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Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity

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Abstract. Fuchsian equations provide a way of constructing large classes of spacetimes whose singularities can be described in detail. In some of the applications of this technique only the analytic case could be handled up to now. This paper develops a method of removing the undesirable hypothesis of analyticity. This is applied to the specific case of the Gowdy spacetimes in order to show that analogues of the results known in the analytic case hold in the smooth case. As far as possible the likely strengths and weaknesses of the method, as applied to more general problems, are displayed.

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1. Introduction

The theory of Fuchsian equations has been applied to analyse singularities in a variety of classes of spacetimes in general relativity. The term 'Fuchsian equations' has not always been used in the literature on this subject and in this paper it denotes a certain class of singular differential equation in a generic way. The existing results in this area will be surveyed briefly below. In this approach spacetimes containing singularities are parametrized by some functions which play the role of data on the singularity. In some cases it was necessary to assume the analyticity of these functions. In other cases smoothness was sufficient. The aim of this paper is to develop ways of removing the analyticity requirement. These will be illustrated by the case of Gowdy spacetimes which represent an ideal laboratory for testing new ideas on the mathematical treatment of the Einstein equations.

It may not be immediately obvious why the apparently technical distinction between analytic (C^ω) functions and smooth (C^∞) functions should be significant with a view to physical applications. There are at least two reasons why it is important. The first is that the physical notion of causality cannot be reasonably formulated within the class of analytic functions, since the unique continuation property of the latter means that the solution of an equation at one point influences its behaviour at all other points, and not only at causally related points. Connected with this is the fact that solutions of an equation do not depend continuously on initial data in any useful sense. For more discussion of these points see [7], particularly section 2.4. The second, which is also a direct consequence of the unique continuation property of analytic functions, is that there is not the same freedom to construct solutions with certain interesting properties within the analytic class. An example of this will be given in section 6 below.

There are a number of results on Fuchsian equations with smooth coefficients in the literature and to start with we need to understand why these do not apply directly to Gowdy spacetimes. The general form of a system of Fuchsian equations for a vector-valued unknown function u is

$$t\frac{\partial u}{\partial t} + N(x)u = tf(t, x, u, D_x u). \tag{1}$$

Here x is a point in some Euclidean space and $D_x u$ is a shorthand for the first order derivatives of u with respect to the spatial variables x. The function f is required to extend continuously to $t = 0$ while the matrix $N(x)$ is required to satisfy some positivity condition, which may depend on the particular context. An example of a condition of this type is that the eigenvalues of the matrix $N(x)$ should have non-negative real parts for all x. Note that the apparently more general system

$$t\frac{\partial u}{\partial t} + N(x)u = t^{\alpha} f(t, x, u, D_x u) \tag{2}$$

with $\alpha > 0$ can be reduced to the form (1) by introducing t^{α} as a new time variable. This results in the matrix $N(x)$ being rescaled by a factor α, but this does not affect its positivity properties.

One approach to proving existence theorems for Fuchsian systems which does not require any analyticity assumption is due to Claudel and Newman [6]. Of course, in the context of data which are merely smooth (or even finitely differentiable) the system must be hyperbolic. This is needed to prove existence for an equation without any singular behaviour in t and the singularity cannot be expected to improve the situation. In the Claudel–Newman theorem it is required that f have an asymptotic expansion about $t = 0$ in integral powers of t and this property is inherited by the solution. The positivity assumption on $N(x)$ is weaker than that mentioned above. It is only required that there be no eigenvalues which are negative integers. An important element of the proof of the theorem is to expand the candidate solution u in powers of t, writing it in the form $u_0 + u_1$ where u_0 is an appropriate polynomial in t and u_1 is a remainder of higher order in t. Then u_1 solves a Fuchsian system where the eigenvalues of $N(x)$ are shifted by an integer in the positive direction. If it is possible to expand to a sufficiently high order then the shifted eigenvalues all have positive real parts. The condition which makes the expansion possible is that a polynomial u_0 can be found which satisfies the equation up to an error of sufficiently high order in t. The only obstruction to this is if the shifted eigenvalue becomes zero at some stage in the process and this is prevented by the assumption made on the eigenvalues of the original matrix. In cases such as the system arising in the analysis of Gowdy spacetimes in [11] the solutions cannot be expanded in integral powers of t. Instead non-integral (and even x-dependent) powers of t and logarithms occur. For this reason the method of [6] does not apply directly. One of the main methods of the present paper is to extend the technique of using expansions of the solution to shift the eigenvalues of $N(x)$ to cases where terms more complicated than integral powers of t occur.

Another approach to proving existence theorems for Fuchsian systems with smooth coefficients is due to Kichenassamy [9]. In that case there is no restriction that solutions have expansions in integral powers of t. It is, however, required that the matrix $N(x)$ be independent of x. This is not satisfied in the Gowdy case. Since the eigenvalues of the matrix $N(x)$ correspond to powers occurring in the expansions, this is a consequence of the dependence of these powers on x in the system coming from the Gowdy spacetimes. Thus the result of [9] does not apply. It might be possible to extend the proofs in [9] to the case of non-constant N, but this will not be attempted here. Both the proofs of Claudel and Newman and of Kichenassamy involve the use of sophisticated techniques from functional analysis, namely semigroup theory and the Yosida approximation, respectively. These will be avoided in the approach developed in the following.

In the paper [5] Anguige treats the case of plane symmetric solutions of the Einstein equations coupled to a perfect fluid. He makes use of a direct energy argument of a type familiar in the theory of regular symmetric hyperbolic systems. It is important in his proof that $N(x)$ is independent of x. This kind of procedure will be generalized in the following and the condition that $N(x)$ should be constant will be removed. In order to apply the energy argument it is necessary to reduce the problem to the case where $N(x)$ is positive definite. This can be achieved by a method related to that used in [6]. The eigenvalue condition used in [6] will be replaced by a condition of formal solvability which abstracts its essential significance.

Next, a brief survey of the literature on applications of Fuchsian equations to general relativity will be given. It appears that the first paper containing an application of this kind was [13], where Moncrief proved the existence of a large class of analytic spacetimes with analytic compact Cauchy horizons. Later, Newman [15, 16] based his work on isotropic singularities and Penrose's Weyl curvature hypothesis on existence theorems for hyperbolic systems with singularities of Fuchsian type. These papers did not include proofs of the required theorems but the necessary proofs were provided in [6]. More recently, results on isotropic singularities for more general matter models were obtained by Anguige and Tod [2–4]. Their theorems require no symmetry assumptions but are confined to a special type of singularities. On the other hand Anguige [5] proved a theorem on the existence of non-isotropic singularities in plane symmetric spacetimes with perfect fluid.

Another line of development of the applications of Fuchsian equations in general relativity starts with the work of Kichenassamy and Rendall [11] on singularities in analytic Gowdy spacetimes. It builds on previous work of Kichenassamy outside general relativity (see [10] and references therein). This direction is continued in the papers [8], [1] and [14] which deal with vacuum models with two spacelike Killing vectors, solutions of the Einstein-scalar field equations and analogues of Gowdy models in string cosmology, respectively. In all these cases analyticity is assumed. A notable feature of the result of [1] is that it makes no symmetry assumptions and so, on the basis of function-counting arguments, concerns general solutions of the Einstein equations with the given matter model.

The paper is organized as follows. In the second section the notion of a formal solution of a Fuchsian system is defined. Assumptions on the coefficients are exhibited which guarantee the existence of a formal solution. They are fulfilled by the first order form of the Gowdy equations introduced in [11]. In the third section this form of the equations is modified slightly so as to obtain a symmetric hyperbolic system. Its formal solvability is shown to follow from that of the original system. The system satisfied by the remainder term which is the difference between a true solution of the system and an approximate solution is computed. The fourth section proves an existence theorem which is general enough to apply to the case of Gowdy spacetimes with sufficiently low velocity ($k < 3/4$). In the fifth section yet another form of the equations is used to cover the remainder of the full low velocity case ($k < 1$). The wider applicablity of the methods of the paper is discussed in the final section.

2. Formal solutions

If the function f in (1) is smooth at $t = 0$ and hence admits an asymptotic expansion about $t = 0$ in integral powers of t then it can be useful to expand it in this way. In the following, a generalization to less smooth functions f will play an important role. This uses the notion of a formal solution of equation (1) which will now be defined. It will also be convenient for the following to introduce a notion of regularity of functions adapted to the given situation. An analogous notion in an analytic context was introduced in [1].

Definition 1. *A function $z(t, x)$ defined on an open subset of $[0, \infty) \times R^N$ and taking values in $\mathbf{R}^K$ is called regular if it is C^∞ for all $t > 0$ and if its partial derivatives (defined for $t > 0$) of any order with respect to the variables $x \in \mathbf{R}^K$ extend continuously to $t = 0$.*

Definition 2. *A finite sequence $(u_1, u_2, \ldots, u_p)$ of functions defined on an open subset U of $[0, \infty) \times \mathbf{R}^n$ containing $\{0\} \times \mathbf{R}^n$ is called a formal solution of order p of (1) if;*

(a) each u_i is regular,
(b) $t\partial_t u_i + N(x)u_i - tf(t, x, u_i, D_x u_i) = O(t^i)$ for all i as $t \to 0$.

Here and in the following the O-symbols are taken in the sense of uniform convergence on compact subsets.

In [11] an iteration was defined which, in the case that the function f has suitable analyticity properties, converges to a solution of (1). It is doubtful if it converges in any useful sense when f is only smooth. However, it can be used to produce a formal solution of any desired order, as will now be shown.

Lemma 2.1. *If the function f is regular and the matrix $N(x)$ is smooth and satisfies an estimate of the form $\|\sigma^{N(x)})\| \leqslant C$ with a constant C for σ in a neighbourhood of zero then for each p the equation (1) has a formal solution of order p which vanishes at $t = 0$.*

Proof. First some definitions from [11] will be recalled. For a regular function u define $F[u] = tf(t, x, u, D_x u)$. Then $F[u]$ is also regular and is $O(t)$ as $t \to 0$ together with all its spatial derivatives. If v is regular and $O(t)$ as $t \to 0$ together with all its spatial derivatives, define $u = H[v] = \int_0^1 \sigma^{N(x)-I} v(\sigma t) \mathrm{d}\sigma$. Then u is regular and $O(t)$ together with all its spatial derivatives and satisfies $(t\partial_t + N)u = v$. Then, if G is defined to be the composition FH, any fixed point v of G defines a solution u of (1) by $u = H[v]$. Let $u_1 = 0$. It defines a formal solution of (1) of order one. It will be shown that defining $u_i = HG^{i-1}[u_1]$ defines a formal solution of order p for each p. Note the relation $u_{i+1} = HFu_i$.

The first defining property of a formal solution is easily proved by induction. The main point is to verify the second property. To do this it will first be shown that $u_{i+1} - u_i = O(t^i)$ for each i, and that spatial derivatives of all orders of the u_i satisfy the corresponding estimates. For $i = 1$ this follows directly from the properties already demonstrated. To prove the general case, consider the equation obtained by forming the difference of the equations satisfied by u_{i+1} and u_i. This gives

$$t\partial_t(u_{i+1} - u_i) + N(x)(u_{i+1} - u_i) = tM_1(u_i - u_{i-1}) + tM_2 D_x(u_i - u_{i-1}) \tag{3}$$

for regular functions M_1 and M_2 of the arguments $t, x, u_i, u_{i-1}, D_x u_i, D_x u_{i-1}$, obtained by applying the mean value theorem to differences. The right-hand side of (3) is $O(t^i)$. Then the fact can be applied that the operator H preserves the set of functions which are $O(t^j)$ for any j. Spatial derivatives can be handled in the same way. To complete the proof of the lemma, consider the relation:

$$t\partial_t u_{i+1} + N(x)u_{i+1} - tf(t, x, u_{i+1}, D_x u_{i+1}) = -t[f(t, x, u_{i+1}, D_x u_{i+1}) - f(t, x, u_i, D_x u_i)]. \tag{4}$$

Using the mean value theorem and the estimates already obtained for $u_{i+1} - u_i$ shows that the right-hand side of (4) is $O(t^{i+1})$. □

Remark. A general criterion for checking the condition on $N(x)$ required to apply this lemma has been given in [1].

Next, some basic equations for the Gowdy spacetimes will be recalled. More details can be found in [11]. Note that we have in mind Gowdy spacetimes with a Cauchy surface diffeomorphic to a three-dimensional torus. Additional complications associated to the occurrence of an axis for Gowdy spacetimes with other topologies are not addressed. The basic unknowns are two real-valued functions $X(t, x)$ and $Z(t, x)$ of two variables. New variables u and v are defined so that the relations

$$Z(t, x) = k(x) \log t + \phi(x) + t^{\epsilon} u(t, x) \tag{5}$$

and

$$X(t, x) = X_0(x) + t^{2k(x)}(\psi(x) + v(t, x)) \tag{6}$$

hold, where k, X_0, ϕ and ψ are given functions. The positive constant ϵ will be restricted later. Next introduce further variables by setting

$$(u_0, u_1, u_2, v_0, v_1, v_2) = (u, t\partial_t u, t u_x, v, t\partial_t v, t v_x). \tag{7}$$

The Gowdy equations imply the following first order system:

$$t\partial_t u_0 = u_1 \tag{8}$$

$$\begin{aligned} t\partial_t u_1 = {} & -2\epsilon u_1 - \epsilon^2 u_0 + t^{2-\epsilon}(k_{xx} \log t + \phi_{xx}) + t\partial_x u_2 \\ & - \exp(-2\phi - 2t^{\epsilon} u_0)\{t^{2k-\epsilon}(v_1 + 2k v_0 + 2k\psi)^2 - t^{2-2k-\epsilon} X_{0x}^2 \\ & - 2t^{1-\epsilon} X_{0x}(v_2 + t\psi_x + k_x(v_0 + \psi)t \log t) \\ & - t^{2k-\epsilon}(v_2 + t\psi_x + 2k_x(v_0 + \psi)t \log t)^2\} \end{aligned} \tag{9}$$

$$t\partial_t u_2 = t\partial_x(u_0 + u_1) \tag{10}$$

$$t\partial_t v_0 = v_1 \tag{11}$$

$$\begin{aligned} t\partial_t v_1 = {} & -2k v_1 + t^{2-2k} X_{0xx} + t\partial_x(v_2 + t\psi_x) + 4k_x(v_2 + t\psi_x)t \log t \\ & + (v_0 + \psi)[2k_{xx} t^2 \log t + 4(k_x t \log t)^2] \\ & + 2t^{\epsilon}(v_1 + 2k v_0 + 2k\psi)(u_1 + \epsilon u_0) \\ & - 2X_{0x} t^{2-2k}(k_x \log t + \phi_x + t^{\epsilon}\partial_x u_0) \\ & - 2t(\partial_x(v_0 + \psi) + 2k_x(v_0 + \psi) \log t)(k_x t \log t + t\phi_x + t^{\epsilon} u_2) \end{aligned} \tag{12}$$

$$t\partial_t v_2 = t\partial_x(v_0 + v_1). \tag{13}$$

Here some minor errors in the equations given in [11] have been corrected†. This system is of the form (2) which implies, as indicated in the introduction, that it can be brought into the form (1) by a change of time variable. This possibility will be used freely without further comment in the following. After the change of time coordinate the system arising from the Gowdy equations satisfies the hypotheses of lemma 2.1 provided $\epsilon < 2k$ and $\epsilon < 2 - 2k$. In particular, the bound on $\sigma^{N(x)}$ was verified directly in [11]. Alternatively, it follows easily from the criterion given in [1]. Hence the above system has a formal solution of any order which vanishes at $t = 0$.

3. The symmetric hyperbolic system

In the following the form of the Gowdy equations introduced in section 2 will be called the first reduced system. Now it will be modified to get a system which, while less convenient for showing formal solvability, is symmetric hyperbolic and therefore appropriate for allowing

† I thank Aurore Cabet for pointing these out.

the theory of hyperbolic equations to be applied. It is obtained from the first reduced system by making the substitutions $u_2 = t\partial_x u_0$ and $v_2 = t\partial_x v_0$ in some places. The result is:

$$t\partial_t u_0 = u_1 \tag{14}$$

$$\begin{aligned} t\partial_t u_1 = {} & -2\epsilon u_1 - \epsilon^2 u_0 + t^{2-\epsilon}(k_{xx}\log t + \phi_{xx}) + t\partial_x u_2 \\ & - \exp(-2\phi - 2t^\epsilon u_0)\{t^{2k-\epsilon}(v_1 + 2kv_0 + 2k\psi)^2 - t^{2-2k-\epsilon}X_{0x}^2 \\ & - 2t^{1-\epsilon}X_{0x}(v_2 + t\psi_x + k_x(v_0+\psi)t\log t) \\ & - t^{2k-\epsilon}(v_2 + t\psi_x + 2k_x(v_0+\psi)t\log t)^2\} \end{aligned} \tag{15}$$

$$t\partial_t u_2 = u_2 + t\partial_x u_1 \tag{16}$$

$$t\partial_t v_0 = v_1 \tag{17}$$

$$\begin{aligned} t\partial_t v_1 = {} & -2kv_1 + t^{2-2k}X_{0xx} + t\partial_x(v_2 + t\psi_x) + 4k_x(v_2 + t\psi_x)t\log t \\ & + (v_0+\psi)[2k_{xx}t^2\log t + 4(k_x t\log t)^2] \\ & + 2t^\epsilon(v_1 + 2kv_0 + 2k\psi)(u_1 + \epsilon u_0) \\ & - 2X_{0x}t^{2-2k}(k_x\log t + \phi_x + t^{\epsilon-1}u_2) \\ & - 2(v_2 + t\partial_x\psi + 2tk_x(v_0+\psi)\log t)(k_x t\log t + t\phi_x + t^\epsilon u_2) \end{aligned} \tag{18}$$

$$t\partial_t v_2 = v_2 + t\partial_x v_1. \tag{19}$$

This system, which will be referred to as the second reduced system, has the advantage of being symmetric hyperbolic but also has two potential disadvantages. One is that the matrix $N(x)$ has been modified, while the other is that possibly dangerous powers of t have been introduced on the right-hand side. The matrix $N(x)$ acquires two negative eigenvalues, which will have to be dealt with by appropriate methods in due course. As far as the other problem is concerned, the power $t^{1+\epsilon-2k}$ is introduced. This power should be positive. This can be achieved subject to the inequalities already assumed for ϵ if and only if $k < 3/4$. This restriction appears unnatural, but will be assumed in this section and the next for the second reduced system.

In fact, although the desired inequalities relating ϵ and $k(x)$ can be ensured by a suitable choice of ϵ at any given point x, this cannot be done at all points simultaneously by a single choice of the constant ϵ. It can be ensured in a neighbourhood of any given point. Solutions will be constructed in local neighbourhoods of this kind and then put together using the domain of dependence to get a solution which is global in x.

Any formal solution $\{u_1, \ldots, u_p\}$ of the first reduced system which vanishes at $t = 0$ is also a formal solution of the second reduced system. For $t\partial_t((u_2 - t\partial_x u_0)_i) = O(t^i)$ and using the fact that the formal solution vanishes at $t = 0$ it follows that $(u_2 - t\partial_x u_0)_i = O(t^i)$. This can then be used to see that the difference terms arising when passing from the first to the second reduced systems are $O(t^i)$, assuming the condition $k < 3/4$. Although it is not of significance for the following it is interesting to note that for $3/4 < k < 1$ the sequence whose element with index i is the element of the formal solution of the first reduced system with index $i + 1$ is a formal solution of the second reduced system.

Given a formal solution it is possible to consider the difference between an actual solution and the formal solution and the equations which this difference satisfies. The hope is that this equation is more tractable analytically than the first and second reduced systems. This is a generalization of the procedure of subtracting a Taylor polynomial of finite order in the case that the solutions are smooth at $t = 0$. Rather than doing this calculation in the specific case of the Gowdy system it will be done for the following more general symmetric hyperbolic Fuchsian system:

$$t\partial_t u + N(x)u + tA^j(t, x, u)\partial_j u = tf(t, x, u). \tag{20}$$

The condition for symmetric hyberbolicity is that the matrices A^j should be symmetric. As before, all coefficients in the equation are assumed regular. If $\{u_i\}$ is a formal solution, let $v_i = t^{-i+1}(u - u_i)$. Let the $O(t^i)$ remainder term in the definition of a formal solution be denoted by R_i. Then

$$t\partial_t u_i + N(x)u_i + tA^j(t, x, u_i)\partial_j u_i = tf(t, x, u_i) + R_i. \tag{21}$$

Subtracting this from the equation for u and rearranging gives

$$\begin{aligned} &t\partial_t v_i + (N(x) + (i-1)I)v_i + tA^j(t, x, u)\partial_j v_i \\ &\quad = -t^{-i+1}[t(A^j(t, x, u) - A^j(t, x, u_i))\partial_j u_i + t(f(t, x, u) - f(t, x, u_i)) - R_i] \end{aligned} \tag{22}$$

Note that $t^{-i+1}R_i = t[t^{-i}R_i]$. Using this and applying the mean value theorem to the differences occurring in the equation for v_i gives

$$t\partial_t v_i + (N(x) + (i-1)I)v_i + tA^j(t, x, u_i + t^i v_i)\partial_j v_i = tg_i(t, x, v_i) \tag{23}$$

for some regular function g_i. Here the dependence of the right-hand side on the given function u_i has been incorporated in the x-dependence of the function g_i. Choosing i large enough ensures that the eigenvalues of $N + (i-1)I$ have positive real parts, or even that the matrix is positive definite. In the Gowdy case this will be referred to as the third reduced system.

4. The basic existence theorem

In this section a local existence theorem will be proved for solutions of the Gowdy equations in a neighbourhood of the initial singularity in the case that the data k, X_0, ϕ and ψ are merely smooth. In the case of analytic data the problem was solved in [11]. Thus, if the data are approximated by a sequence of analytic data $(k_m, X_{0m}, \phi_m, \psi_m)$, a corresponding sequence of analytic solutions is obtained. At the same time formal solutions can be obtained for both the approximate data and the actual data. Denote the former by u_{mi}, where the first index corresponds to the sequence of data and the second to the enumeration of elements of an approximate solution. Denote the latter by u_i, as before. If the approximate solutions are constructed as in the proof of the lemma 2.1 then $u_{mi} \to u_i$ as $m \to \infty$, uniformly on compact subsets. The same is true of the spatial derivatives of these functions of any order. It can be concluded that the sequence of coefficients obtained for the third reduced system is also convergent on compact sets as $m \to \infty$. The sequence of solutions of these equations which we have is only defined *a priori* on a time interval which depends on m. However, using the global existence theorem for the Gowdy equations [12], it is possible to conclude that a sequence of smooth solutions exists on a common time interval. The aim now is to show that this is a Cauchy sequence in a sufficiently strong topology. If that can be done then it will follow that the sequence converges to a limit which is a solution corresponding to the smooth data originally prescribed.

The tool to obtain convergence of the approximations is the technique of energy estimates. This requires some preliminary remarks on linear algebra. Consider the matrix-valued function $N(x)$ in the Fuchsian system. Spatial derivatives of a solution of this system also satisfy a system of the same form, but with a different matrix $N(x)$. Suppose, for instance, we consider the first derivative $D_x u$ of the unknown. The system for the pair $(u, D_x u)$ has a matrix in its singular term which has diagonal blocks $N(x)$ and an off-diagonal block involving $D_x N(x)$. This off-diagonal block does not affect the eigenvalues of the matrix but may well affect whether it is positive definite or not. Since the positive definiteness of matrices like this is important in what follows we adopt a strategy which avoids positivity being lost. In order to implement this it will be assumed that $N(x)$ is positive definite. Then, use the variable $w = KD_x u$ for a positive constant K instead of $D_x u$ itself. For the equation satisfied by (u, w) the matrix of interest is positive definite provided K is chosen sufficiently small. The same trick works for higher derivatives. It suffices to replace the collection of unknowns $\{D^\alpha u\}$ by $w^\alpha = K^{|\alpha|} D^\alpha u$. Let the matrix corresponding to N arising in the system for all these derivatives up to order s be denoted by $N^{(s)}$. By construction it is positive definite.

The standard method of energy estimates (see e.g. [17] or, for a discussion aimed at relativists, [7]) proceeds by estimating the Sobolev norms of solutions. The usual Sobolev norm is given by $\|u\|_{H^s} = (\sum_{|\alpha|\leqslant s} \|D^\alpha u\|^2_{L^2})^{1/2}$. For the present purposes it is convenient to use the equivalent norm $\|u\|_{H^s,K} = (\sum_{|\alpha|\leqslant s} K^{2|\alpha|} \|D^\alpha u\|^2_{L^2})^{1/2}$, where K is, as before, a small enough positive constant.

As a first application of energy estimates, a theorem on the domain of dependence will be proved. Let u and v be two regular solutions of (20) vanishing at $t = 0$. Then their difference satisfies an equation of the following form:

$$\partial_t(u - v) + t^{-1} N(x)(u - v) + A^j(t, x, u)\partial_j(u - v) = M(t, x)(u - v) \qquad (24)$$

where M is a regular function constructed from u and v. Choose two times t_1 and t_2 with $0 < t_1 < t_2$ and let G be the region defined by the inequalities $t_1 \leqslant t \leqslant t_2$ and $|x| \leqslant 2t_2 - t$. Let S_1 and S_2 be its intersections with $t = t_1$ and $t = t_2$ respectively. Now multiply the equation (24) by $e^{-\kappa t}(u - v)$, integrate over G and integrate by parts in the way this is usually done in the derivation of energy estimates. Here κ is a positive constant. The singular term containing N can be discarded, due to its sign, giving an estimate of the form $e^{-\kappa t_2} I_2 \leqslant e^{-\kappa t_1} I_1 + I_G$, where I_1 and I_2 are the L^2 norms of the restrictions of $u - v$ to S_1 and S_2 respectively, and I_G is a volume integral over G which for κ sufficiently large is negative unless $u - v$ is identically zero on G. Letting t_1 tend to zero, so that $I_1 \to 0$, gives a contradiction unless $u - v = 0$ on G. Thus it has been proved that the solutions u and v agree on G. This proves a domain of dependence property for solutions of (20) which can be used for the purpose of gluing together solutions. This means that even if we are interested in producing solutions on manifolds it is enough to solve the problem on $\mathbf{R}^n$. Moreover, it is possible to consider without loss of generality the case of cut-off coefficients and data. By this we mean that there is a compact subset of $\mathbf{R}^n$ such that for x outside this compact set the initial data vanish, the coefficients A^j and f vanish and N is constant.

The aim is now to construct solutions of the third reduced system for Gowdy in the case of smooth coefficients. As already indicated it is enough to do this under the assumption of a cut-off in space. A sequence of functions which is a candidate for a sequence converging to a solution of the third reduced system has already been produced. This sequence is obtained by fixing a value of i sufficiently large that the matrix occurring in the singular term of the third reduced system is positive definite and forming the difference of the solution of the second reduced system corresponding to the data $(k_m, \phi_m, X_{0m}, \psi_m)$ and the function u_{mi}. It will now be shown by using energy estimates that this sequence is bounded in suitable Sobolev spaces and in fact is a Cauchy sequence. To do this, differentiate $\|u\|^2_{H^s,K}$ with respect to t and substitute the third reduced system into the result. This gives

$$d/dt(\|u\|_{H^s,K})^2) = -t^{-1}\langle N^{(s)} u^{(s)}, u^{(s)}\rangle_{L^2,K} + R. \qquad (25)$$

Here $u^{(s)}$ is the collection of all derivatives of u up order s and R is the sum of the terms which arise in the regular case, i.e. in the case where N is identically zero. The first term on the right-hand side is non-positive and may be discarded. The terms in R can be estimated just as in the regular case and this gives a bound for the H^s norm of u, provided $s > n/2 + 1$. Next the difference of successive approximants will be estimated. An attempt to apply the standard techniques in order to show that the sequence is Cauchy only meets one difficulty not present in the regular Cauchy problem. This is due to differences of the matrices N for successive elements of the sequence. This can be overcome in a way similar to that used above where the norms were scaled. The trick is to consider the collection of the derivatives of the functions of the sequence up to order s together with the derivatives of the differences of successive elements of the sequence up to order $s-1$, the derivatives of the differences being multiplied by an additional factor K. This once again ensures that the final matrix obtained is positive definite.

It follows from the above discussions that the sequence of solutions v_i of the third reduced system converges to a solution of the third reduced system corresponding to the original smooth data. This can then be used to define solutions of the second and first reduced systems and finally a solution of the Gowdy system itself corresponding to the data we started with. The construction is such that the interval on which convergence is obtained may depend on s. However standard results about symmetric hyperbolic systems with smooth coefficients show that the solutions for all values of s can be extended to a common time interval. Hence a solution is obtained on that time interval which is C^∞ for $t > 0$. The results of this discussion are summed up in the following theorem, which may be compared with theorem 1 of [9].

Theorem 4.1. *Let $k(x)$, $X_0(x)$, $\phi(x)$ and $\psi(x)$ be C^∞ and assume that $0 < k(x) < 3/4$ for all x. Then there exists a solution of the Gowdy equations with following properties. For each spatial point x there exists an open neighbourhood U_x of x and a number $\epsilon_x > 0$ such that the restriction of the solution to U_x satisfies (5) and (6) with $\epsilon = \epsilon_x$, where u and v are regular and tend to zero as $t \to 0$. The U_x and ϵ_x can be chosen in such a way that the inequalities $2k(y) - 1 < \epsilon_x < \min\{2k(y), 2 - 2k(y)\}$ are satisfied for all x and all $y \in U_x$. Under these conditions the solution is unique.*

Remark. A formulation of the theorem which is equivalent but cleaner can be obtained by replacing the constant ϵ by a function $\epsilon(x)$. Then it would not be necessary to introduce the U_x.

In [9] a theorem was proved concerning high velocity analytic solutions in the case where X_0 is independent of x. The method used to prove theorem 4.1 applies straightforwardly to the high velocity case to give an analogue of the result of [9] in the smooth case. The following theorem results.

Theorem 4.2. *Let $k(x)$, $\phi(x)$ and $\psi(x)$ be C^∞ and let X_0 be a constant. Assume that $k(x) > 0$. Then there exists a solution of the Gowdy equations with the following properties. For each spatial point x there exists an open neighbourhood U_x of x and a number $\epsilon_x > 0$ such that the restriction of the solution to U_x satisfies (5) and (6) with $\epsilon = \epsilon_x$, where u and v are regular and tend to zero as $t \to 0$. The U_x and ϵ_x can be chosen in such a way that the inequality $\epsilon_x < 2k(y)$ is satisfied for all x and all $y \in U_x$. Under these conditions the solution is unique.*

5. Data with intermediate velocity

In the previous section an existence theorem was proved for Gowdy spacetimes under the restriction $0 < k < 3/4$ on the function k. Next it will be shown that using a different ansatz allows the range $1/2 < k < 1$ to be treated. The two together then cover the whole range $0 < k < 1$ for which results were available in the analytic case. The new ansatz involves expanding the function Z to a higher order in t. The ansatz for X remains unchanged. Now Z is of the form

$$Z = k \log t + \phi + \alpha t^{2-2k} + t^{2-2k+\epsilon} u \tag{26}$$

where $\alpha = (2-2k)^{-2}X_{0x}^2$. Re-expressing the Gowdy equations in terms of the new variables u and v and reducing to first order as before leads to an analogue of the second reduced system of section 3. The evolution equations for u_0, u_2, v_0 and v_2 are the same as before and will not be repeated. The modified equation for u_1 is

$$\begin{aligned} t\partial_t u_1 = {} & -2(2-2k+\epsilon)u_1 - (2-2k+\epsilon)^2 u_0 + t^{2k-\epsilon}(k_{xx}\log t + \phi_{xx}) + t\partial_x u_2 \\ & + t^{1-\epsilon}[t\alpha_{xx} - 4k_x \log t(t\alpha_x + t^\epsilon u_2) + 4k_x^2 t(\log t)^2(\alpha + t^\epsilon u_0)] \\ & - \exp(-2\phi - 2\alpha t^{2-2k} - 2t^{2-2k+\epsilon}u_0)\{t^{4k-2-\epsilon}(v_1 + 2kv_0 + 2k\psi)^2 \\ & - 2t^{2k-1-\epsilon}X_{0x}(v_2 + t\psi_x + k_x(v_0+\psi)t\log t) \\ & - t^{4k-2-\epsilon}(v_2 + t\psi_x + 2k_x(v_0+\psi)t\log t)^2\} \end{aligned} \tag{27}$$

and that for v_1 is

$$\begin{aligned} t\partial_t v_1 = {} & -2kv_1 + t^{2-2k}X_{0xx} + t\partial_x(v_2 + t\psi_x) + 4k_x(v_2 + t\psi_x)t\log t \\ & + (v_0+\psi)[2k_{xx}t^2\log t + 4(k_x t\log t)^2] \\ & + 2t^{2-2k+\epsilon}(v_1 + 2kv_0 + 2k\psi)(u_1 + (2-2k+\epsilon)u_0) \\ & + (4-4k)t^{2-2k}(v_1 + 2kv_0 + 2k\psi)\alpha \\ & - 2X_{0x}t^{2-2k}(k_x\log t + \phi_x + t^{2-2k}\alpha_x + t^{1-2k+\epsilon}u_2 \\ & - 2k_x t^{2-2k}\log t(\alpha + t^\epsilon u_0)) - 2(v_2 + t\partial_x\psi + 2tk_x(v_0+\psi)\log t) \\ & \times (k_x t\log t + t\phi_x + t^{3-2k}\alpha_x + t^{2-2k+\epsilon}u_2 - 2k_x t^{3-2k}\log t(\alpha + t^\epsilon u_0). \end{aligned} \tag{28}$$

There is also an obvious analogue of the first reduced system of section 2. The existence of formal solutions of the latter is guaranteed by lemma 2.1 and these give rise to formal solutions of the second reduced system as in section 3.

Consider now the sequence of analytic solutions of the Gowdy equations corresponding to a sequence of analytic approximations to the smooth data of interest. It will be shown that, under the condition that $1/2 < k < 1$, these define a sequence of regular solutions of the second reduced system of this section. To do this it is necessary to show that for each of these solutions the function Z admits an asymptotic expansion of the form (26) and not just of the form (5), which is known *a priori*. To do this it suffices to apply the existence theorem of [11] to the first reduced system of this section with the analytic data.

Once these facts are known, it is straightforward to prove an existence theorem for the second reduced system of this section using the same techniques as were applied to the second reduced system of section 3 provided certain inequalities are satisfied. These are the inequalities which ensure that each term on the right-hand side of the equation contains a positive power of t. Under the assumption that $1/2 < k < 1$ this can be achieved by choosing the positive real number ϵ to satisfy $4k - 3 < \epsilon < 2k - 1$. The following theorem is obtained.

Theorem 5.1. *Let $k(x)$, $X_0(x)$, $\phi(x)$ and $\psi(x)$ be C^∞ and assume that $1/2 < k(x) < 1$ for all x. Then there exists a solution of the Gowdy equations with the following properties. For each spatial point x there exists an open neighbourhood U_x of x and a number $\epsilon_x > 0$ such that the restriction of the solution to U_x satisfies (26) and (6) with $\epsilon = \epsilon_x$, where u and v are regular and tend to zero as $t \to 0$. The U_x and ϵ_x can be chosen in such a way that the inequalities $4k(y) - 3 < \epsilon_x < 2k(y) - 1$ are satisfied for all x and all $y \in U_x$. Under these conditions the solution is unique.*

6. Discussion

The theorems stated in this paper have all concerned Gowdy spacetimes. It is nevertheless clear that many of the arguments are much more generally applicable. At the same time some steps are essentially related to the specific Gowdy case. A general discussion of the procedure will now be given which separates the general from the particular as much as possible. The first step is to make a suitable ansatz for the solutions to be constructed as the sum of an explicit part and a remainder. There may be more than one useful way of doing this. For example, in the Gowdy case equations (5) and (6) were useful for proving one theorem while replacing (5) by (26) allowed a different theorem to be proved. The second step is to reduce the equations to first order. The aim is to produce a system of Fuchsian form for which the theorem of [9] ensures the existence of solutions corresponding to the case where the free functions in the ansatz are analytic. If these free functions are merely smooth the lemma proved in section 2 may be used to show the existence of formal solutions.

The third step is to produce a system which is symmetric hyperbolic and in Fuchsian form. At this stage the matrix $N(x)$ may have negative eigenvalues, as is the case in the Gowdy example. It needs to be shown that the formal solutions already produced define formal solutions of the symmetric hyperbolic system. From this point on the argument proceeds on a general level, with no more details of the Gowdy special case being used.

It is instructive at this stage to consider what difficulties would be likely to arise in an attempt to generalize the results of [1] from the analytic to the smooth case. One problem is to bring the equations into a suitable hyperbolic form by the choice of coordinate or gauge conditions. There was no difficulty of this type in the Gowdy case, where a rather rigid preferred coordinate system is available. In more general cases it will be necessary to choose a form of the reduced Einstein equations carefully from the myriad on offer. If a symmetric hyperbolic system is obtained it is likely to involve a matrix A^0 multiplying the time derivative of the unknown which is not the identity, thus going beyond the case discussed above. Even worse, it may be difficult to ensure that A^0 remains bounded and uniformly positive definite as $t \to 0$. These conditions are very important for the use of energy estimates.

To conclude the paper, an application will be presented where the flexibility of smooth functions is essential. Existence theorems have been proved for Gowdy spacetimes in the low velocity case and, under the condition that X_0 is constant, also in the high velocity case. Using the domain of dependence these can be combined to give a more general class of solutions. To do this consider a smooth function X_0 which is constant on a non-empty open interval I. Now complete this to data (k, X_0, ϕ, ψ) in such a way that $k < 1$ on the closure of the complement of I. Then each point x has a neighbourhood on which one of the existence theorems applies and the resulting local solutions can be put together to produce a solution corresponding to the chosen initial data globally in x. If we tried to do this construction with analytic data then X_0 would have to be globally constant and nothing new would be obtained.

References

[1] Andersson L and Rendall A D 2000 Quiescent cosmological singularities *Preprint* gr-qc/0001047

[2] Anguige K and Tod K P 1999 Isotropic cosmological singularities 1: polytropic perfect fluid spacetimes *Ann. Phys., NY* **276** 257–93

[3] Anguige K and Tod K P 1999 Isotropic cosmological singularities 2: the Einstein–Vlasov system *Ann. Phys., NY* **276** 294–320

[4] Anguige K 2000 Isotropic cosmological singularities 3: the Cauchy problem for the inhomogeneous conformal Einstein–Vlasov equations *Ann. Phys., NY* **282** 395–419

[5] Anguige K 2000 A class of plane symmetric perfect-fluid cosmologies with a Kasner-like singularity *Class. Quantum Grav.* **17** 2117–28

[6] Claudel C M and Newman K P 1998 The Cauchy problem for quasi-linear hyperbolic evolution problems with a singularity in the time *Proc. R. Soc.* A **454** 1073–107

[7] Friedrich H and Rendall A D 2000 The Cauchy problem for the Einstein equations *Einstein's Field Equations and Their Physical Implications. Lecture Notes in Physics 540* ed B G Schmidt (Berlin: Springer)

[8] Isenberg J and Kichenassamy S 1999 Asymptotic behaviour in polarized T^2-symmetric vacuum spacetimes *J. Math. Phys.* **40** 340–52

[9] Kichenassamy S 1996 Fuchsian equations in Sobolev spaces and blow-up *J. Diff. Eqns* **125** 299–327

[10] Kichenassamy S 1996 *Nonlinear Wave Equations* (New York: Marcel Dekker)

[11] Kichenassamy S and Rendall A D 1998 Analytic description of singularities in Gowdy spacetimes *Class. Quantum Grav.* **15** 1339–55

[12] Moncrief V 1980 Global properties of Gowdy spacetimes with $T^2 \times \mathbf{R}$ topology *Ann. Phys., NY* **132** 87–107

[13] Moncrief V 1982 Neighbourhoods of Cauchy horizons in cosmological spacetimes with one Killing vector *Ann. Phys., NY* **141** 83–103

[14] Narita M, Torii T and Maeda K 2000 Asymptotic singular behaviour of Gowdy spacetimes in string theory *Preprint* gr-qc/0003013

[15] Newman R P A C 1993 On the structure of conformal singularities in classical general relativity *Proc. R. Soc.* A **443** 473–92

[16] Newman R P A C 1993 On the structure of conformal singularities in classical general relativity II: evolution equations and a conjecture of K P Tod *Proc. R. Soc.* A **443** 493–515

[17] Taylor M E 1996 *Partial Differential Equations III. Nonlinear Equations* (Berlin: Springer)