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Foundations of Gravitational Lens Theory (Geometry of Light Cones)

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Abstract. The main concepts of gravitational lens theory are introduced on the basis of spacetime geometry without assuming approximations. The singularities of light cones, in particular their caustics, are reviewed as examples of singularities of Lagrangian resp. Legendrian maps. It is indicated how the usual approximate lens theory may be derived from the general framework.

Keywords: gravitational lensing, light cone structure, caustics
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1 Introduction

After the discovery of the double quasar QSO O957 + 561 A, B by Walsh, Carswell and Weymann in 1979, gravitational lensing rapidly developed into a major tool of astrophysics, providing infomation about cosmological parameters, masses and mass distributions on the scales of stars, galaxies, galaxy clusters and that of the universe at large. It enables astronomers to obtain information about dark matter, the structure of quasars and very distant, early generation galaxies up to redshifts of $z \approx 5$.

Usually, gravitational lens theory is based on plausible assumptions and various approximations designed for astrophysical applications. Physical notions and relations are expressed essentially in the framework of classical geometrical optics, with minimal input from general relativity. While such an approach is useful for the intended purpose, it conceals the spacetime-geometrical origin of lensing phenomena. Moreover, by using ab initio simplifications based on intuition, one foregoes the possibility to assess the accuracy of approximations, and one may not even recognize which general relativistic relations are being approximated. Besides, such presentations may render it difficult for relativists, used to think in terms of light cones, timelike world lines and the like, to understand what it's all about.

Be that as it may, here I want to outline how the basic qualitative relations of gravitational lens theory may be introduced as part of Lorentzian spacetime geometry

[a] *I dedicate this paper to George Ellis, with affection and gratitude for fourty years of many stimulating encounters and sharing of ideas.*

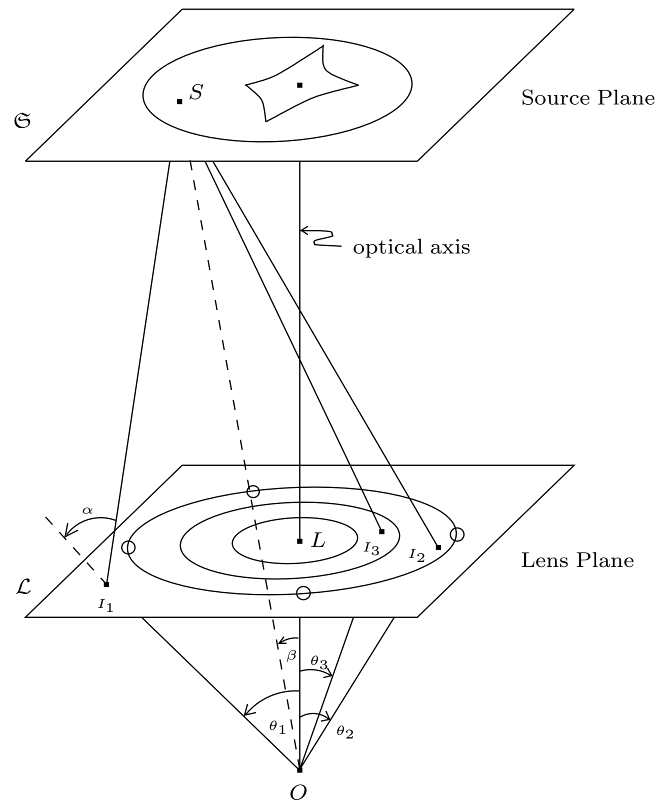

Fig. 1 The lens map according to the standard presentation of lens theory.

without use of perturbation theory. Approximations are introduced only in the last section, intended to indicate the transition to standard, "linear" lens theory where models of lensing mass distributions in an expanding universe are needed which one cannot hope to describe as exact solutions to the gravitational field equations. Thus, the following considerations aim at understanding, not at computing.

The following presentation owes much to collaboration with Ted Newman and Simonetta Fritelli and overlaps with our joint paper [5].

2 The standard formalism

Suppose a point source S (quasar) is seen by an observer O, close to a deflecting mass distribution L (galaxy) and seen by O in three images I_i at angular positions $\vec{\theta_i}$ with respect to the center of the "lens" L (see Fig.1). Since in real cases the deflection angles are very small, the light rays may be approximated by broken straight lines. If the distances OL,OS and LS are denoted as D_L, D_S and D_{LS}, respectively, if Euclidean geometry is applied, and if the angles are represented by tangent vectors to O's sphere of vision, the approximate relations

$$\vec{\beta} = \vec{\theta} - \vec{\bigtriangledown}\Psi(\vec{\theta}) \tag{1}$$

between the (unobservable) "true position" $\vec{\beta}$ of S and the positions $\vec{\theta}$ of its images are easily obtained. In eq.(1),

$$\Psi(\vec{\theta}) = \frac{4GD_{LS}}{c^2 D_L D_S} \cdot \int dM' ln|\vec{\theta} - \vec{\theta}'| \tag{2}$$

is the logarithmic *deflection potential* associated with the mass distribution of L, projected into the *lens plane* $\mathcal{L}$ drawn through some "center" of L orthogonally to the "optical axis" OL. The deflection angle is then

$$\vec{\alpha} = \frac{D_S}{D_{LS}} \vec{\bigtriangledown}\Psi. \tag{3}$$

(Eqs.(2) and (3) express that α is the sum of the well known Einstein deflection angles of the mass elements dM' at $\vec{\theta}'$.)

In order to consider extended sources (and for other reasons) it is useful to introduce also a *source plane* $\mathfrak{S}$ and to study the *lens map*

$$\mathfrak{l} : \mathcal{L} \to \mathfrak{S}, \vec{\theta} \mapsto \vec{\beta} \tag{4}$$

defined by (1). This map is nonlinear and not injective. Generically the critical points of $\mathfrak{l}$ (where the derivative $l_\star$ vanishes) form smooth, non-intersecting, closed *critical curves*; their images, the *caustic curves* of $\mathfrak{l}$, may have cusps and intersect each other.

Consider S as moving around in $\mathfrak{S}$. As long as S is outside of all caustic curves, it has exactly one image I. If S reaches a caustic, there appears another (critical) image which splits into two when S moves on, having crossed the caustic. The caustic curves not only serve to determine the number of images, they are also the positions which lead to particularly bright images.

Maps of the plane into itself have been studied by H. Whitney who showed in 1955 that the only stable singularities of such maps are folds and cusps, as indicated in Fig. 1.

The image positions corresponding to a source at $\vec{\beta}$ are given as the solutions $\vec{\theta}$ of

$$\vec{\bigtriangledown}\phi = 0 \tag{5}$$

where

$$\phi = \frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \Psi(\vec{\theta}). \tag{6}$$

Physically, (5) is an expression of Fermat's principle. In the terminology of catastrophe theory, (5) shows that $\vec{\beta}$ may be considered as a control parameter which determines, via (5) and the "potential" ϕ, the states $\vec{\theta}_i$; whenever $\vec{\beta}$ crosses a caustic, a "catastrophe" occurs. Catastrophe theory provides information about singularities which are stable in *families* of lens maps which arise when the deflection potential is taken to depend on parameters such as the D's in (2), or properties of the lens.

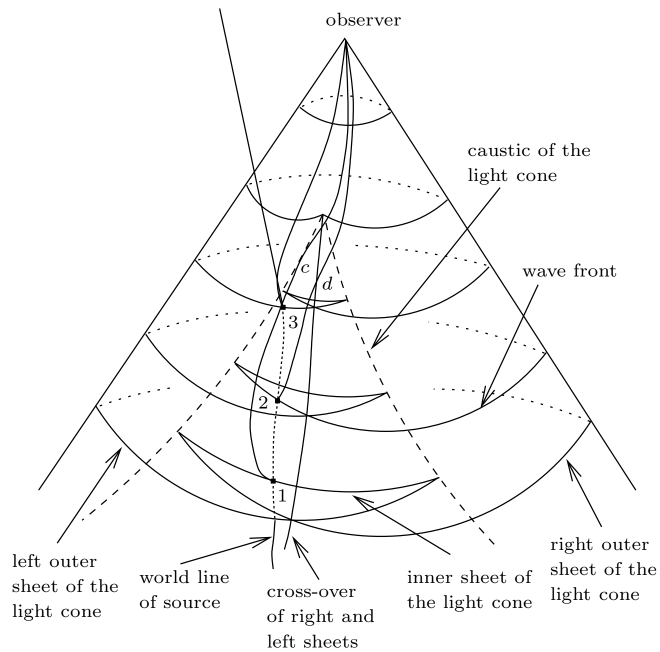

Fig. 2 A typical light cone in (1+2) dimensional spacetime. In (1+3) dimensions, the wave fronts, the caustic and the crossover set are 2dimensional. The world line of a source intersects the light cone at the events marked 1, 2 and 3 which correspond to 3 images of the source seen by the observer; compare Fig. 1.

To travel from S to O, light needs different times along different rays. The arrival time differences between images I_i, I_j are given by

$$c\Delta t_{ij} = \frac{D_L D_S}{D_{LS}} \cdot (1 + z_L) \cdot (\phi(\vec{\theta}_i, \vec{\beta}) - \phi(\vec{\theta}_j, \vec{\beta}). \tag{7}$$

(The redshift z_L of the lens with respect to O, absent in the simple static situation considered so far, has been included to cover the cosmological case.)

The preceding equations can be used to obtain relations between observables and properties of sources and lenses; the formalism can be generalized to several lens planes between source and observer. For a systematic account see, e.g., [1] and the references therein.

We now ask: How does the preceding scenario arise from GR? Which space is shown in Fig. 1, what is meant by the distances if source, deflector and ovserver are situated in an expanding universe?

3 Light cone singularities

To answer the questions just raised, one has to study light cones.

Let (M, g) denote a (time-oriented, (1+3)-dimensional) spacetime. Let C denote the past *null* (half) *cone* in the tangent space M_O of some (observation) event O without its apex, so C has topology $\mathbb{R} \times \mathbb{S}^2$. The *light cone* $\mathfrak{C}$ of O in M is defined as the image of C under the exponential map at O,

$$\exp_O : C \xrightarrow{\gamma} \mathfrak{C}. \tag{8}$$

While C is an ordinary, circular cone, its image $\mathfrak{C}$ in general has two kinds of *singular points*: intersection points where two lightlike geodesic generators of $\mathfrak{C}$ meet, and conjugate points. Intuitively the latter are those events at which "infinitesimally close" generators of $\mathfrak{C}$ "intersect to first order". To define them properly one considers the equation of geodesic deviation

$$\ddot{Y}^a = R^a_{bcd}\, k^b K^c Y^d \tag{9}$$

which defines (connection) vector fields $Y^a(s)$ on geodesics $x^a(s)$ with tangents $K^a(s)$. Then, points conjugate to O on a generator $x^a(s)$ of $\mathfrak{C}$ are those events P for which there exists a (not identically vanishing) Jacobi field $Y^a(s)$ which vanishes at O and P. The events conjugate to O on any generator form the *caustic* of $\mathfrak{C}$, the intersection points constitute the *crossover set* of $\mathfrak{C}$. Fig. 2 shows a light cone in a (1+2) dimensional spacetime.

In a curved spacetime, light cone singularities are not exeptional, but occur generically; thus light cones in general are not immersed submanifolds of M. The deformation of light cones by matter and the ensuing formation of singularities are the origin of gravitational lens phenomena, as Fig. 2 indicates.

At critical points, the map γ defining a light cone - (8) - is not smoothly invertible. To avoid using ill-defined multivalued and non-smooth functions, it turns out to be useful to "lift" light cones (and more generally null hypersurfaces) from spacetime into phase space and to study the resulting projection back into M. The general setting for this procedure will be briefly summarized next.

4 Lagrangian and Legendrian submanifolds, maps and singularities

The cotangent bundle $T^\star M = \{x^a, p_a\}$ of an n-manifold M, e.g. spacetime, carries a canonical 1-form $\kappa = p_a dx^a$ and a symplected 2-form $\omega = d\kappa = dp_a \wedge dx^a$. A submanifold N of $T^\star M$ on which the pull-back of ω vanishes is called *Lagrangian*, and the restriction to N of the canonical projection π of $T^\star M$ onto its base M, $\nu : N \to M$, is said to be a *Lagrangian map.*

Consider next the projectivized cotangent bundle $PT^\star M$ which arises by regarding the momentum coordinates p_a as homogeneous coordinates on each fibre, viewed as an (n-1) dimensional projective space. The canonical 1-form κ on $T^\star M$ defines at each point of $T^\star M$ a contact hyperplane consisting of all tangent vectors to $PT^\star M$ at a point (x, p) which are annihilated by κ. Thus, the total space $PT^\star M$ of the bundle is a contact space. An (n-1) dimensional submanifold Λ of $PT^\star M$ whose tangent vectors are all contained in the contact hyperplanes at their respective positions, is called a *Legendrian submanifold* of $PT^\star M$, its projection $\lambda : \Lambda \to M$ is referred to as a *Legendrian map.*

The usefulness of these concepts in the present context (and similarly in mechanics) is due to the fact that (at least) for $n \leq 5$, all stable singularities of such maps - the local forms of ν respectively λ near points where the rank of the derived maps $\nu_\star$ resp. $\lambda_\star$ are not maximal, have been classified and turned out to be remarkably simple ([2]). For an introduction with examples from physics see, e.g., [12], available as gr-qc/9906065.)

For n = 4 there are, in particular, five types of Legendrian singularities.

As we shall describe below, the map $\mathfrak{l}$ of (4) is Lagrangian, and the map γ of (8) is Legendrian.

As an example of a Legendrian map which will be useful in sec. 6 to illustrate lensing, take $M = \mathbb{R}^3 = \{x, y, z\}$. On that part of $PT^\star M$ where $p_z \neq 0$ we set $p_z = -1$ and take as coordinates x, y, z, p_x, p_y. A Legendrian submanifold is then given by

$$\Lambda : x = 4p_x^3 - 2yp_x, z = 3p_x^4 - yp_x^2, p_y = p_x^2,$$

where p_x and y form coordinates on Λ. (Indeed, on $\Lambda, \kappa = p_x dx + p_y d_y - dz = 0$.) The Legendrian map is

$$\lambda : (p_x, y) \mapsto (x, y, z) \equiv (4p_x^3 - 2yp_x, y, 3p_x^4 - yp_x^2).$$

The critical points of λ form the smooth curve

$$(x, y, z, p_x, p_y) = (-8p_x^3, 6p_x^2, -3p_x^4, p_y, p_x^2).$$

Its image, the caustic curve

$$(x, y, z) = (-8p_x^3, 6p_x^2, -3p_x^4),$$

has a cusp at the origin, where the kernel of $\lambda_\star$ is tangent to the critical curve. The crossover set of the image of Λ in $\mathbb{R}^3$ is the curve $x = 0, z = \frac{1}{4}y^2, y > 0$. The shape of the "surface" $\lambda(\Lambda)$ is qualitatively shown in Fig. 3. At the caustic curves, marked A_2, two sheets of the surface meet in *cusp ridges*. These meet at the *swallowtail singular point* marked A_3, which is also the end point of the crossover curve.

5 Light cones as images of Legendrian maps

The definition (8) of the light cone $\mathfrak{C}$ of an event O can be reformulated as follows. Let U denote a timelike, future-directed unit vector at O, let $\vec{e}$ run trough all unit vectors orthogonal to U, and let s denote that affine parameter on the generator of $\mathfrak{C}$ with initial tangent $\vec{e} - U$ which vanishes at O. $\mathfrak{C}$ is then given by

$$x^a = X^a(\vec{e}, s). \tag{10}$$

(For given $\vec{e}$, s ranges through the maximal interval $0 \leq s < s_m$ for which the corresponding generator exists). If U is the 4-velocity of an observer at O, the event x^a is seen by that observer in the direction $\vec{e}$ at affine distance s. The lift of the map (10) into $PT^\star M$ is defined by

$$\begin{aligned} \hat{\gamma} : C \hookrightarrow \hat{\mathfrak{C}} \subset \; PT^\star M, \\ (\vec{e}, s) \hookrightarrow (X^a(\vec{e}, s), p_a = g_{ab}\partial_s X^b(\ldots)), \end{aligned} \tag{11}$$

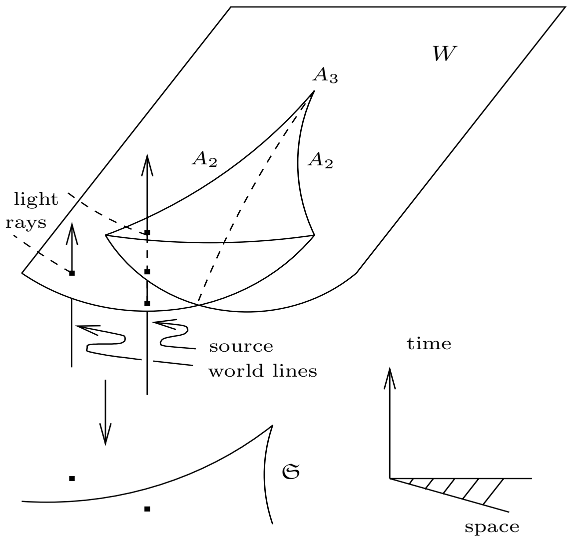

Fig. 3 The range of W of a legendrian map λ with 2 cusp ridges A_2, a swallowtail singularity A_3 and a crossover curve. Further details of the Figure (source world lines, light rays and the surface $\mathfrak{S}$ will be explained in sec. 6 in relation to Fig.4.

where as before the p_a are to be taken as homogeneous fibre coordinates. $\hat{\gamma}$ is an embedding of C into $PT^{\star}M$ as a Legendrian submanifold since $\partial_s X^a$ is lightlike and $\partial_{\vec{e}} X^a$ are Jacobi vectors orthogonal to $\partial_s X^a$. The diagram

$$\begin{array}{ccc} \mathrm{C} & \hookrightarrow & \mathrm{PT^{*}M} \\ & \overset{\gamma}{\searrow} & \downarrow \pi \\ & & \mathfrak{C} \end{array}$$

exhibits the map γ which defines $\mathfrak{C}$ as Legendrian. Therefore the machinery sketched in section 4 applies and provides the five types of stable caustic singularities of light cones. (Canonical forms of these singularities are given in [12]; pictures are presented in [3]and in [4], where they are also explicitly exemplified as optical wavefronts in Minkowski spacetime.)(One subtle point deserves to be mentioned here: Stability in the theory of Legendrian maps is defined with respect to Legendrian perturbations which, if applied to lightcones or lightlike wavefronts, will lead out of the classes of these special Legendrian maps. So far, no proof has been published that stability under appropriately restricted perturbations leads only to the same stable caustics as the general case. This has been stated without proof in [4]; the proof given in [3] has a gap, as pointed out to me by Volker Perlick.)

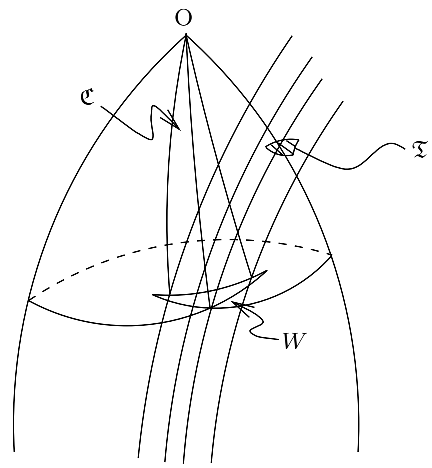

Fig. 4 The light cone $\mathfrak{C}$ of an observation event O and its intersection W with a source hypersurface $\mathfrak{T}$, indicated with suppression of one sapce time dimension. The wave surface W is, in fact, a "surface" such as shown in Fig. 3, intersected by source world lines as indicated in that Figure, which also shows the projection of W onto $\mathfrak{S}$. The intersection of the light rays of Fig. 4 with W correspond bijectively to the points of W in Fig. 3. These points are projected onto $\mathfrak{S}$; points on the same source world line have the same image in $\mathfrak{S}$.

6 Intrinsic definition of the lens map and the emission time function

Now we are ready to answer some of the questions raised at the end of section 2, following [5]. To set the stage, we augment the spacetime to a *kinematical cosmological model* $(M, g,$ U) by introducing a timelike unit vector field U on M representing the mean motion of matter. (For simplicity we disregard peculiar motions here.) As before, O is to represent an observation event, and we assume the observer to participate in the mean motion. Then, the redshift z of sources with respect to (O, U) is defined as a function on C, $z = Z(\vec{e}, s)$. If the model is expanding, z is strictly increasing with s and can be used as a "distance parameter" instead of s. Note that the model $(M, g,$ U) need not have any symmetries.

Suppose we wish to model the lensing of an extended source at redshift z_s or of a collection of point sources with approximately that redshift. The light rays which arrive at O from these sources fill a narrow "subcone" $\mathfrak{c}$ of $\mathfrak{C}$, see Fig. 4. To simplify the geometry without losing significant details we choose, close to the sources, a timelike "source hypersurface" $\mathfrak{T}$ ruled by integral curves of U, and replace the actual source world lines by world lines in $\mathfrak{T}$ in such a way that different source world lines which are connected to O by the same light ray are replaced by one line within $\mathfrak{T}$. (This amounts to projecting the sources onto a "screen" nearly orthogonal to the lines of sight.) Let $y^A, A = 1, 2,$ be spatial, comoving coordinates in $\mathfrak{T}$, and let t be a time coordinate, which measures proper time on the world lines in $\mathfrak{T}$. The 2-parameter family of these world lines, the quotient of $\mathfrak{T}$ by the U-curves, will be called the *source surface* and denoted by $\mathfrak{S}$. The intersection $W = \mathcal{C} \cap \mathfrak{T}$ is a (piece of a) *wave front* associated with the aforementioned subcone $\mathfrak{c}$. ($\mathfrak{T}$ is to be chosen such that at points of W, the tangent planes to W are nearly orthogonal to the light rays towards O in the rest spaces of the sources. The exact configuration does not matter.) The light ray of $\mathfrak{c}$ "seen" at O in the direction $\vec{e}$ meets the wave front W at a unique event $X^a(\vec{e}, s(\vec{e}) = (y^A, t)$, i.e., there is defined a map

$$\lambda : \vec{e} \mapsto (y^A, t) \in W \subset \mathfrak{T} \tag{12}$$

from a disc of S^2 into $\mathfrak{T}$. This map λ is Legendrian. Since $\mathfrak{T}$ is a product $\mathfrak{S} \times \mathbb{R}, \lambda$ decomposes into the (Lagrangian) map

$$\mathfrak{l} : \vec{e} \mapsto (y^A) \in \mathfrak{S} \tag{13}$$

and the function

$$T : \vec{e} \mapsto t. \tag{14}$$

$\mathfrak{l}$ assigns to each direction at O (contained in the subcone) a source position on $\mathfrak{S}$; it thus serves the same purpose as the $\mathfrak{l}$ of (4) in section 2 and will again be called the *lens map*. T assigns to each such $\vec{e}$ the proper time at which the respective light ray was emitted at the source (y^A) , it will be named the *emission time function*. If the cone $\mathfrak{c}$ of light rays intercepts lensing matter, it may contain a part of the caustic of $\mathfrak{C}$. A source (y^A) may then be seen in different images corresponding to different emission times t_i . Thus, T serves essentially the same purpose as the function ϕ in (6).

The preceding analysis implies: The intersection of the caustic of $\mathfrak{C}$ with the wave front W is the singularity of W, and the projection of that into the source surface $\mathfrak{S}$ is the caustic of the lens map. The latter corresponds to the caustic of standard lens theory as used in section 2.

A typical wave front exhibits a swallowtail singularity as shown in Fig. 3. The vertical lines indicate source world lines. The points of the wave front represent events at which photons are emitted which later arrive at O; these points are in one-to-one correspondence with light rays and directions $\vec{e}$ at O (except for points of the crossover curve which correspond to two light rays each). T is a function on the wave front. Fig. 4 also illustrates the role of the caustic in $\mathfrak{S}$ as separating sources with different numbers of images.

7 Distances, fluxes and intensities

The map λ defined in connection with (12) relates a part of the observer's sphere of vision, S^2 , to a part of the wave front W. In the vicinity of a non-critical direction $\vec{e}$, this map is approximated by the tangent map $\lambda_\star$ at $\vec{e}$, which maps the tangent plane $S^2_{\vec{e}}$ bijectively onto the tangent plane to W at the source point (y^A, t) . The image of a little circular source around (y^A, t) is then elliptical. If $\vec{e}$ is a non-degenerate critical point, $\lambda_\star$ maps $S^2_{\vec{e}}$ onto the tangent line of the caustic at $\lambda(\vec{e})$. In this case, the image of a small circular source around $\lambda(\vec{e})$ is a thin filament oriented in the direction of the kernel of $\lambda_\star$ at $\vec{e}$.

To render (some of) the preceding statements quantitative, $\lambda_\star$ will be related to Jacobi fields, defined in connection with (9). For this purpose, we reparametrize the light rays from O to the wave front, setting $\bar{s} = \frac{s}{s(\vec{e})}$, so that $\bar{s} = 0$ at O and $\bar{s} = 1$ at W, and rewrite (10) as

$$x^a = \bar{X}^a(\vec{e}, \bar{s}). \tag{15}$$

Further, let $\vec{\theta}$ be a unit tangent vector of $S^2_{\vec{e}}$, the tangent plane to the observer's sphere of vision. $\vec{\theta}$ uniquely determines a Jacobi field $\bar{Y}(\vec{\theta}, \bar{s}$ on the light ray from O in the direction $\vec{e}$ such that $\bar{Y}(\vec{\theta}, 1)$ is tangent to W at $\lambda(\vec{e})$,

$$\lambda_{\star\vec{e}}(\vec{\theta}) = \bar{Y}(\vec{\theta}, 1). \tag{16}$$

($\bar{Y}(\vec{\theta}, \bar{s})d\phi$ connects the rays determined by $\vec{e}$ and $\vec{e} + \vec{\theta}d\phi$ at the same $\bar{s}$ values.) By construction,

$$|\bar{Y}(\vec{\theta}, 1)| = \frac{dl}{d\phi}, \tag{17}$$

where dl is the distance of the events $\lambda(\vec{e}), \lambda(\vec{e} + \vec{\theta}d\phi)$ on W, which subtends the angle $d\phi$ at the observer. By definition, the ratio in (17) is the"*angular diameter distance* d of the event $x = \lambda(\vec{e})$ from the observer (O, U) in the direction $\vec{e}$ with respect to the transverse direction $\vec{\theta}$." Note that d depends not only on O and x, but also on U, (aberration), on $\vec{e}$ (the direction at which x is seen) and on $\vec{\theta}$ (the transverse direction at which dl is seen).

The $\vec{\theta}$ − dependence exhibits the *distortion* of an image relative to the intrinsic shape of the source's transverse cross section. If, instead of looking at a line element dl, the observer looks at an area dA on W which subtends a solid angle $d\omega$, he can define the " *area distance*

$$r = (\frac{dA}{d\omega})^{\frac{1}{2}} \tag{18}$$

of x from (O, U) in the direction $\vec{e}$." If x is seen by O in different directions $\vec{e}, \vec{e}' \ldots$, the corresponding area distances r, r' can differ considerably; this is called *relative magnification* by lensing. Since the (bolometric) flux F_O of a point source is given in terms of luminosity L_s , redshift z_s and area distance r_s by

$$F_O = \frac{L_s}{4\pi(1 + z_s)^4 r_s^2}, \tag{19}$$

the geometric magnification just referred to is accompanied by an equal *flux magnification.* On the other hand, the observed (bolometric) intensity I_O of an extended, resolved source with intrinsiv intensity I_s depends on redshift only,

$$I_0 = \frac{I_s}{(1 + z_s)^4}. \tag{20}$$

Thus, isophotal curves can be identified in the images, hence relative distortions and magnifications can be measured. (For derivations of (19) and (20) and details see, e.g., [1].

Note that area distance r and redshift z, for an observer (O, U) in a kinematic cosmological model, are defined on the (tangent) null cone C of O, not on the light cone $\mathfrak{C}$, and while r first increases with s or z in a given direction $\vec{e}$, it in general will have maxima and even zeros. In fact, the caustic of $\mathfrak{C}$ consists of exactly those images of C where the area distance r vanishes.

On a light ray, the area distance obeys the *focusing equation* [7]

$$\frac{d^2}{ds^2}r = -(|\sigma|^2 + 4\pi G\rho(1+z)^2)r. \tag{21}$$

$|\sigma|$ denotes the magnitude of the shear of the null cone, ρ the dust matter density. *It is* here and *through the geodesic deviation equation* (9) *that the gravitational field equation enters lens theory.* The shear term in (21) is due to conformal curvature, the matter term is due to Ricci curvature; both are nonlinearly coupled, however. (A cosmological term is not excluded; it does not affect (21).) The focusing equation shows why caustics, magnification etc. generically occur.

8 The problem of computing the lens map

The map λ defined in section 6 depends on the spacetime curvature between sources and observer, which in turn is related to the energy-momentum distribution of matter. Except for the special, ideal case of spherical symmetry ([9], [10], λ presumably cannot be computed exactly. The best one can do is find, under special assumptions on the matter distribution in (part of) spacetime, approximations to λ, analytically or numerically.

One strategy to do that which originated in [6] and in [8], is to first use the geodesic deviation equation to determine the tangent map $\lambda_\star$, in other words to determine λ infinitesimally near one ray, and then secondly to integrate the result to obtain λ itself.

The basis for the first step has been laid already in section 7. Indeed, eq. (16) reads explicitly

$$\bar{Y}(\vec{\theta}, 1) = \theta^B(\frac{\partial y^A}{\partial e^B}\frac{\partial}{\partial y^A} + \frac{\partial t}{\partial e^B}\frac{\partial}{\partial t}). \tag{22}$$

(Remember: $\theta^A = \frac{de^A}{d\phi}$ is the tangent to a curve $\vec{e}(\phi)$ on the observer's sphere of vision which determines the Jacobi vector $\bar{Y}(\vec{\theta}, \bar{s})$ whose value at the wave front is given in (22).) Therefore, once the Jacobi field $\bar{Y}$ has been found, the total differential system

$$dy^A = \mathfrak{l}^A_B(\vec{e})de^B \tag{22a}$$

and the differential

$$dt = T_B(\vec{e})de^B \tag{22b}$$

can be read off (22). On integration, (22a) should result in the lens map $\mathfrak{l}$, eq. (14), and (22b) should give the emission time function T, eq. (15). The problem is thus

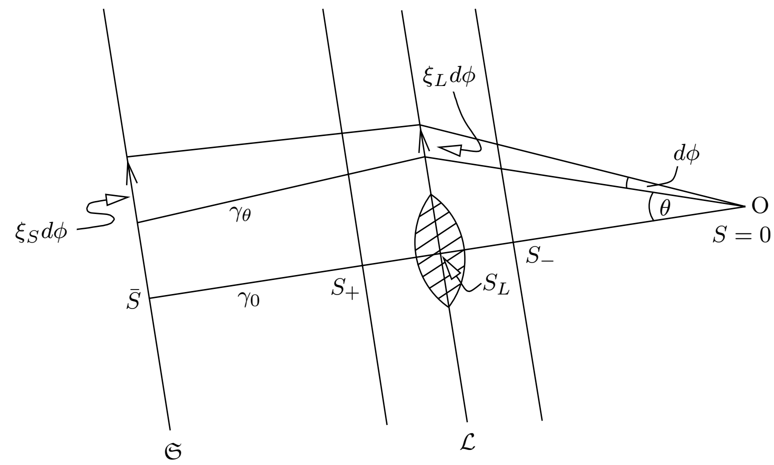

Fig. 5 Light rays from $\mathfrak{S}$ to O. The "central" ray γ_0 is not deflected, in contrast to γ_θ, which is connected to its neighbour by a Jacobi field with values $\xi_L d\phi$, $\xi_S d\phi$.

reduced to the integration of Jacobi's equation (9) for which methods are available ([11], [8]).

The method just indicated can be carried out, at least under some simplifying assumptions, as follows.

Let us consider the lensing situation of section 6, specialized to the case that, between observation event O and source wave frontW, the matter density ρ differs only slightly from a cosmic mean value $\bar{\rho}$, $\frac{\delta\rho}{\bar{\rho}} < 1$, except in a narrow "slab" surrounding a timelike *lens hypersurface* (analogous to the source hypersurface $\mathfrak{T}$) where, in addition to the small fluctuation $\delta\rho$, there is matter with a large density contrast corresponding to a strong lens; see Fig. 5.

In contrast to the wave front W, the intersection $\mathcal{L}$ of the lens hypersurface with the light cone $\mathfrak{C}$ of O is a smooth, spacelike 2-surface if, in agreement with our assumptions, we assume that a caustic of $\mathfrak{C}$ occurs in the past of the slab only.

To follow infinitesimal light beams on $\mathfrak{C}$ backwards from O through $\mathcal{L}$ to W, one introduces on each light ray an orthonormal pair E_1, E_2 of vectors, orthogonal at O to both the 4-velocity of U and the initial direction $\vec{e}$ of the ray and parallel propagated on the ray. The Jacobi vectors on that ray defined by the initial data $Y(O) = 0, D_s Y(O) = \vec{\theta}$ (with $\vec{\theta}$ as in sec. 7), can be written

$$Y = \xi^1 E_1 + \xi^2 e_2 + \xi^0 k.$$

The "screen part" $\xi = \xi^A E_A$ of Y then obeys a *reduced Jacobi equation* $\ddot{\xi} = \mathcal{T} \cdot \xi$, where the 2×2-matrix $\mathcal{T}$ is formed from the curvature tensor. (See [8]) We include in $\mathcal{T}$ only those matter and curvature terms which are due to the small perturbations, not those due to the strong lens. Let $s_- \leq s \leq s_+$ correspond to an interval on a ray where the action of the lens dominates. Outside of it, the rays are affected by the small perturbations only. To determine the reduced Jacobi vector $\xi(s)$, let $\mathfrak{D}(s, s_0)$

denote that solution of

$$\ddot{\mathfrak{D}} = \mathcal{T} \cdot \mathfrak{D} \tag{23}$$

which satisfies the initial conditions

$$\mathfrak{D}(s_0, s_0) = 0, \dot{\mathfrak{D}}(s_0, s_0) = 1.$$

Then,

$$\tilde{\mathfrak{D}}(s_0, s_0) = (\mathfrak{D}(s, 0) - \mathfrak{D}(s, s_0) \cdot \dot{\mathfrak{D}}(s_0, 0)) \cdot \mathfrak{D}^{-1}(s_0, 0) \tag{24}$$

also obeys (23), but with initial data

$$\mathfrak{D}(s_0, s_0) = 1, \dot{\tilde{\mathfrak{D}}}(s_0, s_0) = 0.$$

These matrices serve to "transport" $\xi(s)$ from $O(s = 0)$ to s_-, and from s_+ to the source $S(\bar{s})$, (see Fig. 5):

$$\xi(s_-) = \mathfrak{D}(s_-, 0) \cdot \vec{\theta} \quad (\dot{\xi}(0) = \vec{\theta}),$$
$$\xi(\bar{s}) = \tilde{\mathfrak{D}}(\bar{s}, s_+) \cdot \xi(s_+) + \mathfrak{D}(\bar{s}, s_+) \cdot \dot{\xi}(s_+).$$

The lens action is taken into account through

$$\dot{\xi}(s_+) - \dot{\xi}(s_-) \approx -(1 + z_L) \bigtriangledown_{\xi_L} \vec{\alpha}. \tag{25}$$

This result is obtained by approximating the small, relevant part of the lens surface traversed by rays by a plane and treating its neighbourhood by linear perturbation theory, see [6], [8]; $\vec{\alpha}$ is the deflection angle defined on $\mathcal{L}$, given in eqs. (2), (3). Combining the last three equations and using that $s_+ \approx s_L \approx s_-$, s_L corresponding to the lens surface, one obtains the following approximate relation between $\xi_L = \xi(\bar{s}), \xi_s = \xi(\bar{s})$:

$$\xi_s \approx (\tilde{\mathfrak{D}}(\bar{s}, s_L) + \mathfrak{D}(\bar{s}, s_L)\dot{\mathfrak{D}}(s_L, 0) \cdot \mathfrak{D}^{-1}(s_L, 0)) \cdot \xi_L - (1 + z_L)\mathfrak{D}(\bar{s}, s_L) \bigtriangledown_{\xi_L} \vec{\alpha} \tag{26}$$

The $\mathfrak{D}$-factors in this equation all refer to propagation under the influence of the small fluctuations only; the action of the strong lens is accounted in the differential deflection term only.

We now assume that those matrices which are defined on each ray separately, are nearly independent of *where* a particular ray (of the narrow ray bundle under consideration) passes the lens surface $\mathcal{L}$. Accordingly, we substitute for these matrices their values on one particular ray γ_0 which is not deflected, i.e. on which $\vec{a}$ vanishes. (Such a ray which is not affected by the presence of the strong lens, always exists since the deflection potential Ψ has a minimum.) The matrix $\tilde{\mathfrak{D}}(s, s_L)$ on that ray satisfies (24) with $s_0 = s_L$. Using that simplifies (26) to

$$\xi_s \approx \mathfrak{D}(\bar{s}, 0) \cdot \mathfrak{D}^{-1}(s_L, 0) \cdot \xi_L - (1 + z_L)\mathfrak{D}(\bar{s}, s_L) \bigtriangledown_{\xi_L} \vec{\alpha}.$$

This is the "differential" version of the lens equation aimed at. We now consider that equation for a one-parameter family of rays which begins with the ray γ_0 (where $\vec{a} = 0$) and ends at some chosen ray γ_θ. Such a family is defined by a curve $\vec{e}(\phi)$ on the observer's sphere, the corresponding points on $\mathcal{L}$ and $\mathcal{S}$ trace curves with tangents ξ_L and ξ_s, respectively. Integration of the last equation along such a family of curves gives the *approximate cosmological lens equation* ([13])

$$\vec{y} = \mathfrak{D}(\bar{s}, 0) \cdot \vec{\theta} - (1 + z_L)\mathfrak{D}(\bar{s}, s_L)\vec{\alpha}(\vec{\theta}), \tag{27}$$

where $\vec{\theta}$ and $\vec{\alpha}$ have the same meanings as in eqs. (1) - (3) and $\vec{y}$, according to the construction, stands for coordinates on the source surface $\mathfrak{S}$.

The last equation is more general than the simple lens equation; besides the action of the lens, it takes into account the influence of the small density fluctuations of the matter between lens, source and observer (cosmic rotation and shear). It has first been obtained and discussed in the paper just quoted.

If we put $\vec{\beta} := \mathfrak{D}^{-1}(\bar{s}, 0) \cdot \vec{y}$ and, at the end, neglect cosmic rotation and shear, then the matrices $\mathfrak{D}(\bar{s}, s_L), (1 + z_L)\mathfrak{D}^{-1}(\bar{s}, 0)$ and $\mathfrak{D}(s_L, 0)$ reduce to the unit matrix times the "empty cone, Dyer-Roeder distances" D_{LS}, D_S, D_L, respectively, and (27) simplifies to (1) - (3). These distances depend on the underlying cosmological model - usually a Friedmann model - and on the redshifts of lenses and sources [1].

Now, all ingredients of the simple picture of the standard formalism of section 2 have been obtained from GR, albeit guided by simplifying assumptions from astrophysics.

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