

# RR charges of D2-branes in the WZW model

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## Abstract

We consider the contribution of the  $B$ -field into the RR charge of a spherical D2-brane. Extending a recent analysis of Taylor, we show that the boundary and bulk contributions do not cancel in general. Instead, they add up to an integer as observed by Stanciu. The general formula is applied to compute the RR charges of spherical D-branes of the  $SU(2)$  WZW model at level  $k$  and it shows that these RR charges are only defined modulo  $k + 2$ . We support this claim by studying bound state formation of D0-branes using boundary conformal field theory.

The issue of Ramond-Ramond (RR) charges in background fluxes was raised by Bachas, Douglas and Schweigert [1] (see also [2]). They calculated the RR charges of spherical D2-branes which arise in the WZW model on the group  $SU(2)$  [3], [4], [5]. Two calculations, the first one based on the semi-classical considerations and the second one using the exact 1-point functions in the boundary WZW model, indicated that RR charges of D2-branes are irrational and that their differences are not integral. This observation is in

contradiction with our intuition about the nature  $U(1)$  charges and, hence, gives rise to an interesting puzzle which we discuss in this note. We shall see that there is a natural way to assign a  $\mathbb{Z}_{k+2}$  valued charge to D2-branes. Even though this charge is first motivated within the semi-classical approach, we conjecture that it contains important dynamical information. According to [6], a stack of D0-branes on a large  $SU(2)$  is unstable against decay into spherical D2-branes. Our conjecture is that the integer RR charge of a D2 brane counts the number of its constituent D0-branes and that this number is conserved (*mod*  $k+2$ ) in all physical processes. We provide some evidence for this proposal but it certainly has to undergo further tests.

Let us recall that in the case of flat D-branes the WZ coupling of Ramond-Ramond fields  $C = \sum_i c_i$  on the D-brane world-volume is given by formula [7],

$$I^{flat} = \int_V C \wedge \exp(B + F) \quad , \quad (1)$$

where  $B$  is the bulk  $B$ -field,  $F$  is the curvature of the  $A$ -field on the brane, and  $V$  is the world-volume of the D-brane  $D$ . At this point it is convenient to introduce the special notation  $B_D = B + F$ . We call this combination a *boundary*  $B$ -field. Note that in contrast to  $B$  and  $F$  the field  $B_D$  is gauge invariant since it defines the boundary conditions for the open string action.

Equation (1) implies the following expression for the D0-brane RR charge of a D2-brane,

$$Q_{RR}^{flat} = \int_D B_D \quad . \quad (2)$$

In the case of a contractible D2-brane, the right hand side of equation (2) is quantized in the units of  $2\pi$ . Indeed, the expression for the RR charge is proportional to the flux  $\Phi = 2\pi \int_D B_D$  of the field  $B_D$  through the D-brane. Only the D-branes with integral flux,  $\Phi = 2\pi n$  with  $n$  integer, define consistent boundary conditions for open strings. Thus, the flux quantization condition implies that  $Q_{RR}^{flat}$  is an integer.

In the case of curved D-branes formulas (1) and (2) receive corrections [8],

$$I = \int_V C \wedge \exp(B_D + \frac{1}{2}d)\hat{A}(TD) \frac{1}{f^* \sqrt{\hat{A}(TS)}} \quad . \quad (3)$$

Here  $d$  is the degree two class defining the  $Spin^c$  structure of  $D$ ,  $\hat{A}(TD)$  is the  $A$ -roof genus of the D-brane,  $S$  is the space-time, and  $f : D \rightarrow S$  is the

embedding of the D-brane to the space-time. In this paper we only consider situations when the tangent bundle  $TS$  is trivial. Then, formula (3) simplifies since  $\hat{A}(TS) = 1$  and  $d = c_1(TD)$ ,

$$I = \int_V C \wedge \exp(B_D) \text{Td}(TD) \quad , \quad (4)$$

where  $\text{Td}(TD)$  is the Todd genus of  $D$ . This leads to the following formula for the RR charge,

$$Q_{RR} = \int_D \exp(B_D) \text{Td}(TD). \quad (5)$$

By Atiyah-Singer index theorem, the right hand side is an index of the  $Spin^c$  Dirac operator on the D-brane twisted by the line bundle with the first Chern class represented by the 2-form  $B_D$ . Hence,  $Q_{RR}$  is an integer. In the case of D2-branes formula (5) simplifies,

$$Q_{RR} = \int_D B_D + \frac{1}{2}c_1(TD) = \frac{\Phi}{2\pi} + \frac{1}{2}c_1(TD) \quad . \quad (6)$$

In applying formula (6) to the discussion of D2-branes in the  $SU(2)$  WZW model we begin with the simpler limit in which the level  $k$  is sent to infinity. In this case one is only interested in the small neighborhood of the group unit and one can replace the sphere  $S^3 \cong SU(2)$  by the Euclidean space  $R^3$ . The D-branes are 2-spheres centered at the origin. They carry the flux of the Neveu-Schwarz (NS)  $B$ -field proportional to their radius  $\Phi(r) = 2\pi r$ . Only the spheres with integral radius give rise to consistent open string boundary conditions,  $\Phi(n) = 2\pi n$ . In our case,  $c_1(TD) = 2$ , and we obtain  $Q_{RR}(n) = n + 1$ . Note that in the definition of the Chern class  $c_1(TD)$  we use the complex structure on  $D$  such that the metric defined by  $B_D$  be positive definite.

Let us briefly recall how this conclusion can be obtained from a CFT calculation of the RR charge [1]. Suppose we are dealing with the supersymmetric WZW theory at level  $k+2$ . This model contains currents  $J^a$  satisfying the relations of a level  $k+2$  affine Kac Moody algebra along with a multiplet of free fermionic fields  $\psi^a$  in the adjoint representation of  $su(2)$ . It is well known that one can introduce new bosonic currents

$$J_{\mathbf{b}}^a := J^a + \frac{i}{k} f_{bc}^a \psi^b \psi^c$$

which obey again the commutation relations of a current algebra but now the level is shifted to  $k$ . The fermionic fields  $\psi^a$  commute with the new currents. This means that the theory splits into a product of a level  $k$  WZW model and a theory of three free fermionic fields.

We want to impose gluing conditions  $J^a(z) = \bar{J}^a(\bar{z})$  and  $\psi^a(z) = \pm\bar{\psi}^a(\bar{z})$  along the boundary  $z = \bar{z}$ . This obviously implies  $J_{\mathbf{b}}^a(z) = \bar{J}_{\mathbf{b}}^a(\bar{z})$ . Hence, the boundary states of the theories we are studying, factorize into two well known contributions,

$$|\alpha, \pm\rangle^{\text{susy}} = |\alpha\rangle \otimes |\mathbf{f}, \pm\rangle \quad \text{for} \quad \alpha = 0, \frac{1}{2}, \dots, \frac{k}{2} \quad ,$$

where  $|\alpha\rangle$  is one of the boundary states for a WZW-model at level  $k$  and  $|\mathbf{f}, \pm\rangle$  denote the familiar fermionic boundary states for (anti-)branes in flat space.

The boundary states of the level  $k$  WZW-model can be characterized by the quantities:

$$\langle j|\alpha\rangle = \frac{S_{\alpha j}}{\sqrt{S_{0j}}} = \left(\frac{2}{k+2}\right)^{1/4} \frac{\sin \pi \frac{(2\alpha+1)(2j+1)}{k+2}}{\left(\sin \pi \frac{2j+1}{k+2}\right)^{1/2}} \quad .$$

Here  $|j\rangle$  denotes a spin  $j$  primary field for the currents  $J_{\mathbf{b}}$ .

For a discussion of the fermionic states we refer the reader to the standard literature (see e.g. [9]). Let us only remark that these states  $|\mathbf{f}, \pm\rangle = |\text{NSNS}\rangle \pm |\text{RR}\rangle$  contain contributions from the NSNS and the RR sector. The choice of the sign  $\pm$  distinguishes between branes and anti-branes.

We obtain the RR charge of the brane by computing  $\langle \phi|\alpha, +\rangle^{\text{susy}}$  for an appropriate state  $|\phi\rangle$  from the (RR) sector of the bulk theory.  $|\phi\rangle$  is again a product state containing contributions from the RR ground states of the three free fermions and of the  $(j, \bar{j}) = (1/2, 1/2)$  representation for the currents  $J^a$ . Following [1], we insert this into the formulas above to obtain

$$Q_{RR}(n) = \lim_{k \rightarrow \infty} \frac{\langle 1/2|\alpha, +\rangle^{\text{susy}}}{\langle 1/2|0, +\rangle^{\text{susy}}} = n + 1 \quad .$$

Hence we reproduce precisely the RR charges from the previous discussion based on formula (6).

Now we turn to the case of finite  $k$  where the bulk curvature of  $S^3$  cannot be ignored. It leads to a non-vanishing field strength  $H$  of the bulk NS

$B$ -field. Recently, Taylor observed that for homotopically trivial D2-branes there is an extra contribution to the RR charge of the D2-brane coming from the integral of the  $H$ -field. More precisely, for a D2-brane  $D$  which can be contracted to a point along a 3-manifold  $\Gamma$  with  $\partial\Gamma = D$ , the contribution of the  $H$ -fields to the RR charge of the brane is given by (see (4.1) and (4.4) of [10]),

$$\Delta Q_{RR} = - \int_{\Gamma} H \ . \quad (7)$$

The total RR charge of the D2-brane reads,

$$Q_{RR} = \int_D B_D - \int_{\Gamma} H + \frac{1}{2}c_1(TD) \ . \quad (8)$$

This agrees with the results of Stanciu [11] except from the correction by  $\frac{1}{2}c_1(TD)$ . We can always choose the bulk  $B$ -field such that on  $\Gamma$  one obtains,

$$H = dB. \quad (9)$$

Then, the contributions of  $B$  and  $H$ -fields to  $Q_{RR}$  cancel each other and we obtain,

$$Q_{RR} = \int_D F + \frac{1}{2}c_1(TD) \ . \quad (10)$$

We extrapolate the result of [8] by requiring that  $c_1(TD)$  is obtained using the (almost) complex structure on  $D$  such that the metric defined by  $F$  be positive definite. The extra contribution  $\frac{1}{2}c_1(TD)$  is the correction to Taylor's formula.

In the presence of  $H$ -field the flux integrality condition imposed on the brane is modified [12] and reads (see *e.g.* equation (9) in [3]),

$$\Phi = 2\pi \left( \int_D B_D - \int_{\Gamma} H \right) = 2\pi \int_D F = 2\pi n \quad (11)$$

with  $n$  integer. For the D2-branes of the WZW model the combination of (10) and (11) yields,

$$Q_{RR}(n) = \frac{\Phi}{2\pi} + \frac{1}{2}c_1(TD) = n + 1 \ , \quad (12)$$

which is exactly the same formula as we had in the case  $k \rightarrow \infty$ . In the case of finite  $k$  the difference is that there is only a finite number of spherical D2-branes corresponding to  $n = 1, \dots, (k - 1)$ . These branes carry RR charges

equal to  $Q_{RR} = 2, \dots, k$ , respectively. We provide explicit expressions for  $H$ ,  $B$ ,  $B_D$  and  $F$  for the WZW model in Appendix. We conclude that in Lagrangian approach the self-consistent boundary conditions for open strings necessarily lead to integral RR charges, as expected on general grounds.

There is a new feature of the RR charges on curved backgrounds which follows from our consideration. Since the  $F$ -field is not a gauge invariant object, different choices of  $B$  and  $F$  corresponding to the same value of  $B_D = B + F$  may give rise to different values of  $Q_{RR}$ . In the case of the  $SU(2)$  WZW model the spherical D-branes can be contracted either to the group unit  $e$  along the 3-ball  $\Gamma$  or to the opposite pole  $(-e)$  of  $S^3$  along the 3-ball  $\Gamma'$ . The picture viewed from  $(-e)$  looks exactly the same way as from  $e$  with the exception that the value of the RR charge changes the sign because the 2-form  $F$  changes sign under the transformation  $n \rightarrow (k - n)$  together with  $\Gamma \rightarrow \Gamma'$ , and, hence, one should choose the opposite complex structure on  $D$  such that  $c_1(K_D)$  also changes sign. That is, we obtain another value of the RR charges of the same spherical D2-branes,

$$Q'_{RR}(n) = -((k - n) + 1) = (n + 1) - (k + 2) . \quad (13)$$

Here the shift by  $k$  is due to the change of the 3-ball  $\Gamma$  to  $\Gamma'$  and the extra shift by 2 is due to the change of sign on the canonical bundle when we shift from  $e$  to  $(-e)$ . We conclude that the RR charge is only well defined modulo  $(k + 2)$ . This is a novel feature which arises because the  $H$ -field belongs to a nontrivial cohomology class.

We would like to confirm the conclusions of the Lagrangian approach by computing  $Q_{RR}$  in the CFT picture. The standard prescription suggests computing RR charges from overlaps of closed string states with the boundary state. Unfortunately, in case of finite level  $k$ , such overlaps typically change along the RG-trajectories which are generated by boundary fields. That is, RR charges will not be conserved in physical processes such as formation of bound states of D-branes.

Instead, we shall conjecture that formulas (12-13) give an RG-invariant definition of RR charges which are defined only *mod*  $k + 2$ . We test this conjecture by arguing that an anti-brane at  $-e$  can form as a bound state of  $k + 1$  branes at the origin  $e$ . In this process the state with RR charge  $k + 1$  evolves into the state with RR charge  $-1$ .

We assume that the boundary state  $|0, +\rangle^{\text{susy}}$  describes a brane at  $e$ . Then, the state  $|k/2, +\rangle^{\text{susy}}$  corresponds to an anti-brane at  $-e$ . Indeed, we

can translate branes on the 3-sphere by acting with the currents  $J^a = J_b^a + J_f^a$  where  $J_f^a := -\frac{i}{k} f_{bc}^a \psi^b \psi^c$ . The finite translation which moves D0-branes from  $e$  to  $-e$  corresponds to the element  $-e$  in the group. One can check easily that the first factor  $|0\rangle$  in  $|0\rangle^{\text{susy}} = |0\rangle \otimes |\mathbf{f}, \pm\rangle$  is shifted to  $|k/2\rangle$  (see e.g. [13]). To see the effect of translations on the fermionic factor one should recall that the RR sector contains states with half-integer spins  $(j_f, \bar{j}_f) \in (\frac{1}{2} + \mathbb{Z}) \times (\frac{1}{2} + \mathbb{Z})$  while NSNS states are built up from integer spin only. Since the element  $-e$  is represented by  $\pm 1$  depending on whether it acts on states with integer or half integer spin, we conclude that  $|\mathbf{f}, +\rangle = |\text{NSNS}\rangle + |\text{RR}\rangle$  gets mapped into  $|\mathbf{f}, -\rangle = |\text{NSNS}\rangle - |\text{RR}\rangle$  by the finite translation with  $-e \in SU(2)$ . Hence,  $|k/2, -\rangle^{\text{susy}}$  is the boundary state of a brane at  $-e$  which means that  $|k/2, +\rangle^{\text{susy}}$  describes an anti-brane at  $-e$ .

Now let us look at a stack of  $k + 1$  D0-branes at the origin  $e$ . Following [6], this stack is expected to be unstable against perturbation with  $S_a J^a(x)$  where  $S_a$  are  $k+1$  dimensional representation matrices of  $su(2)$ . This perturbation triggers a decay into some bound state. It is difficult to obtain exact statements on the nature of this bound state but there exists a heuristic rule that seems to produce correct results in condensed matter physics [14] where it is used to identify the low temperature fixed point of the Kondo model.

To be more specific, let us consider open strings that stretch between the stack of  $k + 1$  branes at  $e$  and a single anti-brane at  $e$ . The corresponding states are taken from the space  $V_{k+1} \otimes \mathcal{H}_{e^+, e^-}$  where  $V_{k+1}$  is a  $k+1$ -dimensional vector space and  $\mathcal{H}_{e^+, e^-} = \mathcal{H}_k^0 \otimes \mathcal{H}_-^f$  is the product of the vacuum module for the current algebra at level  $k$  with the fermionic space  $\mathcal{H}_-^f$ . The latter carries an action of the level 2 Kac-Moody algebra generated by  $J_f^a$  and decomposes into a direct sum of  $\mathcal{H}_2^0$  (NS-sector) and  $\mathcal{H}_2^1$  (R-sector). The appearance of the vacuum sector gives rise to the tachyonic mode in the brane-anti-brane system.

On our space  $V_{k+1} \otimes \mathcal{H}_{e^+, e^-}$  there are commuting actions of the  $k + 1$ -dimensional matrices  $S_a$  on the first tensor factor  $V_{k+1}$  and of the currents  $J^a$ . These two symmetries are broken by the perturbation with  $S_a J^a$ . Experience with similar couplings of spin and angular momentum in standard atomic physics leads us to expect that the sum  $S_a + J^a$  has a good chance to be preserved. Note that the generators  $S_a + J_n^a$  which are obtained by shifting all modes  $J_n^a$  of the current  $J^a$  with the same constant  $S_a$ , satisfy the relations of a current algebra at level  $k+2$ . The rule about the conservation of  $S_a + J^a$  has been tested through CFT investigations of the Kondo model [14] and it

does correctly reproduce the results of [6] at large level  $k$ . We shall use it here to determine the open string spectrum of our configuration. Note that  $\mathcal{H}_{e^+,e^-}$  carries a tensor product of the spin  $j_f = 0, \frac{1}{2}$  representations of the fermionic currents with the spin  $j_b = 0$  representation for  $J_b^a$ . These add up to spin  $j = 0, \frac{1}{2}$  representations of the level  $k + 2$  currents  $J^a$ . If we add the spin  $s = k/2$  representation on  $V_{k+1}$  we end up with a spins  $k/2, k/2 \pm 1/2$ . Thus we have shown that the representation of current algebra on  $V_{k+1} \otimes \mathcal{H}_{e^+,e^-}$  generated by  $S_a + J_n^a$  decomposes into representations of spin  $k/2, k/2 \pm 1/2$ .

Our aim is now to compare this to the spectrum of open strings stretching between a single anti-brane at  $-e$  (the conjectured bound state of the original stack) and the anti-brane at  $e$ . Here we employ our previous claim that  $|k/2, +\rangle$  describes the anti-brane at  $-e$ . This implies that the wave functions of open strings stretching between the anti-branes at the poles are taken from the space  $\mathcal{H}_k^{k/2} \otimes \mathcal{H}_-$ . Under the action of  $J^a$  the latter decomposes into irreducibles of spin  $j = k/2, k/2 \pm 1/2$ . This are precisely the spin values we found for  $S_a + J^a$  in the initial configuration.

The analysis supports our claim that  $k + 1$  D0-branes at  $e$  decay into a single anti-brane at  $-e$ . If we want the RR charge to be conserved, we are forced to identify the charges of these configurations. In this sense the CFT considerations seem to confirm our proposal that the RR charge is defined only *mod*  $k + 2$ .

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## Appendix

Here we collect explicit formulas for the fields  $H$ ,  $B$  and  $B_0$  in the case of the  $SU(2)$  WZW model.

We recall [3] that in this case the field  $B_D$  which determines the open string boundary conditions on the D-brane is given by formula,

$$B_D = \frac{k}{8\pi} \text{Tr} \left( dgg^{-1} \frac{1 + Ad(g)}{1 - Ad(g)} dgg^{-1} \right) , \quad (14)$$

where  $k$  is the level on the WZW model. In terms of the Euler angles  $\psi, \theta$

and  $\phi$  this expression can be rewritten as

$$B_D = \frac{k}{2\pi} \sin(2\psi) \sin(\theta) d\theta d\phi .$$

The  $H$ -field is simply equal to the WZW form,

$$H = \frac{k}{12\pi} \text{Tr}(dgg^{-1})^3 = \frac{2k}{\pi} \sin^2(\psi) \sin(\theta) d\psi d\theta d\phi . \quad (15)$$

In terms of the Euler angles, one can solve equation  $H = dB$  with (see equation (2.3) of [1]),

$$B = \frac{k}{\pi} \left( \frac{\sin(2\psi)}{2} - \psi \right) \sin(\theta) d\theta d\phi .$$

Finally, the field  $F$  is obtained as a difference of  $B_D$  and  $B$ ,

$$F = \frac{k}{\pi} \psi \sin(\theta) d\theta d\phi ,$$

with integral over  $D$  given by formula,

$$\int_D F = 2k\psi . \quad (16)$$

For integral D-branes of the WZW model  $\psi_n = \pi n/k$  which yields the contribution to the RR charge equal to  $n$ .

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