

Dynamics of spatially homogeneous locally rotationally symmetric solutions of the Einstein-Vlasov equations

A. D. Rendall

*Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1
D-14476 Golm, Germany*
and

C. Uggla

*Department of Physics, University of Karlstad
S-65188 Karlstad, Sweden*

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Abstract

The dynamics of the Einstein-Vlasov equations for a class of cosmological models with four Killing vectors is discussed in the case of massive particles. It is shown that in all models analysed the solutions with massive particles are asymptotic to solutions with massless particles at early times. It is also shown that in Bianchi types I and II the solutions with massive particles are asymptotic to dust solutions at late times. That Bianchi type III models are also asymptotic to dust solutions at late times is consistent with our results but is not established by them.

1 Introduction

The most popular matter content by far in the study of spatially homogeneous cosmological models is a perfect fluid with linear equation of state (see e.g., the book [13]). It is important to know if the results obtained for this class are structurally stable if we change the matter content. Thus it is of interest to investigate other types of sources. Here we will consider certain diagonalizable locally rotationally symmetric (LRS) spatially homogeneous models with collisionless matter. This class of models was previously studied in the case of massless particles in [10]. Here we will focus on the case with massive particles. We will recast Einstein's field equations into a form so that one part of the boundary of the state space for the massive case can be identified with the state space for the massless case while another

part can be identified with the state space for the corresponding dust equations. (In addition other parts of the boundary have the interpretation of state spaces associated with certain models with distributional matter.) It will be shown that these boundary submanifolds are intimately connected with the early and late time behaviour of the LRS massive collisionless gas models respectively.

The results of our analysis can be summarized as follows. Consideration is restricted to models of Bianchi types I, II and III. This is enough to display a large variety of phenomena. At early times, i.e. close to the initial singularity, the dynamics of solutions with massive particles mimics closely the dynamics for the corresponding symmetry type with massless particles. In particular there are solutions whose behaviour near the singularity is quite different from that of any fluid model of any of these Bianchi types. At late times, i.e. in a phase of unlimited expansion, the general picture is that the dynamics resembles that of a dust model. This is proved for Bianchi types I and II. For type III the results are consistent with dust-like asymptotics but we were not able to prove that this is what happens. If kinetic theory with massive particles always behaved like dust at late times this would provide a justification of the use of a fluid model in that regime.

The outline of the paper is as follows. In section 2 we derive the dynamical system. Sections 3, 4 and 5 analyse the models of types I, II and III respectively, with the main results being stated in Theorems 2.1, 3.1 and 4.1. In section 6 we conclude with some remarks and speculations. An appendix contains some information about dynamical systems which is applied frequently in the paper.

2 A dynamical systems formulation

We will consider LRS models for which the metric can be written in the form

$$ds^2 = -dt^2 + g_{11}(t)(\theta^1)^2 + g_{22}(t)((\theta^2)^2 + (\theta^3)^2), \quad (1)$$

where θ^i are suitable one-forms describing the various symmetry types. The energy-momentum tensor T_{ij} for the Einstein-Vlasov system with massive particles is assumed to be diagonal and is described by

$$\begin{aligned} \rho &= \int f_0(v_i)(m^2 + g^{11}(v_1)^2 + g^{22}((v_2)^2 + (v_3)^2))^{1/2}(\det g)^{-1/2} dv_1 dv_2 dv_3, \\ p_i &= \int f_0(v_i)g^{ii}(v_i)^2(m^2 + g^{11}(v_1)^2 + g^{22}((v_2)^2 + (v_3)^2))^{-1/2}(\det g)^{-1/2} dv_1 dv_2 dv_3, \end{aligned} \quad (2)$$

where ρ is the energy density and $p_i = T^i_i$ the pressure components of the energy-momentum tensor. The function f_0 is determined at some fixed time t_0 by $f_0(v_i) = f(t_0, v_i)$ where f is

the phase space density of particles. The covariant components v_i are independent of time. The function f_0 satisfies the condition $f_0(v_1, v_2, v_3) = F(v_1, (v_2)^2 + (v_3)^2)$.

Some further technical conditions will be imposed on f_0 . It is assumed to be non-negative and have compact support. It is also assumed that the support does not intersect the coordinate planes $v_i = 0$. A function f_0 with this property will be said to have split support. The reason for the assumption of split support will be seen later. In the following it will always be assumed without further comment that the data considered have split support. It follows from the assumptions already made that $f_0(x_i) = f_0(-x_i)$ for $i = 2, 3$. It will be assumed that this also holds for $i = 1$ and functions f_0 with this property will be called reflection-symmetric. This ensures that the form of the phase space density of particles is compatible with a diagonal metric and, in particular, that the energy-momentum tensor is diagonal. For the symmetry types to be considered in the following it then follows that the entire system consisting of geometry and matter is invariant under three commuting reflections. For this reason, solutions where the metric is diagonal and f_0 has the symmetry properties just mentioned will be called reflection-symmetric. A solution is said to be isotropic if $f_0(v_1, v_2, v_3) = F((v_1)^2 + (v_2)^2 + (v_3)^2)$ and if $g_{11} \propto g_{22} \propto g_{33}$ for all time.

The momentum constraints are automatically satisfied for these models. Only the Hamiltonian constraint and the evolution equations are left. Instead of considering a set of second order equations in terms of e.g., a and b , where

$$a^2 = g_{11} \ , \quad b^2 = g_{22} \ , \quad (3)$$

we will reformulate these equations as a first order system of ODEs by introducing a new set of variables. The mean curvature $\text{tr}k$ (where k_{ij} is the second fundamental form) is given by

$$\text{tr}k = -(a^{-1}da/dt + 2b^{-1}db/dt) \ . \quad (4)$$

A new dimensionless time coordinate τ is defined by $-\frac{1}{3} \int_{t_0}^t \text{tr}k(t)dt$ for some arbitrary fixed time t_0 . (We will follow the conventions in [13]. The time variable thus differs by a factor 3 from the one in [10]). In the following a dot over a quantity denotes its derivative with respect to τ . The Hubble variable H is given by $H = -\text{tr}k/3$. Now define the following dimensionless variables:

$$\begin{aligned} z &= m^2/(a^{-2} + 2b^{-2} + m^2) \ , \\ s &= b^2/(b^2 + 2a^2) \ , \\ M_2 &= \sigma_2(a^2/b^4)(\text{tr}k)^{-2} \ , \\ M_3 &= 3\sigma_3b^{-2}(\text{tr}k)^{-2} \ , \\ \Sigma_+ &= -3(b^{-1}db/dt)(\text{tr}k)^{-1} - 1 \ , \end{aligned} \quad (5)$$

where σ_2 is 1 for Bianchi types II, VIII, IX and 0 for Bianchi types I, III and the Kantowski-Sachs (KS) models. The coefficient σ_3 is 1 for types III and VIII. It is -1 for KS and type

IX and 0 for types I, II ¹. These variables lead to a decoupling of the equation for the only remaining dimensional variable H (or equivalently $\text{tr}k$)

$$\dot{H} = -(1 + q)H , \quad (6)$$

where the deceleration parameter q is given by

$$q = 2\Sigma_+^2 + \frac{1}{2}\Omega(1 + R) . \quad (7)$$

The quantity R is defined by

$$R = (p_1 + 2p_2)/\rho , \quad (8)$$

where

$$\begin{aligned} p_1/\rho &= (1 - z)sg_1/h , \\ p_2/\rho &= \frac{1}{2}(1 - z)(1 - s)g_2/h , \\ g_{1,2} &= \int f_0(v_i)(v_{1,2})^2 [z + (1 - z)(s(v_1)^2 + \frac{1}{2}(1 - s)((v_2)^2 + (v_3)^2))]^{-1/2} dv_1 dv_2 dv_3 , \\ h &= \int f_0(v_i) [z + (1 - z)(s(v_1)^2 + \frac{1}{2}(1 - s)((v_2)^2 + (v_3)^2))]^{1/2} dv_1 dv_2 dv_3 . \end{aligned} \quad (9)$$

The assumption of split support ensures that the function $R(s, z)$ is a smooth (C^∞) function of its arguments. The related quantity R_+ defined by

$$R_+ = (p_2 - p_1)/\rho . \quad (10)$$

is a smooth function of s and z for the same reason.

The normalized energy density $\Omega = \rho/(3H^2)$ is determined by the Hamiltonian constraint and, in units where $G = 1/8\pi$, is given by

$$\Omega = 1 - \Sigma_+^2 - M_2 - M_3 . \quad (11)$$

The assumption of a distribution of massive particles with non-negative mass leads to inequalities for R , R_+ and Ω . Firstly, $0 \leq R \leq 1$ with $R = 0$ only when $z = 1$ and $R = 1$ only when $z = 0$. Secondly, $-R \leq R_+ \leq \frac{1}{2}R$ with $R_+ = \frac{1}{2}R$ for $s = 0$ and $R_+ = -R$ for $s = 1$. Thirdly $\Omega \geq 0$. Using these inequalities in equation (7) in turn results in $0 \leq q \leq 2$ for Bianchi types I, II, III and VIII (i.e., the same inequality as for causal perfect fluids, see [13]).

¹The motivation for the variable s comes from more general diagonal models where it is convenient to introduce variables of the type $s_i = g^{ii}/(g^{11} + g^{22} + g^{33})$. s is simply s_1 in the case when $g^{22} = g^{33}$.

The remaining dimensionless coupled system is:

$$\begin{aligned}
\dot{\Sigma}_+ &= -(2-q)\Sigma_+ - \mathcal{S}_+ + \Omega R_+ , \\
\dot{s} &= 6s(1-s)\Sigma_+ , \\
\dot{z} &= 2z(1-z)(1 + \Sigma_+ - 3\Sigma_+s) , \\
\dot{M}_2 &= 2(q - 4\Sigma_+)M_2 , \\
\dot{M}_3 &= 2(q - \Sigma_+)M_3 ,
\end{aligned} \tag{12}$$

where \mathcal{S}_+ is given by

$$\mathcal{S}_+ = -4M_2 - M_3. \tag{13}$$

There are a variety of submanifolds corresponding to different symmetry types:

$$\begin{aligned}
S_{\text{I}} &: M_2 = M_3 = 0 , \\
S_{\text{II}} &: M_2 > 0 , M_3 = 0 , \\
S_{\text{III}} &: M_2 = 0 , M_3 > 0 , \\
S_{\text{KS}} &: M_2 = 0 , M_3 < 0 , \\
S_{\text{VIII}} &: M_2 > 0 , M_3 > 0 , (1-s)M_3 = 6M_2s \\
S_{\text{IX}} &: M_2 > 0 , M_3 < 0 , (1-s)M_3 = -6M_2s .
\end{aligned} \tag{14}$$

The relationship between the various models can be visualized in a symmetry reduction diagram given in Fig. 1 (a collective treatment of the corresponding vacuum models from a Hamiltonian perspective and with the aim of quantizing the models was given in [3]). Note that while this diagram accurately reflects the relationship of the geometry in the different cases, the relationship of the matter content is more subtle when types VIII or IX are involved. This complication does not occur for the Bianchi types studied in detail in this paper and will therefore not be discussed further here.

Note that a non-negative energy density implies that $\Omega \geq 0$, which in turn implies that our variables are bounded for types I,II,III and VIII, since M_2 and M_3 are non-negative and since by definition z and s are bounded. These models expand indefinitely. The KS and type IX models are recollapsing models and since H becomes zero at the point of maximal expansion, the Hubble-normalized variables blow up at this point. However, one can find other variables that are bounded along the lines found in [11]. Neither are the above variables ‘optimal’ for the other LRS models. One can adapt to the particular mathematical features these models exhibit. However, we choose to use the above formulation since the present variables are easier to interpret physically and are naturally generalizable to more general non-LRS models. For simplicity we will from now on study Bianchi types I,II, and III.

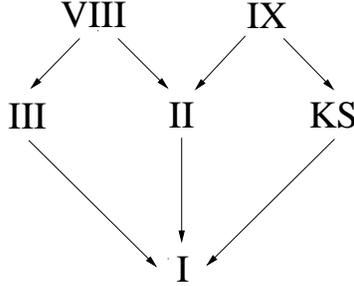


Figure 1: Symmetry reduction diagram for the diagonal LRS models.

It is of interest to note that the metric functions a, b are expressible in terms of s, z in the massive case. The relations are

$$a^2 = z(m^2 s(1-z))^{-1}, \quad b^2 = 2z(m^2(1-s)(1-z))^{-1}. \quad (15)$$

In addition to the symmetry submanifolds there are a number of other boundary submanifolds:

$$\begin{aligned} z &= 0, 1, \\ s &= 0, 1, \\ \Omega &= 0. \end{aligned} \quad (16)$$

The submanifold $z = 0$ corresponds to the massless case. The submanifold $z = 1$ leads to a decoupling of the s -equation, leaving a system identical to the corresponding dust equations. The submanifolds $s = 0, s = 1$ correspond to problems with f_0 being a distribution while $\Omega = 0$ constitutes the vacuum submanifold with test matter. Apart from these solutions there exists an isotropic solution in Bianchi type I characterized by $\Sigma_+ = R_+ = 0$ and a constant value for s that depends on the function f_0 .

Including these boundaries yields compact state spaces for types I, II and III. In order to apply the standard theory of dynamical systems the coefficients must be C^1 on the entire compact state space G of a given model. This is necessary even for uniqueness. In the present case it suffices to show that R and R_+ are C^1 on G , i.e, that they are C^1 for s, z when $0 \leq s \leq 1, 0 \leq z \leq 1$. As has already been pointed out, this follows from the assumption of split support, which even implies the analogous statement with C^1 replaced by C^∞ . It would be possible to get C^1 regularity under the weaker assumption that f_0 vanishes as fast as a sufficiently high power of the distance to the coordinate planes. We have not, however,

examined in detail how large the power would have to be since this is of little relevance to our main concerns in this paper.

Of key importance is the existence of a monotone function in the ‘massive’ interior part of the state space:

$$\begin{aligned} M &= (s(1-s)^2)^{-1/3} z(1-z)^{-1} , \\ \dot{M} &= 2M . \end{aligned} \tag{17}$$

Note that the volume ab^2 is proportional to $M^{3/2}$. This monotone function rules out any interior ω - and α -limit sets and forces these sets to lie on the $s = 0$, $s = 1$, $z = 0$ or $z = 1$ parts of the boundary.

3 Type I models

It is natural to start investigating the type I system since it is a submanifold of the state space of all other symmetry types. The physical state space, G , of these models is given by the region in \mathbf{R}^3 defined by the inequalities $-1 \leq \Sigma_+ \leq 1$, $0 \leq s \leq 1$ and $0 \leq z \leq 1$.

To understand the dynamics of the type I models, it is necessary to determine the stationary points and their stability. The coordinates, in terms of (Σ_+, s, z) , of the various stationary points are the following: $(0, s_0, 0), (\frac{1}{2}, 0, 0), (1, 0, 0), (1, 1, 0), (-1, 1, 0), (1, 0, 1), (1, 1, 1), (-1, 1, 1)$, where s_0 is a particular constant value of s depending on the function f_0 (see [10]). These points are called P_1, \dots, P_8 . (Note that they are numbered differently in the massless case compared to those in [10].) In addition there exist two lines of equilibrium points, $(-1, 0, K), (0, F, 1)$, denoted by L_1, L_2 , where K and F are constant values. The points $P_1, P_2, P_4, P_6, P_7, P_8$ are hyperbolic saddles while P_5 is degenerate, with one zero eigenvalue. The point P_3 is a hyperbolic source. The line L_1 is a transversally hyperbolic saddle while the line L_2 is a transversally hyperbolic sink. (For an explanation of this terminology we refer to the appendix.)

The state space together with equilibrium points and separatrix orbits is depicted in Fig. 2.

The main result in this section is the following theorem:

Theorem 3.1 If a smooth non-vacuum reflection-symmetric LRS solution of Bianchi type I of the Einstein-Vlasov equations for massive particles is represented as a solution of (12) with $M_2 = M_3 = 0$ then for $\tau \rightarrow \infty$ it converges to a point of the line L_2 . For $\tau \rightarrow -\infty$ there exists

- (i) a single (isotropic) solution that converges to P_1 and
- (ii) a one-parameter set of solutions lying on the unstable manifold of P_2 and
- (iii) all remaining solutions belong to a two-parameter set (the generic case) of solutions converging to P_3 .

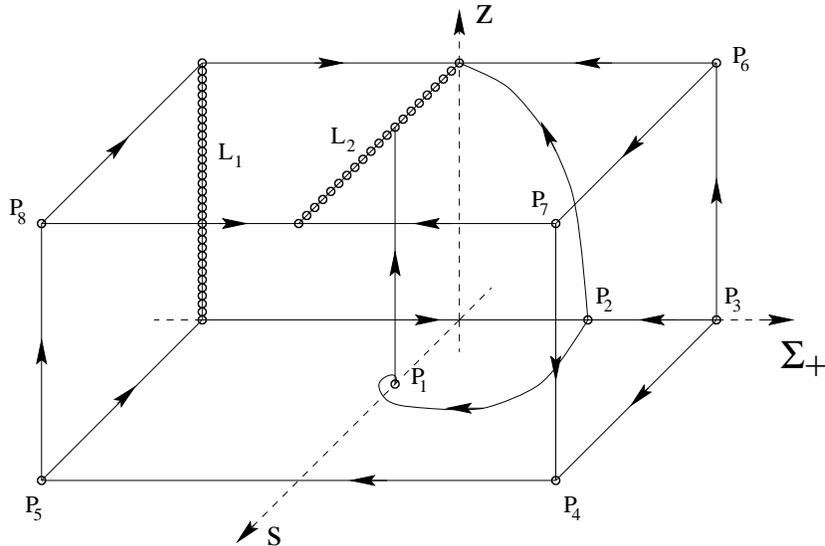


Figure 2: The LRS type I state space together with equilibrium points and separatrix orbits.

This will be proved in a series of lemmas. We refer to [13, 10] for terminology from the theory of dynamical systems.

Lemma 3.1 There exist open neighbourhoods U_1 and U_2 of the point P_3 and the line L_2 respectively such that:

(i) if a solution belongs to U_1 at any time it belongs to U_1 at all earlier times and its α -limit set consists of the point P_3 alone.

(ii) if a solution belongs to U_2 at any time it belongs to U_2 at all later times and its ω -limit point consists of a single point of the line L_2 .

Proof Part (i) follows from the fact that P_3 is a hyperbolic source and the Hartman-Grobman theorem. Part (ii) follows from the fact that L_2 is a transversally hyperbolic sink and the reduction theorem ([10], Theorem A1).

As a step towards analysing the dynamics of the full system we determine the ω -limit points of solutions of the dynamical system on the parts of the boundary of G defined by $s = 0$ and $s = 1$. This information will later be combined with the monotone function M when determining the ω -limit sets of solutions of the full system. In the case of the α -limit sets the monotone function alone accomplishes the same thing.

Lemma 3.2 A solution of the restriction of the system to the part of the boundary of G defined by $s = 1$ for which neither z nor Σ_+ take on one of their limiting values has the

endpoint of L_2 with $s = 1$ as its ω -limit set.

Proof If $\Sigma_+ \geq 0$ at any time, then Σ_+ is decreasing. The rate of decrease remains uniform as long as Σ_+ does not tend to zero. It follows that after a finite time Σ_+ must be strictly less than $1/2$. On the other hand, z is monotone increasing in the region $\Sigma_+ < 1/2$ and the rate of increase remains uniform as long as z does not tend to one. It follows that $z \rightarrow 1$ as $\tau \rightarrow \infty$. If Σ_+ tends to zero in this limit then the conclusion of the lemma holds. Otherwise Σ_+ must become negative at some time. Thus it can be seen that any ω -limit points satisfy $z = 1$ and $-1 \leq \Sigma_+ \leq 0$. From part (ii) of Lemma 3.1 it follows that any solution which enters U_2 has the desired ω -limit set. Since the ω -limit set is a union of orbits, it is possible as a consequence to exclude the points with $z = 1$ and $-1 < \Sigma_+ < 0$ from the ω -limit set. To complete the proof of the lemma it remains only to exclude the point P_8 from the ω -limit set. This point is a hyperbolic saddle of the restriction of the system to $s = 1$ and so it follows from the discussion in the appendix and what has been proved already that it cannot belong to the ω -limit set. For if P_8 belonged to the ω -limit set points of its stable and unstable manifolds would also have to do so, and this has already been ruled out.

Lemma 3.3 A solution of the restriction of the system to the part of the boundary of G defined by $s = 0$ for which neither z nor Σ_+ take on one of their limiting values has the endpoint of L_2 with $s = 0$ as its ω -limit set.

Proof Along any solution of this system z is monotone increasing on the part of the state space of the restricted dynamical system with $\Sigma_+ \neq -1$ and $z(1 - z) \neq 0$. Hence, by the monotonicity principle (see [13]), any ω -limit point must satisfy $z = 1$ or $\Sigma_+ = -1$. However Σ_+ is increasing for Σ_+ close to but not equal to -1 . Hence there can be no ω -limit points with $\Sigma_+ = -1$. It follows that z tends to one as $\tau \rightarrow \infty$ for any solution and any ω -limit point satisfies $z = 1$. From Lemma 3.1, any solution which enters U_2 has the desired ω -limit set. Arguing as in the proof of Lemma 3.3 allows points with $-1 < \Sigma_+ < 0$ and $0 < \Sigma_+ < 1$ to be excluded. The point P_6 , which is a hyperbolic saddle of the restricted system, can be eliminated in the same way as was done in the case of P_8 in the proof of Lemma 3.2 using the results of the discussion in the appendix. Finally, the non-existence of ω -limit points with $\Sigma_+ = -1$, already mentioned above, shows that the endpoint of the line L_1 cannot lie in the ω -limit set.

Lemma 3.4 If a solution lies in the interior of G , then unless it lies on the unstable manifold of P_1 or P_2 its α -limit set consists of the point P_3 .

Proof Consider a solution in the interior of G which does not lie on the unstable manifold of P_1 or P_2 . If it intersects U_1 then by Lemma 3.1 its α -limit set consists of the point P_3 . There can be no other α -limit points in U_1 . Because the function M tends to zero along the solution as $\tau \rightarrow -\infty$ the α -limit set must be contained in the surface $z = 0$. Recall that the surface $z = 0$ corresponds to the case of massless particles which was analysed completely in [10]. (Note that the stationary points were numbered differently in that paper.) Consider the boundary of the surface $z = 0$. Arguing as in the proof of Lemma 3.2, the lines joining

P_3 to P_2 and P_4 can be excluded from the α -limit set. The discussion of the appendix and the fact that P_4 is a hyperbolic saddle with stable manifold $\Sigma_+ = 1$ and unstable manifold the line connecting P_4 to P_5 can be used to exclude that line and the point P_4 itself. The line connecting P_5 to the endpoint of the line L_1 can be excluded in an analogous way, noting that the non-hyperbolic point P_5 is also covered by the discussion of the appendix. The point P_5 is also excluded by this argument. Applying the reduction theorem allows the line joining the endpoint of the line L_1 to P_2 to be excluded together with the endpoint of L_1 . At this stage we can also exclude the point P_2 itself, using the results of the appendix again and the fact that by assumption the solution does not lie on the unstable manifold of P_2 . Thus the only point of the boundary of the set $z = 0$ which can belong to the α -limit set is P_3 . Now suppose that a point of the interior of the surface belongs to the α -limit set. If it is a point of the unstable manifold of P_2 then P_2 also belongs to the α -limit set, in contradiction to what has just been proved. If it is some other point other than P_1 then, using the fact that the α -limit set is a union of orbits and Theorem 3.1 of [10], it follows that P_3 belongs to the α -limit set and we obtain a contradiction again. Finally, if it were P_1 then the results of the appendix would imply that other points of the interior would belong to the α -limit set, and this has just been ruled out.

Lemma 3.5 The ω -limit point of each solution in the interior of G is a point of the line L_2 .

Proof Note first that the function M goes to infinity along any such solution as $\tau \rightarrow \infty$. It follows that any ω -limit point must satisfy $z = 1$, $s = 0$ or $s = 1$. If the solution enters the set U_2 then by part (ii) of Lemma 3.1 the ω -limit set is as claimed. There are no other ω -limit points of any solution in U_2 . Consider now the evolution of Σ_+ on the surface $z = 1$. It either increases from -1 to 0 or decreases from 1 to 0 . Since the ω -limit set is a union of orbits, we conclude that no point of the interior of the surface $z = 1$ or its boundary lines $s = 0$ and $s = 1$ other than the points of the line L_2 can belong to the ω -limit set. Using once more the fact that the ω -limit set is a union of orbits, it is possible to exclude the interior of the surface $s = 1$ from the ω -limit set by Lemma 3.2 and the interior of $s = 0$ by Lemma 3.3. Now all remaining possibilities other than points on L_2 will be excluded successively. The nature of the line L_1 as a transversely hyperbolic saddle suffices to eliminate it, as well as the lines joining it to P_8 and P_2 . The point P_3 , being a hyperbolic source, is clearly ruled out, and with it the lines joining it to P_2 and P_6 . Further applications of the results of the appendix rule out the remaining lines, namely those joining P_8 to P_5 , P_5 to P_4 , P_4 to P_7 and P_7 to P_6 . It follows that the ω -limit set is contained in the line L_2 . Applying the reduction theorem then shows that the ω -limit set is a single point of L_2 .

The results of Lemma 3.4 and Lemma 3.5 together imply Theorem 3.1.

Theorem 3.1 has been formulated entirely in terms of the dynamical systems picture. It should, however, be pointed out that this allows asymptotic expansions for all quantities of geometrical or physical interest near the singularity or in an expanding phase to be obtained

if desired. For example, in an expanding phase in type I the following expansions can be derived:

$$\Sigma_+ = \alpha t^{-1} + o(t^{-1}) \quad (18)$$

$$s = s_0 - \frac{4}{9}s_0(1-s_0)t^{-1} + o(t^{-1}) \quad (19)$$

$$z = 1 - \beta t^{-4/3} + o(t^{-4/3}) \quad (20)$$

$$H = \frac{2}{3}t^{-1} + O(t^{-7/3}) \quad (21)$$

$$\Omega = 1 - \alpha t^{-2} + o(t^{-2}) \quad (22)$$

$$\rho = \frac{4}{9}t^{-2} - \frac{4}{3}\alpha^2 t^{-4} + o(t^{-4}) \quad (23)$$

$$p_1 = O(t^{-10/3}) \quad (24)$$

Here α and β are constants depending on the solution. It should be emphasized that these are not just formal expansions, but rigorous results which emerge from the dynamical systems analysis.

A particular consequence of Theorem 3.1 is that all LRS type I models isotropize at late times. This was already proved by other means in [9], where it was also shown that non-LRS models of Bianchi type I isotropize and have dust-like behaviour for $\tau \rightarrow \infty$.

4 Type II models

The physical state space, G , of the LRS type II models is given by the region in \mathbf{R}^4 defined by the inequalities $M_2 \geq 0$, $0 \leq s \leq 1$, $0 \leq z \leq 1$, and $1 - \Sigma_+^2 - M_2 \geq 0$.

The coordinates, in terms of (Σ_+, s, z, M_2) , of the various stationary points are the following: $(0, s_0, 0, 0), (\frac{1}{2}, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (-1, 1, 0, 0), (1, 0, 1, 0), (1, 1, 1, 0), (-1, 1, 1, 0), (\frac{1}{5}, 1, 0, \frac{6}{25}), (\frac{1}{8}, 0, 1, \frac{3}{64}), (\frac{1}{8}, 1, 1, \frac{3}{64})$, where s_0 is the same particular constant value of s that appeared in the previous type I section. These points are called P_1, \dots, P_{11} (note that they are numbered differently than in [10], in the massless case). In addition there exist two lines of equilibrium points, $(-1, 0, K, 0), (0, F, 1, 0)$, denoted by L_1, L_2 , where K and F are constants. The first eight stationary points and the two lines correspond to points and lines of the same name in the Bianchi I system and their coordinates are obtained by appending a zero to those of the Bianchi I points. The points $P_1, P_2, P_3, P_4, P_6, P_7, P_8, P_9, P_{10}$ are hyperbolic saddles while P_5 is degenerate, with one zero eigenvalue. The point P_{11} is a hyperbolic sink with two real and two complex eigenvalues. The lines L_1 and L_2 are transversally hyperbolic saddles.

To prove results about the global properties of solutions it is helpful to use certain monotone functions. The first is defined for $s < 1$ by

$$Z_1 = (2s/(1-s))^{4/3} M_2 \quad (25)$$

This is obtained by rewriting the function whose time derivative was calculated in equation (23) of [10] in terms of the variables of this paper and observing that it remains monotone in the massive case. It satisfies $\dot{Z}_1 = 2qZ_1$. The second is obtained by combining Z_1 with the monotone function M available for all the Bianchi types considered in this paper. Let $Z_2 = Z_1 M^{-2} = 2^{4/3} s^2 M_2 z^{-2} (1-z)^2$ for $z > 0$. It satisfies $\dot{Z}_2 = 2(q-2)Z_2$. The function Z_1 is defined on the part of the Bianchi II state space where $s \neq 1$ and monotonically increasing except where it vanishes. This is clear if $q \neq 0$. If $q = 0$ it follows that $\Sigma_+ = 0$ and $M_2 = 1$ and at points satisfying these conditions $\dot{\Sigma}_+ \neq 0$. The function Z_2 is defined on the part of the Bianchi II state space where $z > 0$ and is monotonically decreasing except on the set where it vanishes, since $q = 2$ implies $Z_2 = 0$.

Theorem 4.1 If a smooth non-vacuum reflection-symmetric LRS solution of Bianchi type II of the Einstein-Vlasov equations for massive particles is represented as a solution of (12) with $M_3 = 0$, then for $\tau \rightarrow \infty$ it converges to P_{11} . For $\tau \rightarrow -\infty$ there exists

- (i) a one-parameter set of solutions converging to the unstable manifold of P_1 and
- (ii) a three-parameter set of all remaining solutions converging to the heteroclinic cycle on the $z = 0$ submanifold, consisting of the orbits connecting the $z = 0$ endpoint of the line L_1 to P_5 , P_5 to P_4 , P_4 to P_3 on the type I boundary and P_3 to the $z = 0$ endpoint of the line L_1 via the vacuum boundary.

Lemma 4.1 If a solution belongs to the interior of the type II state space then any α -limit point satisfies $z = 0$ and $sM_2 = 0$. Any ω -limit point satisfies $s = 1$ and $(z-1)M_2 = 0$.

Proof From the evolution equation for M it follows that $z = 0$ for any α -limit point and that for any ω -limit point $z = 1$, $s = 0$ or $s = 1$. Next the monotonicity principle will be applied to the functions Z_1 and Z_2 . Applying it to Z_1 on the region where $Z_1 \neq 0$ shows that for any α -limit point $s = 0$ or $M_2 = 0$. It also shows that there are no ω -limit points with $s = 0$. Combining this with the information obtained already shows that any ω -limit point satisfies $z = 1$ or $s = 1$. If $z \neq 1$ then it follows from the monotonicity principle applied to Z_2 that $M_2 = 0$ for any ω -limit point. The monotonicity of Z_1 then implies that $s \rightarrow 1$ as $\tau \rightarrow \infty$.

Lemma 4.2 Consider the dynamical system obtained by restricting the type II system to the plane defined by the conditions $s = 1$ and $z = 1$. If a solution belongs to the interior of the state space for this restricted system then it converges to P_{11} as $\tau \rightarrow \infty$.

Proof The restricted dynamical system is identical with that for type II dust solutions. In [13] it was proved by using a monotone function derived by Hamiltonian methods that for $\tau \rightarrow \infty$ the dust solutions satisfy $\Sigma_+ \rightarrow \frac{1}{8}$ and $M_2 \rightarrow \frac{3}{64}$. Hence it can be concluded that the solution approaches P_{11} as $\tau \rightarrow \infty$.

Lemma 4.3 If a solution lies in the interior of the type II state space then unless it lies on the unstable manifold of P_1 (and this does occur) the α -limit set consists of the heteroclinic cycle described in the statement of Theorem 4.1.

Proof By Lemma 4.1 we know that any α -limit point satisfies $z = 0$. Moreover it satisfies $M_2 = 0$ or $s = 0$. The situation is very similar to that in the massless case treated in [10]

and the proof may be taken over rather directly. It is only necessary to pay attention to the fact that it is the nature of the stationary points in the full massive Bianchi II state space which must be taken into account and that the notation is different.

Suppose that the α -limit set contains a point with $s = 0$ and $M_2 \neq 0$. Then by Lemma 4.3 of [10] it contains the endpoint of L_1 and either P_2 or P_3 . On the other hand, if it contains a point with $M_2 = 0$ then this belongs to the massless Bianchi I state space. Then it must contain one of the points P_1, P_2, P_3, P_4, P_5 or the endpoint of L_1 . To prove the lemma we may assume that the solution does not lie on the unstable manifold of P_1 . If P_1 nevertheless belonged to the α -limit set then this set would have to include points belonging to the unstable manifold of P_1 other than P_1 itself. But these satisfy neither $s = 0$ nor $M_2 = 0$ and so this gives a contradiction. Thus under the given assumptions the α -limit set does not contain P_1 . If the α -limit set contained a point with $M_2 = 0$, $|\Sigma_+| < 1$ and $0 < s < 1$ it would contain P_1 (in its role as ω -limit set for Bianchi type I solutions), leading once more to a contradiction. If the α -limit set contained P_2 then by the results of the appendix it would contain P_1 , which is also not possible. Applying Lemma 4.3 of [10] again allows points with $M_2 \neq 0$ which are not on the vacuum boundary to be excluded from the α -limit set. The straight lines joining P_2 to P_3 and the endpoint of L_1 are excluded as well. The conclusion is that the α -limit set is contained in the heteroclinic cycle mentioned in the statement of Theorem 4.1. It remains to show that it is the whole heteroclinic cycle. This is straightforward to do using the results of the appendix.

Lemma 4.4 If a solution lies in the interior of the type II state space then it converges to the point P_{11} as $\tau \rightarrow \infty$.

Proof Consider any ω -limit point with $z \neq 1$. Then by Lemma 4.1 this point satisfies $s = 1$ and $M_2 = 0$. Any nearby ω -limit points must also satisfy these conditions. If any of these limit points satisfied $z = 0$ then P_4 and P_5 would be ω -limit points of the given solution. Using the saddle point properties of these points then shows that P_7 and P_8 belong to the ω -limit set. Repeating the same argument shows that the endpoint of L_1 with $s = 1$ is an ω -limit point. The fact that this point is a transversely hyperbolic saddle implies that its unstable manifold in the hyperplane $s = 1$ is contained in the ω -limit set. By Lemma 4.2 the ω -limit set also contains P_{11} . Since P_{11} is a hyperbolic sink this contradicts the assumption $z \neq 1$. Thus we conclude that the entire ω limit set is contained in the plane defined by the equations $s = 1$ and $z = 1$. The argument just given rules out the possibility of ω -limit points with $M_2 = 0$. Applying Lemma 4.2 once more shows that the only possible ω -limit point which does not lie on the vacuum boundary is P_{11} . Finally the fact that P_7 and P_8 are hyperbolic saddles can be used to rule out points of the vacuum boundary, thus completing the proof.

5 Type III models

The physical state space, G , of the LRS type III models is given by the region in \mathbf{R}^4 defined by the inequalities $M_3 \geq 0$, $0 \leq s \leq 1$, $0 \leq z \leq 1$, and $1 - \Sigma_+^2 - M_3 \geq 0$.

The coordinates, in terms of (Σ_+, s, z, M_3) , of the various stationary points are the following: $(0, s_0, 0, 0)$, $(\frac{1}{2}, 0, 0, 0)$, $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(-1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 1, 1, 0)$, $(-1, 1, 1, 0)$, $(\frac{1}{2}, 0, 0, \frac{3}{4})$, $(\frac{1}{2}, 0, 1, \frac{3}{4})$, where s_0 the same particular constant value of s that appeared in the previous type I section. These points are called P_1, \dots, P_{10} (note that they are numbered differently than in [10], in the massless case). In addition there exist three lines of equilibrium points, $(-1, 0, K, 0, 0)$, $(0, F, 1, 0, 0)$, $(\frac{1}{2}, 1, z_0, \frac{3}{4})$, denoted by L_1, L_2, L_3 , where K, F and z_0 are constants. The first eight stationary points and the first two lines correspond to points and lines of the same name in the Bianchi I system and their coordinates are obtained by appending a zero to those of the Bianchi I points. The points $P_1, P_2, P_4, P_6, P_7, P_8, P_9$ are hyperbolic saddles while P_5 and P_{10} are degenerate, with one zero eigenvalue each. The point P_3 is a hyperbolic source. The lines L_1 and L_2 are transversally hyperbolic saddles while the line L_3 is degenerate with two zero and two negative eigenvalues.

To prove global results about the global properties of solutions it is useful to note the existence of the following bounded monotone function

$$\begin{aligned} \tilde{M}_3 &= M_3(2 - \Sigma_+)^{-2} , \\ \dot{\tilde{M}}_3 &= 2\tilde{M}_3[(1 - 2\Sigma_+)^2 + \Omega(R + R_+)](2 - \Sigma_+)^{-1} . \end{aligned} \quad (26)$$

Theorem 5.1 If a smooth non-vacuum reflection-symmetric LRS solution of Bianchi type III of the Einstein-Vlasov equations for massive particles is represented as a solution of (12) with $M_2 = 0$, then for $\tau \rightarrow \infty$ it converges to a point of the line L_3 with $z > 0$. For $\tau \rightarrow -\infty$ there exists

- (i) a one-parameter set of solutions lying on the unstable manifold of P_1 and
- (ii) a two-parameter set of solutions lying on the unstable manifold of P_2 and
- (iii) all remaining solutions converge to P_3 .

In all these solutions the scale factor a is monotone increasing at late times.

Lemma 5.1 If a solution belongs to the interior of the type III state space any α -limit point satisfies $M_3 = 0$. Any ω -limit point satisfies $\Sigma_+ = \frac{1}{2}$, $M_3 = \frac{3}{4}$ and $s = 1$.

Proof The continuous function \tilde{M}_3 on the state space must have a maximum and since its gradient never vanishes this maximum can only be attained at points with $M_3 = 1 - \Sigma_+^2$. Computing the derivative of \tilde{M}_3 along the curve in the (M_3, Σ_+) plane defined by this relation shows that the maximum value is $\frac{1}{3}$ and that it is attained when $\Sigma_+ = \frac{1}{2}$ and $M_3 = \frac{3}{4}$. Now we apply the monotonicity principle. Let S be the part of the Bianchi III state space obtained by removing the points with $M_3 = 0$ and those with $\Sigma_+ - \frac{1}{2} = M_3 - \frac{3}{4} = 0$. This is an invariant set for the dynamical system. It will now be shown that \tilde{M}_3 is strictly increasing

along solutions on this set. If $\Sigma_+ \neq \frac{1}{2}$ or if $\Omega(R + R_+) \neq 0$ then this follows immediately from (26). If $\Sigma_+ = \frac{1}{2}$ and $\Omega(R + R_+) = 0$ then

$$\dot{\Sigma}_+ = \frac{3}{4}(M_3 - \frac{3}{4}) \quad (27)$$

This completes the proof that \tilde{M}_3 is strictly increasing on S . The monotonicity principle then shows that any point in the α -limit set must be in the complement of S and such that such that \tilde{M}_3 does not take on its maximum value on \bar{S} there. Hence $M_3 = 0$ there. It also shows that any point in the ω -limit set must be in the complement of S and that \tilde{M}_3 does not take on its minimum value there. Hence in the latter case $\Sigma_+ = \frac{1}{2}$ and $M_3 = \frac{3}{4}$. It follows from this that $\Sigma_+ \rightarrow \frac{1}{2}$ as $\tau \rightarrow \infty$ and the equation for s then implies that $s \rightarrow 1$.

Lemma 5.2 A solution which belongs to the interior of the type III state space converges to a point of the line L_3 with $z > 0$ as $\tau \rightarrow \infty$.

Proof Because of the result of Lemma 5.1 it only remains to prove that z tends to a positive limit as $\tau \rightarrow \infty$. Note first that the evolution equation for s implies an equation of the form $(d/d\tau)(1 - s) = (1 - s)F$ where $F = -6s\Sigma_+$. As τ tends to infinity $F \rightarrow -3$ and a simple comparison argument proves that $1 - s(\tau) = O(e^{(-3+\epsilon)\tau})$ as $\tau \rightarrow \infty$. In particular, $1 - s$ decays exponentially to zero at late times. The evolution equations imply that Ω satisfies the equation:

$$\dot{\Omega}/\Omega = (\Sigma_+ - \frac{1}{2})[(3 - R)\Sigma_+ + \frac{3}{2}(1 + R)] - 2\Sigma_+(R_+ + R) - (M_3 - \frac{3}{4})(1 + R) \quad (28)$$

Note that $R_+ + R \geq 0$ so that the second term on the right hand side is negative. However it is exponentially small at late times since it contains a factor $(1 - s)$ when expressed in terms of the matter quantities. In particular $\dot{\Omega}/\Omega \rightarrow 0$ as $\tau \rightarrow \infty$ and $\Omega^{-1} = O(e^{\epsilon\tau})$ for any $\epsilon > 0$. This means that Ω converges to zero slower than any exponential. In other words, $\Omega e^{\epsilon\tau}$ tends to infinity for any $\epsilon > 0$. Suppose that $\Sigma_+ \geq \frac{1}{2}$ for some solution at some time. Then $M_3 \leq \frac{3}{4}$ and the first and third terms in the expression for $\dot{\Omega}/\Omega$ are positive at late times while the second term is negative. It will now be shown that the third term decays slower than any exponential and thus must eventually dominate the second term. For

$$\Omega = (\frac{1}{4} - \Sigma_+^2) + (\frac{3}{4} - M_3) \leq (\frac{3}{4} - M_3) \quad (29)$$

It follows that $\dot{\Omega}/\Omega > 0$ at late times as long as $\Sigma_+ > \frac{1}{2}$. Since it follows from Lemma 5.3 that $\Omega \rightarrow 0$ as $\tau \rightarrow \infty$ it follows that for any time τ_0 for which $\Sigma_+(\tau_0) > \frac{1}{2}$ there exists a time $\tau > \tau_0$ with $\Sigma_+ = \frac{1}{2}$. When $\Sigma_+ = \frac{1}{2}$ then

$$\dot{\Sigma}_+ = (M_3 - \frac{3}{4})[\frac{3}{4}(1 - R) - \frac{1}{4}(R + R_+)] \quad (30)$$

Now it follows from the evolution equation for z that $1 - z$ cannot approach zero faster than, for instance, $e^{-\tau}$ and the same is then true of $1 - R$. It can be concluded that the first term in

the square bracket on the right hand side of (30) dominates the second at late times. Hence $\frac{1}{2} - \Sigma_+$ must be negative at late times, which in turns implies that z is increasing and that it must tend to a positive limit.

Lemma 5.3 If a solution lies in the interior of the type III state space then unless it lies on the unstable manifold of P_1 or P_2 (and both of these cases occur) the α -limit set consists of the point P_3 .

Proof Note first that it can be concluded as in the proof of Lemma 3.4 that any α -limit point satisfies $z = 0$. Thus, applying Lemma 5.1, it can be identified with a point of the state space for massless type I solutions. Now it is possible to proceed further following the method of proof of Lemma 3.4. Consider the boundary of the state space for massless type I solutions. The point P_3 , being a hyperbolic source in the type III state space, can be excluded as an α -limit point of a solution of type III. It is then possible to successively exclude points of the boundary as in the proof of Lemma 3.4. The facts which need to be used are that all α -limit points satisfy $M_3 = 0$ and $z = 1$ and that the points P_4 , P_5 and the endpoint of L_1 are a hyperbolic saddle, a non-hyperbolic saddle topologically equivalent to a hyperbolic one and a transversely hyperbolic saddle, respectively. At this stage it can be concluded that all α -limit points of solutions of type III are either P_1 , P_2 or points of the unstable manifold of P_1 . For all other points of the interior of the massless type I state space lie on solutions which converge to the hyperbolic source P_3 in the past time direction, and so are excluded. It remains to examine what happens in a neighbourhood of the points P_1 and P_2 , which are both hyperbolic saddles. The unstable manifold of P_2 in the type III state space is three-dimensional and so there are solutions which converge to P_2 as $\tau \rightarrow -\infty$. Any other type III solutions which had P_2 as an α -limit point would have to have α -limit points on the stable manifold of P_2 , which has already been excluded. Hence solutions of type III which do not converge in the past to P_2 cannot have P_2 or a point of its unstable manifold as α -limit points. Thus the only remaining possibility is that solutions lie on the unstable manifold of P_1 and converge to that point in the past. Since the unstable manifold is two-dimensional, solutions of this kind exist.

The results of Lemma 5.2 and Lemma 5.3 together imply all the results of Theorem 5.1 except the last directly. The statement about the scale factor a follows from the fact, derived in the course of the proof of Lemma 5.2, that $\Sigma_+ < \frac{1}{2}$ at late times.

6 Concluding remarks

In this paper we studied the dynamics of solutions of the Einstein-Vlasov equations which are locally rotationally symmetric, reflection-symmetric and of Bianchi types I, II and III. The initial singularities are of four types. There are isotropic singularities which, in the dynamical

systems description used in this paper, are those which converge to the point P_1 as $\tau \rightarrow -\infty$. The general theory of isotropic singularities developed by Anguige and Tod [2, 1] implies as a very special case the occurrence of isotropic singularities in Bianchi models with collisionless matter and information about how many there are. They only developed the theory for massless particles and so in order to apply to the situations considered here it would have to be generalized to the massive case. There are barrel singularities which occur in types I and III but not in type II. In the dynamical systems picture these are the solutions which converge to P_2 as $\tau \rightarrow -\infty$. Fluid models with corresponding symmetries never have barrel singularities and so this is a peculiarity of collisionless matter, both in the case of massive particles studied here and that of massless particles studied in [10]. There is the generic case in types I and III, which concerns solutions which develop from an open dense set of initial data for each of these Bianchi types. These solutions have a cigar singularity and converge to P_3 as $\tau \rightarrow -\infty$. Finally, there are the generic solutions of type II, which have an oscillatory initial singularity.

As far as the late time behaviour is concerned, it is tempting to speculate that behaving like a dust model at late times in an expanding phase may be a general feature of solutions of the Einstein-Vlasov equations with massive particles. We know of no counterexample to this. For the solutions of types I and II treated in this paper it has been proved to be true. For type III the situation appears to be delicate and the occurrence of degenerate stationary points of the dynamical system may require an application of centre manifold theory in order to determine details of the asymptotics. A possible criterion for detecting cases where there may be trouble is as follows. Consider a dust solution which is a candidate for the asymptotic state of solutions of the Einstein-Vlasov equations. If each eigenvalue of the second fundamental form of the homogeneous hypersurfaces, when divided by the mean curvature, is bounded below by a positive constant in the dust solution then it is a strong candidate. Otherwise difficulties are to be expected. This criterion gives a positive recommendation for types I and II and a warning for type III. Thus at least for the models investigated in this paper it is a good guide. Using the information on dust models in chapter 6 of [13] it also gives a positive recommendation for types I, II and VI_0 without the need to restrict to the LRS case.

In this paper a dynamical system has been set up for all LRS Bianchi models of class A as well as for Kantowski-Sachs models and the type III models, which are of class B. We expect that techniques similar to those used here can be applied to analyse Kantowski-Sachs models and LRS models of type VIII and IX. An important feature of all these LRS models is that the Vlasov equation can be solved exactly. This is also true of general Bianchi type I models. Some limited results on the dynamics of Bianchi type I solutions of the Einstein-Vlasov equations which are reflection-symmetric but not necessarily LRS were proved in [9]. A heuristic analysis of reflection-symmetric type I models was given using Hamiltonian techniques in [7], where there are also interesting remarks on the general Bianchi I case. It would be very desirable to have a mathematically rigorous implementation of the ideas of [7].

What can be done in cases where the Vlasov equation cannot be solved exactly? If, as already speculated above, the late time evolution resembles that of a dust solution and if the dust solutions are asymptotically LRS then it may be possible to give a good approximation to the solution of the Vlasov equation in that regime. There is one drawback of this idea as a general tool for Bianchi class A models. Unfortunately there are no LRS spacetimes of Bianchi type VI₀. In the case of fluids there exists a special class of Bianchi VI₀ spacetimes which is often characterized by the rather opaque statement that $n_\alpha^\alpha = 0$. These spacetimes do have a simple geometric characterization which will now be explained. Every Bianchi class A spacetime has a discrete group of isometries whose generators simultaneously reverse two of the invariant one-forms on the group. The special class of Bianchi VI₀ solutions can be characterized by the existence of an additional isometry which reverses just one of the one-forms. It is possible to consider solutions of the Einstein-Vlasov equations with the corresponding type of symmetry. We are, however, not aware that the Vlasov equation can be solved exactly in these special spacetimes. If it could then this might fill the apparent gap in the strategy just suggested.

The oscillatory behaviour observed near the singularity in type II models appears at first sight to indicate that collisionless matter does not fit into the analysis of general spacetime singularities by Belinskii, Khalatnikov and Lifshitz [4]. On the other hand, the fact that in the analysis of Misner [7] using a time-dependent potential we see the phenomenon of walls moving too fast to be caught suggests that the oscillations might go away in general models. This issue requires further work. It could turn out that collisionless matter generically becomes negligible near the singularity, as originally stated for fluids in [4].

To conclude, we mention some further interesting open problems. What happens in the case of a model with two species of particles, one massive and one massless? Of course this could be thought of as a simple cosmological model incorporating both baryonic matter and the microwave background photons. It is related to the two-fluid models which have been analysed in [5]. Mixtures of fluids and kinetic theory could also be considered. We have seen that the Einstein equations with collisionless matter as source may behave very differently from the Einstein equations with a fluid source at early times (and also at late times in the massless case). Under what circumstances are there intermediate stages of the evolution with collisionless matter which can be well described by a fluid? Since the point P_1 is a saddle there are obviously solutions which approach this point and then go away again but is there more that can be said about this issue? What can be said about inhomogeneous models? In [8] Rein analysed the behaviour at early times of solutions of the Einstein-Vlasov equations with spherical, plane and hyperbolic symmetry and massive particles. He identified open subsets of initial data for these symmetry types with a singularity resembling the generic LRS solutions of types I and III. There is an overlap between the results of [8] and those of the present paper. It could be illuminating to attempt a common generalization of these. In any case, it is clear that one of the central challenges of the future in the study of cosmological solutions

of the Einstein-Vlasov equations, or indeed the Einstein equations coupled to any type of matter fields, is to develop techniques which apply to inhomogeneous problems. A thorough understanding of the homogeneous case is likely to be an invaluable guide in addressing it.

A Appendix

In this appendix some general procedures which are useful in determining limit sets of solutions of dynamical systems will be outlined. Let γ be an orbit of a dynamical system and p a stationary point. We will discuss only ω -limit sets, but corresponding statements about α -limit sets follow immediately by reversing the direction of time. We consider the following three statements which may or may not be true for given choices of γ and p .

1. p is an ω -limit point of γ
2. γ lies on the stable manifold of p
3. there are ω -limit points of γ different from p which are arbitrarily close to p and lie on the unstable manifold of p
4. there are ω -limit points of γ different from p which are arbitrarily close to p and lie on the stable manifold of p

In the body of the paper we frequently use certain relations among the statements above which hold under various assumptions on the nature of the stationary point p . Whatever the stationary point, it is always true that the statement 1. is implied by any of the statements 2., 3. or 4. This is a consequence of the elementary fact that the ω -limit set is closed. Now suppose that p is a hyperbolic stationary point. In this case, if 1. is true and 2. is false then both 3. and 4. are true. This follows from Lemma A1 of [10]. Combining these statements we see that for a hyperbolic stationary point there are two mutually exclusive cases under which 1. can hold. Either γ lies on the unstable manifold of p or the ω -limit set contains points of both the stable and unstable manifolds of p arbitrarily close to p . In particular, if p is a hyperbolic source then it cannot be in the ω -limit set of γ and if p is a hyperbolic sink and p is in the ω -limit set of γ it is the whole ω -limit set. If we already have some a priori information about where ω -limit points can lie (due, for instance, to the existence of a monotone function) then this gives more information about where the points on the stable and unstable manifolds whose existence is guaranteed by the general statements above can lie.

Next we consider the case of transversally hyperbolic stationary points. Suppose that p belongs to a manifold of stationary points of dimension d . (Only the case $d = 1$ occurs in this paper.) These points have a zero eigenvalue of multiplicity d . If all other eigenvalues

have non-vanishing real parts then the stationary point is called transversally hyperbolic. (Depending on the signs of the eigenvalues the manifold of stationary points is called a transversally hyperbolic source, sink or saddle.) By the reduction theorem (Theorem A1 of [10]) each of these points lies on an invariant manifold and the restriction of the flow to each invariant manifold is topologically equivalent to that near a hyperbolic stationary point. The arguments for a hyperbolic stationary point adapt easily to give analogous statements for transversally hyperbolic stationary points. In applying these results we can essentially ignore the directions along the manifold of stationary points.

Finally we consider certain other non-hyperbolic stationary points. A result of the type we need was proved in Lemma A2 of [10] but we would like to formulate the statement in a more transparent way here. Consider an isolated stationary point p with a trivial stable manifold and a one-dimensional centre manifold. Using the reduction theorem we see that the unstable manifold divides a neighbourhood of p into two parts on each of which the restriction of the dynamical system is topologically equivalent to the restriction of a dynamical system with a hyperbolic stationary point. Whether the latter system has a saddle or a source depends on a certain sign condition. This condition may be different for the two halves. In the dynamical systems considered in this paper the only example of this is provided by the point P_5 . Only one of the halves belongs to the physical part of the state space and in that half the sign is such that a saddle is obtained. The result of these considerations is that for the arguments in this paper P_5 may be treated just as if it had been a hyperbolic saddle, with the centre manifold taking over the role of the trivial stable manifold.

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