_{\underline{nb}} & \text{if } A=(ab) \\ 0 & \text{if } A=(\alpha\dot\beta) \end{cases} \tag{2.16}$$

Labeling the $E_{8(8)}$ indices $([IJ],A)$ collectively by $\mathcal{A},\mathcal{B},\ldots = 1,\ldots,248$ as in appendix A, the new objects $e^m_{\mathcal{A}}$ and $e_{mn\mathcal{A}}$ together form a rectangular $(8+28)\times 248$ matrix.

The new vielbeine are manifestly covariant w.r.t. local $SO(16)$, thereby enlarging the action of $SO(8)$ of the original theory. While the supersymmetry variation of $e^m_{\mathcal{A}}$ was already given in [4], the variation of $e_{mn\mathcal{A}}$ has so far not been determined. In appendix C we show that

$$\delta e_{mnIJ} = -\tfrac{1}{2}\Gamma^{IJ}_{AB}\omega^A e_{mnB} \qquad \delta e_{mnA} = \tfrac{1}{4}\Gamma^{IJ}_{AB}\omega^B e_{mnIJ} \tag{2.17}$$

with the local $N=16$ supersymmetry parameter

$$\omega^A := \tfrac{1}{4}\Gamma^I_{A\dot A}\bar\varepsilon^I \chi^{\dot A}, \tag{2.18}$$

In deriving this result, a compensating $SO(16)$ rotation must be taken into account to restore the triangular gauge. It is an important consistency check that this compensating rotation comes out to be the same for $e^m_{\mathcal{A}}$ and $e_{mn\mathcal{A}}$, and that the resulting transformation is exactly the same as for both fields, as required by consistency. We can therefore combine the supersymmetry variations of the generalized vielbeine with local $SO(16)$ into the $E_{8(8)}$ covariant form

$$\delta e^m_{\mathcal{A}} = f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}\omega^{\mathcal{B}} e^m_{\mathcal{C}} \qquad \delta e_{mn\mathcal{A}} = f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}\omega^{\mathcal{B}} e_{mn\mathcal{C}} \tag{2.19}$$

We also note that the components $e_{mn\mathcal{A}}$ were not needed in [4] because they cannot appear in the supersymmetry variations of the fermionic fields (the latter depend on the 3-form potential via the 4-index field strengths only).

All fields transform under general coordinate transformations in eleven dimensions. Splitting the $d = 11$ parameter as $\xi^M = (\xi^\mu, \xi^m)$, the transformations generated by ξ^μ take the standard form. For the remaining "internal" coordinate transformations with parameter ξ^m, we have

$$\begin{aligned} \delta B_\mu{}^m &= \mathcal{D}_\mu \xi^m + \xi^n \partial_n B_\mu{}^m \\ \delta B_{\mu mn} &= 2\partial_{[m} \xi^p B_{\mu n]p} + \xi^p \partial_p B_{\mu mn} \end{aligned} \tag{2.20}$$

Furthermore,

$$\begin{aligned} \delta e^m_{\mathcal{A}} &= \xi^p \partial_p e^m_{\mathcal{A}} - \partial_p \xi^m \, e^p_{\mathcal{A}} - \partial_p \xi^p \, e^m_{\mathcal{A}} \\ \delta e_{mn\mathcal{A}} &= \xi^p \partial_p e_{mn\mathcal{A}} + 2\partial_{[m} \xi^p \, e_{n]p\mathcal{A}} - \partial_p \xi^p \, e_{mn\mathcal{A}} \end{aligned} \tag{2.21}$$

Whereas all the $SO(16)$ fields considered previously were inert under tensor gauge transformations $\delta A_{MNP} = 3\partial_{[M} \xi_{NP]}$, the non-invariance of $e_{mn\mathcal{A}}$ under such transformations, due to the appearance of the 3-index field A_{mnp} in its definition, is a new feature. Specifically, under tensor gauge transformations with parameter ξ_{mn} we have

$$\begin{aligned} \delta B_\mu{}^m &= 0 \\ \delta B_{\mu mn} &= \mathcal{D}_\mu \xi_{mn} - 2B_\mu{}^p \partial_{[m} \xi_{n]p} \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \delta e^m_{\mathcal{A}} &= 0 \\ \delta e_{mn\mathcal{A}} &= \partial_p \xi_{mn} \, e^p_{\mathcal{A}} + 2\partial_{[m} \xi_{n]p} \, e^p_{\mathcal{A}} \end{aligned} \tag{2.23}$$

When combined with the previous coordinate transformations, these formulas are very suggestive of a unification of the internal coordinate transformations and tensor gauge transformations, based on combining the internal coordinate transformation parameters ξ^m with the residual tensor gauge parameters ξ_{mn} into a single set (ξ^m, ξ_{np}) of 36 parameters. In the remaining sections we will show how these local symmetries are related to the the maximal nilpotent commuting subalgebra of $E_{8(8)}$.

3 Solution of algebraic constraints on the generalized vielbein

From the expressions (2.13)-(2.16) one can deduce a number of algebraic constraints on the generalized vielbein. They are [12]

$$e^m_A e^n_A - \tfrac{1}{2} e^m_{IJ} e^n_{IJ} = 0 \tag{3.1}$$

$$\begin{aligned} \Gamma^{IJ}_{AB} (e^m_B e^n_{IJ} - e^n_B e^m_{IJ}) &= 0 \\ \Gamma^{IJ}_{AB} e^m_A e^n_B + 4 e^m_{K[I} e^n_{J]K} &= 0 \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
e^{(m}_{IK}e^{n)}_{JK} - \tfrac{1}{16}\delta_{IJ}e^m_{KL}e^n_{KL} &= 0 \\
e^{(m}_{[IJ}e^{n)}_{KL]} + \tfrac{1}{24}e^m_A\Gamma^{IJKL}_{AB}e^n_B &= 0 \\
\Gamma^K_{\dot{A}B}e^{(m}_B e^{n)}_{KL} - \tfrac{1}{14}\Gamma^{IKL}_{\dot{A}B}e^{(m}_B e^{n)}_{KL} &= 0
\end{aligned}
\tag{3.3}
$$

For the new components, we find

$$
e_{mnA}e_{pqA} - \tfrac{1}{2}e_{mnIJ}e_{pqIJ} = 0 \tag{3.4}
$$

$$
\begin{aligned}
\Gamma^{IJ}_{AB}(e_{mnB}e_{pqIJ} - e_{pqB}e_{mnIJ}) &= 0 \\
\Gamma^{IJ}_{AB}e_{mnA}e_{pqB} + 4e_{mnK[I}e_{pqJ]K} &= 0
\end{aligned}
\tag{3.5}
$$

and

$$
e_{mnA}e^p_A - \tfrac{1}{2}e_{mnIJ}e^p_{IJ} = 0 \tag{3.6}
$$

$$
\begin{aligned}
\Gamma^{IJ}_{AB}(e_{mnB}e^p_{IJ} - e^p_B e_{mnIJ}) &= 0 \\
\Gamma^{IJ}_{AB}e_{mnA}e^p_B + 4e_{mnK[I}e^p_{J]K} &= 0
\end{aligned}
\tag{3.7}
$$

whereas no analog of (3.3) exists for the components $e_{mn\mathcal{A}}$. All these relations are proved by decomposing the $SO(16)$ into $SO(8)$ and verifying the vanishing of their components case by case.

The above constraints can be elegantly rewritten in an $E_{8(8)}$ covariant form by means of the projectors onto the invariant subspaces of the tensor product of $E_{8(8)}$ representations $\mathbf{248}\otimes\mathbf{248}$ w.r.t. the decomposition $\mathbf{248}\otimes\mathbf{248} = \mathbf{1}\oplus\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{27000}\oplus\mathbf{30380}$. The relevant projectors are explicitly given in terms of $E_{8(8)}$ structure constants by [13]

$$
(\mathcal{P}_1)_{\mathcal{AB}}{}^{\mathcal{CD}} = \tfrac{1}{248}\,\eta_{\mathcal{AB}}\eta^{\mathcal{CD}}, \tag{3.8}
$$

$$
\begin{aligned}
(\mathcal{P}_{248})_{\mathcal{AB}}{}^{\mathcal{CD}} &= -\tfrac{1}{60}\,f^{\mathcal{E}}{}_{\mathcal{AB}}f_{\mathcal{E}}{}^{\mathcal{CD}}, \\
(\mathcal{P}_{3875})_{\mathcal{AB}}{}^{\mathcal{CD}} &= \tfrac{1}{7}\,\delta^{\mathcal{C}}_{(\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B})} - \tfrac{1}{56}\,\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} - \tfrac{1}{14}\,f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}}f_{\mathcal{EB}}{}^{\mathcal{D})},
\end{aligned}
\tag{3.9}
$$

It is straightforward to see that (3.1), (3.2) (3.4)-(3.7) are equivalent to

$$
\begin{aligned}
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e^n_{\mathcal{D}} &= 0 \\
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e_{pq\mathcal{D}} &= 0 \\
(\mathcal{P}_j)_{\mathcal{AB}}{}^{\mathcal{CD}}e_{mn\mathcal{C}}e_{pq\mathcal{D}} &= 0
\end{aligned}
\tag{3.10}
$$

for $j = 1$ and 248. It takes a little more work to verify that (3.3) can be expressed in the form[3]

$$
(\mathcal{P}_{3875})_{\mathcal{AB}}{}^{\mathcal{CD}}e^m_{\mathcal{C}}e^n_{\mathcal{D}} = 0 \tag{3.11}
$$

[3] A further relation given in [12]

$$
e^{(m}_A e^{n)}_B + \tfrac{1}{32}(\Gamma^{IJ}\Gamma^{KL})_{AB}e^m_{IJ}e^n_{KL} - \tfrac{1}{16}\Gamma^{IJ}_{AC}e^{(m}_C e^{n)}_D\Gamma^{KL}_{DB} = 0
$$

can be shown to be equivalent to (3.3) by means of a Fierz transformation and yields no new information. Therefore, there are no relations involving the projectors $\mathcal{P}_{27000}$ and $\mathcal{P}_{30380}$.

Observe the invariance of the constraints (3.10) and (3.11) under the combined general coordinate and tensor gauge transformations (2.21) and (2.23): the transformed generalized vielbein still obeys all constraints.

We next demonstrate that the $E_{8(8)}$ invariant algebraic relations on the generalized vielbein given above can be solved in terms of an $E_{8(8)}$ matrix $\mathcal{V}$. For the dimensionally reduced theory this is, of course, the expected result [7, 8]), but with the important difference that the dependence on all eleven coordinates is here retained. Thus, $E_{8(8)}$ is already present in eleven dimensions, though not a symmetry of the theory. The existence of $\mathcal{V}$ also clarifies why we end up with the right number of physical degrees of freedom, a fact that cannot be directly ascertained by counting the constraints: being subject to local $SO(16)$ transformations, the matrix $\mathcal{V}$ possesses just the $128 = 248 - 120$ degrees of freedom of the coset $E_{8(8)}/SO(16)$. Thus, the counting works exactly as for the reduced theory.

To corroborate this claim, consider the $E_{8(8)}$ Lie algebra valued matrices

$$\begin{aligned} \tilde{E}^m &:= e^m_{\mathcal{A}}\mathcal{X}^{\mathcal{A}} \equiv \tfrac{1}{2}e^m_{IJ}X^{IJ} + e^m_A Y^A \\ \tilde{E}_{mn} &:= e_{mn\,\mathcal{A}}\mathcal{X}^{\mathcal{A}} \equiv \tfrac{1}{2}e_{mn\,IJ}X^{IJ} + e_{mn\,A}Y^A \end{aligned} \tag{3.12}$$

From the relations presented in the foregoing section we infer that these matrices commute (cf. (3.2), (3.5), (3.7))

$$[\tilde{E}^m\,,\,\tilde{E}^n] = [\tilde{E}^m\,,\,\tilde{E}_{pq}] = [\tilde{E}_{mn}\,,\,\tilde{E}_{pq}] = 0 \tag{3.13}$$

and are nilpotent (cf. (3.1), (3.4), (3.6))

$$\mathrm{Tr}\Big(\tilde{E}^m\tilde{E}^n\Big) = \mathrm{Tr}\Big(\tilde{E}^m\tilde{E}_{pq}\Big) = \mathrm{Tr}\Big(\tilde{E}_{mn}\tilde{E}_{pq}\Big) = 0 \tag{3.14}$$

(ensuring that any linear combination of these matrices has norm zero). Since they are linearly independent (as is most easily checked by setting $e_m{}^a = \delta^a_m$ in the original definition), they form a 36 dimensional abelian nilpotent subalgebra of $E_{8(8)}$. There is only one such algebra, which is unique up to conjugation [11, 10]. Consequently, there must exist an $E_{8(8)}$ matrix $\mathcal{V}$ such that

$$\tilde{E}^m = \mathcal{V}^{-1}Z^m\mathcal{V} \qquad \tilde{E}_{mn} = \mathcal{V}^{-1}Z_{mn}\mathcal{V} \tag{3.15}$$

where Z^m and Z_{mn} are the 8+28 nilpotent generators of $E_{8(8)}$ introduced in the appendix. The assignment of the 8+28 vielbein components to these generators here is uniquely determined by $SL(8,\mathbb{R})$ covariance; its correctness will be confirmed below when we analyze (3.3). Thus,

$$\begin{aligned} e^m_{\mathcal{A}} &= \tfrac{1}{60}\mathrm{Tr}\,(Z^m\mathcal{V}\mathcal{X}_{\mathcal{A}}\mathcal{V}^{-1}) \\ e_{mn\,\mathcal{A}} &= \tfrac{1}{60}\mathrm{Tr}\,(Z_{mn}\mathcal{V}\mathcal{X}_{\mathcal{A}}\mathcal{V}^{-1}) \end{aligned} \tag{3.16}$$

By use of the relation

$$\tfrac{1}{60}\mathrm{Tr}\,\Big(\mathcal{X}^{\mathcal{M}}\mathcal{V}\mathcal{X}_{\mathcal{A}}\mathcal{V}^{-1}\Big) = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \tag{3.17}$$

for the adjoint representation we can write

$$e^m_{\mathcal{A}} = \mathcal{V}^m{}_{\mathcal{A}} \qquad e_{mn\,\mathcal{A}} = \mathcal{V}_{mn\,\mathcal{A}} \tag{3.18}$$

whence the generalized vielbein $(e^m_{\mathcal{A}}, e_{mn\mathcal{A}})$ is actually a rectangular submatrix of $\mathcal{V}$. Let us emphasize once more that $\mathcal{V}$ still depends on eleven coordinates.

At this point, a remark concerning our use of indices is in order. In the appendix, we use "flat" indices $a, b, c = 1, \ldots, 8$ to label the $E_{8(8)}$ generators in the $SL(8,\mathbb{R})$ decomposition. On the other hand, the index m appearing on the l.h.s. of (2.13) should be viewed as "curved" in the sense that it is acted on by internal coordinate transformations. Of course, there is no need for such a distinction in a flat background characterized by $e_m{}^a = \delta^a_m$ and $\mathcal{V} = \mathbf{1}$, whereas the two kinds of indices no longer coincide for curved backgrounds characterized by non-trivial $e_m{}^a$ and A_{mnp}. The nilpotent generators in (3.12) thus represent "curved" analogs of the "flat" generators Z^a and Z_{ab}, and the above relations tell us is that the transition from flat to curved configurations is entirely accounted for by the $E_{8(8)}$ matrix $\mathcal{V}$. This illustrates the enlargement of the geometry in comparison with the conventional description $d = 11$ supergravity, where the achtbein can only be deformed with a $GL(8,\mathbb{R})$ matrix.

To confirm the consistency of the above solution let us analyze the third set of constraints (3.3), which we have not yet discussed. Inspection reveals that the desired relation is equivalent to

$$\mathcal{P}_{3875}(\tilde{E}^m \otimes \tilde{E}^n) = 0 \tag{3.19}$$

Making use of the $E_{8(8)}$ invariance of $\mathcal{P}_{3875}$, we can replace curved by flat indices in this relation. This yields

$$(\mathcal{P}_{3875})_{\mathcal{A}\mathcal{B}}{}^{cd} \equiv \tfrac{1}{7}\,\delta_{\mathcal{A}}^{(c}\delta_{\mathcal{B}}^{d)} - \tfrac{1}{56}\,\eta_{\mathcal{A}\mathcal{B}}\eta^{cd} - \tfrac{1}{14}\,f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(c} f_{\mathcal{E}\mathcal{B}}{}^{d)} = 0 \tag{3.20}$$

By nilpotency, we have $\eta^{cd} = 0$, and contracting the remaining terms with $\mathcal{X}^{\mathcal{B}}$ we see that (2.13) does satisfy (3.3), provided the following relation holds for all $\mathcal{A}$

$$\Big[[\mathcal{X}_{\mathcal{A}}, Z^c]\,, Z^d\Big] = -2\delta_{\mathcal{A}}^{(c} Z^{d)} \tag{3.21}$$

Since the algebra preserves the grading, the relation is trivially satisfied for all generators except $\mathcal{X}_{\mathcal{A}} = Z_a$, which must be checked separately. A quick calculation, using the commutation relations listed in appendix B, shows that the required relation is indeed satisfied. Let us emphasize once more that there is no analog of (3.20) for nilpotent elements conjugate to the Z_{mn}.

Having established the consistency of (3.16) it remains to investigate its uniqueness. It is easy to see that $\mathcal{V}$ is, in fact, *not* unique because (3.16) remains unchanged under the transformation

$$\mathcal{V} \longrightarrow n \cdot \mathcal{V} \qquad \text{for} \qquad n \in \mathcal{N} \tag{3.22}$$

where $\mathcal{N}$ is the Borel subgroup of $E_{8(8)}$ generated by Z^m and Z_{mn}. This remaining non-uniqueness can be fixed by invoking the differential relations

$$e^m_A P^A_\mu = \varepsilon_\mu{}^{\nu\rho} \mathcal{B}_{\nu\rho}{}^m \tag{3.23}$$
$$e_{mnA} P^A_\mu = \varepsilon_\mu{}^{\nu\rho} \mathcal{B}_{\nu\rho mn} \tag{3.24}$$

where $\mathcal{B}_{\mu\nu}{}^m$ was already defined in (2.7), and

$$\mathcal{B}_{\mu\nu mn} := \mathcal{D}_\mu B_{\nu mn} - \mathcal{D}_\nu B_{\mu mn} + 4\partial_{[m} B_{[\mu}{}^p B_{\nu]n]p} + 2\partial_{[m} B_{n]\mu\nu} \tag{3.25}$$

(3.23) thus relates the field strengths to (part of) the $E_{8(8)}$ connection P^A_μ, whose explicit expression in terms of the spin connection and the 4-index field strength is given in formula (22) of [4]. In the reduction to three dimensions these differential constraints become the linear duality relations that allow us to trade the 36 vector fields for their dual scalars.

4 Outlook

The matrix $\mathcal{V}$ plays a role similar to the achtbein $e_m{}^a$, but also incorporates the tensor degrees of freedom from A_{mnp}, as well as the vector fields $B_\mu{}^m$ and $B_{\mu mn}$. We would now like to argue that $\mathcal{V} \in E_{8(8)}/SO(16)$ really *is* the appropriate vielbein encompassing the propagating degrees of freedom of $d = 11$ supergravity, in the same way as the ordinary vielbein, viewed as an element of $GL(d, \mathbb{R})/SO(d)$, describes the graviton degrees of freedom (with a Euclidean signature). To this aim, let us first recall that the internal part of the (inverse densitized) metric is recovered from the generalized vielbein via the $SO(16)$ invariant formula

$$\Delta^{-2} g^{mn} = \tfrac{1}{60} e^m_A e^n_A = \tfrac{1}{120}\left(e^m_A e^n_A + \tfrac{1}{2} e^m_{IJ} e^n_{IJ}\right) \tag{4.1}$$

where the constraint (3.1) was used. The summation on the r.h.s. breaks $E_{8(8)}$ to $SO(16)$ because of the plus sign in front of the second term; with a minus sign, the expression would vanish by the constraints.

Just as for the standard vielbein, and having already introduced this terminology in the foregoing section, we now interpret the indices $\mathcal{M}, \mathcal{N}, \ldots$ and $\mathcal{A}, \mathcal{B}, \ldots$ appearing in (3.18) as "curved" and "flat", respectively. This then immediately suggest the the following generalization of (4.1)

$$\mathcal{G}^{\mathcal{M}\mathcal{N}} := \tfrac{1}{120} \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \mathcal{V}^{\mathcal{N}}{}_{\mathcal{A}} \equiv \tfrac{1}{120}\left(\mathcal{V}^{\mathcal{M}}{}_A \mathcal{V}^{\mathcal{N}}{}_A + \tfrac{1}{2}\mathcal{V}^{\mathcal{M}}{}_{IJ} \mathcal{V}^{\mathcal{N}}{}_{IJ}\right) \tag{4.2}$$

By construction, this metric is invariant under local $SO(16)$, which acts as

$$\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \longrightarrow \mathcal{V}^{\mathcal{M}}{}_{\mathcal{B}} \Sigma^{\mathcal{B}}{}_{\mathcal{A}} \tag{4.3}$$

on the 248-bein with Σ in the $\mathbf{120} \oplus \mathbf{128}$ representation of $SO(16)$ (and depending on all eleven coordinates). As we showed before, a "bosonized" version of local supersymmetry formally extends this to an action of a full local $E_{8(8)}$ acting from the right on $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ (which, however, does not leave $\mathcal{G}^{\mathcal{M}\mathcal{N}}$ inert any more).

Similarly, we now have a combined action of the internal coordinate and tensor gauge transformations on the 248-bein $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$, which is analogous to the action of general coordinate transformations on the standard vielbein. Namely, from the previous formulas (2.21) and (2.23) we can directly read off their action on the 36×248 submatrix of $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$:

$$\begin{aligned} \delta\mathcal{V}^m_{\mathcal{A}} &= \xi^p\partial_p\mathcal{V}^m_{\mathcal{A}} - \partial_p\xi^m\,\mathcal{V}^p_{\mathcal{A}} - \partial_p\xi^p\,\mathcal{V}^m_{\mathcal{A}} \\ \delta\mathcal{V}_{mn\,\mathcal{A}} &= \xi^p\partial_p\mathcal{V}_{mn\,\mathcal{A}} - 2\partial_{[m}\xi^p\,\mathcal{V}_{n]p\,\mathcal{A}} - \partial_p\xi^p\,\mathcal{V}_{mn\,\mathcal{A}} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \delta\mathcal{V}^m_{\mathcal{A}} &= 0 \\ \delta\mathcal{V}_{mn\,\mathcal{A}} &= \partial_p\xi_{mn}\,\mathcal{V}^p_{\mathcal{A}} + 2\partial_{[m}\xi_{n]p}\,\mathcal{V}^p_{\mathcal{A}} \end{aligned} \tag{4.5}$$

This action can be extended to the full 248-bein via

$$\begin{aligned} \delta\mathcal{V} &= \xi^p\partial_p\mathcal{V} + \partial_p\xi^q\left(E_q{}^p - \tfrac{3}{8}\delta^p_q N\right)\mathcal{V} \\ \delta\mathcal{V} &= \partial_p\xi_{qr}E^{pqr}\,\mathcal{V} \end{aligned} \tag{4.6}$$

by means of the $E_{8(8)}$ Lie algebra matrices given in appendix A (although this is usually not done, the general coordinate transformations on the standard vielbein can be cast into an analogous form by use of $GL(d,\mathbb{R})$ matrices). It is important here that this action manifestly preserves $E_{8(8)}$: the transformed vielbein is still an element of $E_{8(8)}$. It is also straightforward to exponentiate the infinitesimal action to a full "diffeomorphism" generated by the pair (ξ^m, ξ_{mn}). Intriguingly, the missing "transport term" with ξ_{mn} suggests a hidden dependence on 28 further coordinates x_{mn}, but it remains to be seen whether such an extension exists. Finally, it is clear that the rigid $E_{8(8)}$ invariance of the dimensionally reduced theory emerges from the above local symmetries in the same way as rigid $GL(d,\mathbb{R})$ symmetry emerges in the torus reduction as a remnant of general coordinate invariance.

Appendix A: The $SO(16)$ decomposition of $E_{8(8)}$.

In the standard $SO(16)$ decomposition $\mathbf{248} \rightarrow \mathbf{120} \oplus \mathbf{128}$, the $E_{8(8)}$ Lie algebra generators are $\mathcal{X}^{\mathcal{A}} = (X^{IJ}, Y^A)$, with $SO(16)$ vector and spinor indices $I, J = 1, \dots, 16$ and $A = 1, \dots, 128$, respectively. They obey

$$\begin{aligned}
[X^{IJ}, X^{KL}] &= 4\,\delta^{I[K}X^{L]J} \\
[X^{IJ}, Y^{A}] &= \tfrac{1}{4}\Gamma^{IJ}_{AB}X^{IJ} \\
[Y^{A}, Y^{B}] &= -\tfrac{1}{2}\Gamma^{IJ}_{AB}Y^{B}
\end{aligned} \tag{A.1}$$

The totally antisymmetric E_8 structure constants $f^{\mathcal{ABC}}$ therefore possess the non-vanishing components

$$f^{IJ,\,KL,\,MN} = -8\,\delta^{I[K}\delta^{L]J}_{MN}, \qquad f^{IJ,\,A,\,B} = -\tfrac{1}{2}\Gamma^{IJ}_{AB} \tag{A.2}$$

The Cartan-Killing form is given by

$$\eta^{\mathcal{AB}} = \frac{1}{60}\mathrm{Tr}\,\mathcal{X}^{\mathcal{A}}\mathcal{X}^{\mathcal{B}} = -\frac{1}{60}f^{\mathcal{A}}{}_{\mathcal{CD}}f^{\mathcal{BCD}} \tag{A.3}$$

with components $\eta^{AB} = \delta^{AB}$ and $\eta^{IJ\,KL} = -2\delta^{IJ}_{KL}$. When summing over antisymmetrized index pairs $[IJ]$, an extra factor of $\frac{1}{2}$ is always understood.

Appendix B: The $SL(8,\mathbb{R})$ decomposition of $E_{8(8)}$.

To recover the $SL(8,\mathbb{R})$ basis of [10], we will further decompose the above representations into representations of the subgroup $SO(8) \equiv (SO(8) \times SO(8))_{\mathrm{diag}} \subset SO(16)$. The indices corresponding to the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$, respectively, will be denoted by a, α and $\dot\alpha$. After a triality rotation the $SO(16)$ vector and spinor representations decompose as

$$\begin{aligned}
\mathbf{16} &\longrightarrow \mathbf{8_s} \oplus \mathbf{8_c} \\
\mathbf{128_s} &\longrightarrow (\mathbf{8_s}\otimes\mathbf{8_c}) \oplus (\mathbf{8_v}\otimes\mathbf{8_v}) = \mathbf{8_v} \oplus \mathbf{56_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \\
\mathbf{128_c} &\longrightarrow (\mathbf{8_v}\otimes\mathbf{8_s}) \oplus (\mathbf{8_c}\otimes\mathbf{8_v}) = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{8_c} \oplus \mathbf{56_c},
\end{aligned} \tag{B.1}$$

respectively. We thus have $I = (\alpha, \dot\alpha)$ and $A = (\alpha\dot\beta, ab)$, and the E_8 generators decompose as

$$X^{[IJ]} \rightarrow (X^{[\alpha\beta]}, X^{[\dot\alpha\dot\beta]}, X^{\alpha\dot\beta})\,, \quad Y^A \rightarrow (Y^{\alpha\dot\alpha}, Y^{ab})\,. \tag{B.2}$$

Next we regroup these generators as follows. The 63 generators

$$E_a{}^b := \tfrac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} + \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{(ab)} - \tfrac{1}{8}\delta^{ab}Y^{cc}\,,$$

for $1 \leq a, b \leq 8$ span an $SL(8,\mathbb{R})$ subalgebra of $E_{8(8)}$. The generator

$$N := Y^{cc}$$

extends this subalgebra to $GL(8,\mathbb{R})$. The remainder of the $E_{8(8)}$ Lie algebra then decomposes into the following representations of $SL(8,\mathbb{R})$:

$$
\begin{aligned}
Z^a &:= \tfrac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha}+Y^{\alpha\dot\alpha})\,,\\
Z_{ab} &:= \tfrac{1}{8}\left(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]}-\Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}\right)+Y^{[ab]}\,,\\
E^{abc} &:= -\tfrac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha}-Y^{\alpha\dot\alpha})
\end{aligned}
\tag{B.3}
$$

and

$$
\begin{aligned}
Z_a &:= -\tfrac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha}-Y^{\alpha\dot\alpha})\,,\\
Z^{ab} &:= -\tfrac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]}-\Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]})+Y^{[ab]}\,,\\
E_{abc} &:= \tfrac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha}+Y^{\alpha\dot\alpha})\,,
\end{aligned}
\tag{B.4}
$$

The Cartan subalgebra is spanned by the diagonal elements ${E_1}^1,\dots,{E_7}^7$ and N, or, equivalently, by $Y^{11},\dots,Y^{88}$. Obviously, the elements ${E_a}^b$ for $a<b$ (or $a>b$) together with the elements (B.3) (or (B.4)) for $a<b<c$ generate the Borel subalgebra of $E_{8(8)}$ associated with the positive (negative) roots of $E_{8(8)}$. Furthermore, these generators are graded w.r.t. the number of times the root α_8 (corresponding to the element N in the Cartan subalgebra) appears, such that for any basis generator X we have $[N,X]=\deg(X)\cdot X$. The degree can be read off from

$$
\begin{aligned}
&[N,Z^a]=3\,Z^a && [N,Z_a] = -3\,Z_a\\
&[N,Z_{ab}]=2\,Z_{ab} && [N,Z^{ab}] = -2\,Z^{ab}\\
&[N,E^{abc}]=E^{abc} && [N,E_{abc}] = -E_{abc}\\
&[N,{E_a}^b]=0
\end{aligned}
$$

The remaining commutation relations are given by

$$
\begin{aligned}
&[Z^a,Z^b]=0 && [Z_a,Z_b]=0\\
&[Z_a,Z^b]={E_a}^b-\tfrac{3}{8}\delta^b_aN
\end{aligned}
$$

$$
\begin{aligned}
&[Z_{ab},Z^c]=0 && [Z_{ab},Z_c]=-E_{abc}\\
&[Z_{ab},Z_{cd}]=0 && [Z_{ab},Z^{cd}]=4\delta^{[c}_{[a}{E_{b]}}^{d]}+\tfrac{1}{2}\delta^{cd}_{ab}N\\
&[Z^{ab},Z^c]=-E^{abc} && [Z^{ab},Z_c]=0
\end{aligned}
$$

$$
\begin{aligned}
&[E^{abc},Z^d]=0 && [E_{abc},Z^d] = 3\delta^d_{[a}Z_{bc]}\\
&[E^{abc},Z_{de}]=-6\delta^{[ab}_{de}Z^{c]} && [E_{abc},Z_{de}] = 0\\
&[E^{abc},E^{def}]=-\tfrac{1}{32}\epsilon^{abcdefgh}Z_{gh} && [E_{abc},E_{def}] = \tfrac{1}{32}\epsilon_{abcdefgh}Z^{gh}\\
&[E^{abc},Z_d]=3\delta^{[a}_dZ^{bc]} && [E_{abc},Z_d] = 0\\
&[E^{abc},Z^{de}]=0 && [E_{abc},Z^{de}] = 6\delta^{de}_{[ab}Z_{c]}\\
&[E^{abc},E_{def}]=-18\delta^{[ab}_{[de}{E_{f]}}^{c]}-\tfrac{3}{4}\delta^{abc}_{def}N
\end{aligned}
$$

$$
\begin{aligned}
[E_a{}^b, Z^c] &= -\delta^c_a Z^b + \tfrac{1}{8}\delta^b_a Z^c & [E_a{}^b, Z_c] &= \delta^b_c Z_a - \tfrac{1}{8}\delta^b_a Z_c \\
[E_a{}^b, Z_{cd}] &= -2\delta^b_{[c} Z_{d]a} - \tfrac{1}{4}\delta^b_a Z_{cd} & [E_a{}^b, Z^{cd}] &= 2\delta^{[c}_a Z^{d]b} + \tfrac{1}{4}\delta^b_a Z^{cd} \\
[E_a{}^b, E^{cde}] &= -3\delta^{[c}_a E^{de]b} + \tfrac{3}{8}\delta^b_a E^{cde} & [E_a{}^b, E_{cde}] &= 3\delta^b_{[c} E_{de]a} - \tfrac{3}{8}\delta^b_a E_{cde} \\
[E_a{}^b, E_c{}^d] &= \delta^b_c E_a{}^d - \delta^d_a E_c{}^b
\end{aligned}
$$

The elements $\{Z^a, Z_{ab}\}$ (or equivalently $\{Z_a, Z^{ab}\}$) span the maximal 36-dimensional abelian nilpotent subalgebra of $E_{8(8)}$ [11, 10].

Finally, the generators are normalized according to

$$
\begin{aligned}
\mathrm{Tr}\,(NN) &= 60\cdot 8\,, \\
\mathrm{Tr}\,(Z^a Z_b) &= 60\,\delta^a_b\,, \\
\mathrm{Tr}\,(Z^{ab} Z_{cd}) &= 60\cdot 2!\,\delta^{ab}_{cd}\,, \\
\mathrm{Tr}\,(E_{abc} E^{def}) &= 60\cdot 3!\,\delta^{def}_{abc}\,, \\
\mathrm{Tr}\,(E_a{}^b E_c{}^d) &= 60\,\delta^d_a \delta^b_c - \tfrac{15}{2}\delta^b_a \delta^d_c\,,
\end{aligned}
$$

with all other traces vanishing.

Appendix C: Supersymmetry variations of the generalized vielbein

The supersymmetry variations of the achtbein and the relevant components of the 3-form potential with our normalization read as follows in the $SO(8)$ basis:

$$
\begin{aligned}
\delta e_m{}^a &= \tfrac{1}{2}\left(\bar\varepsilon_\alpha\,\Gamma^a_{\alpha\dot\beta}\Psi_{m\dot\beta} - \bar\varepsilon_{\dot\alpha}\,\bar\Gamma^a_{\dot\alpha\beta}\Psi_{m\beta}\right)\,, \\
\delta A_{mnp} &= -\tfrac{3}{2}\left(\bar\varepsilon_\alpha\,(\Gamma_{[mn})_{\alpha\beta}\Psi_{p]\beta} - \bar\varepsilon_{\dot\alpha}\,(\bar\Gamma_{[mn})_{\dot\alpha\dot\beta}\Psi_{p]\dot\beta}\right)\,.
\end{aligned}
\tag{C.1}
$$

(The relative minus signs in the second terms on the r.h.s. are due the fact that the Dirac conjugate spinors are appropriate to $d = 3$ and differ from the ones in $d = 11$ by an extra factor Γ^9). These formulas must now be compared with the $SO(16)$ covariant ones in terms of the generalized vielbein. The latter are most conveniently computed in terms of the matrices $\tilde E^m$, $\tilde E_{mn}$ from (3.12), making use of the $SL(8,\mathbb{R})$ decomposition of $E_{8(8)}$ described in the appendix B. In the special $SO(16)$ gauge (2.13), (2.15), (2.16) these matrices take the form

$$
\begin{aligned}
\tilde E^m &= e^m{}_{\mathcal{A}}\,\mathcal{X}^{\mathcal{A}} \;=\; 4\,\Delta^{-1} e_a{}^m\, Z^a\,, \\
\tilde E_{mn} &= e_{mn\,\mathcal{A}}\,\mathcal{X}^{\mathcal{A}} \;=\; 4\,\Delta^{-1} e_m{}^a\, e_n{}^b\, Z_{ab} + A_{mnp}\,\tilde E^p\,,
\end{aligned}
\tag{C.2}
$$

in the upper Borel subalgebra. The $SO(16)$ covariant supersymmetry variations have been presented in (2.19) and can be written as

$$
\delta\,\tilde E^m = \left[\tilde E^m\,,\,\omega\right]\,, \qquad \delta\,\tilde E_{mn} = \left[\tilde E_{mn}\,,\,\omega\right]\,, \tag{C.3}
$$

with ω given by

$$\omega \ := \ \tfrac{1}{4}\left(\Gamma^I_{A\dot{A}}\bar{\varepsilon}^I\chi^{\dot{A}}\,Y^A + \tfrac{1}{2}\,\omega^{IJ}_{\rm comp}\,X^{IJ}\right) \ , \qquad \text{(C.4)}$$

where $\omega^{IJ}_{\rm comp}\,X^{IJ}$ is the compensating $SO(16)$ rotation to restore the triangular gauge of (C.2). Upon decomposition into the $SO(8)$ fields, the first term yields

$$\begin{aligned}
\Gamma^I_{A\dot{A}}\bar{\varepsilon}^I\chi^{\dot{A}}\,Y^A \ &= \ 2\left(\bar{\varepsilon}_\alpha\Psi_{a\alpha} + \bar{\varepsilon}_{\dot\alpha}\Psi_{a\dot\alpha}\right)\,(Z^a{+}Z_a) \qquad \text{(C.5)}\\
&\quad + \left(\bar{\varepsilon}_\alpha\Gamma^{ab}_{\alpha\beta}\Psi_{b\beta} + \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{ab}_{\dot\alpha\dot\beta}\Psi_{b\dot\beta}\right)\,(Z^a{+}Z_a)\\
&\quad - \tfrac{1}{2}\,\left(\bar{\varepsilon}_\alpha\Gamma^{abc}_{\alpha\dot\beta}\,\Psi_{c\dot\beta} + \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{abc}_{\dot\alpha\beta}\,\Psi_{c\beta}\right)\,(Z_{ab}{+}Z^{ab})\\
&\quad - \tfrac{1}{2}\,\left(\bar{\varepsilon}_\alpha\Gamma^{ab}_{\alpha\beta}\,\Psi_{c\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{ab}_{\dot\alpha\dot\beta}\,\Psi_{c\dot\beta}\right)\,(E^{abc}{+}E_{abc})\\
&\quad - \left(\bar{\varepsilon}_\alpha\Gamma^{a}_{\alpha\dot\beta}\,\Psi_{b\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{a}_{\dot\alpha\beta}\,\Psi_{b\beta}\right)\,\left(E_a{}^b{+}E_b{}^a{-}\tfrac{3}{4}\delta^{ab}\,N\right) \ ,
\end{aligned}$$

whereas $\omega^{IJ}_{\rm comp}\,X^{IJ}$ is determined in such a way as to rotate ω back into the upper Borel subalgebra. The resulting ω will then preserve the special gauge choice (C.2). Explicitly, it takes the form

$$\begin{aligned}
\omega \ &\equiv \ \Gamma^I_{A\dot{A}}\bar{\varepsilon}^I\chi^{\dot{A}}\,Y^A + \tfrac{1}{2}\omega^{IJ}_{\rm comp}\,X^{IJ} \qquad \text{(C.6)}\\
&= \ 4\left(\bar{\varepsilon}_\alpha\Psi_{a\alpha} + \bar{\varepsilon}_{\dot\alpha}\Psi_{a\dot\alpha}\right)\,Z^a + 2\left(\bar{\varepsilon}_\alpha\Gamma^{ab}_{\alpha\beta}\Psi_{b\beta} + \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{ab}_{\dot\alpha\dot\beta}\Psi_{b\dot\beta}\right)\,Z^a\\
&\quad - \left(\bar{\varepsilon}_\alpha\Gamma^{abc}_{\alpha\dot\beta}\,\Psi_{c\dot\beta} + \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{abc}_{\dot\alpha\beta}\,\Psi_{c\beta}\right)\,Z_{ab}\\
&\quad - \left(\bar{\varepsilon}_\alpha\Gamma^{ab}_{\alpha\beta}\,\Psi_{c\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{ab}_{\dot\alpha\dot\beta}\,\Psi_{c\dot\beta}\right)\,E^{abc}\\
&\quad - \left(\bar{\varepsilon}_\alpha\Gamma^{a}_{\alpha\dot\beta}\,\Psi_{b\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{a}_{\dot\alpha\beta}\,\Psi_{b\beta}\right)\,\left(2E_a{}^b{-}\tfrac{3}{4}\delta^{ab}\,N\right)
\end{aligned}$$

From (C.3) we then find the supersymmetry variation:

$$\begin{aligned}
\delta\,\tilde{E}^m \ &= \ -2\Delta^{-1}\,Z^a\,\left(\bar{\varepsilon}_\alpha\Gamma^{m}_{\alpha\dot\beta}\,\Psi_{a\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{m}_{\dot\alpha\beta}\,\Psi_{a\beta}\right) \qquad \text{(C.7)}\\
&\quad - 2\Delta^{-1}Z^a\,e_a{}^m\left(\bar{\varepsilon}_\alpha\Gamma^{b}_{\alpha\dot\beta}\,\Psi_{b\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\bar{\Gamma}^{b}_{\dot\alpha\beta}\,\Psi_{b\beta}\right)
\end{aligned}$$

$$\begin{aligned}
\delta\,\tilde{E}_{mn} \ &= \ 4\Delta^{-1}\,Z_{ab}\,e_m{}^a\,\left(\bar{\varepsilon}_\alpha\,\Gamma^{b}_{\alpha\dot\beta}\Psi_{n\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\,\bar{\Gamma}^{b}_{\dot\alpha\beta}\Psi_{n\beta}\right) \qquad \text{(C.8)}\\
&\quad - 2\Delta^{-1}\,Z_{ab}\,e_m{}^a e_n{}^b\,\left(\bar{\varepsilon}_\alpha\,\Gamma^{c}_{\alpha\dot\beta}\Psi_{c\dot\beta} - \bar{\varepsilon}_{\dot\alpha}\,\bar{\Gamma}^{c}_{\dot\alpha\beta}\Psi_{c\beta}\right)\\
&\quad - \tfrac{3}{2}\,\tilde{E}^p\,\left(\bar{\varepsilon}_\alpha\,(\Gamma_{[mn})_{\alpha\beta}\Psi_{p]\beta} - \bar{\varepsilon}_{\dot\alpha}\,(\bar{\Gamma}_{[mn})_{\dot\alpha\dot\beta}\Psi_{p]\dot\beta}\right)\\
&\quad + A_{mnp}\,\delta\tilde{E}^p
\end{aligned}$$

and thus agreement with (C.1), since the one-but-last term is just $\delta A_{mnp}\tilde{E}^p$.

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