HYPER VERSION

Received: *April 4, 2000*, Accepted: *May 3, 2000*

Vertex operators for the supermembrane

Arundhati Dasgupta, Hermann Nicolai and Jan Plefka

Albert-Einstein-Institut
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1, D-14476 Golm, Germany
E-mail: dasgupta@aei-potsdam.mpg.de, nicolai@aei-potsdam.mpg.de, plefka@aei-potsdam.mpg.de

Abstract: We derive the vertex operators that are expected to govern the emission of the massless $d = 11$ supermultiplet from the supermembrane in the light cone gauge. We demonstrate that they form a representation of the supersymmetry algebra and reduce to the type-IIA superstring vertex operators under double-dimensional reduction, as well as to the vertices of the $d = 11$ superparticle in the point-particle limit. As a byproduct, our results can be used to derive the corresponding vertex operators for matrix theory and to describe its linear coupling to an arbitrary $d = 11$ supergravity background. Possible applications are discussed.

Keywords: Superstrings and Heterotic Strings, M-Theory, p-branes, M(atrix) Theories.

JHEP05(2000)007

Contents

1. Introduction

The fundamental supermembrane [1] has many features that make it an attractive candidate for a fundamental description of M-Theory at the microscopic level (see e.g. [2] for many further references). As special limits, it contains the type-II superstrings [3] as well as the $d = 11$ superparticle [4] and is thereby also related to maximal $d = 11$ supergravity [5]. Furthermore, matrix theory can be obtained as a regularization of the fundamental supermembrane [6]. The theory thus sits atop the main contenders for a unified theory of quantum gravity, but actually possesses even more degrees of freedom. This is obvious for the superparticle, where one retains only the degrees of freedom corresponding to the $d = 11$ supermultiplet, discarding all internal excitations of the membrane. In the superstring truncation, which can be obtained at the kinematical level by a simple procedure called double-dimensional reduction [7], one keeps the infinite tower of perturbative excited superstring states, but loses the true M-Theory degrees of freedom. However, one still recovers in this way both the IIA and IIB superstrings if one keeps the winding modes and associated BPS multiplets [8, 9]. Finally, maximally supersymmetric matrix theory, which was proposed as a candidate for M-Theory in the light cone gauge [10, 11], does capture the non-perturbative degrees of freedom, but only finitely many (and misses the winding states of the membrane). At least in the opinion of the present authors, the successes of the matrix theory proposal are really rooted in the supermembrane origin of the theory. In particular, supermembrane theory naturally accounts for all aspects related to longitudinal degrees of freedom, which have to be guessed in matrix theory because supersymmetric-Yang-Mills theory does not "know" about an 11-th dimension.

Why is it, then, that supermembrane theory has not gained wider acceptance, despite all its appealing features? One obvious reason is the intrinsic nonlinearity of the theory that makes it much harder to deal with than the superstring, and that has until now blunted all attempts to make meaningful calculations *at the quantum level* (of course, there is much work on classical and semiclassical aspects of the $d = 11$ supermembrane, see e.g. [12]). The supersymmetric SU(N)-matrix theory, on the other hand, does have the advantage of being rigorously defined as a model of quantum mechanics (for finite N), and at the same time being an intrinsically non-perturbative approximation, but it, too, suffers from a host of unsolved problems, especially concerning the existence and precise nature of the $N \to \infty$ limit. All matrix theory calculations performed so far are consequently limited in scope; for instance, scattering amplitudes have only been calculated in the eikonal regime where no longitudinal momentum transfer is allowed [13, 14]. A recent test of the R^4 corrections has failed to reproduce the structures predicted by string theory [15].

A further difficulty with supermembrane theory is that we have at present very little idea of what the sensible objects are to consider and the relevant quantities to compute. This question is related to our lack of understanding as to what the fundamental supermembrane degrees of freedom really are. Owing to the continuity of the supermembrane spectrum [16] (independently realized in [17]) there appears to be no analog of the perturbative excited superstring states, even though the supermembrane has far more degrees of freedom. A crucial insight, occasioned by the matrix proposal, was that the excitations of the theory are to be associated with multi-particle rather than one-particle states [10]. The degeneracy of the membrane with regard to string-like deformations suggests a similar picture [18]. The only sensible one-particle-like excitations of the theory appear to be the ones associated with the massless $d = 11$ supermultiplet. However, it does not seem to be possible to set up the usual perturbative scheme based on Fock space quantization, or even to assign a definite "membrane number" to a given supermembrane configuration.

In this paper we take a step in the direction of making supermembrane theory "more computable". By generalizing previous work on superstring theory [3] and the more recent construction of the $d = 11$ superparticle vertex operators [19], we have succeeded in identifying the supermembrane vertex operators that are expected to govern the emission of the massless $d = 11$ multiplet from the supermembrane. By construction, our vertex operators contain all previous ones, but they also furnish new

JHEP05(2000)007

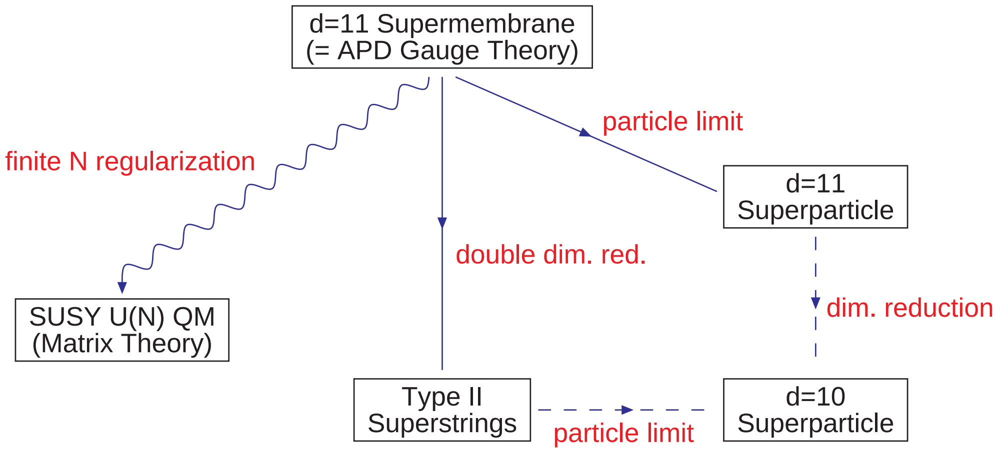

Figure 1: The hierarchy of theories.

information. Namely, as a byproduct of the present construction, we are able to solve two outstanding and closely related problems of matrix theory: the construction of matrix vertex operators and the coupling of the matrix model to a non-trivial $d = 11$ supergravity background in the light cone gauge. In this way, we are now in a position to investigate how the various presently available results on superstring and matrix model amplitudes as well as the non-perturbative results of [20, 21, 19] embed into supermembrane theory.

The figure displays how the various theories and their vertex operators are contained in the supermembrane. The embedding of vertex operators corresponding to the dashed lines was already studied in [19].

2. The supermembrane as a supersymmetric gauge theory of area preserving diffeomorphisms

Supermembrane theory was originally formulated as a covariant theory coupled to an arbitrary background satisfying the equations of motion of $d = 11$ supergravity [1]. There are eleven bosonic target space coordinates $X^M = (X^a, X^\pm)$ (where indices $a, b, \ldots = 1, \ldots, 9$ label the transverse dimensions), and 32 fermionic fields Θ, which transform as $\mathrm{SO}(1, 10)$ spinors, but are world-volume scalars. All of these fields depend on the membrane world-volume coordinates $(\tau, \sigma^1, \sigma^2)$. Like with superstring theory, the supermembrane action simplifies dramatically when one imposes the light cone gauge $X^+ = p^+\tau$ and $\Gamma^+\Theta = 0$ (in the following we shall set $p^+ = 1$ for simplicity, moreover $\Gamma^\pm = (\Gamma^{10} \pm \Gamma^0)/\sqrt{2}$). These conditions reduce the number of bosonic degrees of freedom to the nine transverse ones, and halve the number of fermionic degrees of freedom to the 16 components of an $\mathrm{SO}(9)$ spinor θ.

An important property of the light cone gauge fixed theory is its invariance under a residual infinite-dimensional group, the group of area preserving diffeomorphisms

(APDs) [22] (whose analog for string theory simply consists of the constant shifts along the space-like world-sheet coordinate σ^1). The canonical constraint associated with the APDs is actually necessary to eliminate one further bosonic degree of freedom in order to balance the number of bosonic and fermionic degrees of freedom on shell, as is required for a supersymmetric theory. For any two functions $A(\sigma^1, \sigma^2)$ and $B(\sigma^1, \sigma^2)$ on the membrane the APD Lie bracket is given by

$$\{A, B\} := \varepsilon^{rs} \partial_r A \, \partial_s B \,, \tag{2.1}$$

where $\partial_r := \partial/\partial\sigma^r$. In this interpretation, one views the coordinates (σ^1, σ^2) not as providing coordinates for the membrane, but rather as a parametrization of the APD Lie algebra elements.

This residual invariance can be exploited to reformulate the light cone supermembrane theory as a supersymmetric gauge theory of area preserving diffeomorphisms [6], thereby establishing the link between the supermembrane and maximally extended supersymmetric-Yang-Mills theory. To this aim one introduces (by hand) an APD gauge field ω, such that in the gauge $\omega = 0$ one reobtains the original supermembrane action in the light cone gauge. The resulting theory coincides with the dimensional reduction of maximally supersymmetric-Yang-Mills theory *to one (time) dimension*, i.e. a model of supersymmetric quantum mechanics, but with an infinite-dimensional gauge group (for finite dimensional gauge groups, these models were originally derived in [23]). The APD gauge field ω then coincides with the time component of the gauge field of dimensionally reduced super-Yang-Mills theory.

The supersymmetric lagrangian of the APD gauge theory reads

$$\mathcal{L} = \frac{1}{2}(DX^a)^2 - i\theta D\theta - \frac{1}{4}\left\{X^a, X^b\right\}^2 - i\theta\,\gamma_a\{X^a, \theta\}\,, \tag{2.2}$$

where

$$D\mathcal{O} = \partial_0 \mathcal{O} - \{\omega, \mathcal{O}\} \tag{2.3}$$

and where the infinitesimal area preserving diffeomorphisms

$$\sigma^r \longrightarrow \sigma^r + \epsilon^{rs}\partial_s\,\xi(\sigma) \tag{2.4}$$

act on the fields as $\delta X^a = \{\xi, X^a\}$, $\delta\theta = \{\xi, \theta\}$ and $\delta\omega = \partial_0\xi + \{\xi, \omega\}$.

The lagrangian (2.2) is invariant under the supersymmetry variations

$$\begin{aligned} \delta X^a &= -2\epsilon\,\gamma^a\theta\,, \\ \delta\theta &= i\,DX\cdot\gamma\,\epsilon - \frac{i}{2}\left\{X^a, X^b\right\}\,\gamma_{ab}\epsilon + \eta\,, \\ \delta\omega &= -2\epsilon\theta\,. \end{aligned} \tag{2.5}$$

As expected from the $d = 11$ origin of the model, there are still 32 supersymmetry parameters. These are split into two 16-component SO(9) spinors η and ϵ. Following established usage, we will refer to them as *linear supersymmetry* (parametrized by η) and *non-linear supersymmetry* (parametrized by ϵ) transformations, respectively. The linear supersymmetry transformations obviously affect only the zero modes.

The equations of motion that follow from the above action are

$$0 = D^2 X^a - \left\{\left\{X^a, X^b\right\}, X^b\right\} - i\{\theta, \gamma^a \theta\}\,, \tag{2.6}$$

$$0 = D\theta + \{\gamma \cdot X, \theta\}\,, \tag{2.7}$$

$$0 = \{DX^a, X^a\} - i\{\theta, \theta\}\,. \tag{2.8}$$

The last of these equations, obtained by varying the gauge field ω, is the constraint associated with the APD gauge invariance on the membrane.

As shown in [6], the above model can be approximated by a supersymmetric SU(N)-matrix model, such that the full theory is (formally) recovered by taking the limit $N \to \infty$. The essential ingredient here is the result that the group of APDs can be approximated by SU(N). This statement, first established for spherical membranes in [22] and for toroidal ones in [24, 25], actually holds for membranes of arbitrary topology [26]. The prescription for obtaining a matrix model from the above lagrangian is simple: just replace the target space fields by SU(N) matrices according to

$$\begin{aligned} X^a\left(\tau, \sigma^1, \sigma^2\right) &\longrightarrow \mathbf{X}^a_{mn}(\tau) \equiv \sum_A X^{aA}(\tau)\, Y^A_{mn}\,, \\ \theta_\alpha\left(\tau, \sigma^1, \sigma^2\right) &\longrightarrow \theta^\alpha_{mn}(\tau) \equiv \sum_A \theta^A_\alpha(\tau)\, Y^A_{mn} \end{aligned} \tag{2.9}$$

with $m, n = 1, \ldots, N$ labeling the entries of the $N \times N$-hermetian matrices $\mathbf{X}^a$ and θ^α, and a (hermitian and orthonormal) basis $\{Y^A \mid A = 1, \ldots, N^2 - 1\}$ of the SU(N) Lie algebra. Furthermore, the APD Lie bracket gets replaced by a matrix commutator $\{\cdot\,,\cdot\} \to i\,[\cdot\,,\cdot]$. This is all that is needed to get the matrix model, proposed as a candidate for a microscopic description of M-Theory in the light cone gauge [10]. We will return to matrix theory in section 5, to show how our results can be exploited to derive vertex operators for the matrix model, and to couple the matrix model to a non-trivial $d = 11$ background.

An equally important property of the supermembrane is that it contains the superstring as a special truncation. The embedding is achieved by identifying the membrane target space coordinate X^9 with the world-volume coordinate σ^2, a procedure called "double-dimensional reduction" [7]. Setting $X^9 = \sigma^2$ and letting all other fields only depend on σ^1, with i, j labeling the first eight transverse directions, the action (2.2) collapses to[1]

$$\mathcal{L}_{\mathrm{DDR}} = \frac{1}{2}\left(\partial_0 X^i\right)^2 - \frac{1}{2}\left(\partial_1 X^i\right)^2 - i\,\theta \partial_0\,\theta + i\,\theta\gamma^9\partial_1\theta \tag{2.10}$$

[1]Since $\sigma^2 \in [0, R)$, there are also the winding modes associated with the compactification on the circle.

which is just the Green Schwarz light cone lagrangian of the IIA superstring.[2] As we will see, the superstring vertex operators can be recovered from those of the supermembrane by an analogous procedure.

3. The vertex operators

The massless states of the supermembrane (and the supersymmetric matrix model) are expected to yield a massless multiplet of $d = 11$ supergravity, containing the graviton, the three-form gauge potential and the gravitino (see [27] for progress in establishing the existence of such states). We are therefore interested in constructing candidates for vertex operators that would describe the emission of these massless states from the supermembrane (due to the continuity of the supermembrane mass spectrum, there appear to be no discrete excited supermembrane states). Clearly, an essential consistency requirement for such operators is that they should coincide with the corresponding ones of the $d = 11$ superparticle [19], as well as with the full superstring vertex operators upon double-dimensional reduction. We note in passing that the leading θ contribution to the (covariant) gravitino vertex operator has already been used in computations of membrane instanton effects in [28].

To arrive at closed expressions for the vertex operators, we follow the strategy that was already successfully employed in the construction of superstring vertex operators [3], and more recently the construction of vertex operators for the $d = 11$ superparticle [19]. Namely, one exploits the fact that under the above supersymmetries the vertex operators should vary into one another, such that the transformations can be thrown onto the corresponding variations of the polarizations as they follow from $d = 11$ supergravity. Schematically, we thus have

$$\begin{aligned} \delta V_h &= V_{\delta\psi[h]} \,, \\ \delta V_C &= V_{\delta\psi[C]} \,, \\ \delta V_\psi &= V_{\delta h} + V_{\delta C} \end{aligned} \tag{3.1}$$

up to total derivatives (while the total derivatives in [19] were always derivatives w.r.t. time, they here appear both as $D(\cdots)$ and $\{\cdot\,,\cdot\}$). The variations to be performed on the l.h.s. of these expressions are the ones of the supersymmetric APD gauge theory given in (2.5) above, whereas the variations on the r.h.s. are those induced by $d = 11$ supergravity on the various polarizations. By $\delta\psi[h]$ and $\delta\psi[C]$ we here

[2]For the SO(9) Clifford algebra, we choose the following representation:

$$\gamma^9 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \Gamma^i \\ \bar{\Gamma}^i & 0 \end{pmatrix} ,$$

where $\Gamma^i_{\alpha\dot{\beta}}$ and $\bar{\Gamma}^i_{\dot{\alpha}\beta}$ are the standard SO(8) Γ-matrices.

designate the terms in the gravitino variation depending on the graviton and three-form polarizations h and C, respectively. A detailed explanation of the general procedure can for instance be found in [19].

An alternative route to arrive at our results would be to start from the covariant $d = 11$ supermembrane [1], whose background coupling is explicitly known in *superspace*. This approach would yield the fully covariant vertices, which should then reduce to the vertices presented above in the light cone gauge. However, obtaining the component form of this action by constructing the superspace vielbein and tensor gauge field in terms of the component fields to all orders in θ appears to be a prohibitively difficult task: to date the expansion is only known up to order θ^2 for general backgrounds [30] (incidentally, the fully covariant vertex operators are not even known for the GS superstring or the superparticle). The light-cone approach adopted here proves to be far more efficient because the expansion in θ already terminates at order five — as opposed to order 32 for the covariant expressions. This demonstrates again the drastic simplification of the fermion sector in the light-cone gauge, already seen for the flat background action.

Let us now present the results, and then comment on their derivation and the various consistency checks which we have performed to ascertain the correctness of these expressions. All vertex operators come with a factor $\exp(-ik \cdot X)\exp(ik^- \tau)$, where k_a is the (transverse) momentum of the state emitted. Following standard practice in string theory [29], we will set $k^+ \equiv k_- = 0$ in order to avoid the appearance of the longitudinal target space coordinate $X^-(\tau, \sigma^1, \sigma^2)$ in the exponential.[3] Furthermore we shall often disregard the extra factor $\exp(ik^-\tau)$ in our considerations, except in those places where it gives extra contributions from integrating by parts the time derivative operator D.

The vertex operators are contracted with the polarizations corresponding to the massless states of $d = 11$ supergravity. Choosing the gauge conditions $h_{a-} = h_{--} = h_{+-} = 0$, $C_{ab-} = C_{a+-} = 0, \psi_- = \tilde{\psi}_- = 0$, and splitting the remaining polarizations according to their longitudinal content, we have

$$\begin{aligned} \text{graviton} : & \, (h_{ab}, h_{a+}, h_{++}) \, , \\ \text{three-form} : & \, (C_{abc}, C_{ab+}) \, , \\ \text{gravitino} : & \, \left(\psi_a, \psi_+; \tilde{\psi}_a, \tilde{\psi}_+\right) . \end{aligned} \tag{3.2}$$

Note that again we have 32 spinor components in accordance with the $d = 11$ origin of the model, namely 16 components for ψ and $\tilde{\psi}$ each. We will also use the gauge

[3] We are aware that this choice of frame is somewhat questionable, although widely adopted: with it, the transverse momentum components must become complex. In addition, there are inverse factors of $1/k^+$ in some of the compensating transformations; fortunately, these drop out due to the gauge invariance of the vertices.

invariant combinations

$$F_{abcd} := 4k_{[a}C_{bcd]}\,, \qquad F_{abc+} := 3k_{[a}C_{bc]+} + k_+C_{abc}\,. \tag{3.3}$$

The polarizations are subject to the following physical state constraints [19]:

$$\begin{aligned} k^a h_{ab} = h_{aa} = 0 = k^a C_{abc}\,, \\ \gamma^a\,\tilde\psi_a = \gamma^b k_b\,\tilde\psi_a = k^a\,\tilde\psi_a = 0 = k^a\,\psi_a\quad, \\ \gamma^a\,\psi_a = \tilde\psi_+\,, \qquad k^b\gamma_b\,\psi_a = k^-\tilde\psi_a\,. \end{aligned} \tag{3.4}$$

For the bilinears in the sixteen-component real spinors θ we introduce the notation

$$R^{abc} = \frac{1}{12}\,\theta\,\gamma^{abc}\theta\,, \qquad R^{ab} = \frac{1}{4}\,\theta\,\gamma^{ab}\theta\,. \tag{3.5}$$

The key SO(9) Fierz identity for θ reads

$$\theta_\alpha\,\theta_\beta = \frac{1}{2}\delta_{\alpha\beta}\,\delta^{(2)}(0) + \frac{1}{32}\,\gamma^{ab}_{\alpha\beta}\,\theta\gamma_{ab}\,\theta + \frac{1}{96}\,\gamma^{abc}_{\alpha\beta}\,\theta\gamma_{abc}\theta\,. \tag{3.6}$$

The singular $\delta_{\alpha\beta}\,\delta^{(2)}(0)$ term here arises if one assumes the standard canonical anticommutation relations for the fermionic operators θ_α. Fortunately, however, this term drops out in all the manipulations performed in this work and is thus irrelevant to our final expressions.

Let us now state the main results of this paper and describe its derivation in the next chapter. The graviton vertex operator is given by

$$\begin{aligned} V_h = h_{ab}\Big[&DX^a\,DX^b - \{X^a,X^c\}\,\{X^b,X^c\} - i\theta\gamma^a\,\{X^b,\theta\} - \\ &- 2DX^a\,R^{bc}\,k_c - 6\{X^a,X^c\}\,R^{bcd}\,k_d + 2R^{ac}\,R^{bd}\,k_c\,k_d\Big]\,e^{-ik\cdot X}\,, \end{aligned} \tag{3.7}$$

$$V_{h_+} = -2h_{a+}\,\big(DX^a - R^{ab}k_b\big)\,e^{-ik\cdot X}\,, \tag{3.8}$$

$$V_{h_{++}} = h_{++}\,e^{-ik\cdot X}\,. \tag{3.9}$$

For the vertex operator corresponding to the three-form potential, we find

$$\begin{aligned} V_C = &-C_{abc}\,DX^a\,\{X^b,X^c\}\,e^{-ik\cdot X} + \\ &+F_{abcd}\bigg[\Big(DX^a - \frac{2}{3}R^{ae}\,k_e\Big)\,R^{bcd} - \frac{1}{2}\,\{X^a,X^b\}\,R^{cd} - \\ &- \frac{1}{96}\,\{X^e,X^f\}\,\theta\gamma^{abcdef}\theta\bigg]\,e^{-ik\cdot X}\,, \end{aligned} \tag{3.10}$$

$$V_{C_+} = C_{ab+}\,\big(\{X^a,X^b\} + 3R^{abc}\,k_c\big)\,e^{-ik\cdot X}\,. \tag{3.11}$$

Finally, for the gravitino vertex operators, we obtain

$$\begin{aligned}
V_\Psi = \psi_a \Big[\Big(DX^a - 2R^{ab}\, k_b + \gamma_c \, \{X^c, X^a\} \Big) \theta \Big] \, e^{-ik\cdot X} + \\
+ \tilde{\psi}_a \Big[\gamma \cdot DX \Big(DX^a - 2R^{ab}\, k_b + \gamma_c \, \{X^c, X^a\} \Big) \theta + \\
+ \frac{1}{2} \gamma_{bc} \, \big\{X^b, X^c\big\} \, \big(DX^a - \big\{X^a, X^d\big\} \, \gamma^d \big) \, \theta + \\
+ 8 \gamma_b \theta \, \big\{X^b, X^c\big\} \; R^{cad}\, k_d + \frac{5}{3} \gamma_{bc} \theta \, \big\{X^b, X^c\big\} \, R^{ad}\, k_d + \\
+ \frac{4}{3} \gamma_{bc} \theta \, \Big(\big\{X^a, X^b\big\} \; R^{cd} + \big\{X^c, X^d\big\} \, R^{ab} \Big) k_d + \\
+ \frac{2}{3} i \Big(\gamma_b \theta \{X^a, \theta\} \gamma^b \theta - \theta \{X^a, \theta\} \theta \Big) + \frac{8}{9} \gamma^b \theta \, R^{ac}\, R^{bd}\, k_c\, k_d \Big] \, e^{-ik\cdot X} \,,
\end{aligned} \tag{3.12}$$

$$V_{\Psi_+} = - \Big[\psi_+ \theta + \tilde{\psi}_+ \Big(\gamma^a DX^a + \frac{1}{2} \gamma^{ab} \, \big\{X^a, X^b\big\} \Big) \, \theta \Big] \, e^{-ik\cdot X} \,. \tag{3.13}$$

Concerning the above vertices one should keep in mind that relaxing the frame choice $k^+ = 0$, we would have to cope with extra terms involving the longitudinal component $k^+ X^-$ not only in the exponential, but also in the prefactors multiplying the exponential. Secondly, when passing to the quantum theory we must be prepared to modify the vertices by extra "renormalizations" as would be the case for composite operators in any interacting quantum field theory (such as QCD). However, such modifications are very tightly constrained in that they must not only preserve the symmetry properties to be discussed below, but also reduce to the standard normal-ordering prescription in the superstring limit.

4. Consistency checks

The complete expressions given above were arrived at by exploiting a number of constraints and consistency requirements. There are altogether four of these, which follow from (*i*)-gauge invariance, (*ii*)-dimensional reduction, (*iii*)-linear supersymmetry, and (*iv*)-non-linear supersymmetry. We will now discuss these in turn. A further (and quite tedious) check, which we have not performed, would be to verify the covariance of the vertices under Lorentz boosts in eleven dimensions, using the supermembrane boost generators constructed in [25].

4.1 Gauge invariance

Gauge invariance of the vertices requires that they be left unchanged under the following transformations,

$$\delta h_{ab} = k_{(a} \xi_{b)} \,, \qquad \delta h_{a+} = \frac{1}{2} \left(k_a \xi_+ + k_+ \xi_a \right) \,, \qquad \delta h_{++} = k_+ \xi_+ \,, \tag{4.1}$$

$$\delta C_{abc} = 3k_{[a}\xi_{bc]}\,, \qquad \delta C_{ab+} = 2k_{[a}\xi_{b]+} + k_+\xi_{ab}\,, \tag{4.2}$$

$$\begin{aligned} \delta\psi_a &= k_a\epsilon\,, \qquad \delta\tilde{\psi}_a = k_a\eta\,, \\ \delta\psi_+ &= k_+\epsilon\,, \qquad \delta\tilde{\psi}_+ = k_+\eta \end{aligned} \tag{4.3}$$

which are induced on the polarization tensors by the corresponding gauge symmetries of $d = 11$ supergravity. The transformations listed, respectively, correspond to (linearized) coordinate transformations (with parameter ξ^a), to tensor gauge transformations (with parameter $\xi_{ab} = -\xi_{ba}$), and to the inhomogeneous (field independent) part of the supersymmetry transformations. Gauge invariance holds only on-shell, because in order to establish it, we will have to make use of the equations of motion (2.6), (2.7) and (2.8).

The invariance under tensor gauge transformations is manifest in the transverse sector, except for the first term in V_C, which transforms as

$$\begin{aligned} \delta_\xi V_C &= \Big[-\xi_{bc}\, k\cdot DX\, \left\{X^b, X^c\right\} - 2\xi_{bc}\, DX^b \left\{X^c, k\cdot X\right\}\Big]\, e^{-ik\cdot X + ik^-\tau} = \\ &= -k^-\, \xi_{bc}\, \left\{X^b, X^c\right\}\, e^{-ik\cdot X+ik^-\tau} \end{aligned} \tag{4.4}$$

upon partial integration. This precisely cancels the variation of V_{C_+} as

$$\delta_\xi V_{C_+} = 2i\,\xi_{b+}\, \left\{e^{-ik\cdot X+ik^-\tau}, X^b\right\} + k^-\,\xi_{ab}\, \left\{X^a, X^b\right\}\, e^{-ik\cdot X+ik^-\tau}\,, \tag{4.5}$$

where the first term in (4.5) is a total derivative.

The graviton vertex requires a little more work: replacing h_{ab} by $k_{(a}\xi_{b)}$ in (3.7) we see that several terms drop out by antisymmetry. For the remaining ones, we get

$$\begin{aligned} \delta_\xi V_h = \Big[& D(k\cdot X)\left(D(\xi\cdot X) - R^{ab}\xi_a k_b\right) - \{k\cdot X, X^c\}\{\xi\cdot X, X^c\}\,\times \\ & \times 3\{k\cdot X, X^c\}R^{abc}\xi_a k_b - \frac{i}{2}\theta\xi_a\gamma^a\{k\cdot X,\theta\} - \frac{i}{2}\theta k_a\gamma^a\{\xi\cdot X,\theta\}\Big]e^{-ik\cdot X}\,. \end{aligned} \tag{4.6}$$

Next we integrate by parts the terms involving $k\cdot X$; this yields

$$\begin{aligned} \delta_\xi V_h = ie^{-ik\cdot X}\Big[& -D^2(\xi\cdot X) + \{X^c,\{X^c,\xi\cdot X\}\} + \frac{i}{2}\{\theta,\xi_a\gamma^a\theta\} \\ & + \frac{1}{2}D\theta\gamma^{ab}\theta\,\xi_a\,k_b - \frac{1}{2}\theta k\cdot\gamma\,\{\xi\cdot X,\theta\} + \frac{1}{2}\,\{X^c,\theta\}\gamma^{abc}\theta\,\xi_a\,k_b\Big] \\ & + k^-\,\left(D(\xi\cdot X) - R^{ab}\xi_a k_b\right)\, e^{-ik\cdot X}\,. \end{aligned} \tag{4.7}$$

The terms in the first two lines vanish by making use of the equations of motion of X^a (2.6) and θ (2.7), whereas the last term is seen to cancel with the gauge transformations of the longitudinal graviton vertices

$$\delta_\xi V_{h_+} = -\Big[\xi^-\,k^- + k^-\,\left(D(\xi\cdot X) - R^{ab}\xi_a k_b\right)\Big]\, e^{-ik\cdot X}\,, \tag{4.8}$$

$$\delta_\xi V_{h_{++}} = \xi^-\,k^-\,e^{-ik\cdot X}\,. \tag{4.9}$$

For the gravitino vertex again several terms drop out by antisymmetry, and we are left with

$$
\begin{aligned}
\delta_\xi V_\Psi = & -i\eta e^{-ik\cdot X}\Big[D\theta + \gamma^a\{X^a,\theta\}\Big] + k^-\,\eta\theta\, e^{-ik\cdot X} + \\
& + \epsilon e^{-ik\cdot X}\Big[\gamma^a DX^a\left(D(k\cdot X) + \gamma^b\left\{X^b, k\cdot X\right\}\right)\theta + \\
& \qquad + \frac{1}{2}\gamma^{ab}\left\{X^a,X^b\right\}\left(D(k\cdot X) - \{k\cdot X, X^c\}\gamma^c\right)\theta\Big] + \\
& + \frac{2}{3} i e^{-ik\cdot X}\Big[\epsilon\theta\,\theta\{k\cdot X,\theta\} - \epsilon\gamma^a\theta\,\theta\gamma^a\{k\cdot X,\theta\}\Big]\,. \qquad (4.10)
\end{aligned}
$$

The first line is just the fermionic equation of motion plus a term that cancels against the gauge transformation of V_{Ψ_+}. The remaining terms can likewise be shown to vanish on shell after some integrations by part, and use of the Fierz identity:

$$
\epsilon\gamma^a\{\theta,\theta\}\theta - \epsilon\{\theta,\theta\}\theta = \frac{1}{2}\,\epsilon\theta\,\{\theta,\theta\} - \frac{1}{2}\,\epsilon\gamma_a\theta\,\{\theta,\gamma^a\theta\}\,. \qquad (4.11)
$$

4.2 Reductions

There are two reductions which provide stringent consistency checks. The first arises from the comparison of our vertex operators with those of the superparticle recently determined in [19]. In this truncation one stays in eleven dimensions, but discards all internal degrees of freedom, such that the variables (X^a,θ) no longer depend on the coordinates (σ^1,σ^2), but only on τ. Accordingly, one simply drops the terms involving the APD Lie bracket $\{\cdot,\cdot\}$ in all expressions. Although this looks like a rather trivial truncation, it still yields a good deal of the information required; in particular, quartic and quintic fermionic terms are not affected by it at all, as they are independent of X^a. This allows us to take over the pertinent expressions from [19] and thereby to fix many terms without further ado.

To check the agreement of our vertices with those of superstring theory (which are also listed in [19]) after double-dimensional reduction is more subtle, not least because some "obvious" guesses turn out to be incorrect. In this truncation one retains the infinite tower of (perturbative) massive superstring states together with the BPS states (the winding states of the membrane), but $d = 11$ covariance is lost. Demanding the doubly reduced vertices to agree with those of superstring theory then fixes the terms involving the APD Lie brackets, which cannot be determined from the superparticle vertex operators. It is most remarkable that, despite the absence of any factorization in eleven dimensions, our vertices do factorize in precisely the required way after dimensional reduction. Furthermore, they combine the contributions originating from the $\mathrm{R}\otimes\mathrm{R}$ and the $\mathrm{NS}\otimes\mathrm{NS}$ sectors, which superstring theory treats separately, into unified expressions.

As already mentioned, upon double-dimensional reduction, the APD brackets either vanish, or become derivatives w.r.t. the remaining string worldsheet coordinate

$\sigma \equiv \sigma^1$, such that

$$\{X^i, X^j\} = 0\,, \qquad \{X^i, X^9\} = \partial_1 X^i\,. \tag{4.12}$$

Adopting the gauge $\omega = 0$, we must then regroup all terms containing derivatives ∂_0 and ∂_1 in such a way that the derivatives appear only in the left- or right-moving combinations $\partial_\pm \equiv \partial_0 \pm \partial_1$, as required by consistency. The SO(9) spinors θ must be decomposed into SO(8) spinors according to

$$\theta(\tau,\sigma) = \begin{pmatrix} S_\alpha(\tau,\sigma) \\ \tilde{S}_{\dot\alpha}(\tau,\sigma) \end{pmatrix}. \tag{4.13}$$

From the equations of motion (2.7), or from the reduced action (2.10), it immediately follows that $\partial_- S = \partial_+ \tilde{S} = 0$. Therefore, in the reduction the spinor θ decomposes into the the left- and right-moving free fermions of IIA superstring theory. It is easy to see that

$$R^{ij} = \frac{1}{4} S\Gamma^{ij} S + \frac{1}{4}\tilde{S}\Gamma^{ij}\tilde{S}\,, \qquad R^{i9} = \frac{1}{2}\tilde{S}\Gamma^i S\,, \tag{4.14}$$

$$R^{ijk} = \frac{1}{6} S\Gamma^{ijk}\tilde{S}\,, \qquad R^{ij9} = \frac{1}{12} S\Gamma^{ij} S - \frac{1}{12}\tilde{S}\Gamma^{ij}\tilde{S} \tag{4.15}$$

in terms of SO(8) spinors. Let us emphasize once more that the superparticle reduction ensures that quartic and quintic fermionic terms work by themselves, so the tests performed below concern only terms containing the APD bracket.

For the $d = 10$ graviton h_{ij}, an NS $\otimes$ NS field, the double-dimensional reduction gives

$$\begin{aligned}(V_h)_{\mathrm{DDR}} = h_{ij}\Big[&\partial_0 X^i \partial_0 X^j - \partial_1 X^i \partial_1 X^j - \frac{1}{2}\partial_0 X^i \left(S\Gamma^{jm}S + \tilde{S}\Gamma^{jm}\tilde{S}\right) k_m + \\ &+ \frac{1}{2}\partial_1 X^i \left(S\Gamma^{jm}S - \tilde{S}\Gamma^{jm}\tilde{S}\right) k_m + \frac{1}{4} S\Gamma^{im}S\, \tilde{S}\Gamma^{jn}\tilde{S} k_m k_n\Big] e^{-ik\cdot X} = \\ = h_{ij}&\left(\partial_+ X^i - \frac{1}{2} S\Gamma^{im} S k_m\right)\left(\partial_- X^j - \frac{1}{2}\tilde{S}\Gamma^{jn}\tilde{S} k_n\right) e^{-ik\cdot X}\,. \end{aligned} \tag{4.16}$$

This is the desired result, see e.g. [19, section 4.1]. For the $R \otimes R$ vector field h_{i9}, we obtain

$$h_{i9}\Big[-i\theta\gamma^i\partial_1\theta + 2\partial_0 X^i R^{m9} k_m + \partial_1 X^j R^{ijm} k_m + 2R^{im} R^{9n} k_m k_n\Big] e^{-ik\cdot X}\,. \tag{4.17}$$

Again the quartic terms are easily seen to agree. To get rid of the derivatives on θ, which are absent in the superstring vertices, we make use of the superstring equations of motion $\partial_1 S = \partial_0 S$ and $\partial_1 \tilde{S} = -\partial_0 \tilde{S}$, and integrate the resulting expression by parts. After a little algebra we arrive at the desired result:

$$k_i h_{j9}\Big[S\Gamma^{ij}\Gamma^k \tilde{S}\,\partial_- X^k - S\Gamma^k\Gamma^{ij}\tilde{S}\,\partial_+ X^k\Big] e^{-ik\cdot X}\,. \tag{4.18}$$

The three-form C_{abc} gives rise to the R$\otimes$R field C_{ijk} and the NS$\otimes$NS field C_{ij9} in the reduction to ten dimensions, and the corresponding vertices must again be checked separately. Dimensional reduction of (3.10) yields

$$\begin{aligned}(V_C)_{\mathrm{DDR}} = & -C_{ij9}\,\partial_0\,X^i\partial_1X^j\,e^{-ik\cdot X} + \\ & +F_{ijk9}\left[\frac{3}{4}\left(\partial_0X^i-\frac{2}{3}R^{im}k_m\right)R^{jk9}-\frac{1}{6}R^{9m}R^{ijk}k_m-\frac{1}{4}\partial_1X^iR^{jk}\right]e^{-ik\cdot X} + \\ & +F_{ijkl}\left[\left(\partial_0X^i-\frac{2}{3}R^{im}k_m\right)R^{jkl}-\frac{1}{48}\partial_1X^m\,\theta\,\gamma^{ijklm9}\theta\right]e^{-ik\cdot X}\,. \end{aligned}\tag{4.19}$$

The superstring vertices involving the R$\otimes$R field C_{ijk} can be deduced from the formulas listed in [19, section 4.1]. They are given by (dropping the quartic fermion terms)

$$\begin{aligned}&\frac{1}{48}F_{ijkl}\left[S\Gamma^{ijkl}\Gamma^m\tilde{S}\,\partial_-X^m+\tilde{S}\Gamma^{ijkl}\Gamma^mS\,\partial_+X^m\right]e^{-ik\cdot X} = \\ &\quad = F_{ijkl}\left[\frac{1}{6}\,\partial_0X^l\,S\Gamma^{ijk}\tilde{S}-\frac{1}{24}\,\partial_1X^m\,S\Gamma^{ijklm}\tilde{S}\right]e^{-ik\cdot X}\end{aligned}\tag{4.20}$$

which indeed agrees with the result derived before. The agreement for the NS$\otimes$NS vertex involving C_{ij9} is verified similarly.

More work is required to check the gravitino vertex. Most of the terms can be guessed correctly by making the "obvious" substitutions, such as

$$\begin{pmatrix}\partial_\pm X^i & (\Gamma^i\tilde{S})_\alpha \\ \partial_\mp X^i & (\bar{\Gamma}^iS)_{\dot\alpha}\end{pmatrix}\longrightarrow\left(\gamma^aDX^a\pm\frac{1}{2}\gamma^{ab}\left\{X^a,X^b\right\}\right)\theta\,.\tag{4.21}$$

The substitutions for the terms cubic in θ and containing an APD bracket are more tricky. Under double-dimensional reduction

$$\begin{aligned}&6\tilde{\psi}_a\gamma_b\,\theta\,R^{acd}k_c\left\{X^b,X^d\right\}\longrightarrow \\ &\quad\longrightarrow\frac{1}{2}\left(\tilde{\psi}_{i\alpha}\Gamma^j_{\alpha\dot\beta}\tilde{S}_{\dot\beta}+\tilde{\psi}_{i\dot\alpha}\Gamma^j_{\dot\alpha\beta}S_\alpha\right)\left(S\Gamma^{im}S-\tilde{S}\Gamma^{im}\tilde{S}\right)\times \\ &\quad\quad\times k_m\,\partial_1X^j+\left(\tilde{\psi}_{i\alpha}S_\alpha-\tilde{\psi}_{i\dot\alpha}\tilde{S}_{\dot\alpha}\right)S\Gamma^{ijm}\tilde{S}k_m\,\partial_1X^j\,.\end{aligned}\tag{4.22}$$

Only the terms on the first line of the r.h.s. agree with the corresponding ones for the superstring. To eliminate the unwanted terms, we must add two further terms to the gravitino vertex, viz.

$$\tilde{\psi}_a\gamma_{bc}\,\theta\,R^{ad}k_d\left\{X^b,X^c\right\}\tag{4.23}$$

and

$$\tilde{\psi}_a\gamma_{bc}\,\theta\left(R^{cd}\left\{X^a,X^b\right\}+R^{ab}\left\{X^c,X^d\right\}\right)k_d\tag{4.24}$$

and, by a judicious choice of coefficients try to cancel them. This is indeed possible, if one makes use of the following SO(8) Fierz identities

$$\tilde{\psi}_{i\alpha} S_\alpha \, S\Gamma^{imj} \tilde{S} k_m \, \partial_1 X^j = -\frac{1}{2} \tilde{\psi}_{i\alpha} \Gamma^j_{\alpha\dot{\beta}} \tilde{S}_{\dot{\beta}} \, S\Gamma^{im} S k_m \, \partial_1 X^j + \tag{4.25}$$

$$+\frac{1}{4} \tilde{\psi}_{i\alpha} \Gamma^j_{\alpha\dot{\beta}} \tilde{S}_{\dot{\beta}} \Big(S\Gamma^{ij} S \, k \cdot \partial_1 X + S\Gamma^{jm} S k_m \, \partial_1 X^i \Big)$$

and

$$\tilde{\psi}_{i\alpha} \Gamma^j_{\alpha\dot{\beta}} \tilde{S}_{\dot{\beta}} \, \tilde{S}\Gamma^{im} \tilde{S} k_m \, \partial_1 X^j = \frac{1}{3} \tilde{\psi}_{i\alpha} \Gamma^j_{\alpha\dot{\beta}} \tilde{S}_{\dot{\beta}} \left(\tilde{S}\Gamma^{ij} \tilde{S} \, k \cdot \partial_1 X + \tilde{S}\Gamma^{jm} \tilde{S} k_m \, \partial_1 X^i \right). \tag{4.26}$$

To summarize: the comparison with the $d = 11$ superparticle and $d = 10$ superstring vertices constrains the possible terms so tightly that we are left with unique expressions for the supermembrane vertex operators. The final test is then provided by supersymmetry.

4.3 Linear supersymmetry

The first consistency check under supersymmetry involves the variation of the vertex operators (3.7), (3.10) and (3.12) under the linear transformations

$$\delta X^a = \delta\omega = 0 \qquad \text{and} \qquad \delta\theta = \eta \tag{4.27}$$

which should induce the homogenous supergravity variations (neglecting longitudinal polarizations) [19]

$$\delta h_{ab} = -\tilde{\psi}_{(a}\gamma_{b)}\eta \,, \qquad \delta h_{a+} = -\frac{1}{\sqrt{2}} \psi_a \eta \,, \tag{4.28}$$

$$\delta C_{abc} = \frac{3}{2} \tilde{\psi}_{[a}\gamma_{bc]}\eta \,, \qquad \delta C_{ab+} = \sqrt{2} \, \psi_{[a}\gamma_{b]}\eta \,, \tag{4.29}$$

$$\delta\psi_a = k_b \, h_{ca} \, \gamma^{bc}\eta + \frac{1}{72} \left(\gamma_a{}^{bcde} \, F_{bcde} - 8\gamma^{bcd} \, F_{abcd} \right) \eta \,,$$

$$\delta\tilde{\psi}_+ = -\frac{\sqrt{2}}{72} \gamma^{abcd} \eta \, F_{abcd} \,, \qquad \delta\psi_+ = \delta\tilde{\psi}_a = 0 = \delta h_{++} \tag{4.30}$$

of the polarizations. As before we work in the kinematical sector where $k^+ = 0$.

Performing the variation (4.27) on the transverse graviton vertex (3.7) yields

$$\delta V_h = k_b \, h_{ca} \eta \, \gamma^{bc} \Big[DX^a - 2R^{ad} \, k_d - \gamma^d \, \left\{ X^a, X^d \right\} \theta \Big] \, e^{-ik\cdot X} -$$

$$-h_{ab} \Big[\left\{ X^a, k \cdot X \right\} \eta\gamma^b\theta + i\eta\gamma^a \left\{ X^b, \theta \right\} \Big] \, e^{-ik\cdot X} = -V_{\delta\psi[h]} \,, \tag{4.31}$$

where the two terms in the second line cancel via a partial integration.

Next we turn to the transverse 3-from vertex whose variation yields

$$\delta V_C = F_{abcd}\Big[\frac{1}{6}\left(DX^a - \frac{2}{3}R^{ae}\,k_e\right)\eta\gamma^{bcd}\theta - \frac{1}{36}\eta\,\gamma^{ae}\theta\,\theta\gamma^{bcd}\theta\,k_e -$$

$$- \frac{1}{4}\left\{X^a, X^b\right\}\eta\,\gamma^{cd}\theta - \frac{1}{48}\left\{X^e, X^f\right\}\eta\gamma^{abcdef}\theta\Big]\,e^{-ik\cdot X} =$$

$$= F_{abcd}\Big[\frac{1}{6}DX^a\,\eta\,\gamma^{bcd}\theta - \frac{1}{4}\left\{X^a, X^b\right\}\eta\gamma^{cd}\theta - \frac{1}{48}\left\{X^e, X^f\right\}\eta\,\gamma^{abcdef}\theta\Big]\,e^{-ik\cdot X} -$$

$$-\frac{1}{36}R^{af}\,k_f\Big[\eta\gamma_a{}^{bcde}\theta\,F_{bcde} + 8\eta\,\gamma^{bcd}\theta\,F_{abcd}\Big]\,e^{-ik\cdot X}\,, \tag{4.32}$$

where we have made use of the Fierz identity

$$F_{abcd}\,\eta\,\gamma^{ae}\theta\,\theta\gamma^{bcd}\theta k_e = -F_{abcd}\Big[\eta\,\gamma^{abc}\theta\,\theta\gamma^{de}\theta\,k_e + \frac{1}{4}\theta\gamma^{abcde}\eta\,\theta\gamma^{ef}\theta\,k_f\Big] \tag{4.33}$$

on the terms of order θ^3. This result is to be compared with the vertex operators of the varied gravitino polarizations

$$V_{\delta\psi_a} + \frac{1}{\sqrt{2}}V_{\delta\tilde{\psi}_+} = \frac{1}{72}\Big(\eta\,\gamma_a{}^{bcde}\,F_{bcde} + 8\eta\,\gamma^{bcd}\theta\,F_{abcd}\Big)\times$$

$$\times\Big(DX^a - 2R^{ab}\,k_b - \gamma^c\left\{X^a, X^c\right\}\Big)\theta -$$

$$-\frac{1}{72}F_{abcd}\,\eta\gamma^{abcd}\left(\gamma\cdot DX + \frac{1}{2}\gamma^{ab}\left\{X^a, X^b\right\}\right)\theta\,e^{-ik\cdot X} \tag{4.34}$$

which is easily shown to equal (4.32).

Finally we examine the linear supersymmetry variation of the gravitino vertex, which due to its size and the required heavy use of Fierz rearrangements in the computation is considerably more involved.

The variation of the ψ_a vertex yields

$$\delta V_\psi = \psi_a\left[\eta\,DX^a - \gamma_b\eta\,\left\{X^a, X^b\right\} - \eta R^{ab}\,k_b - 3\gamma_b\eta\,R^{abc}\,k_c\right], \tag{4.35}$$

where we made use of the Fierz identity

$$\psi_a\theta\,\theta\gamma^{ab}\eta\,k_b = -\psi_a\eta\,R^{ab}\,k_b + 3\psi_a\gamma_b\eta\,R^{abc}\,k_c \tag{4.36}$$

ignoring longitudinal polarizations. From the longitudinal supergravity variations (4.28) and (4.29) and the V_{h_+} and V_{C_+} vertices of (3.8) and (3.11) we see that (4.35) reads $\delta V_\psi = -(V_{\delta h_+} + V_{\delta C_+})/\sqrt{2}$ as expected.

For the more involved $\tilde{\psi}_a$ vertex let us analyze the resulting terms order by order in θ to keep the resulting expressions in a manageable size. At zeroth order in θ one finds

$$\delta V_{\tilde{\psi}}\Big|_{\theta^0} = \tilde{\psi}_{(a}\gamma_{b)}\eta\left[DX^a DX^b - \left\{X^a, X^c\right\}\left\{X^b, X^c\right\}\right] +$$

$$+\frac{3}{2}\,\tilde{\psi}_{[a}\gamma_{bc]}\eta\,DX^a\left\{X^b, X^c\right\} + \tilde{\psi}_a\eta\,DX^b\,\left\{X^b, X^a\right\}\,. \tag{4.37}$$

In the first line we can already recognize the θ independent terms of the transverse graviton and three-form vertex.

For the terms quadratic in θ let us first look at the term which survives the particle reduction already discussed in [19]

$$\begin{aligned}\delta\Big[-2\tilde{\psi}_a\,\gamma\cdot DX\,\theta\,R^{ab}\,k_b\Big] &= -2\tilde{\psi}_{(a}\gamma_{b)}\eta\,DX^a\,R^{bc}\,k_c - 6k_{[a}\tilde{\psi}_b\gamma_{cd]}\eta DX^a\,R^{bcd} - \\ &\quad -\tilde{\psi}_a\gamma_b\eta\,R^{ab}\,k\cdot DX - 3\tilde{\psi}_a\eta\,R^{abc}\,DX^b\,k_c \end{aligned}\tag{4.38}$$

while for the remaining genuine membrane-like terms involving the APD bracket one finds

$$\begin{aligned}&\delta\bigg[8\tilde{\psi}_a\gamma_b\theta\,\left\{X^b,X^c\right\}\,R^{cad}\,k_d + \frac{5}{3}\tilde{\psi}_a\gamma_{bc}\theta\,\left\{X^b,X^c\right\}R^{ad}\,k_d + \\ &\quad +\frac{4}{3}\tilde{\psi}_a\gamma_{bc}\theta\Big(\left\{X^a,X^b\right\}\,R^{cd} + \left\{X^c,X^d\right\}R^{ab}\Big)k_d + \\ &\quad +\frac{2}{3}i\tilde{\psi}_a\,\Big(\gamma_b\theta\,\{X^a,\theta\}\gamma^b\theta - \theta\,\{X^a,\theta\}\theta\Big)\bigg] = \\ &= -6\tilde{\psi}_{(a}\gamma_{b)}\eta\,\left\{X^a,X^c\right\}\,R^{bcd}\,k_d + 3k_{[a}\,\tilde{\psi}_b\gamma_{cd]}\eta\,\left\{X^a,X^b\right\}R^{cd} - \\ &\quad -\frac{1}{2}k_a\,\tilde{\psi}_b\gamma_{cde}\eta\,\Big(\left\{X^a,X^b\right\}R^{cde} - 3\left\{X^b,X^c\right\}R^{dea} - \\ &\qquad\qquad - 3\left\{X^e,X^a\right\}R^{bcd} - 3\left\{X^d,X^e\right\}R^{abc}\Big) + \\ &\quad +3\tilde{\psi}_a\gamma_b\eta\,\{k\cdot X,X^c\}R^{abc} + i\tilde{\psi}_a\gamma_b\eta\,\theta\gamma^b\{\theta,X^a\} + \\ &\quad +2k_{[a}\tilde{\psi}_{b]}\eta\,\left\{X^a,X^c\right\}R^{bc} - i\tilde{\psi}_a\eta\,\{\theta,X^a\}\,, \end{aligned}\tag{4.39}$$

where we made use of several Fierz rearrangements, in which one also invokes the physical state constraints (3.4) of the gravitino. Now the first line of the variations in (4.38) and (4.39) respectively together produce two of the three θ^3 terms in the transverse graviton (3.7) and 3-from (3.10) vertex. Moreover the missing θ^3-term of the three-from vertex is actually given by the second line of the right hand side of (4.39) as

$$\begin{aligned}&\delta F_{abcd}\bigg(-\frac{1}{96}\left\{X^e,X^f\right\}\,\theta\gamma^{abcdef}\theta\bigg) = \\ &\quad = -\frac{1}{16}k_a\,\tilde{\psi}_b\gamma_{cd}\eta\left\{X^e,X^f\right\}\,\theta\gamma^{abcdef}\theta = \\ &\quad = -\frac{1}{48}k_a\,\tilde{\psi}_b\gamma_{abcdefg}\eta\left\{X^c,X^d\right\}\,\theta\gamma^{efg}\theta = \\ &\quad = -\frac{1}{2}k_a\,\tilde{\psi}_b\gamma_{cde}\eta\Big(\left\{X^a,X^b\right\}R^{cde} - 3\left\{X^b,X^c\right\}R^{dea} - \\ &\qquad\qquad - 3\left\{X^e,X^a\right\}R^{bcd} - 3\left\{X^d,X^e\right\}R^{abc}\Big)\,, \end{aligned}\tag{4.40}$$

where we first dualized the gamma matrices and thereafter reduced $k_a\,\tilde{\psi}_b\gamma_{abcdefg}\eta$ to expressions with three index gamma matrices via the physical state constraints of the gravitino (3.4). The missing term $-i\theta\gamma^{(a}\{X^{b)},\theta\}$ of the graviton vertex is found from (4.38) and (4.39) by first partially integrating the first term of the last line of (4.39)

$$\begin{aligned} &3\tilde{\psi}_a\gamma_b\eta\{k\cdot X, X^c\}R^{abc}\, e^{-ik\cdot X} = \\ &\quad = \left[-\frac{i}{2}\tilde{\psi}_a\gamma_b\eta\,\theta\gamma^{ab}\gamma^c\{\theta, X^c\} + i\tilde{\psi}_{[a}\gamma_{b]}\eta\,\theta\gamma^a\left\{\theta, X^b\right\}\right]e^{-ik\cdot X}\,, \end{aligned} \tag{4.41}$$

where we have also made use of the identity $\gamma^{abc} = \gamma^{ab}\gamma^c - 2\gamma^{[a}\,\delta^{b]c}$. Now adding the second term of the last line of (4.39) to (4.41) yields the desired symmetrized expression $i\tilde{\psi}_{(a}\gamma_{b)}\eta\,\theta\gamma^a\{\theta, X^b\}$. What remains to be shown, however, is that the first term in (4.41) cancels. This is achieved by again partially integrating the first term of the second line of (4.38)

$$-\tilde{\psi}_a\gamma_b\,\eta\,R^{ab}\,k\cdot DX\,e^{-ik\cdot X} = \frac{i}{2}\tilde{\psi}_a\gamma_b\eta\,\theta\,\gamma^{ab}D\theta\,e^{-ik\cdot X} - k^-\,\tilde{\psi}_a\gamma_b\eta\,R^{ab}\,e^{-ik\cdot X} \tag{4.42}$$

which thus cancels the first term in (4.41) upon using the equation of motions for θ of (2.7).

Putting it all together we arrive at the final result

$$\begin{aligned} \delta V_\Psi &= -V_{\delta h} - V_{\delta C} - \frac{1}{\sqrt{2}}V_{\delta h_+} - \frac{1}{\sqrt{2}}V_{\delta C_+} + \\ &\quad + \tilde{\psi}_a\eta\,DX^b\left\{X^b, X^a\right\} - 2k_{[a}\,\tilde{\psi}_{b]}\eta\,R^{ac}\,\left\{X^b, X^c\right\} - i\tilde{\psi}_a\eta\,\theta\{\theta, X^a\} - \\ &\quad - 3\tilde{\psi}_a\eta\,R^{abc}\,DX^b\,k_c + \frac{4}{3}\tilde{\psi}_a\eta\,R^{abc}\,R^{bd}\,k_c\,k_d - k^-\,\tilde{\psi}_a\gamma_b\eta\,R^{ab}\,. \end{aligned} \tag{4.43}$$

The remaining terms in the second and third line are associated with the longitudinal parts of the vertex operators whose polarization components vanish by our gauge choices: for instance, it is easy to see that the terms multiplying $\tilde{\psi}_a\eta$ arise in the $d = 11$ supersymmetry variation of C_{a+-} and therefore belong to the vertex operator for C_{a+-} which may now be read off as

$$\begin{aligned} V_{C_{+-}} = C_{a+-}\Big[&DX^b\left\{X^b, X^a\right\} - i\theta\{\theta, X^a\} - 3R^{abc}DX^bk_c + \\ &+ \{X^a, X^c\}\,R^{cd}k_d - \frac{i}{2}\theta\,\gamma^{ac}\{\theta, X^c\} + \frac{4}{3}R^{abc}R^{bd}k_ck_d\Big]e^{-ik\cdot X}\,. \end{aligned} \tag{4.44}$$

Also, the gauge invariance of $V_{C_{+-}}$ may be checked easily. It is important to realize that these longitudinal operators do appear in the variations even if their polarizations have been set to zero. Compensating gauge transformations are not relevant here, as the vertex operators are inert under these transformations.

4.4 Non-linear supersymmetry

The non-linear supersymmetry transformations on the vertex operators give further consistency checks. They constitute the ϵ dependent transformations as given in (2.5). We restate them here, and denote the transformations as $\tilde{\delta}$. The APD brackets play a major role in the non-linear supersymmetry of the supermembrane coordinates and mark a difference from the superparticle. This also makes these transformations non trivial.

$$\tilde{\delta}X^a = -2\epsilon\gamma^a\theta\,, \qquad \tilde{\delta}\theta = i\,DX\cdot\gamma\,\epsilon - \frac{i}{2}\left\{X^a, X^b\right\}\,\gamma_{ab}\epsilon\,,$$
$$\tilde{\delta}DX^a = -2\epsilon\gamma^a D\theta + 2\{\epsilon\theta, X^a\}\,. \tag{4.45}$$

The corresponding transformations for supergravity wave functions are [19][4]

$$\tilde{\delta}h_{ab} = \epsilon\gamma_{(a}\psi_{b)}\,, \tag{4.46}$$
$$\tilde{\delta}\psi_a = -k_{[+}h_{b]a}\gamma^b\epsilon\,, \tag{4.47}$$
$$\tilde{\delta}\tilde{\psi}_a = k_{[c}h_{b]a}\gamma^{cb}\epsilon + \frac{1}{72}\Big(\gamma_a^{bcde}F_{bcde} + 24\gamma^{bcd}k_bC_{acd} + 4\gamma^{bcd}k_aC_{bcd}\Big)\epsilon\,, \tag{4.48}$$
$$\tilde{\delta}C_{abc} = \frac{3}{2}\epsilon\gamma_{[ab}\psi_{c]}\,. \tag{4.49}$$

We have quoted the transformations of the transverse components only, as we shall only need those in the following discussion. In fact, here we present only the transformation of the transverse graviton vertex, and show that the terms remarkably combine to give the expected gravitino vertices and total derivative terms. The graviton vertex (3.7) under non-linear supersymmetry gives:

$$\begin{aligned}\tilde{\delta}V_h = h_{ab}\Bigg[&4\left(-\epsilon\gamma^aD\theta + \{\epsilon\theta, X^a\}\right)DX^b + 4\left(\{\epsilon\gamma^a\theta, X^c\} + \{X^a, \epsilon\gamma^c\theta\}\right)\,\left\{X^b, X^c\right\} -\\ &-2\left\{X^b, DX^c\right\}\epsilon\gamma^c\gamma^a\theta - \left\{X^b, \left\{X^e, X^f\right\}\right\}\epsilon\gamma^{ef}\gamma^a\theta -\\ &-iDX^a\left(DX^e\epsilon\gamma^e + \frac{1}{2}\left\{X^e, X^f\right\}\epsilon\gamma^{ef}\right)\gamma^{bc}\theta k_c -\\ &-i\{X^a, X^c\}\left(DX^e\epsilon\gamma^e + \frac{1}{2}\left\{X^e, X^f\right\}\epsilon\gamma^{ef}\right)\gamma^{bcd}k_d\theta\Bigg]e^{-ik\cdot X} +\\ &+ih_{ab}\tilde{\delta}\theta\gamma^a\left\{X^b, e^{-ik\cdot X}\right\}\theta + 2i\epsilon k\cdot\gamma\theta\;V_h\,.\end{aligned} \tag{4.50}$$

We ignore terms of order θ^3 for simplicity at present. The terms in the third and fourth line of the above equation yield most of the relevant terms. They can be combined as:

$$\begin{aligned}&-ih_{ab}\left(DX^e\epsilon\gamma^e + \frac{1}{2}\left\{X^e, X^f\right\}\epsilon\gamma^{ef}\right)\gamma^{bd}k_d\left(DX^a + \gamma^c\left\{X^c, X^a\right\}\right)\theta -\\ &-ih_{ab}\left\{X^b, e^{-ik\cdot X}\right\}\tilde{\delta}\theta\gamma^a\theta\,.\end{aligned} \tag{4.51}$$

[4]We stress once more that the compensating gauge transformations considered in [19], which are singular in k^+, are redundant as they vanish when contracted into the corresponding vertex operators by the gauge invariance of the latter.

After a few manipulations in which we commute the γ^{bd} to the left to contract it with ϵh_{ab}, we get the following from (4.51):

$$
\begin{aligned}
ih_{a[b}k_{d]}\epsilon\gamma^{db}&\left(DX\cdot\gamma+\frac{1}{2}\left\{X^e,X^f\right\}\gamma^{ef}\right)\left(DX^a+\gamma^c\left\{X^c,X^a\right\}\right)\theta+\\
&+2ih_{ab}k^-\epsilon\gamma^b\Big(DX^a+\gamma^c\left\{X^c,X^a\right\}\Big)\theta-\\
&-2h_{ab}\epsilon\gamma^aD\Big[\Big(DX^b+\gamma^c\left\{X^c,X^a\right\}\Big)\theta e^{-ik\cdot X}\Big]+\\
&+h_{ab}\Big[4\epsilon\gamma^aD\theta DX^b-2\epsilon\gamma^a\gamma^c\{X^b,DX^c\}-4\{\epsilon\gamma^a\theta,X^c\}\left\{X^b,X^c\right\}-\\
&\qquad-4\{X^a,\epsilon\gamma^c\theta\}\left\{X^b,X^c\right\}-2i\epsilon k\cdot\\
&\qquad\cdot\gamma\Big(DX^aDX^b-\left\{X^a,X^c\right\}\left\{X^b,X^c\right\}\Big)\theta+\\
&\qquad+2\Big\{X^a,\left\{X^c,X^b\right\}\Big\}\epsilon\gamma^c\theta-2\{X^e,\{X^a,X^c\}\}\epsilon\gamma^{ebc}\theta\Big]e^{-ik\cdot X}+\\
&+4h_{ab}\epsilon\,\theta DX^a\left\{e^{-ik\cdot X},X^b\right\}-ih_{ab}\left\{X^b,e^{-ik\cdot X}\right\}\tilde{\delta}\theta\gamma^a\theta-\partial_rW'^r\,.
\end{aligned}
\tag{4.52}
$$

Here $\partial_rW'^r$ comes from the partial integration of terms proportional to $\{X^a,e^{-ik\cdot X}\}$, where $W'^r=2\epsilon^{rs}h_{ab}(\epsilon\gamma^b\gamma^e\theta\partial_sX^eDX^a+\epsilon\gamma^e\gamma^b\gamma^c\theta\partial_sX^e\left\{X^c,X^a\right\})e^{-ik\cdot X}$. Unlike the superparticle case considered in [19], where $\dot{\theta}=\ddot{X}=0$ the total derivative term in the second line of (4.52) involves quite a few terms proportional to $D\theta,D^2X$ and $\{DX,X\}$ not present for the superparticle, and it is remarkable that the non-linear supersymmetry variation of the supermembrane yields all the required derivatives. We use the equations of motion given in (2.8) extensively in the above and in particular, we take $D^2X^a+\{X^c,\{X^a,X^c\}\}=0$ at this order in θ. Substituting (4.52), in (4.50), we get

$$
\begin{aligned}
\tilde{\delta}V_h=&\;\tilde{\delta}\psi_a\left(DX^a+\gamma^c\left\{X^c,X^a\right\}\right)\theta+\\
&+\tilde{\delta}\tilde{\psi}_a\left(DX\cdot\gamma+\frac{1}{2}\left\{X^b,X^c\right\}\gamma^{bc}\right)\left(DX^a+\gamma^d\left\{X^d,X^a\right\}\right)\theta+\\
&+h_{ab}\epsilon\Big[-\left\{X^a,\left\{X^e,X^f\right\}\right\}\gamma^{efb}-2\left\{X^e,\left\{X^a,X^f\right\}\right\}\gamma^{ebf}\Big]\theta-\\
&-\epsilon DW^0-\epsilon\partial_rW^r\,.
\end{aligned}
\tag{4.53}
$$

The term in the third line of (4.53) is easily seen to vanish by Jacobi identity. The gravitino vertex given in (3.12) is also clearly recovered to this order in θ (from (4.47) and (4.48), $\tilde{\delta}\psi\ \propto\ k^-h_{ab}\gamma^b\epsilon$, $\tilde{\delta}\tilde{\psi}\ \propto k_{[d}h_{b]a}\gamma^{bd}\epsilon$). Also, $W^r=W'^r-4h_{ab}\epsilon^{rs}\partial_sX^aDX^b\epsilon\theta$. The functions

$$
\begin{aligned}
W^0&=2h_{ab}\gamma^a\left(DX^b+\gamma^c\left\{X^c,X^b\right\}\right)\theta e^{-ik\cdot X}\,,\\
W^r&=2\epsilon^{rs}h_{ab}\epsilon\gamma^e\gamma^a\partial_sX^e\left(DX^b+\gamma^c\left\{X^c,X^b\right\}\right)\theta e^{-ik\cdot X}
\end{aligned}
\tag{4.54}
$$

are also expected to obey certain transformation properties under supersymmetry as given for the superparticle in [19]. However, we have not checked for them, and it shall be interesting to investigate them in the future.

The variation of the graviton vertex into the gravitino vertex to order θ^3 involves more tedious computations, and we refrain from checking for all the terms. However, it is easy to see that the following terms arises in the variation

$$h_{ab}\left(4\epsilon\gamma^a D\theta R^{bc}k_c + 2\epsilon\gamma^a\theta\theta\gamma^{bc}D\theta k_c - 4i\epsilon\gamma^a\theta DX\cdot kR^{bc}k_c\right)e^{-ik\cdot X}\,.$$

This term can be combined into a total derivative and the variation of ψ, as

$$\tilde{\delta}\psi_a(-2\theta R^{ac}\,k_c)e^{-ik\cdot X} + D\Big(4h_{ab}\epsilon\gamma^a\theta R^{bc}\,k_c\,e^{-ik\cdot X}\Big)\,.$$

Thus the vertex for ψ_a is recovered to all orders in θ here.

5. Applications to M(atrix) theory

Our results immediately imply two important applications to matrix theory. Firstly, we now have the lagrangian for the light cone supermembrane in a weak background, as the vertex operators represent nothing but the linear coupling of the background fields to the supermembrane coordinates X^a and θ. Hence

$$\mathcal{L}_{\text{weak}} = \mathcal{L} + V_{h(X)} + V_{h_+(X)} + V_{h_{++}(X)} + V_{C(X)} + V_{C_+(X)} + V_{\Psi(X)} + V_{\Psi_+(X)}\,, \quad (5.1)$$

where $\mathcal{L}$ denotes the supermembrane lagrangian in flat space (2.2) and where one writes the vertex operators of (3.7)–(3.13) in configuration space, e.g.

$$V_{h_+(X)} = -2\left(DX^a - R^{ab}\frac{\partial}{\partial X^b}\right)h_{a+}(X) \quad (5.2)$$

for the linear coupling to the background field $h_{+a}(X)$. We stress that we now know this action to *all* orders in θ, which is to be contrasted with the results on the covariant supermembrane in general background fields [30] where the action was derived to all orders in the background fields, but only up to order θ^2 in the membrane fermions.[5] Clearly our results immediately carry over to matrix theory: one needs only repeat the usual matrix model regulation [22, 6] of the light-cone supermembrane using the prescription of (2.9)

$$X^a(\tau,\sigma_1,\sigma_2) \longrightarrow \mathbf{X}^a_{mn}(\tau)\,, \qquad \theta^\alpha(\tau,\sigma_1,\sigma_2) \longrightarrow \theta^\alpha_{mn}(\tau) \quad (5.3)$$

with $n, m = 1,\ldots,N$ labeling the entries of the $N\times N$ hermitian matrices $\mathbf{X}^a$ and θ^α. Moreover the APD Lie bracket gets replaced by a matrix commutator $\{\cdot\,,\cdot\} \to i[\cdot\,,\cdot]$.

[5] In [31] the covariant membrane action for the $AdS_4\times S_7$ and $AdS_7\times S_4$ backgrounds was obtained to all orders in θ. It would be interesting to compare these results to ours by taking the action of [31] to the light-cone.

The only subtlety in replacing world-space integrals by traces occurs in expressions of higher than second order in the matrices $D\mathbf{X}^a$, θ^a, $[\mathbf{X}^a, \mathbf{X}^b]$, $[\theta, \mathbf{X}^b]$ and $\exp[-i\,k\cdot\mathbf{X}]$, where we must deal with ordering ambiguities under the trace. However, in order to maintain the defining transformation properties of the vertex operators under gauge symmetry and supersymmetry discussed in section 3 for the matrix theory regulation it is sufficient to replace the world-space integral by a *symmetrized trace*, i.e.

$$\frac{1}{4\pi}\int d^2\sigma(\cdots) \longrightarrow \frac{1}{N}\,\mathrm{STr}[\cdots]\,, \tag{5.4}$$

where it is understood that the symmetrization in the trace is to be performed over the set of matrices ($D\mathbf{X}^a$, θ^a, $[\mathbf{X}^a, \mathbf{X}^b]$, $[\theta, \mathbf{X}^b]$, $\exp[-i\,k\cdot\mathbf{X}]$). The vertex operators obtained in this way may be compared to the results of Taylor and Van Raamsdonk [32], who derived certain expressions for the energy-momentum tensor, the membrane current and supercurrent of matrix theory up to quadratic order in θ and (partially) up to linear order in transverse space derivatives ∂_a (related to the k_a in the momentum picture). Their results are based on a one-loop matrix theory computation for general block-diagonal matrix backgrounds. Happily, we find agreement with their results to the order that they have computed.[6] However, there are additional operators in the matrix theory picture of [32] coupling to the background fields h_{a-}, h_{--}, C_{ab-}, C_{a+-}, Ψ_-, which we have gauged to zero.[7]

Besides the background field matrix theory action obtained above, we believe that another interesting application of the supermembrane vertex operators lies in a new definition of scattering amplitudes in matrix theory. Conventionally these are evaluated by computing an effective background field action through a fluctuation expansion around diagonal matrix backgrounds obeying the classical equations of motion. The obtained effective action is then Fourier transformed and sandwiched between polarization states in order to obtain genuine S-matrix element in momentum space [14]. This approach only allows for the computation of amplitudes in the eikonal (zero momentum transfer) limit. Moreover it completely neglects bound-state effects as it models the complicated-matrix theory ground state by semiclassical diagonal-matrix configurations.

With the matrix theory vertex operators at hand a much more natural definition of n-particle scattering amplitudes is given by the path integral

$$\begin{aligned}\mathcal{A}_{\mathcal{H}_1\ldots\mathcal{H}_n} &= \Big\langle \prod_{j=1}^{n}\int d\tau_j\ \mathrm{STr}\Big(V_{\mathcal{H}_j}\,[\mathbf{X}^a(\tau_j),\theta(\tau_j)]\Big)\Big\rangle = \\ &= \int \mathcal{D}[\mathbf{X}^a,\theta]\prod_{j=1}^{n}\int d\tau_j\ \mathrm{STr}\Big(V_{\mathcal{H}_j}[\mathbf{X}^a(\tau_j),\theta(\tau_j)]\Big)\,e^{i\,S_{MT}[\mathbf{X},\theta]}\,,\end{aligned} \tag{5.5}$$

[6]But there seems to be a mismatch in one order θ^2 term in the three-form vertex (membrane current).

[7]Our result for $V_{C_{+-}}$ of (4.44) also agrees with [32] to the order that they have computed.

where $\mathcal{H}_j$ denotes the polarization and momentum of the j-th particle. It remains to be seen whether a (perturbative) evaluation of (5.5) makes sense, because in contrast to the superstring or the superparticle we are now dealing with the computation of expectation values of composite operators in an *interacting* theory. However, the definition (5.5) overcomes the restriction to the eikonal sector of the conventional approach, it should include large-N and bound state effects and manifestly obeys supersymmetric Ward identities, which is far from obvious in the conventional approach. Also at least for the further reduction to the zero-dimensional IKKT matrix model of IIB theory [33] a numerical evaluation of scattering amplitudes along the lines of [34] may now become feasible.

6. Outlook

In this paper we have demonstrated that the supermembrane and the associated supersymmetric APD gauge theory contain the type-II superstrings and the matrix model not only at the level of the action, but also at the level of the vertex operators expected to describe various physical processes. Although a full quantum treatment of the supermembrane or the equivalent supersymmetric APD gauge theory still seems difficult, we can now explore the theory much further *at the dynamical level* by matching it in the appropriate domains with the simpler subtheories that must be consistently contained in it. In particular, we have in mind the following comparisons:

- The $d = 11$ superparticle reduction has been used in [20] to determine the non-perturbative contributions to the R^4 corrections to the effective string action in terms of non-holomorphic Eisenstein series (also computed in [21]). Remarkably, this calculation makes use of only a single term in the graviton vertex (3.7), namely the zero mode of $h_{ab}R^{ac}R^{bd}k_ck_d$ (the coefficient of the fermionic quadrilinear is easily seen to coincide with the linearized Riemann tensor). The resulting infinite sum over D instanton contributions can be alternatively viewed as a sum over BPS multiplets [35]. However, in order to arrive at a finite result a divergent term must be discarded "by hand" [19, 35]. This infinity should disappear when the M-theory degrees of freedom are properly taken into account.

- As already pointed out in the foregoing section, the matrix theory vertex operators afford an entirely novel approach to the computation of scattering amplitudes. In particular, it should now be possible to determine these beyond the eikonal regime. The computation of R^4 corrections within the framework of matrix theory will have to be re-examined.

- Superstring amplitudes should emerge in the superstring limit. While the matrix theory scattering amplitude (5.5) is one way to approximate the APD gauge theory path integral

$$\int \mathcal{D}[X^a, \theta] \prod_j \int d\tau_j \, d^2\sigma_j V_{\mathcal{H}_j}[X^a(\tau_j, \sigma_j), \theta(\tau_j, \sigma_j)] e^{i S_{APD}} \,, \tag{6.1}$$

the superstring amplitudes are obtained in a very different limit of the same expression. In that approximation one looks at the regions where the membrane degenerates into a multi-string configuration, and the vertex operator insertions reduce to superstring vertex operators, as we have shown. In this way, one should also be able to recover *multi-string vertex operators* (see e.g. [36] and references therein) from the quantum supermembrane.

Finally, we would like to emphasize once more the intrinsic multi-particle nature of the theory, which is the main conceptual difference between supermembrane and superstring theory: it appears to be impossible to tackle supermembrane theory by first defining one-particle excitations, and subsequently second-quantizing it so as to obtain its multi-particle states. Therefore, unlike for superstring theory, the conventional Fock space quantization breaks down. An interesting consequence of this conclusion is that there should not exist any analog of the vertex operators corresponding to excited (massive) string states.

Acknowledgments

We are grateful to M.B. Green for correspondence. H.N. would like to thank the CERN Theory Division, where this work was first presented, for hospitality.

References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75; *Properties of the eleven-dimensional super membrane theory*, *Ann. Phys. (NY)* **185** (1988) 330.

[2] M.J. Duff, *Supermembranes*, `hep-th/9611203`;
B. de Wit, *Supermembranes and super matrix models*, `hep-th/9902051`.

[3] M.B. Green and J.H. Schwarz, *Covariant description of superstrings*, *Phys. Lett.* **B 136** (1984) 367; *Properties of the covariant formulation of superstring theories*, *Nucl. Phys.* **B 243** (1984) 285.

[4] L. Brink and J.H. Schwarz, *Quantum superspace*, *Phys. Lett.* **B 100** (1981) 310.

[5] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[6] B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys.* **B 305** (1988) 545.

[7] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, *Superstrings in $D = 10$ from supermembranes in $D = 11$*, *Phys. Lett.* **B 191** (1987) 70.

[8] J.H. Schwarz, *An* $\mathrm{SL}(2,\mathbb{Z})$ *multiplet of type-IIB superstrings*, *Phys. Lett.* **B 360** (1995) 13 [`hep-th/9508143`]; *The power of M-theory*, *Phys. Lett.* **B 367** (1996) 97 [`hep-th/9510086`];
P.S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 30 [`hep-th/9508154`].

[9] M. Abou-Zeid, B. de Wit, D. Lust and H. Nicolai, *Space-time supersymmetry, IIA/B duality and M-theory*, *Phys. Lett.* **B 466** (1999) 144 [`hep-th/9908169`].

[10] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [`hep-th/9610043`].

[11] D. Bigatti and L. Susskind, *Review of matrix theory*, `hep-th/9712072`;
T. Banks, *Tasi lectures on matrix theory*, `hep-th/9911068`;
I.W. Taylor, *The M(atrix) model of M-theory*, `hep-th/0002016`.

[12] M.J. Duff, R.R. Khuri and J.X. Lu, *String solitons*, *Phys. Rep.* **259** (1995) 213 [`hep-th/9412184`].

[13] K. Becker and M. Becker, *A two-loop test of M(atrix) theory*, *Nucl. Phys.* **B 506** (1997) 48 [`hep-th/9705091`];
S. Paban, S. Sethi and M. Stern, *Constraints from extended supersymmetry in quantum mechanics*, *Nucl. Phys.* **B 534** (1998) 137 [`hep-th/9805018`];
D. Kabat and I.W. Taylor, *Linearized supergravity from matrix theory*, *Phys. Lett.* **B 426** (1998) 297 [`hep-th/9712185`].

[14] J.C. Plefka, M. Serone and A.K. Waldron, *The matrix theory S-matrix*, *Phys. Rev. Lett.* **81** (1998) 2866 [`hep-th/9806081`]; *$D = 11$ sugra as the low energy effective action of matrix theory: three form scattering*, *J. High Energy Phys.* **11** (1998) 010 [`hep-th/9809070`].

[15] R. Helling, J. Plefka, M. Serone and A. Waldron, *Three graviton scattering in M-theory*, *Nucl. Phys.* **B 559** (1999) 184 [`hep-th/9905183`].

[16] B. de Wit, M. Lüscher and H. Nicolai, *The supermembrane is unstable*, *Nucl. Phys.* **B 320** (1989) 135.

[17] A.V. Smilga, *Super Yang-Mills quantum mechanics and supermembrane spectrum* in *Supermembranes and physics in $2+1$ dimensions*, M.J. Duff, C.N. Pope and E. Sezgin eds., World Scientific, Singapore 1990.

[18] H. Nicolai and R. Helling, *Supermembranes and M(atrix) theory*, `hep-th/9809103`.

JHEP05(2000)007

[19] M.B. Green, M. Gutperle and H.H. Kwon, *Light-cone quantum mechanics of the eleven-dimensional superparticle*, *J. High Energy Phys.* **08** (1999) 012 [hep-th/9907155].

[20] M.B. Green, M. Gutperle and P. Vanhove, *One loop in eleven dimensions*, *Phys. Lett.* **B 409** (1997) 177 [hep-th/9706175].

[21] J.G. Russo and A.A. Tseytlin, *One-loop four-graviton amplitude in eleven-dimensional supergravity*, *Nucl. Phys.* **B 508** (1997) 245 [hep-th/9707134].

[22] J. Goldstone, unpublished;
J. Hoppe, *Quantum theory of a relativistic surface*, in *Proc. int. workshop on constraint's theory and relativistic dynamics*, G. Longhi and L. Lusanna eds., World Scientific, Singapore 1987.

[23] M. Claudson and M.B. Halpern, *Supersymmetric ground state wave functions*, *Nucl. Phys.* **B 250** (1985) 689;
M. Baake, M. Reinicke and V. Rittenberg, *Fierz identities for real Clifford algebras and the number of supercharges*, *J. Math. Phys.* **26** (1985) 1070;
R. Flume, *On quantum mechanics with extended supersymmetry and nonabelian gauge constraints*, *Ann. Phys. (NY)* **164** (1985) 189.

[24] J. Hoppe, *$Diff_A T^2$ and the curvature of some infinite dimensional manifolds*, *Phys. Lett.* **B 215** (1988) 706;
D.B. Fairlie, P. Fletcher and C.K. Zachos, *Trigonometric structure constants for new infinite algebras*, *Phys. Lett.* **B 218** (1989) 203.

[25] B. de Wit, U. Marquard and H. Nicolai, *Area preserving diffeomorphisms and supermembrane Lorentz invariance*, *Comm. Math. Phys.* **128** (1990) 39.

[26] M. Bordemann, E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and* gl(N), *$N \to \infty$ limits*, *Comm. Math. Phys.* **165** (1994) 281 [hep-th/9309134].

[27] A.V. Smilga, *Witten index calculation in supersymmetric gauge theory*, *Nucl. Phys.* **B 266** (1986) 45;
P. Yi, *Witten index and threshold bound states of D-branes*, *Nucl. Phys.* **B 505** (1997) 307 [hep-th/9704098];
S. Sethi and M. Stern, *D-brane bound states redux*, *Comm. Math. Phys.* **194** (1998) 675 [hep-th/9705046];
M. Porrati and A. Rozenberg, *Bound states at threshold in supersymmetric quantum mechanics*, *Nucl. Phys.* **B 515** (1998) 184 [hep-th/9708119];
J. Fröhlich and J. Hoppe, *On zero mass ground states in supermembrane matrix models*, hep-th/9701119;
M.B. Halpern and C. Schwartz, *Asymptotic search for ground states of* SU(2) *matrix theory*, *Int. J. Mod. Phys.* **A 13** (1998) 4367 [hep-th/9712133];
J. Fröhlich, G.M. Graf, D. Hasler, J. Hoppe and S.T. Yau, *Asymptotic form of zero*

energy wave functions in supersymmetric matrix models, *Nucl. Phys.* **B 567** (2000) 231 [hep-th/9904182];
G. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Comm. Math. Phys.* **209** (2000) 77 [hep-th/9803265];
V.G. Kac and A.V. Smilga, *Normalized vacuum states in $N = 4$ supersymmetric Yang-Mills quantum mechanics with any gauge group*, *Nucl. Phys.* **B 571** (2000) 515 [hep-th/9908096];
J. Hoppe and J. Plefka, *The asymptotic ground state of* SU(3) *matrix theory*, hep-th/0002107.

[28] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158];
J.A. Harvey and G. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[29] M.B. Green and J.H. Schwarz, *Supersymmetrical dual string theory, 2. Vertices and trees*, *Nucl. Phys.* **B 198** (1982) 252.

[30] B. de Wit, K. Peeters and J. Plefka, *Superspace geometry for supermembrane backgrounds*, *Nucl. Phys.* **B 532** (1998) 99 [hep-th/9803209].

[31] B. de Wit, K. Peeters, J. Plefka and A. Sevrin, *The M-theory two-brane in $AdS_4 \times S^7$ and $AdS_7 \times S^4$*, *Phys. Lett.* **B 443** (1998) 153 [hep-th/9808052].

[32] I.W. Taylor and M.V. Raamsdonk, *Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds*, *J. High Energy Phys.* **04** (1999) 013 [hep-th/9812239]; *Multiple D0-branes in weakly curved backgrounds*, *Nucl. Phys.* **B 558** (1999) 63 [hep-th/9904095].

[33] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[34] W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M-theory*, *Phys. Lett.* **B 431** (1998) 31 [hep-th/9803117];
W. Krauth and M. Staudacher, *Finite Yang-Mills integrals*, *Phys. Lett.* **B 435** (1998) 350 [hep-th/9804199]; *Eigenvalue distributions in Yang-Mills integrals*, *Phys. Lett.* **B 453** (1999) 253 [hep-th/9902113];
W. Krauth, J. Plefka and M. Staudacher, *Yang-Mills integrals*, *Class. and Quant. Grav.* **17** (2000) 1171 [hep-th/9911170];
J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large-N dynamics of dimensionally reduced 4d* SU(N) *super Yang-Mills theory*, hep-th/0003208.

[35] B. de Wit and D. Lüst, *BPS amplitudes, helicity supertraces and membranes in M-theory*, *Phys. Lett.* **B 477** (2000) 299 [hep-th/9912225].

[36] A. Neveu and P. West, *Symmetries of the interacting gauge covariant bosonic string*, *Nucl. Phys.* **B 278** (1986) 601; *Cycling, twisting and sewing in the group theoretic approach to strings*, *Comm. Math. Phys.* **119** (1988) 585.