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Slowly rotating two–fluid neutron star model

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Abstract. We study stationary axisymmetric configurations of a star model consisting of two barotropic fluids, which are uniformly rotating at two different rotation rates. Analytic approximate solutions in the limit of slow rotation are obtained with the classical method of Chandrasekhar, which consists of an expansion of the solution in terms of the rotation rate, and which is generalized to the case of two fluids in order to apply it to the present problem. This work has a direct application to neutron star models, in which the neutron superfluid can rotate at a different speed than the fluid of charged components. Two cases are considered, the case of two non–interacting fluids, and the case of an interaction of a special type, corresponding to the vortices of the neutron superfluid being completely pinned to the second fluid. The special case of the equation of state $P \propto \rho^2$ is solved explicitly as an illustration of the foregoing results.

Key words: hydrodynamics – stars: neutron – stars: rotation

1. Introduction

More than 30 years after the discovery of the pulsar phenomenon and its identification with rotating neutron stars (Gold 1968), there exists today a considerable body of observational data (Lyne & Graham–Smith 1998), but also still a number of uncertainties and open questions about the theoretical model for pulsars, mainly due to the extremely dense (and therefore poorly known) state of matter implied (Glendenning 1997).

One of the generally agreed characteristics of neutron stars is the existence of a superfluid neutron component. This is not only predicted by calculations from nuclear physics (Ainsworth et al. 1989), but also agrees with observed features of pulsar behavior, like the very long relaxation times, from months up to years, after a glitch (a sudden increase of angular velocity of the order of $\Delta\Omega/\Omega \lesssim 10^{-6}$). All the charged parts of a neutron star (nuclei, protons and electrons) can be treated as a single normal fluid, and are predicted to be "locked" together in a state of corotation (Easson 1979) on sufficiently long timescales. In contrast, the neutron superfluid can have a different rotation even on very long timescales, and so one is naturally led to consider a neutron star model consisting of two independent fluids, an approach that was first adopted by Baym et al. (1969). This model has since been the basis of our understanding of the glitch behavior and the subsequent post–glitch relaxation observed in pulsars (Anderson & Itoh 1975; Alpar et al. 1984; Sedrakian et al. 1995b; Link & Epstein 1996).

Apart from being inviscid and therefore forming an independent fluid component, a superfluid is moreover constrained to be in a state of irrotational flow, and consequently its rotation can only be achieved by the presence of quantized vortices. These vortices will interact with the fluid of charged components (Feibelman 1971; Sauls et al. 1982; Epstein & Baym 1988; Jones 1990, 1991; Link & Epstein 1991; Sedrakian & Sedrakian 1995a), giving rise to an effective friction force on a moving vortex, and they can even be completely pinned to the Coulomb lattice of nuclei that forms the crust of the neutron star. A consequence is, that the vortices will not corotate with the superfluid and will therefore be subject to the Magnus force orthogonal to their relative velocity with respect to the superfluid. These forces will balance each other, which leads to an effective interaction between the two fluids.

The long–term slowdown of the neutron star's rotation rate, which is caused by the loss of energy in form of electromagnetic radiation, has many important consequences. The global slowdown tends to decrease the ellipticity of the equilibrium shape of the neutron star. This leads to the buildup of stress forces in the solid crust, which can get suddenly released in form of a starquake. This has been proposed by Ruderman (1969) as one of the first models in order to explain glitches, and has since been a subject of great interest, directly as a model for glitches (Baym & Pines 1971; Heintzmann et al. 1973; Ruderman 1991; Link et al. 1998), or at least as a trigger for some other glitch–mechanism via the energy liberated in such a starquake event (Link & Epstein 1996). Another aspect of the global slowing down has been pointed out by Reisenegger (1995): the decrease of the centrifugal force leads to a global compression of the neutron star matter (consisting of neutrons, protons and electrons). But the equilibrium composition (with respect to β reactions) of this plasma depends on the density, and so a global compression drives the plasma out of equilibrium. This has some possibly observable consequences, e.g., on the emission of neutrinos and on the evolution of the temperature of neutron stars.

These consequences have been examined from the point of view of a global slowdown of the whole neutron star, but it has to be noted that in the two–fluid model, it is primarily the fluid of charged components that gets slowed down, while the superfluid neutrons will significantly lag behind and continue to turn at a faster rotation rate. It has been remarked recently (Carter et al. 1999), that this could lead to a new mechanism to induce stress forces in the crust, due to an increasing deficit of centrifugal buoyancy. The model for the driven deviation from chemical equilibrium also has to be refined according to the two–fluid picture. Not only is there a global compression, but also a relative displacement of the two mass distributions with respect to each other, as the difference of their rotation rates increases. For example, when the two fluids have been in β equilibrium in the state of corotation, the slowdown of one fluid changes its ellipticity and therefore moves volume elements of that fluid to regions with a different equilibrium composition, so that they are no longer in a state of equilibrium with the second fluid.

The purpose of this paper is to study the consequences of the two fluids having different rotation rates on the mass distribution of the star. Even in the case of a single rotating, self–gravitating fluid, it is impossible to obtain exact analytic solutions, and one has to rely either on numerical treatments or on analytic approximations (e.g., see Tassoul 1978). In the present work we will develop a generalization of the analytic approximation of Chandrasekhar (Chandrasekhar 1933; Tassoul 1978) to the case of a barotropic two–fluid star. This method consists of an expansion of the rotating solution around the static solution in terms of the rotation rate. Using this method, we will obtain an expression for the stationary mass distribution of a barotropic two–fluid star up to second order in the two rotation rates. The obvious limitations of this approach are that the rotation rates have to be small compared to their "natural" scale, and that both have to be of the same order of magnitude. These conditions are in general satisfied in the case of neutron stars. The fact that we considered stationary solutions is no real restriction either, as the slowdown of pulsars takes place on very long timescales. Therefore it should be possible to describe it as a quasi–stationary process, passing through a series of stationary states.

The plan of this paper is the following. In Sect. 2 we define the Newtonian general model of a barotropic two–fluid star, and in Sect. 3 we further specialize this general model in the context of neutron stars. In Sect. 4 we generalize and apply the classical method of Chandrasekhar to this two–fluid star, which allows us to reduce the problem to a set of ordinary differential equations. Sect. 5 is devoted to the boundary conditions necessary to obtain the complete solution, which is given in Sect. 6. Sect. 7 is concerned with some consequences of the solution, like the change in ellipticity and moment of inertia. In Sect. 8 we discuss an effect that we call "rotational coupling", which is the fact that changes of the rotation speed of one fluid influence the rotation of the other fluid via the gravitational potential, even if the two fluids are supposed to be strictly non–interacting. Sect. 9 gives an illustration of the foregoing results in the completely analytically solvable case of a special polytropic equation of state. Sect. 10 summarizes this work.

Table 1. The system of the chosen "natural" units, R and ρ_0 are respectively the radius and the central density of the non–rotating configuration

Quantity	Unit
Length	R
Density	ρ_0
Time	$1/\sqrt{4\pi G\rho_0}$
Frequency	$\sqrt{4\pi G\rho_0}$
Mass	$\rho_0 R^3$
Moment of Inertia	$\rho_0 R^5$
Gravitational Potential	$4\pi G\rho_0 R^2$
Pressure	$4\pi G\rho_0^2 R^2$
Angular Momentum	$\sqrt{4\pi G}\rho_0^{3/2} R^5$
Force/Volume	$4\pi G\rho_0^2 R$

2. The two–fluid model

We want to describe a star consisting of two independent fluids in Newtonian gravitation. We distinguish a fluid denoted by the subscript c, that will represent the globally neutral fluid of charged components of a neutron star (nuclei of the crust, protons and electrons), and a fluid denoted by the subscript s, that will describe the superfluid of free neutrons. We will also refer to the fluid of charged components as the "normal fluid", as opposed to the superfluid. So the basic description of our model consists of the Euler equations for the two fluids:

$$\begin{aligned}\rho_s\left(\partial_t v_s^i + v_s^j \nabla_j v_s^i\right) &= -\nabla^i P_s - \rho_s \nabla^i \phi + f_s^i\,,\\ \rho_c\left(\partial_t v_c^i + v_c^j \nabla_j v_c^i\right) &= -\nabla^i P_c - \rho_c \nabla^i \phi + f_c^i\,,\end{aligned} \tag{1}$$

where ∂_t denotes the partial derivative with respect to time, ρ_α, P_α, v_α^i and f_α^i are the respective mass density, pressure, velocity and force per volume of each of the two fluids, and α is the "chemical index" ($\alpha = s, c$). ϕ is the gravitational potential, which is related to the total density $\rho \equiv \rho_c + \rho_s$ by Poisson's equation

$$\nabla^2 \phi = 4\pi G\rho, \tag{2}$$

where G is Newton's constant.

We consider only stationary axisymmetric configurations, with the two fluids rotating uniformly with respective angular velocities Ω_c and Ω_s, i.e., $\boldsymbol{v}_\alpha \equiv \boldsymbol{\Omega}_\alpha \times \boldsymbol{r}$. In the subsequent analysis we work with dimensionless quantities, measuring length scales in units of the radius R, densities in units of the central density ρ_0 of the non–rotating configuration and time in units of $1/\sqrt{4\pi G\rho_0}$. Table 1 shows a summary of the employed fundamental and derived units.

In order to avoid unnecessary complications of notation, we will in the following keep the same symbols for the dimensionless variables, with the exception of the rotation rates Ω_α, which we will now denote ε_α. This is in order to emphasize the fact that we are considering *slow rotations* with respect to the natural scale of Ω (see Table 1; this scale is in general still bigger than the Keplerian rotation rate $\Omega_K^2 = 4\pi G\overline{\rho}/3$, where $\overline{\rho}$ is the mean density), and therefore ε_α represents a small parameter, i.e., $\varepsilon_\alpha \ll 1$.

The rescaled Euler equations (1) take the form

$$\frac{1}{\rho_s}\nabla^i P_s + \nabla^i(\phi - \frac{1}{2}\varepsilon_s^2\varpi^2) \;=\; \frac{1}{\rho_s} f_s^i\,,$$
$$\frac{1}{\rho_c}\nabla^i P_c + \nabla^i(\phi - \frac{1}{2}\varepsilon_c^2\varpi^2) \;=\; \frac{1}{\rho_c} f_c^i\,, \qquad (3)$$

where ϖ is the cylindrical radius, and Poisson's equation (2) in the new variables reads

$$\nabla^2\phi = \rho\,. \qquad (4)$$

The fundamental assumption in our treatment is that each of the two fluids obeys a *barotropic* equation of state (EOS), i.e., $P_\alpha = P_\alpha(\rho_\alpha)$. This allows us to write the terms $\nabla^i P_\alpha/\rho_\alpha$ in (3) as the gradient of a function $-\psi_\alpha$, say, that is defined as

$$-\,\psi_\alpha \equiv \int^{P_\alpha} \frac{dp}{\rho_\alpha(p)}\,. \qquad (5)$$

As we will work in the approximation of $T = 0$, the quantity $-\psi_a$ is equivalent to the enthalpy per mass unit and to the chemical potential per mass unit, and in our subsequent analysis it will play the role of an effective potential.

3. The two–fluid model for neutron stars

In the present section we will specialize the general two–fluid model of the previous section to the case of a neutron star. The "normal" fluid of charged components (c) is supposed to be corotating with the crust on short timescales, because of the strong magnetic field that "locks" all charged components to the crust (Easson 1979). The independent component (s) is a perfect fluid that coexists with the normal fluid without any viscous interaction, but we will allow for an indirect interaction via the superfluid vortices. We neglect all magnetic and thermal influences, as we are mainly interested in the effects of rotation. While the assumption of uniform rotation is probably quite realistic for the normal fluid, the superfluid neutrons could perfectly well be in state of differential rotation ($\nabla^i\Omega_s \neq 0$), even under the condition of stationarity, but for simplicity we will assume it to be uniformly rotating.

As we are interested in stationary solutions, we will also neglect the external forces acting on the neutron star, which, for isolated neutron stars, are due to electromagnetic radiation and lead to the long–term slowdown of the rotation rate of the crust. This approximation is easily justified, as the timescales of mechanical displacements of the neutron star matter due to a change in rotation is much shorter than the typical slowdown–timescale $\Omega/\dot\Omega$, which is of the order of 10^6 years.

Part of the normal fluid, namely the solid crust, is not really a fluid, but we could still approximately describe it as a fluid subject to anisotropic volume–forces, namely the stress forces due to the solidity. This means that the force density f_c^i acting on the normal fluid would not only consist of the opposite interaction term $-f_s^i$, but also of an extra term f_a^i due to the anisotropic stress forces, i.e., we would have

$$f_c^i = f_a^i - f_s^i\,. \qquad (6)$$

The fact that there is no temperature–dependence in the bulk EOS is an excellent approximation in the neutron–star context, as the actual temperatures (for not extremely young neutron stars) are some orders of magnitudes below the Fermi temperature. Additionally, as we assume two independently conserved barotropes, we also neglect possible "chemical interactions" between the two fluids via β reactions, which transform neutrons into protons and electrons and vice versa ($n \rightleftharpoons p + e + \bar\nu_e$). But the nature of the involved β reactions in neutron star matter (namely modified Urca) seems to be rather slow, i.e., the chemical equilibration timescales are of the order of several years for not very young neutron stars (Haensel 1992) and therefore much longer than the dynamical timescales under consideration. So the above approximation should be rather viable as long as we do not consider evolutions on very long timescales, where inevitable effects of transfusion would have to be included in the analysis (e.g., see Langlois et al. 1998).

We still need to specify the nature of the interaction force f_s^i. The conditions of stationarity *and* different rotation rates do not allow a dissipative interaction between the superfluid vortices and the normal fluid, so we are basically left with two possible types of interaction, the case of completely *pinned* vortices, e.g., as obtained by Epstein & Baym (1988), and the case of quasi–*free* vortices, as suggested by the results of Jones (1991). The pinned case should still be a good approximation even if vortex–creep is effective (that is, the vortices jump from pinning site to pinning site, as they are pushed by the Magnus force), whenever the creep–timescale is long compared to the dynamical timescale, so that the quasi–stationary mass distribution in the creep case should not differ from the pinned case. The pinned case leads to an interaction caused to the Magnus force acting on the vortices, which is given by

$$f_M^i = \rho_s(\varepsilon_s - \varepsilon_c)\varepsilon_s \nabla^i\varpi^2\,. \qquad (7)$$

This supposes a parallel lattice of vortices. We will follow this common assumption, which has been shown to be valid under certain conditions by Ruderman & Sutherland (1974). In the free case we have

$$f_s^i = 0\,, \qquad (8)$$

so we can treat the two cases (7) and (8) together, writing

$$f_s^i = -\delta_p\, f_M^i\,, \qquad (9)$$

where the "pinning switch" δ_p is 1 in the pinned case and 0 in the free case.

We arrive at the following form for the two Euler equations (3):

$$\nabla^i\left(-\psi_s + \phi - \frac{1}{2}\varepsilon_s^2\varpi^2 + \delta_p\,\varepsilon_s(\varepsilon_s - \varepsilon_c)\varpi^2\right) = 0, \qquad (10)$$

$$\nabla^i\left(-\psi_c + \phi - \frac{1}{2}\varepsilon_c^2\varpi^2\right) \;=\; \delta_p\,\kappa(r)\varepsilon_s(\varepsilon_s - \varepsilon_c)\nabla^i\varpi^2 + \frac{1}{\rho_c} f_a^i + \mathcal{O}(\varepsilon^4)\,, \qquad (11)$$

where we have defined

$$\kappa(r) \equiv \frac{\rho_s^{(0)}(r)}{\rho_c^{(0)}(r)}\,, \tag{12}$$

$\rho_\alpha^{(0)}$ being the zeroth order density distributions, that is, of the non–rotating configuration.

We see that the right–hand side of (11) has to be the gradient of some scalar function. Looking at the pinning term (containing δ_p) of this equation, we see that this term alone can in general not be written as a gradient, because of the factor $\kappa(r)$. This shows that in general the pinning force cannot be compensated without the presence of the anisotropic stress force f_a^i, which is provided by the solidity of the crust, as has already been noticed in the literature (e.g., Ruderman 1991).

There is however a special case that has the advantage of being analytically tractable, where the pinning force *can* be compensated by the gradient force on the left–hand side alone, without including any anisotropic stress forces. This is obviously the case when $\kappa(r)$ is a constant. As with our preceding assumption of uniformity of Ω_s, this case is not necessarily realistic for neutron stars, but it is still of interest since it provides qualitative insight in the behavior of the system in the pinned case. It corresponds to the limiting case of a very ductile crust that does not develop any notable shear stress and deforms like a fluid under the applied Magnus force. On the other hand, contrary to a fluid it is able to keep the vortices from moving relative to the crust.

The condition of constant $\kappa(r) = \kappa$ does not restrict the choice of the EOS of *both* fluids, but only fixes the EOS of the second fluid with respect to the chosen EOS for the first fluid by the relation

$$P_c(\rho_c) = \frac{1}{\kappa} P_s(\kappa\rho_c)\,. \tag{13}$$

In the following we set $f_a^i = 0$ and postpone the difficult problem of including anisotropic stress forces to future work, so we restrict our analysis to the two above mentioned completely "fluid" cases:

(i) **free** vortices ($\delta_p = 0$)
(ii) **pinned** vortices ($\delta_p = 1$)
(with the EOS subject to (13), such that $\rho_s^{(0)}/\rho_c^{(0)} = \kappa$ is a constant)

From Eqs. (10), (11) and $f_a^i = 0$ we obtain the effective potentials

$$\psi_s \;=\; \phi - \frac{\varpi^2}{2}\left(\varepsilon_s^2 - 2\delta_p\varepsilon_s(\varepsilon_s - \varepsilon_c)\right) + C_s\,, \tag{14}$$

$$\psi_c \;=\; \phi - \frac{\varpi^2}{2}\left(\varepsilon_c^2 + 2\kappa\delta_p\varepsilon_s(\varepsilon_s - \varepsilon_c)\right) + C_c\,, \tag{15}$$

where the C_α are constants in *space*, but they may depend on the rotation rates ε_α. One can see that the pinned case ($\delta_p = 1$) introduces mixed terms $\varepsilon_s\varepsilon_c$, while in the free case ($\delta_p = 0$) the only non–zero terms are the diagonal ones, that is ε_α^2.

The pressure P_α should be a monotonic function of density ρ_a, and so we see from (5) that ψ_α should also be a monotonic function of ρ_α. This relation can therefore be globally inverted, so that the density ρ_α can be uniquely written as a function of the effective potential ψ_α in the form

$$\rho_\alpha = \rho_\alpha(\psi_\alpha)\,, \tag{16}$$

a relation that will be important for the subsequent analysis.

4. Generalized Chandrasekhar expansion

It will be convenient, in order to obtain more compact expressions, to introduce a matrix notation in the fluid indices. One will effectively recover the usual Chandrasekhar type of terms known from the case of one fluid (see Chandrasekhar 1933; Tassoul 1978), with the scalar perturbation quantities replaced by symmetric 2×2 matrices. We write the effective potentials as follows:

$$\psi_\alpha = \phi - \frac{\varpi^2}{2}\underline{\varepsilon}\cdot\widehat{Z}_\alpha\cdot\underline{\varepsilon} + C_\alpha(\underline{\varepsilon})\,, \tag{17}$$

where the "centrifugal" matrices $\widehat{Z}_\alpha$ are defined as

$$\widehat{Z}_s \;\equiv\; \begin{pmatrix} 1 & \\ & 0\end{pmatrix} - \delta_p\begin{pmatrix} 2 & -1\\ -1 & 0\end{pmatrix}\,,$$
$$\widehat{Z}_c \;\equiv\; \begin{pmatrix} 0 & \\ & 1\end{pmatrix} + \kappa\delta_p\begin{pmatrix} 2 & -1\\ -1 & 0\end{pmatrix}\,. \tag{18}$$

By writing $\widehat{M}$ we indicate that the quantity M is a symmetric 2×2 matrix in the fluid indices with components $M^{\alpha\beta}$, and $\underline{\varepsilon}$ is the vector with components ε_α.

Following the standard method of Chandrasekhar, we expand all quantities up to second order in the rotation parameter $\underline{\varepsilon}$ around the non-rotating configuration. Because of the symmetry under parity, i.e., $\underline{\varepsilon}\to-\underline{\varepsilon}$, there can be no terms of first order in $\underline{\varepsilon}$. The second–order term is a quadratic form in $\underline{\varepsilon}$ and therefore the definition of the coefficient matrix is ambiguous. We can fix this ambiguity by the additional condition that the matrices occurring in the expansions have to be symmetric.

We work in spherical coordinates r and $u\equiv\cos(\theta)$ (where, of course, θ is defined with respect to the axis of rotation) and so for the fluid densities $\rho_\alpha(r,u)$ this expansion reads

$$\rho_\alpha(r,u) = \rho_\alpha^{(0)}(r) + \delta\rho_\alpha(r,u) \quad \text{with} \quad \delta\rho_\alpha = \underline{\varepsilon}\cdot\widehat{\rho}_\alpha\cdot\underline{\varepsilon}\,. \tag{19}$$

We expand the other quantities ϕ, C_α and ρ in the same way, with the respective second order coefficient matrices $\widehat{\phi}$, $\widehat{C}_\alpha$ and $\widehat{\rho}$ (where of course $\widehat{\rho} = \widehat{\rho}_s + \widehat{\rho}_c$).

It is important to note that the additive constants C_α depend in general on the rotation rates $\underline{\varepsilon}$. We can absorb the additive constant $C_\alpha^{(0)}$ into the definition of $\rho_\alpha(\psi_\alpha)$, so for convenience we can set $C_\alpha^{(0)} = 0$, but we have to keep track of the $\mathcal{O}(\underline{\varepsilon}^2)$ correction $\underline{\varepsilon}\cdot\widehat{C}_\alpha\cdot\underline{\varepsilon}$.

In order to obtain the relations between $\widehat{\rho}_\alpha$ and $\widehat{\phi}$ to second order in $\underline{\varepsilon}$, we expand $\rho_\alpha(\psi_\alpha)$ around the non–rotating configuration $\psi_\alpha^{(0)} = \phi^{(0)}$:

$$\rho_\alpha(\psi_\alpha) \;=\; \rho_\alpha(\phi^{(0)}) - k_\alpha\,\underline{\varepsilon}\cdot\left(\widehat{\phi} - \frac{\varpi^2}{2}\widehat{Z}_\alpha + \widehat{C}_\alpha\right)\cdot\underline{\varepsilon}\,, \tag{20}$$

where

$$k_\alpha \equiv -\left.\frac{d\rho_\alpha(\psi)}{d\psi}\right|_{\phi^{(0)}} . \tag{21}$$

Order by order comparison between (20) and (19) together with the condition of symmetric matrices leads to the identifications

$$\begin{aligned} \rho_\alpha(\phi^{(0)}) &= \rho_\alpha^{(0)}(r)\,, \\ \widehat{\rho}_\alpha &= -k_\alpha\left(\widehat{\phi} - \frac{\varpi^2}{2}\widehat{Z}_\alpha + \widehat{C}_\alpha\right)\,, \end{aligned} \tag{22}$$

which further allows us to write the "structure function" k_α simply as

$$k_\alpha(r) = -\frac{d\rho_\alpha^{(0)}}{d\phi^{(0)}}\,. \tag{23}$$

The total density perturbation coefficient is found from (22) to be

$$\widehat{\rho} = -k\widehat{\phi} + \frac{3\varpi^2}{2}\widehat{K}(r) - \widehat{D}(r)\,, \tag{24}$$

where we have defined $k \equiv k_c + k_s$ and the matrices

$$\widehat{K}(r) \equiv \frac{1}{3}\left(k_s\widehat{Z}_s + k_c\widehat{Z}_c\right) \quad \text{and} \quad \widehat{D}(r) \equiv k_s\widehat{C}_s + k_c\widehat{C}_c\,. \tag{25}$$

Surprisingly, the matrix $\widehat{K}$ is found (using the definitions of κ and k_α, (12) and (23)) to be the same in the free (i) and the pinned (ii) case, namely

$$\widehat{K} = \frac{1}{3}\begin{pmatrix} k_s & 0 \\ 0 & k_c \end{pmatrix}\,. \tag{26}$$

Inserting (24) into Poisson's equation (4), one finally obtains the partial differential equation for the second order corrections $\widehat{\phi}$ of the gravitational potential,

$$\nabla^2\widehat{\phi} + k\,\widehat{\phi} = \frac{3\varpi^2}{2}\,\widehat{K}(r) - \widehat{D}(r)\,. \tag{27}$$

Using the decomposition of $\widehat{\phi}(r,u)$ in the orthogonal basis of Legendre polynomials, we can reduce this partial differential equation to an infinite series of ordinary differential equations. We write

$$\widehat{\phi}(r,u) = \sum_{l=0}^{\infty} P_{2l}(u)\,\widehat{\phi}_{2l}(r)\,, \tag{28}$$

where we only need to sum over Legendre polynomials with even index, assuming equatorial symmetry. Using the well known differential equation for the Legendre Polynomials, the Laplace operator acting on $\widehat{\phi}$ is seen to reduce to

$$\nabla^2\widehat{\phi}(r,u) = \sum_{l=0} P_{2l}(u)\mathcal{D}_{2l}\,\widehat{\phi}_{2l}(r)\,, \tag{29}$$

where the differential operator $\mathcal{D}_n$ is defined as

$$\mathcal{D}_n \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{n(n+1)}{r^2}\,. \tag{30}$$

Using the orthogonality property of the Legendre polynomials together with the fact that $3\varpi^2 = 2\,r^2(1-P_2(u))$ leads to the following series of ordinary differential equations

$$\begin{aligned} \mathcal{D}_0\widehat{\phi}_0 + k\,\widehat{\phi}_0 &= +r^2\widehat{K} - \widehat{D}\,, \\ \mathcal{D}_2\widehat{\phi}_2 + k\,\widehat{\phi}_2 &= -r^2\widehat{K}\,, \\ \mathcal{D}_{2l}\widehat{\phi}_{2l} + k\,\widehat{\phi}_{2l} &= 0 \quad \text{for} \quad l \geq 2\,. \end{aligned} \tag{31}$$

In order to solve these equations, one must specify the appropriate boundary conditions, which we consider in the next section.

5. Boundary conditions

The first restriction on the solutions of (31) comes from the requirement that the $\widehat{\phi}_{2l}$ should be regular functions in $r=0$, and therefore the left hand side of the differential equation has to be regular in the origin too. This leads to the conditions

$$\begin{aligned} \widehat{\phi}_{2l}(0) &= 0 \quad \text{for} \quad l \geq 1, \\ \widehat{\phi}'_{2l}(0) &= 0 \quad \text{for} \quad l \geq 0\,. \end{aligned} \tag{32}$$

The prime stands for derivatives with respect to r. Another boundary condition is obtained by matching the solution for the gravitational potential inside the star to the solution ϕ_E outside the star. The external solution is normalized conventionally by $\lim_{r\to\infty}\phi_E = 0$, and satisfies $\nabla^2\phi_E = 0$. Its expansion in terms of Legendre polynomials, and up to second order in $\underline{\varepsilon}$ has therefore the following form:

$$\phi_E(r,u) = \frac{\kappa^{(0)}}{r} + \underline{\varepsilon}\cdot\left(\sum_{l=0}\frac{\widehat{\kappa}_{2l}}{r^{2l+1}}P_{2l}(u)\right)\cdot\underline{\varepsilon} + \mathcal{O}(\varepsilon^4)\,. \tag{33}$$

Taking into account the deviation of the star from sphericity, the surface can be expressed as

$$R(u) = 1 + \underline{\varepsilon}\cdot\left(\sum_{l=0}\widehat{R}_{2l}P_{2l}(u)\right)\cdot\underline{\varepsilon} + \mathcal{O}(\varepsilon^4)\,, \tag{34}$$

where the radius of the non–rotating configuration $R^{(0)} = 1$ in our units (see Table 1). The matching conditions are given by

$$\begin{aligned} \phi(R(u),u) &= \phi_E(R(u),u)\,, \\ \phi'(R(u),u) &= \phi'_E(R(u),u)\,. \end{aligned} \tag{35}$$

The deviation of the derivative normal to the surface from a simple radial derivative is of order $\mathcal{O}(\underline{\varepsilon}^4)$, so we can neglect it.

Expanding these matching conditions up to second order and using the fact that ${\phi^{(0)}}''(1) + 2{\phi^{(0)}}'(1) = 0$ yields the following boundary condition for the $\widehat{\phi}_{2l}$:

$$\widehat{\phi}'_{2l}(1) + (2l+1)\,\widehat{\phi}_{2l}(1) = 0\,. \tag{36}$$

It is interesting to note that this condition was found without ever specifying the actual surface of matching. The $\widehat{R}_{2l}$ were in fact completely arbitrary apart from the restriction to be small compared to ε^{-2}, such that the development (34) makes sense. This shows that the obtained boundary relation for the $\widehat{\phi}_{2l}$ is a rather robust consequence of the matching to the vacuum solution itself. One could in fact find the $\widehat{R}_{2l}$ which specify the actual surface of the star up to second order in $\underline{\varepsilon}$ in terms of the $\widehat{\phi}_{2l}$ by the obvious definition

$$\rho\left(R(u), u\right) = 0 \tag{37}$$

which then leads to the expression for the surface up to $\mathcal{O}(\varepsilon^2)$ in the form

$$R(u) = 1 - \frac{1}{\rho^{(0)\prime}(1)}\,\underline{\varepsilon}\cdot\left(\sum_{l=0}\widehat{\rho}_{2l}(1)P_{2l}(u)\right)\cdot\underline{\varepsilon}\,. \tag{38}$$

For the individual fluids we can find the $\rho_\alpha = 0$ surfaces in the same way:

$$R_\alpha(u) \;=\; R_\alpha^{(0)} - \frac{1}{\rho_\alpha^{(0)\prime}(R_\alpha^{(0)})} \times\underline{\varepsilon}\cdot\left(\sum_{l=0}\widehat{\rho}_{\alpha,2l}(R_\alpha^{(0)})\,P_{2l}(u)\right)\cdot\underline{\varepsilon}\,. \tag{39}$$

It has already been recognized by various authors that this type of expansion eventually becomes singular in the vicinity of the star's surface (see Smith 1975; Tassoul 1978, and references therein). The zeroth order term of $\rho^{(0)}(r) + \underline{\varepsilon}\cdot\widehat{\rho}\cdot\underline{\varepsilon}$ obviously becomes zero on the non–rotating star's radius $r = 1$, and so the $\mathcal{O}(\varepsilon^2)$ correction can no longer be considered as being small with respect to the zeroth order term. Due to this fact the value for $\rho(r, u)$ is locally valid only as long as one stays away from the surface, and so the definition of $\widehat{R}_{2l}$ via (37) seems rather unreliable. Therefore it is important that the boundary condition (36) does not depend on the actual form of the boundary surface.

We note that for the case $l = 0$ we still need two more conditions in order to fix all the four free parameters of the solutions $\widehat{\phi}_0(r)$ and $\widehat{\rho}_{\alpha,0}$ ($\widehat{C}_\alpha$ and the two free parameters for a solution of a differential equation of second order). These conditions are obtained by invoking the requirement of mass conservation for each of the two fluids:

$$\int_{V_\alpha} d^3x\,\rho_\alpha(r,u) = \int_{V_\alpha^{(0)}} d^3x\,\rho_\alpha^{(0)}(r)\,. \tag{40}$$

The fact that $\rho_\alpha^{(0)}(r)$ vanishes in $R_\alpha^{(0)}$ leads to

$$\int_{V_\alpha} d^3x\rho_\alpha \;=\; \int_{V_\alpha^{(0)}} d^3x\rho_\alpha^{(0)}(r) +\underline{\varepsilon}\cdot\left(\int_{V_\alpha^{(0)}} d^3x\,\widehat{\rho}_\alpha\right)\cdot\underline{\varepsilon} + \mathcal{O}(\varepsilon^4)\,. \tag{41}$$

Because of the orthogonality property of the Legendre polynomials and $P_0(u) = 1$, any integral of the type $\int_{-1}^1 du P_{2l}(u)$ vanishes for $l \neq 0$, so that the condition of mass conservation simply reduces to

$$\int_0^{R_\alpha^{(0)}} dr\, r^2\widehat{\rho}_{\alpha,0}(r) = 0\,. \tag{42}$$

As mentioned by Heintzmann et al. (1973) in the case of *one* fluid, the integral constraint of *total* mass conservation can be reduced, with the help of Poisson's equation (4), to a differential boundary condition on ϕ_0, namely

$$\widehat{\phi}_0'(1) = 0\,, \tag{43}$$

but in the case of two fluids considered here, we still have to use one of the two integral constraints (42), in order to fix the second constant.

If one wanted to consider a transfusive type of model (see Langlois et al. 1998), one would effectively have only (43) and would still need some other prescription in order to fix the remaining constant, and thereby the respective transfusive mass transfer between the two fluids.

6. Formal solution

The prescription of the boundary conditions not only completely specifies the solutions of our series of ordinary differential equations (31), but it even restricts nearly all of them to be zero. For $l \geq 2$, $\widehat{\phi}_{2l}$ is given by a homogeneous differential equation of second order, subject to the boundary conditions (32) and (36). Only one of the two fundamental solutions can be chosen to be regular in the origin, so we have the freedom of only one multiplicative constant in order to satisfy (36), which can in general only be zero. All the solutions are trivial whenever the differential equation is homogeneous. This is the case for all the $\widehat{\phi}_{2l}$ with $l \geq 2$ (but also for those matrix-elements in the cases $l = 0$ and $l = 1$, for which the inhomogeneous term, that is the corresponding matrix–element of $\widehat{K}$ and $\widehat{D}$, is zero).

So the formal solution of the problem consists of the following density perturbation coefficients (see (22))

$$\begin{aligned}\widehat{\rho}_{\alpha,0}(r) \;&=\; -k_\alpha(r)\left(\widehat{\phi}_0(r) - \frac{r^2}{3}\widehat{Z}_\alpha + \widehat{C}_\alpha\right)\,,\\ \widehat{\rho}_{\alpha,2}(r) \;&=\; -k_\alpha(r)\left(\widehat{\phi}_2(r) + \frac{r^2}{3}\widehat{Z}_\alpha\right)\,,\end{aligned} \tag{44}$$

with the $\widehat{\phi}_0(r)$ and $\widehat{\phi}_2(r)$ solutions of (31), subject to the conditions of regularity (32), continuity with the external potential (36), and mass conservation (42).

7. Expansion, ellipticity and moment of inertia

We will now discuss some of the consequences of the obtained formal solution up to second order in ε for the densities $\rho_\alpha(r, u)$. For simplicity, we restrict our attention in this section to mass distributions ρ_α with a simply connected topology, that is to say, which possess only one boundary surface for each fluid, namely the outer surface, and so we have $\rho_\alpha^{(0)\prime}(R_\alpha^{(0)}) \leq 0$. From the expression for the respective boundary surfaces (39) we see that there is a uniform expansion of the fluid as a whole of amount $\underline{\varepsilon}\cdot(-\widehat{\rho}_{\alpha,0}(R_\alpha^{(0)})/\rho_\alpha^{(0)\prime}(R_\alpha^{(0)}))\cdot\underline{\varepsilon}$, and superposed on this a term proportional to $P_2(u)$, which leads to the ellipticity of the surface. At the equator $P_2(u) = -1/2$ and at the poles $P_2(u) = +1$, so we get the general expression for the ellipticity:

$$\sigma_\alpha = \frac{3}{2}\frac{\left(-\underline{\varepsilon}\cdot\widehat{\rho}_{\alpha,2}(R_\alpha^{(0)})\cdot\underline{\varepsilon}\right)}{R_\alpha^{(0)}\,|\rho_\alpha^{(0)\prime}(R_\alpha^{(0)})|}\,. \tag{45}$$

From (44) and the regularity condition $\widehat{\phi}_2(0) = 0$, we see that $\widehat{\rho}_2(0) = 0$, and so the relative change of the central density is given by

$$\frac{\delta\rho_\alpha(0)}{\rho_\alpha^{(0)}(0)} = \frac{\underline{\varepsilon} \cdot \widehat{\rho}_{\alpha,0}(0) \cdot \underline{\varepsilon}}{\rho_\alpha^{(0)}(0)} \,. \tag{46}$$

The respective fluid volumes are given by

$$V_\alpha = V_\alpha^{(0)} + 4\pi R_\alpha^{(0)\,2} \, \frac{\underline{\varepsilon} \cdot \widehat{\rho}_{\alpha,0}(R_\alpha^{(0)}) \cdot \underline{\varepsilon}}{|\rho_\alpha^{(0)\prime}(R_\alpha^{(0)})|} \,. \tag{47}$$

Finally, we write the change of the two moments of inertia in the form

$$I_\alpha = I_\alpha^{(0)} + \underline{\varepsilon} \cdot \widehat{I}_\alpha \cdot \underline{\varepsilon} \,, \tag{48}$$

where $\widehat{I}_\alpha$ is given by

$$\widehat{I}_\alpha = \int_{V_\alpha^{(0)}} d^3x \, \varpi^2 \widehat{\rho}_\alpha(r,u) \,. \tag{49}$$

We note that the integral is done only over the unperturbed, spherical volume $V_\alpha^{(0)}$, because the corrections due to the form of the boundary surface are of order $\mathcal{O}(\varepsilon^4)$, which is due to the same cancellation as has already been encountered in the density integration (41). Further evaluation leads to

$$\widehat{I}_\alpha = \frac{8\pi}{3} \int_0^{R_\alpha^{(0)}} dr \, r^4 \left(\widehat{\rho}_{\alpha,0}(r) - \frac{1}{5} \, \widehat{\rho}_{\alpha,2}(r) \right) \,. \tag{50}$$

8. Rotational coupling

In this section we investigate a consequence of the dependence of the moments of inertia on the rotation rates $\underline{\varepsilon}$, which is expressed in equation (48). The moment of inertia of one fluid also depends on the rotation of the *second* fluid, which leads to what can be called "rotational coupling". This effect is still present in the free case, where the only way the two fluids communicate with each other is via the gravitational potential ϕ: changing the rotation rate of the fluid α changes its mass distribution ρ_α and therefore also ϕ, which in its turn will change the mass distribution of the second fluid ρ_β. As we saw above, this effect takes place on the order $\mathcal{O}(\varepsilon^2)$.

Let us consider the angular momentum, which in our units (see Table 1) is given by

$$L_\alpha = \left(I_\alpha^{(0)} \; + \underline{\varepsilon} \cdot \widehat{I}_\alpha \cdot \underline{\varepsilon} \right) \, \varepsilon_\alpha \; + \mathcal{O}(\varepsilon^5) \,. \tag{51}$$

If we want to express the rotation rates ε_α in terms of the angular momenta L_α, it suffices to invert this relation and we obtain

$$\varepsilon_\alpha = \varepsilon_\alpha^{(1)} \left(1 - \frac{\underline{\varepsilon}^{(1)} \cdot \widehat{I}_\alpha \cdot \underline{\varepsilon}^{(1)}}{I_\alpha^{(0)}} \right) \,, \tag{52}$$

where we have defined the first order rotation rate by

$$\varepsilon_\alpha^{(1)} \equiv \frac{L_\alpha}{I_\alpha^{(0)}} \,, \tag{53}$$

which is the rotation rate for a given angular momentum L_α, if we kept the mass distribution fixed to the value of the non–rotating case. We see that in (52), at the order $\mathcal{O}(\varepsilon^3)$, we were allowed to replace ε_α by $\varepsilon_\alpha^{(1)}$. This is the explicit relation for $\varepsilon_\alpha(\varepsilon_s^{(1)}, \varepsilon_c^{(1)})$, or equivalently $\varepsilon_\alpha(L_s, L_c)$. Here we see again the effect of the rotational coupling between the two fluids, namely the change of the rotation rate of one fluid if we change the angular momentum of the other fluid. This mutual dependence explicitly reads as

$$\frac{\partial\varepsilon_\alpha}{\partial L_\beta} = \left(I_\alpha^{(0)} - \underline{\varepsilon}^{(1)} \cdot \widehat{I}_\alpha \cdot \underline{\varepsilon}^{(1)} \right) \delta_{\alpha\beta} - 2\varepsilon_\alpha^{(1)} \left(\widehat{I}_\alpha \cdot \underline{\varepsilon}^{(1)} \right)_\beta \,. \tag{54}$$

The effect is of order $\mathcal{O}(\varepsilon^2)$ and its actual importance is determined by the coefficient $\widehat{I}_\alpha / I_\alpha^{(0)}$, which depends on the EOS.

Let us take a look at a particular case, where we change the angular momentum L_c without changing L_s, corresponding to what happens in a real neutron star, for example when we consider the loss of angular momentum of the normal fluid due to electromagnetic radiation. In this case we can express the change of angular velocity of the superfluid with respect to the change of the normal fluid as

$$\frac{d\varepsilon_s}{d\varepsilon_c} = \frac{\partial\varepsilon_s/\partial L_c}{\partial\varepsilon_c/\partial L_c} = -2\varepsilon_s^{(1)} \frac{\left(\widehat{I}_s \cdot \underline{\varepsilon}^{(1)} \right)_c}{I_s^{(0)}} + \mathcal{O}(\varepsilon^4) \,. \tag{55}$$

9. Exact solution for the polytrope $P \propto \rho^2$

In the previous sections we have obtained formal solutions, and all quantities have been expressed in terms of $\widehat{\phi}_0$ and $\widehat{\phi}_2$, which satisfy the differential equations (31). The purpose of this section is to consider a special case for which these equations can be explicitly solved, and that is the case of the two fluids obeying a polytropic EOS of the type

$$P_\alpha = \frac{\rho_\alpha^2}{2k_\alpha} \,, \tag{56}$$

where for the moment the k_α is just a fluid–specific positive constant. We can see that the two EOS (56) satisfy the relation (13) with $\kappa = k_s/k_c$, so we can study the free and the pinned case for this special EOS. The solutions $\rho_\alpha^{(0)}(r)$ for the non–rotating case will satisfy $\rho_s^{(0)} = \kappa\rho_c^{(0)}$. This relation tells us that both fluids share the same boundary surface, which is therefore the star's surface, and so $R_s^{(0)} = R_c^{(0)} = R = 1$.

We start by the zeroth order approximation, that is the non–rotating configuration of the two–fluid star. The equation of hydrostatic equilibrium in the non–rotating case reads as

$$\frac{1}{\rho_\alpha^{(0)}} \nabla^i P_\alpha = -\nabla^i \phi^{(0)} \,, \tag{57}$$

and for the EOS (56) it has the solution

$$\rho_\alpha^{(0)} = -k_\alpha \left(\phi^{(0)} + C_\alpha^{(0)} \right) \,. \tag{58}$$

Using the definition (23) of the structure function $k_\alpha(r)$, we see it is equal the constant k_α defined in the EOS (56). For the total density we find

$$\rho^{(0)} = -k\phi^{(0)} + C^{(0)} \,. \tag{59}$$

Putting this expression into Poisson's equation (4), we recover the same Lane–Emden equation we would get for *one* polytrope of the form $P = \rho^2/2k$, namely

$$\nabla^2 \rho^{(0)}(r) + k\rho^{(0)}(r) = 0\,, \tag{60}$$

even if the combined system of the two fluids cannot be described as a barotrope at all, i.e., $P(\rho_s, \rho_c) \equiv P_s + P_c$ cannot be written as a function of ρ alone. The above equation, subject to the boundary condition $\rho^{(0)}(0) = 1$ (which is due to our choice of units), has the following solution:

$$\rho^{(0)}(r) = \frac{\sin(r\sqrt{k})}{(r\sqrt{k})}\,, \quad \text{for} \quad r \le 1\,, \tag{61}$$

which implies that

$$k = \pi^2\,. \tag{62}$$

This is not too surprising, as it is well known that in the case of a static polytrope with polytropic index 2 there exists a simple proportionality relation between the star's radius R and the coefficient k, the radius being in fact degenerate with respect to the star's mass. As we are working in units where $R = 1$, this also fixes the numerical value of k. Due to the proportionality relation $\rho_s^{(0)} = \kappa \rho_c^{(0)}$ and $\kappa = k_s/k_c$, we obtain for the respective densities

$$\rho_\alpha^{(0)}(r) = \frac{k_\alpha}{\pi^2}\,\frac{\sin(r\pi)}{r\pi}\,. \tag{63}$$

We come now to the corrections of order $\mathcal{O}(\varepsilon^2)$, determined by the coefficients $\widehat{\phi}_0$ and $\widehat{\phi}_2$ that are the solutions of (31). The regular homogeneous solution is found in terms of the spherical Bessel function $j_n(x)$ to be

$$\widehat{\varphi}_{2l}(r) = \widehat{A}\, j_{2l}(r\pi)\,. \tag{64}$$

Particular solutions are found by inspection, and so we obtain the exact solution to (31) in the form

$$\begin{aligned} \widehat{\phi}_0(r) &= \widehat{A}_0 j_0(r\pi) + \frac{\widehat{K}}{\pi^2}\left(r^2 - \frac{6}{\pi^2}\right) - \frac{\widehat{D}}{\pi^2}\,, \\ \widehat{\phi}_2(r) &= \widehat{A}_2 j_2(r\pi) - \frac{\widehat{K}}{\pi^2} r^2\,, \\ \widehat{\phi}_{2l}(r) &= 0\,, \quad \text{for} \quad l \ge 2\,, \end{aligned} \tag{65}$$

where the remaining constants $\widehat{A}_0$, $\widehat{A}_2$ and $\widehat{D}$ are to be determined by the boundary conditions (36) and (43), which finally yields

$$\widehat{\phi}_0(r) = \frac{\widehat{K}}{\pi^2}\left(2 j_0(r\pi) + r^2 - 1\right)\,, \tag{66}$$

$$\widehat{\phi}_2(r) = \frac{\widehat{K}}{\pi^2}\left(5 j_2(r\pi) - r^2\right)\,. \tag{67}$$

Inserting the obtained $\widehat{\phi}$ into the equation (44) for the $\widehat{\rho}_\alpha$ and invoking the mass conservation condition (42) for the individual fluids determines the remaining constants $\widehat{C}_\alpha$. For the sake of completeness we will write the complete solution (44) after putting all the pieces together:

$$\begin{aligned} \widehat{\rho}_{\alpha,0}(r) &= -k_\alpha \left\{ \frac{\widehat{K}}{\pi^2}\left(2\, j_0(r\pi) + r^2 - \frac{3}{5} - \frac{6}{\pi}\right) + \widehat{Z}_\alpha\left(\frac{1}{5} - \frac{r^2}{3}\right)\right\}, \\ \widehat{\rho}_{\alpha,2}(r) &= -k_\alpha \left\{ \frac{\widehat{K}}{\pi^2}\left(5\, j_2(r\pi) - r^2\right) + \frac{r^2}{3}\widehat{Z}_\alpha \right\}\,, \end{aligned} \tag{68}$$

while the total density perturbation coefficients $\widehat{\rho}_{2l}$ can be written more compactly,

$$\begin{aligned} \widehat{\rho}_0 &= -\widehat{K}\left(2\, j_0(r\pi) - \frac{6}{\pi^2}\right)\,, \\ \widehat{\rho}_2 &= -5\,\widehat{K}\, j_2(r\pi)\,. \end{aligned} \tag{69}$$

Using this explicit solution we can evaluate the coefficients that determine the rotational coupling (54) discussed in Sect. 8. The integration (50) over the explicit solutions (68) yields

$$\frac{\widehat{I}_\alpha}{I_\alpha^{(0)}} = a\,\widehat{K} + b\,\widehat{Z}_\alpha\,, \tag{70}$$

with the coefficients

$$\begin{aligned} a &= \frac{9}{\pi^2 - 6}\left(3 - \frac{\pi^2}{5} - \frac{\pi^4}{175}\right)\,, \\ b &= \frac{3\pi^6}{175(\pi^2 - 6)}\,. \end{aligned}$$

The expression (55), which applies to the particular case where $dL_s = 0$, $dL_c \neq 0$, can now be obtained explicitly as

$$\frac{d\varepsilon_s}{d\varepsilon_c} = -2a\,k_c\,\varepsilon_s^{(1)}\varepsilon_c^{(1)} \tag{71}$$

in the free case (i), and

$$\frac{d\varepsilon_s}{d\varepsilon_c} = -2\left(b(\varepsilon_s^{(1)})^2 + a k_c \varepsilon_s^{(1)}\varepsilon_c^{(1)}\right) \tag{72}$$

in the pinned case (ii).

10. Conclusions

We have considered stationary axisymmetric configurations of two fluids rotating uniformly with different rotation rates. The analytical method of Chandrasekhar, known from the classical problem of a single rotating fluid, has been generalized to the two–fluid case. By applying this method we have obtained the formal solution of the respective equilibrium mass distributions for the two fluids in terms of the two functions $\widehat{\phi}_0(r)$ and $\widehat{\phi}_2(r)$, which are the solutions of the ordinary differential equations (31). In order to fully determine these solutions, one needs to specify an EOS for the two fluids. The case of the special polytropic EOS $P \propto \rho^2$ is solved as an example in Sect. 9. A genuine effect of the two–fluid model is pointed out in Sect. 8, namely the fact that the gravitational potential communicates changes in rotation speed and mass distribution between the two fluids.

Further effort would be necessary in order to include the effects of solidity of the crust, so that one could analyze the buildup of stress forces in the crust, including the case of pinned vortices in its generality, without the present restriction of (13). Further investigations will also be concerned with the implications of the present results on the deviation from chemical equilibrium and thus heating and neutrino emission. Finally, a general relativistic description would be desirable, as the mass concentration and rotation rates of neutron stars clearly exceed the range for which a Newtonian treatment can be accurate.

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