A framework for perturbations and stability of differentially rotating stars

Horst R. Beyer
Max Planck Institute for Gravitational Physics,
Albert Einstein Institute,
D-14476 Golm, Germany

October 24, 2002

Abstract

The paper provides a new framework for the description of linearized adiabatic lagrangian perturbations and stability of differentially rotating newtonian stars. In doing so it overcomes problems in a previous framework by Dyson and Schutz and provides the basis of a rigorous analysis of the stability of such stars. For this the governing equation of the oscillations is written as a first order system in time. From that system the generator of time evolution is read off and a Hilbert space is given where it generates a strongly continuous *group*. As a consequence the governing equation has a well-posed initial value problem. The spectrum of the generator relevant for stability considerations is shown to be equal to the spectrum of an operator polynomial whose coefficients can be read off from the governing equation. Finally, we give for the first time sufficient criteria for stability in the form of inequalities for the coefficients of the polynomial. These show that a negative canonical energy of the star does not necessarily indicate instability. It is still unclear whether these criteria are strong enough to prove stability for realistic stars.

1 Introduction

The study of oscillations of stars is an important and exciting field of current astrophysics. For instance through period-luminosity and period-radius relationships variable stars provide important ‘yardsticks’ for distance measurements in the universe. Their observation yield important information about the interior of stars, like the equation of state of the matter, which is otherwise hard to obtain. Further, Neutron star pulsations may be a source of gravitational radiation detectable for experiments like LIGO, VIRGO and GEO 600 in the near future.

On the other hand, it is probably fair to say that there has not been very much work on the mathematical foundations of the theory of stellar oscillations. In the non relativistic limit there is a well-known framework for the description of oscillations of nonrotating stars [9], [36] (see [2] for a rigorous version). It is important to note that even on that level it turns out that the governing operator of the spheroidal oscillations belongs to a class of operators which were apparently (apart from special cases considered in [13]) previously unconsidered in operator theory [4]. Somewhat surprisingly it also turned out that differently to radial oscillations these operators don't have a compact resolvent. Hence the well developed perturbation theory for such operators cannot be applied and a corresponding theory for the new type of operators still has to be developed. For rotating stars there is little known about the relevant operators apart from abstract properties (like the symmetry, semiboundedness and continuity of associated operators [27, 22]), indication of a continuous part in the spectrum [24, 6] and instabilities caused by so called 'r-modes' (also called 'quasi-toroidal modes') [1, 26]. To my knowledge there is no consideration of these operators in sufficient detail. Indeed, a large part of the present paper considers the more modest first step of such an investigation, namely to *identify* and determine which operators *should be* considered in that case.

The governing equation for linearized adiabatic oscillations of a stationary differentially rotating perfect-fuid star in an inertial frame (t, x) is [27]

$$\frac{\partial^2 \xi}{\partial t^2} + B' \frac{\partial \xi}{\partial t} + C' \xi = 0 \ , \tag{1}$$

where

$$B' \frac{\partial \xi}{\partial t} := 2\,(v \cdot \nabla)\, \frac{\partial \xi}{\partial t} \ , \tag{2}$$

$$\begin{aligned} C'\xi := & - \frac{1}{\rho} \nabla \left(p\,\Gamma_1 \nabla \cdot \xi\right) + (v \cdot \nabla)^2 \xi + \frac{1}{\rho} \left[-\nabla\left((\nabla p)\cdot \xi\right) + (\nabla \cdot \xi)\,\nabla p\right] \\ & - \nabla \Phi_\xi + \sum_{j,k=1}^{3} \left(\frac{1}{\rho} \frac{\partial^2 p}{\partial x_j \partial x_k} + \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) \xi_k \, e_j \ , \end{aligned} \tag{3}$$

$$\Phi_\xi(t, x) := -G \int_\Omega \frac{[\nabla \cdot (\rho \xi)](t, y)}{|x - y|}\, d^3y \ , \ \psi(x) := -G \int_\Omega \frac{\rho(y)}{|x - y|}\, d^3y \ , \ x \in \Omega \ , \tag{4}$$

ξ is the Lagrangian displacement vector field, for $j \in \{1, 2, 3\}$ the symbol e_j denotes the canonical unit vector in the direction of x_j, Ω is the (bounded open) volume of the star, and v, p, ρ, Γ_1 are the velocity field, pressure, density and the adiabatic index functions of the background star satisfying the equations of momentum and mass conservation

$$(v \cdot \nabla)\, v = -\left(\frac{1}{\rho} \nabla p + \nabla \psi \right) \ , \ \nabla \cdot (\rho v) = 0 \ . \tag{5}$$

and an equation of state. In addition to (1) the variation δp of the pressure has to vanish at the boundary $\partial\,\Omega$ of the star, i.e.,

$$\lim_{y \to x} \left(p\,\Gamma_1 \nabla \cdot \xi \right)(y) = 0 \tag{6}$$

for all $x \in \partial\,\Omega$.

The remarkable paper [12] of Dyson and Schutz provides a framework for deciding the stability of the solutions of (1). In the following this paper is referred to as DS. Compared to previous frameworks given in [27], [22] the main step forward in that paper is the fact that it relates the stability of the system directly to the growth properties of the perturbations in time as is usual for nonrotating stars (see, e.g., [2], [3], [7], [9], [25], [36]). Moreover the paper shows that these growth properties are governed by spectral properties of the generator of time evolution. This greatly simplifies the stability discussion. Unfortunately, the approach still has some drawbacks, and in the present paper will be given a varied framework which overcomes those problems. Moreover here are given for the first time sufficient criteria for stability in the form of inequalities which have to be satisfied by the coefficients of an operator polynomial. These criteria show that a negative canonical energy does not necessarily indicate an instability of the star. It is still unclear whether these criteria are strong enough to prove stability for realistic stars.

A rough discussion of the approach of Dyson and Schutz is now given. The paper considers axisymmetric solutions of the form

$$\xi(t,x) = exp(im\varphi)\, \xi_m(t,r,\theta) \ , \tag{7}$$

where r, θ, φ are spherical coordinates and $m \in \mathbb{Z}$. Inserting this ansatz into (1) leads to an equation of the same structure with induced operators B'_m C'_m. The index m is supressed in the following discussion. A Hilbert space H' (here X) for the data is chosen such that, both, B' becomes *continuous and antisymmetric* and C' becomes symmetric. In the nonrotating limit H' goes over into the usual Hilbert space used in the stability discussion for spherically symmetric stars. A physically reasonable condition on the background model is given which leads to a lower bounded C'. Assuming that condition C' is substituted by its so called Friedrichs extension. This is an abstractly defined self-adjoint extension which exists for *every* densely defined linear symmetric and semibounded operator in Hilbert space (see e.g. [30] Vol. II). [1] In the standard way the resulting wave equation is written as a first order system in time for $\xi(r,\theta,t)$ and $(\partial\xi/\partial t)(r,\theta,t)$. The initial value problem of the system is studied. The Hilbert space of the data is chosen as H'^2 with the induced 'euclidean' scalar product. However it is noticed that this is physically not meaningful, because the scalar product has no physical interpretation and is not even dimensionally correct. From the first order system the linear operator T generating time evolution is read of and it is shown that its spectrum is equal to the spectrum of a quadratic operator polynomial generated by B' and C' [28, 33]. Moreover the resolvent of T can be given in terms of the inverses of the operator polynomial. That information along with estimates on the resolvent of the operator polynomial are used to give an estimate on the spectrum of T. In general that estimate is not strong enough to decide the question of stability of the system. From these estimates it is further shown that there is a solution of the initial value problem for the system corresponding to elements of the domain of T. The uniqueness of the solution is not shown. The authors remark that they could not show that T is the generator of a strongly continuous semigroup and as a consequence the results of standard semigroup theory could *not* be used. Finally, the completeness of normal modes of the system is discussed.

From the description the reader might have noticed that in the derivation of these results only abstract properties like 'continuity', 'symmetry', 'semiboundedness' and 'self-adjointness' of B' and C' play a role. This is indeed true and is the reason why that approach is called here a 'framework'. The same also applies to the approach here. As a

[1] The importance of choosing a *self-adjoint* extension of C' can be seen in the limit of no rotation. In [2], [3] it is shown that for polytropic stars with a polytropic index $n < 1$ there is an infinite number of different self-adjoint extensions which all lead to a well-posed initial value problem for the wave equation.

consequence these frameworks can be used to describe a lot more physical systems than stellar oscillations. The main ingredient for such an application is a system of wave equations which is second order in time (with or without first order time derivatives) and which is not explicitly time dependent. The 'coefficients' in that system can be (not necessarily local) linear operators with certain abstract properties. For this reason we abandon in Section 2 any reference to rotating stars and just consider *abstract* wave equations of type (1). Having this in mind might also provide a better understanding of some of the statements below.

After this digression the discussion of DS is continued. The main problem of the approach comes from the chosen Hilbert space along with a scalar product which is not related to any physical quantity and not dimensionally correct. Of course the latter could be remedied by first introducing a dimensionless time coordinate. But experience tells that this should not be essential at such an early stage. Also it is known that the use of a suitable Hilbert space decides whether semigroup theory can be applied or not. So it is very likely that the use of H'^2 is responsible for the fact that semigroup theory could not be applied. Indeed a different choice of the Hilbert space will turn out to be the key to the results of this paper.

Another point which was not addressed in DS is the fact that in addition to (1) the boundary condition (6) has to be satisfied that the Lagrangian variation δp of the pressure vanishes at the surface of the star. [25, 9] Indeed it has been shown in [2, 3] for the limit of no rotation that for a polytropic equation of state with polytropic index $n < 1$ there is a infinite number of different self-adjoint extensions of C', which all lead to different initial value formulations for the wave equation. Moreover it has been shown that the condition of a vanishing δp at the surface of the star picks exactly one of these self-adjoint extensions. Of course the choice of the Friedrichs extension of C', is equivalent to posing a boundary condition. But because of the abstractness of this extension it is not obvious and has to be investigated whether it is compatible with (6). To my knowledge this has been shown only for the case of *radial* oscillations of spherically symmetric stars in [2]. This point will not be pursued any further in this paper.

The approach in this paper is similar to that of Dyson and Schutz. The point of departure is in the choice of the Hilbert space for the initial data of the first order system. Here a space Y is chosen , which is in general a proper subspace of H'^2. Moreover a different and dimensionally correct scalar product is chosen. The square of the induced norm of the initial data is a positive definite part of the corresponding canonical energy of the system.[16, 17] For this C' is split into sum of a strictly positive self-adjoint operator A and a 'rest' C. Of course such a decomposition is not unique but it can be shown (see Lemmas 14 and 15 in Section 2) that trivial rescalings all lead to the same set Y along with equivalent norms on Y. In particular such changes lead to theories which are related by a similarity transformation and hence the outcome of the stability discussion is not affected. In general the canonical energy *cannot* be used as a norm for Y because it is not always positive definite. In situations where it is C can be chosen as zero. In the limit of no rotation where $B' = 0$ and the operator C' is semibounded the approach here reduces to the approach in [30] Vol. II (see the proposition at the beginning of chapter X.13) for classical wave equations.

A major consequence of the change is that it allows the use of semigroup theory which is a standard and well developed tool in particular in the theory of partial differential equations (see e.g. [10],[14],[20], [19],[29], [32] and the cited references therein). This simplifies the reasoning a lot, because it can be and will be built on those results. In particular here the operator G_+ which corresponds to T in DS generates a strongly continuous *group* of bounded transformations and hence the well-posedness of the initial value problem for the first order system follow from abstract semigroup theory. At the same time a considerable generalization is achieved. The operator B' (here denoted by iB) can be unbounded and not antisymmetric. Moreover C' has not to be assumed symmetric. The restrictions imposed on these operators are the following. The operator C' has to be of the form $A + C$ where A is some densely defined and strictly positive self-adjoint operator in X and C is a relatively bounded perturbation of the positive square root $A^{1/2}$ of A. In addition B' has to be a relatively bounded perturbation of $A^{1/2}$ with relative bound smaller than 1. Finally, B' has to be antisymmetric or continuous, but not necessarily both. All these conditions are trivially satisfied for the case of axisymmetric solutions of (1) considered by Dyson and Schutz. Whether this generalization is sufficient to provide a framework for (1) and not only for its axisymmetric form is not yet clear. For this it seems necessary that C' given by (3) is semibounded and this is still open. The reason for considering also more general situations with nonantisymmetric B' and nonsymmetric C' is that the framework here will also be used in a future paper in the stability discussion of the Teukolsky equations on a Kerr background where this is the case. [35] A further important advantage of the approach here is that it can be shown (see Theorem 3) that the dominating part of G_+ (but in general not G_+ itself) is *self-adjoint*. Using perturbation theory this gives important information on the spectrum of G_+ and is also the basis of the proof that G_+ is the generator of a strongly continuous group of bounded transformations. (See Theorem 7) Further it is the basis for another result (see Corollary 12) having no counterpart in DS namely the conservation of the 'canonical energy' E. On the other hand it turns out that the spectrum of G_+ is the *same* as of T. In particular that spectrum is given by the spectrum of the *same* operator polynomial $C' - \lambda B' + \lambda^2, \lambda \in \mathbb{C}$ (See Theorem)

A plausible definition for the stability of a rotating star is the following. The system is stable if and only if the semigroup $T_+(t), t \in [0, \infty)$ generated by G_+ is bounded. Note that this definition is invariant to similarity transformations and hence not so sensitive to changes of Hilbert space like one only invariant under unitary transformations.[2] From semigroup theory one has then the following.

1. The system is *unstable* if G_+ has a spectral value with real part smaller than zero.

2. For a *stable* system the corresponding spectrum of G_+ is contained in the closed right half plane of the complex plane.

3. From only the fact that the spectrum of G_+ is part of the closed right half plane of the complex plane, it *cannot* be concluded that the system is stable. [3]

4. The system is *stable* if the *real part* of all 'expectation values'

$$(\xi|G_+\xi) \tag{8}$$

[2]Such a definition would be given for instance by the demand that the semigroup should be *contractive*, i.e., that the norms of the semigroup elements are $\leqslant 1$.

[3] For a counterexample compare for instance the note after the proof of Corollary 9.

is positive ($\geqslant 0$) for all elements ξ from the domain (or a core) of G_+.[4]

Point 1 gives a sufficient but not necessary condition for instability. Note that this condition is invariant under similarity transformations. Moreover because of Theorem 13 it is equivalent to the condition that there is complex number λ with real part smaller than zero such that

$$C' - \lambda B' + \lambda^2 \tag{9}$$

is not bijective. This reduces in the nonrotating case to the condition that C' is strictly negative, which is a well known sufficient condition for instability.[2] An important final observation is that from the existence of such a λ follows the existence of an element ξ from the Hilbert space such that the corresponding function of norms $|T_+(t)\xi|, t \in \mathbb{R}$ grows exponentially for large times.[5] Hence the existence of such a λ leads to a much stronger kind of instability.

Point 4 gives a sufficient but not necessary condition for stability. It is appealing because it is of the form of an inequality, which is more easily accessible than the spectrum of G_+. On the other hand it is strong and *not* invariant to similarity transformations. It turns out to be equivalent to C' being strictly positive, i.e., that the spectrum of this operator consists only of positive real numbers different from zero. Note that this reduces to a known sufficient condition for stability in the nonrotating case. But for such stars it can be satisfied only for radial oscillations (for instance this is the case for constant $\Gamma_1 > 4/3$), but not for nonradial oscillations.[6] [2, 3, 7] Note that in the limit of no rotation the trivial toroidal oscillations give rise to solutions of (1,6) whose norm increases linear in time for large times. Hence applying the stability definition above to that limit would lead to an 'unstable star'. Of course, these oscillations can be excluded in that case just by considering a reduced operator.

The following two stability criteria are new. They will be proven in Theorem 17.

1. If B' and C' are such that

$$< \xi|C'\xi > -\frac{1}{4} < \xi|B'\xi >^2 \geqslant 0 \tag{10}$$

for all ξ from the domain of C' such that $\|\xi\| = 1$ then the spectrum of G_+ is purely imaginary.

2. If the operator

$$C' - \frac{ib}{2}B' - \frac{b^2}{4} \tag{11}$$

is positive for some $b \in \mathbb{R}$ then the spectrum of G_+ is purely imaginary and there are $K \geqslant 0$ and $t_0 \geqslant 0$ such that

$$|T_+(t)| \leqslant Kt \tag{12}$$

for all $t \geqslant t_0$.

[4]Then G_+ generates a *contraction* semigroup.

[5]This is easily seen for instance by using Theorem 4.1 in chapter 4 of [29]. Here it is important to remember that in general the spectrum of G_+ does not only consist of 'eigenvalues' (for which this statement is of course trivially satisfied) but also values $\mu \in \mathbb{C}$ for which the map $G_+ - \mu$ is just not onto. Such values are often from a continuous part of the spectrum.

[6]This is obvious since the spectrum of the trivial toroidal oscillations is $\{0\}$. But in [3] it has also been shown that 0 is in the spectrum of spheroidal oscillations.

Note for the first point that $-(1/4) < \xi | B'\xi >^2$ is *positive*, because of the antisymmetry of B'. Also note in this connection that in DS it has been shown that $C' - (1/4)B'^2$ is bounded from below *uniformly in m*. Unfortunately, in general this does not imply (10).

It is still unclear whether these criteria are strong enough to prove stability for realistic stars. On the other hand the second criterium has been sucessfully applied in the stability discussion of the Kerr metric where the master equation governing perturbations is of the form (1), too.[5]

2 The framework

This section developes the initial value formulation for abstract differential equations of the form (1). It is self-contained and necessarily very technical. The reader who is not interested in the excessive mathematical details given here is referred to the introduction. The used nomenclature can be found in standard textbooks on Functional analysis.[30] Vol. I, [31, 38]

Before going into the mathematical details it is explained about the meaning of the individual results of this section. The section is based on the assumptions General Assumption 1 and General Assumption 4 on three operators A, B and C. A different form of General Assumption 1 which is more convenient for applications can be given in the obvious way using Lemma 18. Definition 2 gives the Hilbert space Y which is used here instead of the Hilbert space in DS. A rigorous form (55) of (1) along with the existence and uniqueness of the solution corresponding to initial values is given in Theorem 11. Corollary 12 gives the corresponding 'energy' along with an identity for its time derivative. The analogue G_+ of the generator T in DS is given in Definition 5. Theorem 3 proves that the 'dominating parts' of G_+ are self-adjoint. In Theorem 7 it is proved that under General Assumption 1 and General Assumption 4, both, G_+ and $-G_+$ are generators of strongly continuous semigroups T_+ and T_-, resp. Theorem 13 shows the identity of the spectrum of G_+ with the spectrum of an operator polynomial generated by the operators B and $A + C$. Lemmas 14 and 15 show that certain simple rescalings of A and C which formally leave invariant (55) lead to theories which are related by a similarity transformation. Theorem 16 shows for a special case how these rescalings can be used to derive a better estimate for the growth of T_+ and T_- than the one induced by (30) in Lemma 6. Theorem 17 gives sufficient criteria for stability in the form of inequalities which have to be satisfied by the coefficients of the operator polynomial. Part (ii) of this Theorem has been sucessfully applied in the discussion of the stability of the Kerr metric. [5]

The rest of this section contains the mathematical details.

Assumption 1 *In the following let* $(X, < | >)$ *be a non trivial complex Hilbert space. Denote by* $\| \|$ *the norm induced on* X *by* $< | >$. *Further let* $A : D(A) \to X$ *be a densely defined linear self-adjoint operator in* X *for which there is an* $\varepsilon \in (0, \infty)$ *such that*

$$< \xi | A\xi > \geqslant \varepsilon < \xi | \xi > \tag{13}$$

for all $\xi \in D(A)$. *Denote by* $A^{1/2}$ *the square root of* A *with domain* $D(A^{1/2})$. *Further let be* $B : D(A^{1/2}) \to X$ *a linear operator in* X *such that for some* $a \in [0, 1)$ *and* $b \in \mathbb{R}$

$$\|B\xi\|^2 \leqslant a^2 \|A^{1/2}\xi\|^2 + b^2 \|\xi\|^2 \tag{14}$$

for all $\xi \in D(A^{1/2})$*. Finally, let* $C : D(A^{1/2}) \to X$ *be linear and such that for some real numbers* c *and* d

$$\|C\xi\|^2 \leqslant c^2 \|A^{1/2}\xi\|^2 + d^2\|\xi\|^2 \tag{15}$$

for all $\xi \in D(A^{1/2})$*.*

Note that as a consequence of (13) the spectrum of A is contained in the interval $[\varepsilon, \infty)$. Hence A is in particular positive and bijective and there is a uniquely defined linear and positive selfadjoint operator $A^{1/2} : D(A^{1/2}) \to X$ such that $(A^{1/2})^2 = A$. That operator is the so called *square root of* A. Further note that from its definition and the bijectivity of A follows that $A^{1/2}$ is in particular bijective. This can be concluded for instance as follows. By using the fact that $A^{1/2}$ commutes with A it easy to see that for every $\lambda \in [0, \varepsilon^{1/2})$ by $(A^{1/2} + \lambda)(A - \lambda^2)^{-1}$ there is given the inverse to $A^{1/2} - \lambda$. Hence the spectrum of $A^{1/2}$ is contained in the interval $[\varepsilon^{1/2}, \infty)$. All these facts will be used later on.

Definition 2 *We define*

$$Y := D(A^{1/2}) \times X \tag{16}$$

and $(\,|\,) : Y^2 \to \mathbb{C}$ *by*

$$(\xi|\eta) :=< A^{1/2}\xi_1|A^{1/2}\eta_1 > + < \xi_2|\eta_2 > \tag{17}$$

for all $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in Y$*.*

Then we have the following

Theorem 3 **(i)** $(Y, (\,|\,))$ *is a complex Hilbert space.*

(ii) *The operator* $H : D(A) \times D(A^{1/2}) \to Y$ *in* Y *defined by*

$$H\xi := (-i\xi_2, iA\xi_1) \tag{18}$$

for all $\xi = (\xi_1, \xi_2) \in D(A) \times D(A^{1/2})$ *is densely-defined, linear and self-adjoint.*

(iii) *The operator* $\hat{B} : D(H) \to Y$ *defined by*

$$\hat{B}\xi := (0, -B\xi_2) \tag{19}$$

for all $\xi = (\xi_1, \xi_2) \in D(H)$ *is linear. If* B *is symmetric then* $\hat{B}$ *is symmetric, too. If* B *is bounded then* $\hat{B}$ *is bounded, too, and the corresponding operator norms* $\|B\|$ *and* $|\hat{B}|$ *satisfy*

$$|\hat{B}| \leqslant \|B\| \,. \tag{20}$$

(iv) *The sum* $H + \hat{B}$ *is closed. If* B *is symmetric then* $H + \hat{B}$ *is self-adjoint.*

(v) *The operator* $V : Y \to Y$ *defined by*

$$V\xi := (0, iC\xi_1) \tag{21}$$

for all $\xi = (\xi_1, \xi_2) \in Y$ *is linear and bounded. The operator norm* $|V|$ *of* V *satisfies*

$$|V| \leqslant (c^2 + d^2/\epsilon)^{1/2} \,. \tag{22}$$

Proof: (i): Obviously, $(\,|\,)$ defines a hermitean sesquilinear form on Y^2. That $(\,|\,)$ is further positive definite follows from the positive definiteness of $<\,|\,>$ and the injectivity of $A^{1/2}$. Finally, the completeness of $(Y, |\,|)$, where $|\,|$ denotes the norm on Y induced by $(\,|\,)$, follows from the completeness of $(X, \|\,\|)$ together with the fact that $A^{1/2}$ has a *bounded* inverse. Here it is essentially used that 0 is not contained in the spectrum of A. (ii): That $D(A) \times D(A^{1/2})$ is dense in Y is an obvious consequence of the facts that $D(A)$ is a core for $A^{1/2}$ (see e.g. Theorem 3.24 in chapter V.3 of [23]) and that $D(A^{1/2})$ is dense in X. The linearity of H is obvious. Also the symmetry of H follows straighforwardly from the symmetry of $A^{1/2}$. By that symmetry one gets further for any $\xi = (\xi_1, \xi_2) \in D(H^*)$ and any $\eta = (\eta_1, \eta_2) \in D(H)$:

$$\begin{aligned} (H^*\xi|\eta) &= < (H^*\xi)_1 | A\eta_1 > + < (H^*\xi)_2 | \eta_2 > \\ = (\xi|H\eta) &= < -i\xi_2 | A\eta_1 > + < iA^{1/2}\xi_1 | A^{1/2}\eta_2 > \end{aligned} \tag{23}$$

and from this by using that A is bijective and $A^{1/2}$ is self-adjoint that $\xi_1 \in D(A)$ and

$$(H^*\xi)_1 = -i\xi_2 \ , \ (H^*\xi)_2 = iA\xi_1 \ . \tag{24}$$

Hence H is an extension of H^* and thus $H = H^*$. (iii): The linearity of $\hat{B}$ is obvious. Also it is straightforward to see that $\hat{B}$ is symmetric if B is symmetric. If B is bounded then

$$|\hat{B}\xi|^2 = \|B\xi_2\|^2 \leqslant \|B\|^2 \|\xi_2\|^2 \leqslant \|B\|^2 |\xi|^2 \tag{25}$$

for all $\xi = (\xi_1, \xi_2) \in D(H)$. Hence $\hat{B}$ is also bounded and $|\hat{B}|$, $\|B\|$ satisfy the claimed inequality. (iv): Obviously, (14) implies

$$|\hat{B}\xi|^2 \leqslant a^2 |H\xi|^2 + b^2 |\xi|^2 \tag{26}$$

for all $\xi \in D(H)$. From this it is easily seen that $H + \hat{B}$ is closed (see, e.g., [18], Lemma V.3.5). Moreover in the case that B (and hence by (iii) also $\hat{B}$) is symmetric (26) implies according to the *Kato-Rellich* Theorem (see, e. g., Theorem X.12 in [30] Vol. II) that $H + \hat{B}$ is self-adjoint. For the application of these theorems the assumption $a < 1$ made above is essential. (v) The linearity of V is obvious. For any $\xi = (\xi_1, \xi_2) \in Y$ one has

$$\begin{aligned} |V\xi|^2 &= \|iC\xi_1\|^2 \leqslant c^2 \|A^{1/2}\xi_1\|^2 + d^2 \|\xi_1\|^2 \\ &= c^2 \|A^{1/2}\xi_1\|^2 + d^2 \|(A^{1/2})^{-1} A^{1/2}\xi_1\|^2 \leqslant (c^2 + d^2/\epsilon)\, |\xi|^2 \ . \end{aligned} \tag{27}$$

In the last step it has been used that

$$\|(A^{1/2})^{-1}\| \leqslant 1/\sqrt{\varepsilon} \ . \tag{28}$$

This follows by an application of the spectral theorem (see, e.g. Theorem VIII.5 in [30] Vol. I) to $A^{1/2}$. Since ξ is otherwise arbitrary from (27) follows the boundedness of V and the claimed inequality.□

Assumption 4 *In the following we assume in addition that B is symmetric or bounded.*

Note that condition (14) is trivially satisfied if B is bounded. We define:

Definition 5

$$G_+ := -i(H + \hat{B} + V) \ , \ G_- := i(H + \hat{B} + V) \ . \tag{29}$$

then

Lemma 6 *The operators G_+ and G_- are closed and quasi-accretive. In particular*

$$Re(\xi|G\xi) \geqslant -(\mu_B + |V|)\,(\xi|\xi) \tag{30}$$

for $G \in \{G_+, G_-\}$ and all $\xi \in D(H)$. Here Re denotes the real part and

$$\mu_B := \begin{cases} 0 & \text{if } B \text{ is symmetric} \\ \|B\| & \text{if } B \text{ is bounded} \end{cases} . \tag{31}$$

Proof: That G_+ and G_- are closed is an obvious consequence of (iv) and (v) of the previous theorem. Further if B is symmetric one has because of (iv) and (v) of the preceeding theorem

$$Re(\xi|G_\pm\xi) = \mp Re(\xi|iV\xi) \geqslant -|(\xi|iV\xi)| \geqslant -|V|\,(\xi|\xi) \tag{32}$$

for all $\xi \in D(H)$. Similarly, if B is bounded one has because of (ii),(iii),(iv), (20)

$$Re(\xi|G_\pm\xi) = \mp Re(\xi|i(\hat{B}+V)\xi) \geqslant -|(\xi|i(\hat{B}+V)\xi)| \geqslant -(\|B\| + |V|)\,(\xi|\xi) \tag{33}$$

for all $\xi \in D(H)$. Hence in both cases G_+ and G_- are quasi-accretive.□

Theorem 7 *The operators G_+ and G_- are infinitesimal generators of strongly continuous semigroups $T_+ : [0,\infty) \to L(Y,Y)$ and $T_- : [0,\infty) \to L(Y,Y)$, respectively. If $\mu_\pm \in \mathbb{R}$ are such that*

$$Re(\xi|G_\pm\xi) \geqslant -\mu_\pm\,(\xi|\xi) \tag{34}$$

for all $\xi \in D(H)$ the spectra of G_+ and G_- are contained in the half-plane $[-\mu_+,\infty) \times \mathbb{R}$ and $[-\mu_-,\infty) \times \mathbb{R}$, respectively, and

$$|T_+(t)| \leqslant \exp(\mu_+ t)\ ,\ \ |T_-(t)| \leqslant \exp(\mu_- t) \tag{35}$$

for all $t \in [0,\infty)$.

Proof: Obviously, by the Lumer-Phillips theorem (see, e.g., Theorem X.48 in Vol. II of [30]) and the preceeding lemma the theorem follows if we can show that there is a real number $\lambda < \min\{-\mu_+, -\mu_-\}$ such that $G_\pm - \lambda$ has a dense range in Y. For that proof let be ξ some element of $D(H)$ and λ any real number such that $|\lambda| \geqslant |V|^2$. Then we get from the symmetry of H

$$|(H - i\lambda)\xi|^2 = |H\xi|^2 + \lambda^2|\xi|^2 \tag{36}$$

and

$$|(H - i\lambda)\xi| \geqslant \max\{|H\xi|, |\lambda|^{1/2}|V\xi|\}\ . \tag{37}$$

Using these identities together with (14)

$$\begin{aligned} |(\hat{B}+V)\xi|^2 &\leqslant |\hat{B}\xi|^2 + 2|\hat{B}\xi|\,|V\xi| + |V\xi|^2 \\ &\leqslant a^2|H\xi|^2 + 2|\hat{B}\xi|\,|V\xi| + (b^2 + |V|^2)|\xi|^2 \\ &\leqslant a^2|H\xi|^2 + 2a|H\xi|\,|V\xi| + (b + |V|)^2|\xi|^2 \\ &\leqslant a^2|(H-i\lambda)\xi|^2 + 2a|(H-i\lambda)\xi|\,|V\xi| + [(b+|V|)^2 - a^2\lambda^2]|\xi|^2 \\ &\leqslant a(a + 2|\lambda|^{-1/2})|(H-i\lambda)\xi|^2 + [(b+|V|)^2 - a^2\lambda^2]|\xi|^2\ . \end{aligned} \tag{38}$$

Hence for any real λ with

$$|\lambda| > \max\{|V|^2, 4(1-a)^{-2}, (b+|V|)/a, |\mu_+|, |\mu_-|\} \ , \tag{39}$$

where we assume without restriction that $a > 0$, we get

$$|(\hat{B}+V)\xi| \leqslant a'|(H-i\lambda)\xi| \tag{40}$$

where a' is some real number from $[0,1)$. Since $\xi \in D(H)$ is otherwise arbitrary, we conclude that

$$(\hat{B}+V)(H-i\lambda)^{-1} \tag{41}$$

defines a bounded linear operator on Y with operator norm smaller than 1. Since

$$H+\hat{B}+V-i\lambda = \left(1+(\hat{B}+V)(H-i\lambda)^{-1}\right)(H-i\lambda) \tag{42}$$

we conclude that $H+\hat{B}+V-i\lambda$ is bijectice and hence also that $G_+ - \lambda$ and $G_- - \lambda$ are both bijective. Hence the theorem follows.□

We note that General Assumption 4 has been used only to conclude that G_+ and G_- are *both* quasi-accretive. Now it is easy to see that if B is in addition such that iB is quasi-accretive (but not necessarily bounded or antisymmetric) then $-i\hat{B}$ and hence also G_+ are quasi-accretive, too. As a consequence we have the following

Corollary 8 *Instead of General Assumption 4 let B be such that iB is quasi-accretive. Then G_+ is the infinitesimal generator of a strongly continuous semigroup $T_+ : [0,\infty) \to L(Y,Y)$. If $\mu_+ \in \mathbb{R}$ is such that*

$$Re(\xi|G_+\xi) \geqslant -\mu_+\,(\xi|\xi) \tag{43}$$

for all $\xi \in D(H)$ the spectrum of G_+ is contained in the half-plane $[-\mu_+,\infty)\times\mathbb{R}$ and

$$|T_+(t)| \leqslant \exp(\mu_+ t) \tag{44}$$

for all $t \in [0,\infty)$.

Theorem 7 has the following

Corollary 9 **(i)** *By*

$$T(t) := \begin{cases} T_+(t) & \textit{for } t \geqslant 0 \\ T_-(-t) & \textit{for } t < 0 \end{cases} \tag{45}$$

for all $t \in \mathbb{R}$ there is defined a strongly continuous group $T : \mathbb{R} \to L(Y,Y)$.

(ii) *For every $t_0 \in \mathbb{R}$ and every $\xi \in D(G_+)$ there is a uniquely determined differentiable map $u : \mathbb{R} \to Y$ such that*

$$u(t_0) = \xi \tag{46}$$

and

$$u'(t) = -G_+u(t) \tag{47}$$

for all $t \in \mathbb{R}$. Here $'$ denotes differentiation of functions assuming values in Y.

(iii) *The function* $(u|u) : \mathbb{R} \to \mathbb{R}$ *defined by*

$$(u|u)(t) := (\, u(t)|u(t)\,)\ , t \in \mathbb{R} \tag{48}$$

is differentiable and

$$(u|u)'(t) = -2Re\,(\, u(t)|G_+u(t)\,) \tag{49}$$

for all $t \in \mathbb{R}$.

Proof: The corollary follows from Theorem 7 by standard results of semigroup theory. For instance, see section 1.6 in [29] for (i) and section IX.3 in [23] for (ii). (iii) is an obvious consequence of (ii). □

Note in particular the *the special case* [7] that there is a non trivial element η in the kernel of $A + C$ for which there is $\xi \in D(A)$ such that

$$(A + C)\xi = -iB\eta\ . \tag{50}$$

Then by

$$u(t) := (\xi + t\eta, \eta)\ ,\ t \in \mathbb{R} \tag{51}$$

there is given a growing solution of (47).

The following lemma is needed in the formulation of the subsequent theorem.

Lemma 10 *By*

$$\|\xi\|_{A^{1/2}} := \|A^{1/2}\xi\|\ ,\ \xi \in D(A^{1/2}) \tag{52}$$

there is defined a norm $\|\ \|_{A^{1/2}}$ *on* $D(A^{1/2})$. *Moreover*

$$W_1 := (D(A^{1/2}), \|\ \|_{A^{1/2}}) \tag{53}$$

is complete.

Proof: The lemma is a trivial consequence of the completeness of X and the bijectivity of $A^{1/2}$.□

Theorem 11 *Let be* $t_0 \in \mathbb{R}$, $\xi \in D(A)$ *and* $\eta \in D(A^{1/2})$. *Then there is a uniquely determined differentiable map* $u : \mathbb{R} \to W_1$ *with*

$$u(t_0) = \xi \text{ and } u'(t_0) = \eta \tag{54}$$

and such that $u' : \mathbb{R} \to X$ *is differentiable with*

$$(u')'(t) + iBu'(t) + (A + C)u(t) = 0 \tag{55}$$

for all $t \in \mathbb{R}$.

[7]Such cases are easy to construct.

Proof: For this let be $v = (v_1, v_2) : \mathbb{R} \to Y$ be such that

$$v(t_0) = (\xi, \eta) \tag{56}$$

and

$$v'(t) = -G_+ v(t) \ , t \in \mathbb{R} \ . \tag{57}$$

Such v exists according to Corollary 9 (ii). Using the continuity of the canonical projections of Y onto W_1 and X it is easy to see that $u := v_1$ is a differentiable map into W_1 such that $u' : \mathbb{R} \to X$ is differentiable and such that (54), (55) are both satisfied. On the other hand if $u : \mathbb{R} \to W_1$ has the properties stated in the corollary it follows by the continuity of the canonical imbeddings of W_1, X into Y that $w := (u, u')$ satisfies both equations (56) and (57). Then $u = v_1$ follows by Corollary 9 (ii). □

Corollary 12 *In addition to the assumptions made let C be in particular bounded.* [8] *Further let $u : \mathbb{R} \to W_1$ be differentiable with a differentiable derivative $u' : \mathbb{R} \to X$ and such that (55) holds. Finally, define $E_u : \mathbb{R} \to \mathbb{R}$ by*

$$E_u(t) := \frac{1}{2} \left(< u'(t) | u'(t) > + < u(t) | (A + Re(C)) u(t) > \right) \ . \tag{58}$$

Then E_u is differentiable and

$$E'_u(t) = \begin{cases} -Im < u(t) | Im(C) u'(t) > & \text{for symmetric } B \\ \frac{1}{2} < u'(t) | Im(B) u'(t) > -Im < u(t) | Im(C) u'(t) > & \text{for bounded } B \end{cases} \tag{59}$$

for all $t \in \mathbb{R}$, where for any bounded linear operator F on X:

$$Re(F) := \frac{1}{2} \left(F + F^* \right) \ , \ Im(F) := \frac{1}{2i} \left(F - F^* \right) \ . \tag{60}$$

Proof: For this define $v := (u, u')$. Then according to the preceeding proof v satisfies (57). *For a symmetric B* it follows by Corollary 9 and Theorem 3 (iv) that

$$\begin{aligned} (v|v)'(t) &= 2Re \left(v(t) | iVv(t) \right) \\ &= - < u'(t) | Cu(t) > - < Cu(t) | u'(t) > \\ &= - < u | Re(C) u >' (t) - 2Im < u(t) | Im(C) u'(t) > \end{aligned} \tag{61}$$

for all $t \in \mathbb{R}$. In the last step it has been used that u is also differentiable with the same derivative viewed as map with values in X. This follows from the fact the canonical imbedding of W_1 into X is continuous since $A^{1/2}$ is bijective. Further the definition

$$< u | Re(C) u > (t) := < u(t) | Re(C) u(t) > \ , \ t \in \mathbb{R} \tag{62}$$

for the map $< u | Re(C) u >: \mathbb{R} \to \mathbb{R}$ has been used. Obviously, (59) follows from (61) by using definition (58). In this step also the symmetry of $A^{1/2}$ is used together with the fact that u assumes values in $D(A)$. *For a bounded B* by Corollary 9 and Theorem 3 (ii) follows that

$$\begin{aligned} (v|v)'(t) = \ & 2Re \left(v(t) | i(\hat{B} + V) v(t) \right) \\ = \ & 2Im < u'(t) | Bu'(t) > \\ & - < u'(t) | Cu(t) > - < Cu(t) | u'(t) > \\ = \ & 2 < u'(t) | Im(B) u'(t) > \\ & - < u | Re(C) u >' (t) - 2Im < u(t) | Im(C) u'(t) > \end{aligned} \tag{63}$$

[8] Note that in this case (15) is trivially satisfied.

for all $t \in \mathbb{R}$. Obviously, (59) follows from (61) by using definition (58). $\square$

The next theorem relates the spectrum of G_+ to the spectrum of the so called *operator polynomial* $A + C - \lambda B - \lambda^2$, where λ runs through the complex numbers. [28, 33]

Theorem 13 *Let λ be some complex number.*

(i) *Then $H + \hat{B} + V - \lambda$ is not injective if and only if $A + C - \lambda B - \lambda^2$ is not injective. If $H + \hat{B} + V - \lambda$ is not injective then*

$$\ker(H + \hat{B} + V - \lambda) = \{(\xi, i\lambda\xi) : \xi \in \ker(A + C - \lambda B - \lambda^2)\} \ . \tag{64}$$

(ii) *Further $H + \hat{B} + V - \lambda$ is bijective if and only if $A + C - \lambda B - \lambda^2$ is bijective. If $H + \hat{B} + V - \lambda$ is bijective then for all $\eta = (\eta_1, \eta_2) \in Y$:*

$$(H + \hat{B} + V - \lambda)^{-1}\eta = (\xi, i(\lambda\xi + \eta_1)) \ , \tag{65}$$

where

$$\xi = (A + C - \lambda B - \lambda^2)^{-1}[(B + \lambda)\eta_1 - i\eta_2] \ . \tag{66}$$

Proof: (i) If $H + \hat{B} + V - \lambda$ is not injective and $\xi = (\xi_1, \xi_2) \in \ker(H + \hat{B} + V - \lambda)$ it follows from the definitions in theorem 3 that

$$\xi_2 = i\lambda\xi_1 \ , \ (A + C - \lambda B - \lambda^2)\xi_1 = 0 \tag{67}$$

and hence also that $A + C - \lambda B - \lambda^2$ is not injective. If $A + C - \lambda B - \lambda^2$ is not injective it follows again from the definitions in theorem 3 that

$$(H + \hat{B} + V - \lambda)(\xi, i\lambda\xi) = 0 \tag{68}$$

and hence also that $H + \hat{B} + V - \lambda$ is not injective. (ii) If $H + \hat{B} + V - \lambda$ is bijective it follows by (i) that $A + C - \lambda B - \lambda^2$ is injective. For $\eta \in X$ and $\xi = (\xi_1, \xi_2) := (H + \hat{B} + V - \lambda)^{-1}(0, i\eta)$ it follows from the definitions in theorem 3 that

$$(A + C - \lambda B - \lambda^2)\xi_1 = \eta \tag{69}$$

and hence that $A + C - \lambda B - \lambda^2$ is also surjective. If $A + C - \lambda B - \lambda^2$ is bijective it follows by (i) that $H + \hat{B} + V - \lambda$ is injective. Further if $\eta = (\eta_1, \eta_2) \in Y$ and ξ is defined by (66) it follows from the definitions in theorem 3 that

$$(H + \hat{B} + V - \lambda)(\xi, i(\lambda\xi + \eta_1)) = \eta \tag{70}$$

and hence that $H + \hat{B} + V - \lambda$ is also surjective. $\square$

Lemma 14 *Let be $\varepsilon' < \varepsilon$ and*

$$A' := A - \varepsilon' \ , \ C' := C + \varepsilon' \ . \tag{71}$$

Then

(i)

$$D(A'^{1/2}) = D(A^{1/2}) \tag{72}$$

and for all $\xi \in D(A^{1/2})$

$$\|A^{1/2}\xi\|^2 = \|A'^{1/2}\xi\|^2 + \varepsilon'\|\xi\|^2 \; . \tag{73}$$

(ii) *The operators* A', B *and* C' *satisfy*

$$\begin{aligned} < \xi|A'\xi > \quad &\geqslant \quad (\varepsilon - \varepsilon') < \xi|\xi > \\ \|B\xi\|^2 \quad &\leqslant \quad a^2\|A'^{1/2}\xi\|^2 + (a^2\varepsilon' + b^2)\|\xi\|^2 \\ \|C'\xi\|^2 \quad &\leqslant \quad |c| \left[|c| + 2|\varepsilon'| \, (\varepsilon - \varepsilon')^{-1/2} \right] \|A'^{1/2}\xi\|^2 + \\ & \quad\quad \left[|\varepsilon'| + (c^2|\varepsilon'| + d^2)^{1/2} \right]^2 \, \|\xi\|^2 \end{aligned} \tag{74}$$

for all $\xi \in D(A^{1/2})$.

Proof: (i) First, since $\varepsilon' < \varepsilon$ by (71) there is defined a linear self-adjoint and positive operator A' in X. Obviously, using the symmetry of $A^{1/2}$ and $A'^{1/2}$ (73) follows for all elements of $D(A)$. From this (72) and (73) follow straightforwardly by using the facts that $D(A)$ is a core for both, $A^{1/2}$ and $A'^{1/2}$ (see e.g. Theorem 3.24 in chapter V.3 of [23]), that X is complete and that both operators, $A^{1/2}$ and $A'^{1/2}$ are closed. (ii) The first two inequalities are obvious consequences of the corresponding ones in General Assumption 1, the definition (71) and of (73). For the proof of the third we notice that from the first inequality along with an application of the spectral theorem (see, e.g. Theorem VIII.5 in [30] Vol. I) to $A'^{1/2}$ follows that

$$\|(A'^{1/2})^{-1}\| \leqslant 1/\sqrt{\varepsilon - \varepsilon'} \; . \tag{75}$$

Further from General Assumption 1 and (73) one gets

$$\|C\xi\|^2 \leqslant c^2\|A'^{1/2}\xi\|^2 + (c^2|\varepsilon'| + d^2)\|\xi\|^2 \; . \tag{76}$$

for all $\xi \in D(A^{1/2})$. From these inequalities we get

$$\begin{aligned} \|C'\xi\|^2 \;&\leqslant\; \|C\xi\|^2 + 2|\varepsilon'| \, \|C\xi\| \, \|\xi\| + \varepsilon'^2\|\xi\|^2 \\ &\leqslant\; c^2\|A'^{1/2}\xi\|^2 + (\varepsilon'^2 + c^2|\varepsilon'| + d^2)\|\xi\|^2 + 2|\varepsilon'| \, \|C\xi\| \, \|\xi\| \\ &\leqslant\; c^2\|A'^{1/2}\xi\|^2 + \left[|\varepsilon'| + (c^2|\varepsilon'| + d^2)^{1/2} \right]^2 \|\xi\|^2 + 2|\varepsilon'| \, |c| \, \|A'^{1/2}\xi\| \, \|\xi\| \\ &\leqslant\; |c| \left[|c| + 2|\varepsilon'| \, (\varepsilon - \varepsilon')^{-1/2} \right] \|A'^{1/2}\xi\|^2 + \left[|\varepsilon'| + (c^2|\varepsilon'| + d^2)^{1/2} \right]^2 \|\xi\|^2 \end{aligned} \tag{77}$$

for all $\xi \in D(A^{1/2})$ and hence the third inequality.□

As a consequence of (ii) the sequence X, A', B, C' satisfies General Assumption 1. The corresponding Y given by Definition 2 is because of (i) again given by (16). Moreover the corresponding norm $|\,|'$ on Y turns out to be equivalent to $|\,|$. More precisely one has for every $\varepsilon' \leqslant 0$

Lemma 15

$$|\,|' \leqslant |\,| \leqslant \varepsilon^{1/2} \, (\varepsilon - \varepsilon')^{-1/2} \, |\,|' \tag{78}$$

and for every bounded linear operator F *on* Y*:*

$$\varepsilon^{-1/2} \, (\varepsilon - \varepsilon')^{1/2} \, |F|' \leqslant |F| \leqslant \varepsilon^{1/2} \, (\varepsilon - \varepsilon')^{-1/2} \, |F|' \; . \tag{79}$$

Proof: The first inequality is a straightforward consequence of (73) and (75). The second inequality is a straightforward implication of the first. □.

Note that the $G_\pm$ corresponding to the the sequence X, A', B, C' are the same for all ε' (ε' drops out of the definition). Moreover as a consequence of the preceding lemma the the topologies induced on Y are equivalent. Hence the generated groups are the same, too. This will be used in the following important special case.

Theorem 16 *Let be $A = A_0 + \varepsilon$, where A_0 is a densely defined linear positive self-adjoint operator and let be $C = -\varepsilon$. Then*

$$|T_\pm(t)| \leqslant e\,\varepsilon^{1/2}\,t\,\exp(\mu_B t) \tag{80}$$

for all $t \geqslant \varepsilon^{-1/2}$.

Proof: For this let be $\varepsilon' \in [0, \varepsilon)$ and define A' and C' as in Lemma 14. Hence

$$A' = A_0 + \varepsilon - \varepsilon'\ ,\ C' = -(\varepsilon - \varepsilon')\ . \tag{81}$$

Then from Theorem 3(v), Lemma 6 and Theorem 7 we conclude that

$$|T_\pm(t)|' \leqslant \exp\left([\mu_B + (\varepsilon - \varepsilon')^{1/2}]\,t\right) \tag{82}$$

for all $t \in \mathbb{R}$ and hence by Lemma 15 that

$$|T_\pm(t)| \leqslant \varepsilon^{1/2}\,(\varepsilon - \varepsilon')^{-1/2}\,\exp\left([\mu_B + (\varepsilon - \varepsilon')^{1/2}]\,t\right)\ . \tag{83}$$

For $t \geqslant \varepsilon^{-1/2}$ we get from this (80) by choosing

$$\varepsilon' := \varepsilon - t^{-2}\ .\square \tag{84}$$

Note that in this special case (58) is conserved and positive.

We are now giving stability criteria.

Theorem 17 *In addition let B and C be both symmetric.*

(i) *Let A, B and C be such that*

$$< \xi|(A + C)\xi > + \frac{1}{4} < \xi|B\xi >^2 \geqslant 0 \tag{85}$$

for all $\xi \in D(A)$ with $\|\xi\| = 1$. Then the spectrum of iG_+ is real.

(ii) *In addition let B and C be both bounded and let $A + C + (b/2)B - (b^2/4)$ be positive for some $b \in \mathbb{R}$. Then the spectrum of iG_+ is real and there are $K \geqslant 0$ and $t_0 \geqslant 0$ such that*

$$|T(t)| \leqslant K|t| \tag{86}$$

for all $|t| \geqslant t_0$.

Proof: (i): First from General Assumption 1 and the assumed symmetry of B and C follows that, both, by $A^{-1/2}BA^{-1/2}$ and $A^{-1/2}CA^{-1/2}$ there is given a bounded symmetric and hence (by the theorem of Hellinger and Toplitz) also self-adjoint operator on X. Hence

$$A(\lambda) := \lambda^2 A^{-1} + \lambda A^{-1/2}BA^{-1/2} - \left(1 + A^{-1/2}CA^{-1/2}\right) \ , \ \lambda \in \mathbb{C} \tag{87}$$

defines a self-adjoint operator polynomial in $L(X, X)$. In addition one has $A^{-1} \geqslant 1/\varepsilon$. Further for every $\xi \in D(A^{1/2})$ and $\lambda \in \mathbb{C}$

$$< \xi|A(\lambda)\xi >=< \eta|A^{1/2}A(\lambda)A^{1/2}\eta >= - < \eta|(A + C - \lambda B - \lambda^2)\eta > \tag{88}$$

where $\eta := A^{-1/2}\xi \in D(A)$. Now (85) implies that the roots of the polynomial $< \eta|(A + C - \lambda B - \lambda^2)\eta >, \lambda \in \mathbb{C}$ are real. Hence by (88) the roots of $< \xi|A(\lambda)\xi >, \lambda \in \mathbb{C}$ are real, too. Since $\xi \in D(A^{1/2})$ is otherwise arbitrary and $D(A^{1/2})$ is dense in X this implies also that $< \xi|A(\lambda)\xi >$ has only real roots for all $\xi \in X$. Hence (see [28], Lemma 31.1) the polynomial $A(\lambda), \lambda \in \mathbb{C}$ is weakly hyperbolic and has therefore a real spectrum. As a consequence $A(\lambda)$ is bijective for all non real λ. Now for any such λ

$$A + C - \lambda B - \lambda^2 = -A^{1/2}A_{|}(\lambda)A^{1/2}_{|} \ , \tag{89}$$

where $A^{1/2}_{|}$ denotes the restriction of $A^{1/2}$, both, to $D(A)$ in domain and $D(A^{1/2})$ in range and $A_{|}(\lambda)$ denotes the restriction of $A(\lambda)$ to $D(A^{1/2})$, both, in domain and in range. For this note that $A(\lambda)$ leaves $D(A^{1/2})$ invariant. Further from the bijectivity of $A^{1/2}$, $A(\lambda)$ and (87) follows the bijectivity of $A^{1/2}_{|}$ and $A_{|}(\lambda)$, respectively and hence by (89) that $A + C - \lambda B - \lambda^2$ is bijective. This is true for all non real λ and hence it follows by Theorem 13 that the spectrum of iG_+ is real. (ii) So let B and C be both bounded and let $A + C + (b/2)B - (b^2/4)$ be positive for some $b \in \mathbb{R}$. In addition let be ϵ some real number greater than zero and define

$$A' := A + C + (b/2)B - (b^2/4) + \varepsilon \ , \ C' := -\varepsilon \ , \ B' := B - b \ . \tag{90}$$

First it is observed that

$$D(A'^{1/2}) = D(A^{1/2}) \tag{91}$$

and that there exist nonvanishing real constants K_1 and K_2 such that

$$K_1^2\|A^{1/2}\xi\|^2 \leqslant \|A'^{1/2}\xi\|^2 \leqslant K_2^2\|A^{1/2}\xi\|^2 \tag{92}$$

for every $\xi \in D(A^{1/2})$. This can be proved as follows. Obviously, by the symmetry of $A^{1/2}$ and $A'^{1/2}$, the Cauchy-Schwarz inequality, the boundedness of B, C, $A^{-1/2}$ and $A'^{-1/2}$ follows the existence of nonvanishing real constants K_1 and K_2 such that (92) is valid for all $\xi \in D(A)$. Since $D(A)$ is a core for both, $A^{1/2}$ and $A'^{1/2}$ (see e.g. Theorem 3.24 in chapter V.3 of [23]) from that inequality follows (91) and (92) for all $\xi \in D(A^{1/2})$. Note that to conclude this it is used that X is complete and that, both, $A^{1/2}$ and $A'^{1/2}$ are closed.

Obviously, from the assumptions made follows that also A', B' and C' instead of A, B and C, respectively, satisfy General Assumption 1 and General Assumption 4. Hence by Theorem 16 follows that

$$|T'_{\pm}(t)|' \leqslant e\,\varepsilon^{1/2}\,t \tag{93}$$

for all $t \geqslant \varepsilon^{-1/2}$, where primes indicate quantities whose definition uses one or more of the operators A', B' and C' instead of A, B and C. In addition (91) and (92) imply $Y = Y'$ as well as the equivalence of the norms $|\,|$ and $|\,|'$. Now define the auxiliar transformation $S_0 : Y' \to Y$ by

$$S_0\xi := (\xi_1, \xi_2 - i(b/2)\xi_1) \tag{94}$$

for all $\xi = (\xi_1, \xi_2) \in Y'$. Obviously, S_0 is bijective and bounded with the bounded inverse S_0^{-1} given by $S_0^{-1}\xi := (\xi_1, \xi_2 + i(b/2)\xi_1)$ for all $\xi = (\xi_1, \xi_2) \in Y$. In addition define $S_\pm : [0, \infty) \to L(Y, Y)$ by

$$S_\pm(t) := \exp(\mp ibt/2) S_0 T'_\pm(t) S_0^{-1} \ , \tag{95}$$

for all $t \in [0, \infty)$. Obviously, $S_\pm$ defines a strongly continuous semigroup with the corresponding generator

$$S_0\, G'_\pm\, S_0^{-1} \pm i\frac{b}{2} = G_\pm \ . \tag{96}$$

This implies $S_\pm = T_\pm$ and by (93) and (95) the existence of $K \geqslant 0$ and $t_0 \geqslant 0$ such that (86) is valid for all $|t| \geqslant t_0$. Finally, from this follows by the Theorem of Hille-Yosida-Phillips that the spectrum of iG_+ is real.□

Lemma 18 *Let D be a core for A. Further let be $B_0 : D \to X$ a linear operator in X such that for some real numbers a_0 and b_0*

$$\|B_0\xi\|^2 \leqslant a_0^2 < \xi|A\xi > + b_0^2\, \|\xi\|^2 \tag{97}$$

for all $\xi \in D$. Then there is a uniquely determined linear extension $\bar{B}_0 : D(A^{1/2}) \to X$ of B_0 such that

$$\|\bar{B}_0\xi\|^2 \leqslant a_0^2\, \|A^{1/2}\xi\|^2 + b_0^2\, \|\xi\|^2 \tag{98}$$

for all $\xi \in D(A^{1/2})$. If B_0 is in addition symmetric $\bar{B}_0$ is symmetric, too.

Proof: First we notice that D is a core for $A^{1/2}$, too. Obviously, since $D(A)$ is a core for $A^{1/2}$ (see e.g. Theorem 3.24 in chapter V.3 of [23]) this follows if we can show that the closure of the restriction of $A^{1/2}$ to D extends the restriction of $A^{1/2}$ to $D(A)$. To prove this let ξ be some element of $D(A)$. Since D is a core for A there is a sequence $\xi_0, \xi_1 \dots$ of elements of D converging to ξ and at the same time such that $A\xi_0, A\xi_1 \dots$ converges to $A\xi$. Since $A^{1/2}$ has a bounded inverse it follows from this that $A^{1/2}\xi_0, A^{1/2}\xi_1 \dots$ vonverges to $A^{1/2}\xi$. Since ξ can be chosen otherwise arbitrarily it follows that the closure of the restriction of $A^{1/2}$ to D extends the restriction of $A^{1/2}$ to $D(A)$ and hence that D is a core for $A^{1/2}$. Hence for any $\xi \in D(A^{1/2})$ there is a a sequence $\xi_0, \xi_1 \dots$ in D converging to ξ and at the same time such that $A^{1/2}\xi_0, A^{1/2}\xi_1 \dots$ is converging to $A^{1/2}\xi$. Hence by (97) along with the completeness of X follows the convergence of the sequence $B_0\xi_1, B_0\xi_2 \dots$ to some element $\bar{B}\xi$ of X and

$$\|\bar{B}\xi\|^2 \leqslant a_0^2\, \|A^{1/2}\xi\|^2 + b_0^2\, \|\xi\|^2 \ . \tag{99}$$

Moreover if $\xi'_0, \xi'_1 \dots$ is another sequence having the same properties as $\xi_0, \xi_1 \dots$ by (97) follows that

$$\bar{B}\xi = \lim_{n\to\infty} B_0\xi_n = \lim_{n\to\infty} B_0\xi'_n \ . \tag{100}$$

From this it easily seen that by defining

$$\bar{B} := (D(A^{1/2}) \to X \, , \, \xi \mapsto \bar{B}\xi) \tag{101}$$

there is also given a linear map. Hence the existence of a linear extension of B_0 satisfying (98) is shown. Moreover from the definition it is obvious that $\bar{B}$ is symmetric if B_0 is in addition symmetric. If on the other hand $\bar{B}_0$ is a linear extension of B_0 satisfying (98) and ξ and ξ_1, ξ_2 are as above from (98) follows that

$$\bar{B}_0\xi = \lim_{n\to\infty} B_0\xi_n \ . \tag{102}$$

Finally, since ξ can be chosen otherwise arbitrarily from this follows $\hat{B}_0 = \hat{B}$.□

3 Discussion and results

This paper provides a rigorous framework for the description of linearized adiabatic lagrangian perturbations and stability of differentially rotating newtonian stars using semigroup theory. Problems of a previous framework by Dyson and Schutz are overcome and a basis for a rigorous analysis of the stability of such stars is provided. The spectrum of the oscillations is shown to coincide with the spectrum of an operator polynomial whose coefficients can be read off from the equation governing the oscillations about the equilibrium configuration. Moreover, for the first time sufficient criteria for stability are given in form of inequalities for the coefficients of that polynomial. These show that a negative canonical energy of the star does not necessarily indicate instability.

It is still unclear whether these criteria are strong enough to prove stability for realistic stars. On the other hand the second criterium has been sucessfully applied in the (on first sight seemingly unrelated case of the) stability discussion of the Kerr metric where the master equation governing perturbations is of the form (1), too.[5] Another similarity of that case to the cases considered here is the fact that the corresponding operators C' and B'^2 there are such that $C' - (1/4)B'^2$ is positive whereas here this combination is semibounded as has been shown in DS.

Also the determination of the spectrum of the operator polynomial $C' - \lambda B' + \lambda^2, \lambda \in \mathbb{C}$ for some special case would be very useful. It is likely that this cannot be done for a physically relevant case. But it is also likely that the outcome to *qualitative* questions like

- Does one have uniform stability in m?
- Does a continuous part occur in the oscillation spectrum?

only depends on *structural* properties of the operators C' and B'. So from C' probably only the highest order derivatives are relevant and details of the equation of state should be unimportant. From this point of view even the highly idealized case of a spherical background model with a truncated C' along with a non constant velocity field v would be interesting to consider.

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