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The exponential law: monopole detectors, Bogoliubov transformations, and the thermal nature of the Euclidean vacuum in $\mathbb{RP}^3$ de Sitter spacetime

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Abstract. We consider scalar field theory on the $\mathbb{RP}^3$ de Sitter spacetime ($\mathbb{RP}^3$dS), which is locally isometric to de Sitter space (dS) but has spatial topology $\mathbb{RP}^3$. We compare the Euclidean vacua on $\mathbb{RP}^3$dS and dS in terms of three quantities that are relevant for an inertial observer: (a) the stress–energy tensor; (b) the response of an inertial monopole particle detector; (c) the expansion of the Euclidean vacuum in terms of many-particle states associated with static coordinates centred at an inertial worldline. In all these quantities, the differences between $\mathbb{RP}^3$dS and dS turn out to fall off exponentially at early and late proper times along the inertial trajectory. In particular, (b) and (c) yield at early and late proper times in $\mathbb{RP}^3$dS the usual thermal result for the de Sitter Hawking temperature. This conforms to what one might call an exponential law: in expanding locally de Sitter spacetimes, differences due to global topology should fall off exponentially in the proper time.

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1. Introduction

Observable consequences of the large-scale topology of the universe are a subject of increasing interest. It has been recognized for some time that a topologically nontrivial universe can produce multiple images of individually identifiable objects in the sky, and our not having seen such images sets bounds on the potential scale of nontrivial topology in our universe [1–4]. More recently, it was recognized that nontrivial topology can also leave an imprint on the cosmic microwave background, through the quantum mechanical origin of density inhomogeneities and the subsequent Sachs–Wolfe effect, and the observational bounds obtained in this way could in fact be more stringent [5, 6].

The purpose of this paper is to explore another situation in which quantum fields in a curved spacetime feel the large-scale topology: we consider the experiences of an inertial observer coupled to a quantum field in a spacetime that is locally de Sitter, but has spatial topology $\mathbb{RP}^3$ instead of the usual S^3. We consider a free scalar field, and we assume the field to be in

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the globally regular vacuum state that is induced by the Euclidean vacuum on de Sitter space. As the unconventional spatial topology lies beyond the cosmological horizon of the inertial observer, this problem illustrates how quantum fields can probe large-scale topology that is classically unobservable by virtue of topological censorship [7]. Also, as the cosmological horizon of the observer does not coincide with a bifurcate Killing horizon, this problem sheds light on the role of the bifurcate Killing horizon in the thermal effects experienced by an inertial observer in de Sitter space [8].

We shall find, from the analysis of a monopole particle detector, as well as from a Bogoliubov transformation between the Euclidean vacuum and the vacuum natural for the inertial observer, that the experiences of the inertial observer are not identical to those of an inertial observer in the Euclidean vacuum in de Sitter space. However, in the limit of early or late proper times along the observer trajectory, the differences vanish exponentially, and the experiences of the observer become asymptotically thermal in the usual de Sitter Hawking temperature. We also compute the renormalized stress–energy tensor, finding that it reduces to that in de Sitter space in the limit of early and late times on each inertial trajectory. These results conform to what one might call an exponential law: in expanding locally de Sitter spacetimes, differences due to global topology should fall off exponentially in the proper time. From the viewpoint of the absence of a bifurcate Killing horizon, qualitatively similar results have been previously found on the single-exterior eternal black hole known as the $\mathbb{RP}^3$ geon [9] and on the conformal boundary of the $(2+1)$-dimensional single-exterior black hole known as the $\mathbb{RP}^2$ geon [10].

The remainder of the paper is as follows. In section 2 we briefly review the properties of four-dimensional de Sitter spacetime, which we denote by dS, and the quotient construction of a spacetime, denoted by $\mathbb{RP}^3$dS, which has the same local geometry but whose spatial topology is $\mathbb{RP}^3$. Scalar field theory and the Euclidean vacua on these spacetimes are introduced in section 3, and the stress–energy tensor on $\mathbb{RP}^3$dS is evaluated by point-splitting methods. Section 4 constructs the Bogoliubov transformation on $\mathbb{RP}^3$dS, and the particle detector is analysed in section 5. Section 6 contains a brief summary and discussion. An evaluation of the stress–energy tensor by conformal methods, in the special case of a conformal field, is given in the appendix.

We work in Planck units, $\hbar = c = G = 1$. A metric with signature $(-{+}{+}{+})$ is called Lorentzian and a metric with signature $({+}{+}{+}{+})$ Riemannian. All scalar fields are global sections of a real line bundle over the spacetime (i.e. we do not consider twisted fields). Complex conjugation is denoted by an overline.

2. de Sitter spacetime and $\mathbb{RP}^3$dS

In this section we briefly review the geometry of four-dimensional de Sitter spacetime (dS) and its quotient space $\mathbb{RP}^3$dS. The main purpose of the section is to establish the notation and to introduce the coordinate systems that will be used with the quantum field theory.

2.1. de Sitter spacetime

Four-dimensional de Sitter space is a Lorentzian spacetime form of positive sectional curvature. It can be represented as the hyperboloid

$$H^{-2} = -U^2 + V^2 + X^2 + Y^2 + Z^2 \tag{2.1}$$

in five-dimensional Minkowski space with the global coordinates (U, V, X, Y, Z) and the metric

$$ds^2 = -dU^2 + dV^2 + dX^2 + dY^2 + dZ^2. \tag{2.2}$$

The parameter $H > 0$ is the inverse of the radius of curvature of the embedded hypersurface. The spacetime is Lorentzian, and it solves Einstein's equations with the cosmological constant $\Lambda = 3H^2$. The Ricci scalar is $R = 12H^2$. The spacetime is globally hyperbolic, with spatial topology S^3, and a global (3 + 1) foliation is provided, for example, by the spacelike hypersurfaces of constant U. The (connected component of the) isometry group is (the connected component of) $O(4, 1)$. We denote this spacetime by dS.

If x and y denote points in dS, we define

$$\mathcal{Z}(x, y) := H^2\eta_{ab}\, X^a(x)\, X^b(y), \tag{2.3}$$

where $X^a(x)$ and $X^a(y)$ are the five-dimensional Minkowski coordinates of the points on the hyperboloid (2.1), and η_{ab} is the five-dimensional Minkowski metric (2.2). $\mathcal{Z}(x, y)$ is clearly invariant under the isometries of dS, and it encodes almost all the isometry-invariant information about the relative location of x and y. In particular, x is on the lightcone of y if and only if $\mathcal{Z}(x, y) = 1$. For more detail, see for example [11].

dS admits several coordinatizations that are adapted to different isometry subgroups. Of relevance to this paper are two: hyperspherically symmetric coordinates and static coordinates. We now exhibit these.

We introduce on dS the chart $(t, \chi, \theta, \varphi)$ by

$$U = H^{-1}\sinh(Ht), \tag{2.4a}$$

$$V = H^{-1}\cosh(Ht)\cos\chi, \tag{2.4b}$$

$$Z = H^{-1}\cosh(Ht)\sin\chi\cos\theta, \tag{2.4c}$$

$$X = H^{-1}\cosh(Ht)\sin\chi\sin\theta\cos\varphi, \tag{2.4d}$$

$$Y = H^{-1}\cosh(Ht)\sin\chi\sin\theta\sin\varphi. \tag{2.4e}$$

The metric reads

$$ds^2 = -dt^2 + H^{-2}\cosh^2(Ht)\, d\Omega_3^2, \tag{2.5}$$

where $d\Omega_3^2$ is the metric on the unit 3-sphere,

$$d\Omega_3^2 := d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.6}$$

The angles (χ, θ, φ) form a standard set of hyperspherical coordinates on S^3, and the coordinate singularities of this chart on S^3 can be handled in the standard way. When (χ, θ, φ) is understood in this extended sense as a global coordinatization of S^3, the chart $(t, \chi, \theta, \varphi)$ and the metric (2.5) are global on dS with $-\infty < t < \infty$. The worldlines at constant (χ, θ, φ) are timelike geodesics, and the proper time along them is t.

The coordinates $(t, \chi, \theta, \varphi)$ make manifest the $O(4)$ isometry subgroup whose orbits are at constant t. Conversely, the (3 + 1) foliation of dS given by these coordinates is uniquely specified by the choice of a particular $O(4)$ isometry subgroup.

It is useful to introduce the conformal time η,

$$\eta := 2\arctan(e^{Ht}), \tag{2.7}$$

which takes the values $0 < \eta < \pi$. As $\cosh(Ht) = 1/\sin\eta$, the metric (2.5) takes the form

$$ds^2 = \frac{1}{H^2\sin^2\eta}\left[-d\eta^2 + d\Omega_3^2\right]. \tag{2.8}$$

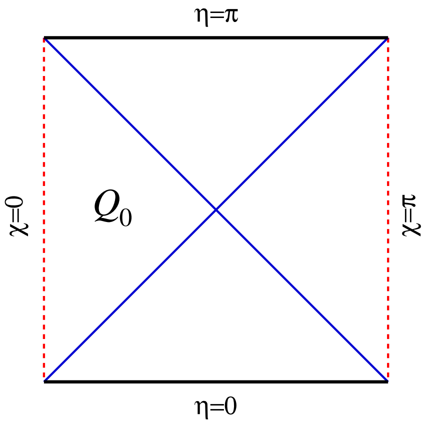

Figure 1. A conformal diagram of dS. The coordinates shown are (η, χ), and those suppressed are (θ, φ). For $0 < \chi < \pi$, each point in the diagram represents a suppressed 2-sphere of radius $H^{-1} \sin\chi/(\sin\eta)$; at $\chi = 0$ and π, each point in the diagram represents a point in the spacetime. The quadrant Q_0, covered by the static coordinates $(\sigma, r, \theta, \varphi)$, is at $\cos\chi > |\cos\eta|$. The involution J, introduced in the text, consists of the reflection $(\eta, \chi) \mapsto (\eta, \pi - \chi)$ about the vertical axis, followed by the antipodal map $(\theta, \varphi) \mapsto (\pi - \theta, \varphi + \pi)$ on the suppressed 2-sphere.

The coordinates (η, χ) are therefore appropriate for a conformal diagram in which (θ, φ) are suppressed. Such a conformal diagram is shown in figure 1.

We now turn to the static coordinates. Let Q_0 be the quadrant $V > |U|$ of dS. In Q_0, we introduce the chart $(\sigma, r, \theta, \varphi)$ by

$$U = H^{-1}\sqrt{1 - H^2r^2}\,\sinh(H\sigma), \tag{2.9a}$$

$$V = H^{-1}\sqrt{1 - H^2r^2}\,\cosh(H\sigma), \tag{2.9b}$$

$$Z = r\cos\theta, \tag{2.9c}$$

$$X = r\sin\theta\cos\varphi, \tag{2.9d}$$

$$Y = r\sin\theta\sin\varphi. \tag{2.9e}$$

The metric takes the static form

$$\mathrm{d}s^2 = -\left(1 - H^2r^2\right)\mathrm{d}\sigma^2 + \left(1 - H^2r^2\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2, \tag{2.10}$$

where $\mathrm{d}\Omega_2^2$ is the metric on the unit 2-sphere,

$$\mathrm{d}\Omega_2^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.11}$$

For $0 < r < H^{-1}$, the set (r, θ, φ) forms a standard set of three-dimensional polar coordinates, and the coordinate singularity at $r = 0$ and at the singularities of the spherical coordinates (θ, φ) on the 2-spheres of constant r can be handled in the standard way. When (r, θ, φ) is understood in this extended sense as a global coordinatization of $\mathbb{R}^3$, with $0 \leqslant r < H^{-1}$, the metric (2.5) with $-\infty < \sigma < \infty$ is global on Q_0. In the chart $(\eta, \chi, \theta, \varphi)$, Q_0 is the region $\cos\chi > |\cos\eta|$, as shown in the conformal diagram in figure 1. The transformation between the charts reads

$$Hr = \frac{\sin\chi}{\sin\eta}, \tag{2.12a}$$

$$H\sigma = -\operatorname{arctanh}\left(\frac{\cos\eta}{\cos\chi}\right). \tag{2.12b}$$

Q_0 has topology $\mathbb{R}^4$. As seen from figure 1, it is globally hyperbolic, and the hypersurfaces of constant σ are Cauchy surfaces for Q_0 (but not for dS). The curve at $r = 0$ is a timelike geodesic in dS, and σ is the proper time along this geodesic: the static coordinates $(\sigma, r, \theta, \varphi)$ are centred around the worldline of an inertial observer at $r = 0$. The boundary of Q_0, at $r \to H^{-1}$, is the cosmological horizon for this observer, and the Killing vector ∂_σ, which is timelike in Q_0, becomes null at the horizon. The horizon therefore has an infinite redshift.

In the quadrant $V > -|U|$, or $\cos\chi < -|\cos\eta|$, a similar static chart can be introduced with the obvious modifications. The future and past quadrants, $U > |V|$ and $U < -|V|$, can be covered by charts in which (2.9*a*) and (2.9*b*) are replaced by

$$U = \pm H^{-1}\sqrt{H^2r^2 - 1}\cosh(H\sigma), \qquad (2.13a)$$

$$V = H^{-1}\sqrt{H^2r^2 - 1}\sinh(H\sigma) \qquad (2.13b)$$

with the upper (lower) sign in (2.13*a*) in the future (past) quadrant. The metric in the future and past quadrants takes the form (2.10) with $r > H^{-1}$.

As any timelike geodesic in dS can be mapped to any other by an isometry, a static metric of the form (2.10) can be introduced in a quadrant of the spacetime centred around any timelike geodesic. The horizon of the static coordinates is in this sense observer dependent.

2.2. The quotient spacetime $\mathbb{RP}^3$dS

On the five-dimensional Minkowski space (2.2), consider the map

$$\tilde{J}: \quad (U, V, X, Y, Z) \mapsto (U, -V, -X, -Y, -Z). \qquad (2.14)$$

We denote by J the map that $\tilde{J}$ induces on dS. In the coordinates $(\eta, \chi, \theta, \varphi)$, we have

$$J: \quad (\eta, \chi, \theta, \varphi) \mapsto (\eta, \pi - \chi, \pi - \theta, \varphi + \pi). \qquad (2.15)$$

J is an involutive isometry, it acts without fixed points and properly discontinuously, and it preserves both space and time orientation. The quotient space dS$/J$ is a space- and time-orientable Lorentzian manifold. We refer to this quotient space as $\mathbb{RP}^3$ de Sitter space and denote it by $\mathbb{RP}^3$dS.

$\mathbb{RP}^3$dS is globally hyperbolic, with spatial topology $\mathbb{RP}^3$. The chart $(\eta, \chi, \theta, \varphi)$ can be reinterpreted as a global chart $\mathbb{RP}^3$dS, provided the angles are understood in the sense of hyperspherical coordinates on $\mathbb{RP}^3$ and not on S^3: with this reinterpretation, equation (2.5) gives the global metric on $\mathbb{RP}^3$dS. A conformal diagram in which the coordinates (θ, φ) are

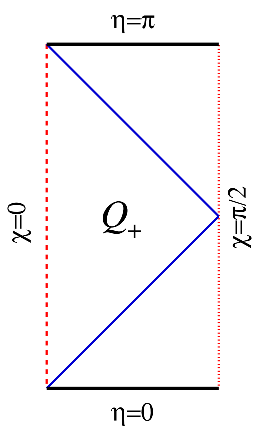

Figure 2. A conformal diagram of $\mathbb{RP}^3$dS. The region $\chi < \frac{1}{2}\pi$ is identical to that in the diagram of figure 1, each point with $0 < \chi < \frac{1}{2}\pi$ representing a suppressed 2-sphere in the spacetime, and each point at $\chi = 0$ representing a point in the spacetime. At $\chi = \frac{1}{2}\pi$, each point in the diagram represents a suppressed $\mathbb{RP}^2$. The region Q_+, covered by the static coordinates $(\sigma, r, \theta, \varphi)$, is at $\cos\chi > |\cos\eta|$.

suppressed is shown in figure 2. As seen in the figure, one can represent $\mathbb{RP}^3$dS by taking the region $0 \leqslant \chi \leqslant \frac{1}{2}\pi$ of dS and identifying at $\chi = \frac{1}{2}\pi$ the antipodal points on the 2-spheres coordinatized by (θ, φ).

The isometry group of $\mathbb{RP}^3$dS is $\mathbb{Z}_2 \times O(4)$, as induced by the largest subgroup of $O(4, 1)$ that commutes with $\tilde{J}$. In the coordinates $(\eta, \chi, \theta, \varphi)$ on $\mathbb{RP}^3$dS, the $O(4)$ factor acts trivially on η, while the nontrivial element of the $\mathbb{Z}_2$ factor acts trivially on the angles and sends η to $\pi - \eta$. The connected component of the isometry group is $SO(4)$. It follows that the $(3 + 1)$ foliation of $\mathbb{RP}^3$dS provided by the coordinates $(\eta, \chi, \theta, \varphi)$ is a geometrically distinguished one: it is the only foliation in which the spacelike hypersurfaces are orbits of the connected component of the isometry group.

As J maps the quadrants $V > |U|$ and $V < -|U|$ of dS onto each other, these two quadrants project onto a region of $\mathbb{RP}^3$dS that is isometric to a single quadrant. We denote this region of $\mathbb{RP}^3$dS by Q_+, and we introduce on it the chart $(\sigma, r, \theta, \varphi)$ induced by the chart (2.9) on Q_0. Equation (2.10) then gives a globally defined metric on Q_+, and the chart $(\sigma, r, \theta, \varphi)$ gives an explicit isometry between Q_+ and Q_0. The line $r = 0$ in Q_+ is a timelike geodesic that is orthogonal to the distinguished foliation of $\mathbb{RP}^3$dS.

From the isometries of $\mathbb{RP}^3$dS it is immediately obvious that a static metric of the form (2.10) could be introduced in a wedge in $\mathbb{RP}^3$dS centred around any timelike geodesic orthogonal to the distinguished foliation. It is straightforward to show that a similar static metric could also be introduced around the timelike geodesics that are not orthogonal to the distinguished foliation.

3. Scalar field quantization and the Euclidean vacuum

We now turn to the quantum theory of a real scalar field ϕ. In this section we recall the definition and some characteristic properties of the Euclidean vacuum on dS [11–15] and discuss the induced vacuum on $\mathbb{RP}^3$dS.

3.1. Euclidean vacuum on de Sitter

The massive scalar field action on a general curved spacetime is

$$S = -\tfrac{1}{2} \int \sqrt{-g}\, \mathrm{d}^4x \big[g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + (\mu^2 + \xi R)\phi^2\big], \tag{3.1}$$

where μ is the mass, R is the Ricci scalar and ξ is the curvature coupling constant. Specializing to dS, we have $R = 12H^2$. We assume $\mu^2 + 12\xi H^2 > 0$, and we define the effective mass as $\tilde{\mu} := \sqrt{\mu^2 + 12\xi H^2}$.

The field equation reads

$$\big(\Box - \tilde{\mu}^2\big)\phi = 0, \tag{3.2}$$

where $\Box$ denotes the scalar Laplacian on dS. The (indefinite) inner product, evaluated on a hypersurface of constant t, is

$$(\phi_1, \phi_2) := \mathrm{i}H^{-3}\cosh^3(Ht) \int_{S^3} \sin^2\chi\, \sin\theta\, \mathrm{d}\chi\, \mathrm{d}\theta\, \mathrm{d}\varphi\, \overline{\phi_1}\, \overleftrightarrow{\partial_t}\, \phi_2\,. \tag{3.3}$$

The spatial dependence of the field equation (3.2) can be separated by the hyperspherical harmonics Q_{nlm}, which are eigenfunctions of the Laplacian on the unit 3-sphere with the eigenvalue $-(n^2 - 1)$: here $n = 1, 2, \ldots,$ and the degeneracy described by the indices l and

m is n^2. For more detail about the harmonics, see for example [16]. The remaining, time-dependent equation can then be solved in terms of associated Legendre functions. For the normalized positive-frequency mode functions, we choose

$$\phi_{nlm} := \mathrm{e}^{-\mathrm{i}\nu\pi/2} \sqrt{\frac{\pi H^2 \Gamma(n+\frac{1}{2}-\nu)}{4\Gamma(n+\frac{1}{2}+\nu)}}\, Q_{nlm} \sin^{3/2}(\eta)\big[P^{\nu}_{n-\frac{1}{2}}(-\cos\eta) - (2\mathrm{i}/\pi) Q^{\nu}_{n-\frac{1}{2}}(-\cos\eta)\big], \tag{3.4}$$

where $P^{\nu}_{n-\frac{1}{2}}$ and $Q^{\nu}_{n-\frac{1}{2}}$ are the associated Legendre functions on the cut [17–19] and ν is one of the solutions of

$$\nu^2 = \tfrac{9}{4} - \tilde{\mu}^2 H^{-2}. \tag{3.5}$$

Which of the two solutions of (3.5) is chosen for ν is immaterial, as the two choices give equivalent mode functions. The resulting vacuum, which we denote by $|0_{\mathrm{E}}\rangle$, is known as the Euclidean vacuum or the Chernikov–Tagirov vacuum [12–14].

The vacuum $|0_{\mathrm{E}}\rangle$ is uniquely characterized by the properties that its two-point function $G^{+}_{\mathrm{dS}}(x, x')$ is invariant under the connected component of the isometry group of dS, and that the only singularity of $G^{+}_{\mathrm{dS}}(x, x')$ occurs when x' is on the lightcone of x [11]. Explicitly, we have

$$G^{+}_{\mathrm{dS}}(x, x') = A H^2 \tilde{F}\big(\tfrac{1}{2}[1 + \mathcal{Z}_{\epsilon}(x, x')]\big), \tag{3.6}$$

where $\tilde{F}$ is the hypergeometric function [17]

$$\tilde{F}(z) := {}_2F_1\big(\tfrac{3}{2}+\nu, \tfrac{3}{2}-\nu; 2; z\big), \tag{3.7}$$

and the numerical factor A is given by

$$A := \frac{\tilde{\mu}^2 H^{-2} - 2}{16\pi \cos\pi\nu} \tag{3.8}$$

for $\tilde{\mu}^2 H^{-2} \neq 2$, and in the special case $\tilde{\mu}^2 H^{-2} = 2$ by the limiting value of (3.8), $A = 1/(16\pi^2)$. Here $\mathcal{Z}_{\epsilon}(x, y)$ is equal to $\mathcal{Z}(x, y)$ (equation (2.3)), but understood near $\mathcal{Z}(x, y) = 1$ in a sense that gives $G^{+}_{\mathrm{dS}}(x, x')$ the correct singularity structure on the lightcone [14]: we can represent $\mathcal{Z}_{\epsilon}$, for example, by

$$\mathcal{Z}_{\epsilon}(x, y) := \mathcal{Z}(x, y) - \mathrm{i}\epsilon[U(x) - U(y)] - \epsilon^2[U(x) - U(y)]^2 \tag{3.9}$$

where $\epsilon \to 0_{+}$.

The renormalized stress–energy tensor in $|0_{\mathrm{E}}\rangle$ is by construction invariant under the isometries of dS, and hence proportional to the metric tensor. In particular, the energy density measured by an inertial observer is constant along the observer trajectory and the same for every observer. The explicit expression for the stress–energy tensor can be found, for example, in [20].

dS can be regarded as a Lorentzian section of a complex spacetime that admits the round 4-sphere as a Riemannian section. The Feynman propagator in $|0_{\mathrm{E}}\rangle$ then analytically continues to the unique Green function on the Riemannian section. This property is the origin of the name 'Euclidean vacuum' for $|0_{\mathrm{E}}\rangle$.

3.2. Euclidean vacuum on $\mathbb{RP}^3$dS

The above quantization on dS adapts to $\mathbb{RP}^3$dS with the obvious modifications. In the inner product (3.3), the spatial integration is now over $\mathbb{RP}^3$. The spatial dependence of the field equation (3.2) is separated by the harmonics on the unit $\mathbb{RP}^3$: these harmonics are constructed by taking from the set $\{Q_{nlm}\}$ those that are invariant under the antipodal map, projecting to $\mathbb{RP}^3$, and multiplying by $\sqrt{2}$ to achieve the correct normalization†. In the normalized positive-frequency mode functions, we choose the time dependence as in (3.4). We denote the resulting vacuum by $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$.

As $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ is induced by $|0_\mathrm{E}\rangle$ under the projection dS $\to$ $\mathbb{RP}^3$dS, the two-point functions in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ are obtained from those in $|0_\mathrm{E}\rangle$ by the method of images. For example, for the positive-frequency Wightman function $G^+_{\mathbb{RP}^3\mathrm{dS}}(x, x')$ in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$, we have

$$G^+_{\mathbb{RP}^3\mathrm{dS}}(x, x') = G^+_{\mathrm{dS}}(x, x') + G^+_{\mathrm{dS}}(x, J(x')), \tag{3.10}$$

where x and x' on the two sides of the equation are understood as points in dS or $\mathbb{RP}^3$dS in the obvious way. It follows that all the two-point functions in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ are invariant under the connected component of the isometry group of $\mathbb{RP}^3$dS.

$\mathbb{RP}^3$dS can be regarded as a Lorentzian section of a complex spacetime using the formalism of (anti)holomorphic involutions [22, 23], and its Riemannian section can then be defined as a certain $\mathbb{Z}_2$ quotient of the round 4-sphere. By method-of-images techniques similar to those used in [9], one sees that the Feynman propagator in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ analytically continues to the unique Green function on the Riemannian section. We therefore refer to $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ as the Euclidean vacuum on $\mathbb{RP}^3$dS.

Using (3.10), it is straightforward to compute the stress–energy tensor in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ by the point-splitting method [14]. The contribution from the first term on the right-hand side of (3.10) is identical to the stress–energy tensor in dS. The remaining contribution, arising from the second term on the right-hand side of (3.10), is finite without additional renormalization, and it is clearly invariant under the isometries of $\mathbb{RP}^3$dS. Denoting this contribution by $\Delta T_{\mu\nu}$, we find that its nonvanishing mixed components in the coordinates $(t, \chi, \theta, \varphi)$ are

$$\Delta T^t{}_t = AH^4\big[3\xi\tilde{F}(z) + \tfrac{3}{2}(1-4\xi)z\tilde{F}'(z)\big], \tag{3.11a}$$

$$\Delta T^i{}_j = AH^4\big\{\big[3\xi + (4\xi-1)\tilde{\mu}^2H^{-2}\big]\tilde{F}(z) + \big[\tfrac{1}{2}(16\xi-3)z + 1 - 6\xi\big]\tilde{F}'(z)\big\}\delta^i{}_j, \tag{3.11b}$$

where the Latin indices denote the spatial coordinates (χ, θ, φ), $z := -\sinh^2(Ht)$ and $\tilde{F}'(z) := \mathrm{d}\tilde{F}(z)/\mathrm{d}z$.

It is clear from (3.11) that the energy density measured by an inertial observer is not constant along the observer trajectory. However, it follows from the expansions of hypergeometric functions [17] that all the components of $\Delta T^\mu{}_\nu$ fall off exponentially in t at large $|t|$, the details of the falloff depending on the parameters. Therefore, in the distant past and future of each observer trajectory, the stress–energy tensor in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ is exponentially asymptotic to the stress–energy tensor in $|0_\mathrm{E}\rangle$.

As a special case, consider the massless conformally coupled field, for which $\xi = \frac{1}{6}$ and $\mu = 0$. Then $\nu = \frac{1}{2}$, $A = 1/(16\pi^2)$ and $\tilde{F}(z) = 1/(1-z)$. In the coordinates $(t, \chi, \theta, \varphi)$, we obtain

$$\Delta T^\mu{}_\nu = \frac{H^4}{32\pi^2\cosh^4(Ht)}\,\mathrm{diag}\big(1, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\big). \tag{3.12}$$

In the appendix we independently verify the result (3.12) by techniques that take advantage of the conformal relation between $\mathbb{RP}^3$dS and the $\mathbb{RP}^3$ version of the Einstein static universe.

† For more details, see [21].

4. Bogoliubov transformation on $\mathbb{RP}^3$dS

In this section we use a Bogoliubov-transformation technique to examine the experiences of an inertial observer in $\mathbb{RP}^3$dS, under the assumption that the worldline of the observer is normal to the distinguished spacelike foliation. In subsection 4.1 we review the quantization in the spacetime region covered by the static coordinates, centred around the worldline of the observer, and we recall the construction of the Boulware-like vacuum $|0_{\text{BdS}}\rangle$ in this region. In subsection 4.2 we express $|0_{\mathbb{RP}^3\text{E}}\rangle$ in terms of the excited states built on $|0_{\text{BdS}}\rangle$ and interpret the result in terms of particles seen by the observer.

4.1. Quantization in the static coordinates

In this subsection we review the quantization of a real scalar field ϕ in the spacetime covered by the static metric (2.10). As explained in section 2, we can interpret this spacetime as the quadrant Q_0 in dS, or as the region Q_+ in $\mathbb{RP}^3$dS.

The (indefinite) inner product, evaluated on a hypersurface of constant σ, reads

$$(\phi_1, \phi_2) := \mathrm{i} \int_{S^2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \int_0^\infty r^2 \, \mathrm{d}r^* \, \overline{\phi_1} \; \overleftrightarrow{\partial}_\sigma \; \phi_2, \tag{4.1}$$

where r^* is the tortoise coordinate,

$$r^* := \frac{1}{2H} \ln\left(\frac{1+Hr}{1-Hr}\right), \tag{4.2}$$

having the range $0 \leqslant r^* < \infty$. Separating the field equation (3.2) by the ansatz

$$\phi = (4\pi\omega)^{-1/2} r^{-1} R_{\omega l}(r) \, \mathrm{e}^{-\mathrm{i}\omega\sigma} Y_{lm}(\theta, \varphi), \tag{4.3}$$

where Y_{lm} are the spherical harmonics†, the equation for the radial function $R_{\omega l}(r)$ becomes

$$0 = \left[\frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} + \omega^2 - \left(1 - H^2 r^2\right)\left(\tilde{\mu}^2 - 2H^2 + \frac{l(l+1)}{r^2}\right)\right] R_{\omega l}. \tag{4.4}$$

The one-dimensional differential operator in (4.4) is essentially self-adjoint with respect to the Schrödinger-type inner product $\int_0^\infty \mathrm{d}r^* \, \overline{R_1} R_2$ for $l > 0$, and for $l = 0$ we choose for this operator the self-adjoint extension whose (generalized) eigenfunctions vanish at $r^* = 0$. The spatial parts of the wavefunctions (4.3) are then the (generalized) eigenfunctions of the essentially self-adjoint spatial part of the wave operator in the field equation (3.2) [25], which, in particular, means that $R_{\omega l}$ are proportional to $(r^*)^{l+1}$ at small r^*. It follows by standard techniques‡ that for each l, the spectrum of ω^2 is continuous and spans the positive real axis.

We choose the positive-frequency mode functions to have $\omega > 0$, and we denote the resulting vacuum by $|0_{\text{BdS}}\rangle$. As these mode functions are positive frequency with respect to the timelike Killing vector ∂_σ, which generates the inertial motion along the geodesic at $r = 0$, an observer moving along this geodesic experiences $|0_{\text{BdS}}\rangle$ as her physical no-particle state: $|0_{\text{BdS}}\rangle$ is analogous to the Boulware vacuum on the exterior Schwarzschild, and to the Rindler vacuum in a Rindler wedge on Minkowski space. For a complete orthonormal set of positive-frequency modes, we choose

$$u_{\omega lm} := \mathrm{e}^{\mathrm{i}(l+|m|)\pi/2} (4\pi\omega)^{-1/2} r^{-1} R_{\omega l} \, \mathrm{e}^{-\mathrm{i}\omega\sigma} Y_{lm}, \tag{4.5}$$

† We use the Condon–Shortley phase convention (see, for example, [24]), in which $Y_{l(-m)}(\theta, \varphi) = (-1)^m \overline{Y_{lm}(\theta, \varphi)}$ and $Y_{lm}(\pi - \theta, \varphi + \pi) = (-1)^l Y_{lm}(\theta, \varphi)$.

‡ When $l = 0$ and $\tilde{\mu}^2 < 2H^2$, equation (4.4) is analysed, for example, in [26]. In other cases the analysis is standard by the non-negativity of the potential term in (4.4).

where the functions $R_{\omega l}$ are real-valued and normalized so that their asymptotic form at large r^* is

$$R_{\omega l} \sim 2\cos(\omega r^* + \delta_{\omega l}), \qquad r^* \to \infty, \tag{4.6}$$

where $\delta_{\omega l}$ is a real-valued phase shift. The orthonormality relation reads

$$(u_{\omega lm}, u_{\omega' l' m'}) = \delta_{ll'}\delta_{mm'}\delta(\omega - \omega'), \tag{4.7}$$

with the complex conjugates satisfying a similar relation with a minus sign, and the mixed inner products vanishing.

We expand the quantized field as

$$\phi = \sum_{lm} \int_0^\infty \mathrm{d}\omega \left[b_{\omega lm} u_{\omega lm} + b^\dagger_{\omega lm} \overline{u_{\omega lm}} \right], \tag{4.8}$$

where $b_{\omega lm}$ and $b^\dagger_{\omega lm}$ are the annihilation and creation operators associated with the mode $u_{\omega lm}$. The vacuum $|0_{\mathrm{BdS}}\rangle$ satisfies by definition

$$b_{\omega lm}|0_{\mathrm{BdS}}\rangle = 0. \tag{4.9}$$

4.2. Bogoliubov transformation

We now consider the above quantization in the static coordinates as having been performed in the region Q_+ of $\mathbb{RP}^3$dS. We wish to write the vacuum induced on Q_+ by $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ in terms of $|0_{\mathrm{BdS}}\rangle$ and the excitations created by $\{b^\dagger_{\omega lm}\}$. Much of our analysis builds on the transformations developed for dS in [27].

Rather than computing directly the Bogoliubov transformation between the sets $\{\phi_{nlm}\}$ (3.4) and $\{u_{\omega lm}\}$ (4.5), we take advantage of the observation that the modes ϕ_{nlm} are analytic functions in η in the lower half of the strip $0 < \mathrm{Re}(\eta) < \pi$ in the complex η-plane, and that they are bounded as $\mathrm{Im}(\eta) \to -\infty$ in this strip [28]. Following Unruh [29], we can therefore find a set of modes that share the vacuum $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ by forming from $\{u_{\omega lm}\}$ and their complex conjugates linear combinations that are analytically continued across the horizons with $\mathrm{Im}(\eta) < 0$, and globally well defined on $\mathbb{RP}^3$dS. We call these modes W-modes.

The construction of the W-modes follows closely that in the Rindler-type spacetime in [9]. We coordinatize $\mathbb{RP}^3$dS by $(\eta, \chi, \theta, \varphi)$ in the sense explained in section 2. The region of $\mathbb{RP}^3$dS covered by the static coordinates $(\sigma, r, \theta, \varphi)$ is then $\chi - \frac{1}{2}\pi < |\eta - \frac{1}{2}\pi|$, and the embedding is given by (2.12). Near the horizon in the static region, $\chi - \frac{1}{2}\pi \to |\eta - \frac{1}{2}\pi|$, $u_{\omega lm}$ is asymptotically proportional to

$$\left\{ \mathrm{e}^{\mathrm{i}\delta_{\omega l}} \left[\tan\left(\tfrac{1}{2}(\eta - \chi)\right)\right]^{-\mathrm{i}\omega/H} + \mathrm{e}^{-\mathrm{i}\delta_{\omega l}} \left[\tan\left(\tfrac{1}{2}(\eta + \chi)\right)\right]^{-\mathrm{i}\omega/H} \right\} Y_{lm}. \tag{4.10}$$

Continuing the asymptotic expression (4.10) past the horizon $\eta = \chi$ into the past region of $\mathbb{RP}^3$dS, in the lower half-plane in η, we obtain

$$\left\{ \mathrm{e}^{\mathrm{i}\delta_{\omega l}}\, \mathrm{e}^{-\pi\omega/H} \left[\tan\left(\tfrac{1}{2}(\chi - \eta)\right)\right]^{-\mathrm{i}\omega/H} + \mathrm{e}^{-\mathrm{i}\delta_{\omega l}} \left[\tan\left(\tfrac{1}{2}(\chi + \eta)\right)\right]^{-\mathrm{i}\omega/H} \right\} Y_{lm}. \tag{4.11}$$

In order to have the asymptotic form of a mode that is globally well defined in the past region, one needs to add to (4.11) its image under (2.15), which is

$$(-1)^l \left\{ \mathrm{e}^{\mathrm{i}\delta_{\omega l}}\, \mathrm{e}^{-\pi\omega/H} \left[\tan\left(\tfrac{1}{2}(\chi + \eta)\right)\right]^{\mathrm{i}\omega/H} + \mathrm{e}^{-\mathrm{i}\delta_{\omega l}} \left[\tan\left(\tfrac{1}{2}(\chi - \eta)\right)\right]^{+\mathrm{i}\omega/H} \right\} Y_{lm}. \tag{4.12}$$

Continuing the sum of (4.11) and (4.12) back to the static region, matching the asymptotic form to a linear combination from the set $\{u_{\omega lm}\}$, and normalizing, we recover the modes

$$W_{\omega lm} := \frac{1}{\sqrt{2\sinh(\pi\omega/H)}} \left(\mathrm{e}^{\pi\omega/2H}\, u_{\omega lm} + \mathrm{e}^{-\pi\omega/2H}\, \overline{u_{\omega l(-m)}} \right). \tag{4.13}$$

A continuation to and from the future region of $\mathbb{RP}^3$dS instead of the past region is similar and leads again to (4.13). The set $\{W_{\omega lm}\}$ provides the desired complete orthonormal set of W-modes.

The quantized field can be expanded in terms of the W-modes as

$$\phi = \sum_{lm} \int_0^\infty \mathrm{d}\omega \left(d_{\omega lm} W_{\omega lm} + d^\dagger_{\omega lm} \overline{W_{\omega lm}} \right), \tag{4.14}$$

where $d_{\omega lm}$ and $d^\dagger_{\omega lm}$ are, respectively, the annihilation and creation operators associated with the mode $W_{\omega lm}$. The vacuum of the W-modes is by construction $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$,

$$d_{\omega lm}|0_{\mathbb{RP}^3\mathrm{E}}\rangle = 0. \tag{4.15}$$

Comparing the expansions (4.8) and (4.14), and using (4.13), we see that the Bogoliubov transformation between the operators reads

$$b_{\omega lm} = \frac{1}{\sqrt{2\sinh(\pi\omega/H)}} \left(\mathrm{e}^{\pi\omega/2H}\, d_{\omega lm} + \mathrm{e}^{-\pi\omega/2H}\, d^\dagger_{\omega l(-m)} \right). \tag{4.16}$$

Suppressing ω and l, and proceeding as in [9], we obtain

$$\begin{aligned} |0_{\mathbb{RP}^3\mathrm{E}}\rangle = {} & \frac{1}{\sqrt{\cosh(r_\omega)}} \left(\sum_{q=0}^{\infty} \frac{(2q-1)!!\exp(-\pi\omega q/H)}{\sqrt{(2q)!}} |2q\rangle_0 \right) \\ & \times \prod_{m>0}^{\infty} \left(\frac{1}{\cosh(r_\omega)} \sum_{q=0}^{\infty} \exp(-\pi\omega q/H) |q\rangle_m |q\rangle_{(-m)} \right), \end{aligned} \tag{4.17}$$

where

$$\tanh(r_\omega) := \exp(-\pi\omega/H), \tag{4.18}$$

and $|q\rangle_m$ denotes the normalized state with q excitations in the static mode labelled by m (and the suppressed indices ω and l),

$$|q\rangle_n := (q!)^{-1/2} \left(b^\dagger_m \right)^q |0_{\mathrm{BdS}}\rangle. \tag{4.19}$$

The notation in (4.17) is adapted to the tensor product structure of the Hilbert space over the modes: the state $|q\rangle_m |q\rangle_{(-m)}$ contains q excitations both in the mode m and in the mode $-m$. The vacuum $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ therefore contains excitations with $m \neq 0$ in pairs whose members only differ in the sign of m.

For generic operators with support in the static region, or even with support only on the inertial trajectory at $r = 0$, the expectation values in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ are clearly not thermal. However, suppose that $\hat{A}$ is an operator with support in the static region, such that $\hat{A}$ does not couple to the modes $u_{\omega lm}$ with $m = 0$, and for each triplet (ωlm) with $m \neq 0$, $\hat{A}$ only couples to one of the modes $u_{\omega lm}$ and $u_{\omega l(-m)}$. It is easily seen from (4.17), as in the Rindler analysis in [9], that the expectation values of $\hat{A}$ are thermal in the temperature $T = H/(2\pi)$.

The mode functions $u_{\omega lm}$ are unlocalized in σ. However, it is straightforward to adapt the above analysis to wavepackets localized partially in both σ and ω, as in the Rindler case discussed in [9]. In the static region, one find thermal expectation values in the temperature $T = H/(2\pi)$ for any operator whose support is localized at asymptotically early or late values of σ.

5. Inertial particle detector in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$

We now turn to the experiences of an inertial monopole particle detector [14, 29–31] in $\mathbb{RP}^3$dS in the vacuum $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$. As $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ is invariant under the isometries of $\mathbb{RP}^3$dS, we can without loss of generality consider the detector trajectory whose one preimage on the hyperboloid (2.1) is

$$U = H^{-1}\sinh(H\tau)\cosh\gamma, \tag{5.1a}$$
$$V = H^{-1}\cosh(H\tau), \tag{5.1b}$$
$$Z = H^{-1}\sinh(H\tau)\sinh\gamma, \tag{5.1c}$$
$$X = 0, \tag{5.1d}$$
$$Y = 0, \tag{5.1e}$$

where γ is a non-negative parameter and τ is the proper time along the trajectory. Geometrically, γ is the hyperbolic angle between the trajectory at $\tau = 0$ and the normal to the spacelike hypersurface belonging to the distinguished foliation of $\mathbb{RP}^3$dS. For $\gamma = 0$, the trajectory is orthogonal to the distinguished foliation at all τ. For $\gamma > 0$, the trajectory is nowhere orthogonal to this foliation, but it becomes asymptotically orthogonal as $|\tau| \to \infty$. We shall consider on a par both this trajectory in $\mathbb{RP}^3$dS and the well known case of the trajectory (5.1) in dS.

In first-order perturbation theory, the probability for the detector becoming excited is [14, 29–31]

$$c^2 \sum_{E>0} |\langle\langle E|\boldsymbol{m}(0)|0\rangle\rangle|^2 \mathcal{F}(E), \tag{5.2}$$

where c is the coupling constant, $\boldsymbol{m}(\tau)$ is the detector's monopole moment operator, $|0\rangle\rangle$ is the ground state of the detector, the sum is over all the excited states $|E\rangle\rangle$ of the detector, and the detector response function $\mathcal{F}(E)$ is given by

$$\mathcal{F}(E) := \int d\tau \int d\tau'\, e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau')). \tag{5.3}$$

For the trajectory (5.1) in dS, equation (3.6) yields

$$G^+_{\mathrm{dS}}(x(\tau), x(\tau')) = AH^2 \tilde{F}\big(\cosh^2[H(\tau-\tau')/2 - i\epsilon]\big). \tag{5.4}$$

For the trajectory in $\mathbb{RP}^3$dS, equations (3.6) and (3.10) give

$$G^+_{\mathbb{RP}^3\mathrm{dS}}(x(\tau), x(\tau')) = G^+_{\mathrm{dS}}(x(\tau), x(\tau')) + \Delta G^+(\tau, \tau'), \tag{5.5}$$

where

$$\Delta G^+(\tau, \tau') = AH^2 \tilde{F}\left(\tfrac{1}{2}[1 + \tilde{\mathcal{Z}}_\epsilon(\tau, \tau')]\right) \tag{5.6}$$

with

$$\tilde{\mathcal{Z}}_\epsilon(\tau, \tau') := -\cosh[H(\tau+\tau')] - 2\sinh^2\gamma\ \sinh(H\tau)\sinh(H\tau') - i\epsilon(\tau-\tau'). \tag{5.7}$$

For $\gamma = 0$, the imaginary part in (5.7) can be dropped, as the argument of $\tilde{F}$ in (5.6) is then always negative: the geometrical reason is that in this case the trajectory (5.1) in dS and its image under J have a spacelike separation, which guarantees that $\Delta G^+(\tau, \tau')$ is necessarily nonsingular. For $\gamma > 0$, on the other hand, the imaginary part in (5.7) is needed to specify the singularity structure in $\Delta G^+(\tau, \tau')$ when the argument of $\tilde{F}$ in (5.6) takes the value 1.

Consider now the familiar case of the detector (5.1) in dS. $G^+_{\mathrm{dS}}(x(\tau), x(\tau'))$ (equation (5.4)) is independent of γ, and it depends on τ and τ' only through the difference

$\tau - \tau'$, as the case must be by the invariance of $|0_{\rm E}\rangle$ under the connected isometries of dS. If the detector is adiabatically turned on in the asymptotic past and off in the asymptotic future, the total response function $\mathcal{F}_{\rm dS}(E)$ is infinite, which reflects the fact that the excitation rate is constant and nonvanishing along the trajectory: the excitation rate in unit proper time is recovered by leaving out one of the integrals in (5.3). For $\tilde{\mu}^2 H^{-2} = 2$, one recovers for the excitation rate the Planckian result at the temperature $T = H/(2\pi)$ [14],

$$\frac{\mathcal{F}_{\rm dS}(E)}{(\text{unit proper time})} = \frac{E}{2\pi\left(e^{2\pi E/H} - 1\right)}. \tag{5.8}$$

Consider then the detector in $\mathbb{RP}^3$dS. As $\Delta G^+(\tau, \tau')$ (equation (5.6)) depends on τ and τ' not only through the difference $\tau - \tau'$ but also through the individual values, the excitation probability per unit proper time is not a constant along the trajectory, and this probability also depends on γ. The detector therefore senses the distinction between the vacua $|0_{\rm E}\rangle$ and $|0_{\mathbb{RP}^3{\rm E}}\rangle$, and it also senses its velocity with respect to the distinguished foliation of $\mathbb{RP}^3$dS. However, if τ and τ' are both large and positive, or if they are both large and negative, $\tilde{\mathcal{Z}}_\epsilon(\tau, \tau')$ (equation (5.7)) is large and negative, and $\Delta G^+(\tau, \tau')$ tends to zero as $|\tau + \tau'| \to \infty$ [17]. In the asymptotic future, or in the asymptotic past, the detector therefore responds as in $|0_{\rm E}\rangle$. For $\gamma = 0$, this is the result one would have expected from the Bogoliubov transformation of section 4.

6. Summary and discussion

We have shown that the Euclidean vacua of a free scalar field on the spacetimes dS and $\mathbb{RP}^3$dS are distinguishable to an inertial observer who couples to the field through a monopole detector, or to an observer who can measure the field stress–energy tensor. In the special case of an inertial observer whose worldline on $\mathbb{RP}^3$dS is orthogonal to the distinguished foliation, we arrived at a similar conclusion by constructing the Bogoliubov transformation between the modes that define the Euclidean vacuum and the modes that are of positive frequency with respect to the observer's natural time coordinate. However, we also saw that the differences between dS and $\mathbb{RP}^3$dS become exponentially small in the distant past or future on an inertial observer worldline, and in these limits the observer thus sees the Euclidean vacuum on $\mathbb{RP}^3$dS as a thermal bath in the usual de Sitter Hawking temperature. This result conforms to the central tenet of inflationary cosmology, namely, that the physics in an exponentially expanding spacetime should become indistinguishable from physics in de Sitter space exponentially fast: what falls off exponentially in our case are the effects of the unconventional spatial topology on the quantum field.

While our particle detector analysis accommodated an arbitrary inertial observer in $\mathbb{RP}^3$dS, we only performed the Bogoliubov transformation for an inertial observer whose worldline is orthogonal to the distinguished foliation. As any inertial worldline in $\mathbb{RP}^3$dS has a neighbourhood covered by the static metric (2.10), such that the worldline is at $r = 0$, the case of a nonorthogonal trajectory would also be in principle amenable to a Bogoliubov transformation analysis. One would expect the correlations in the counterpart of (4.17) to be more complicated for a nonorthogonal trajectory, but one would expect a wavepacket analysis to also show thermality in the limit of early and late times in this case. Finding the Bogoliubov transformation explicitly for a nonorthogonal trajectory is, however, more difficult, and we shall not pursue this question further here.

Yet another way to investigate the experiences of an inertial observer in $\mathbb{RP}^3$dS is through the complex analytic properties of the Feynman propagator in $|0_{\mathbb{RP}^3{\rm E}}\rangle$. As mentioned in section 3, $\mathbb{RP}^3$dS can be regarded as a Lorentzian section of a complex spacetime whose

Riemannian section is a certain $\mathbb{Z}_2$ quotient of the round 4-sphere, and the Feynman propagator in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ continues to the unique Green function on this Riemannian section. A set of coordinates covering the Riemannian section can be obtained from the static coordinates $(\sigma, r, \theta, \varphi)$ on Q_+ by setting $\sigma = -\mathrm{i}\tilde{\sigma}$, provided the coordinates $(\tilde{\sigma}, r, \theta, \varphi)$ are identified as

$$(\tilde{\sigma}, r, \theta, \varphi) \sim (\tilde{\sigma} + 2\pi/H, r, \theta, \varphi) \sim (\pi/H - \tilde{\sigma}, r, \pi - \theta, \varphi + \pi). \tag{6.1}$$

The first identification in (6.1) is just as for dS, and this identification implies for the Feynman propagator in $|0_{\mathrm{E}}\rangle$ complex analytic properties that correspond to thermality in the de Sitter Hawking temperature $H/(2\pi)$ [32]. The second identification in (6.1) is specific to $\mathbb{RP}^3$dS. One can argue that the complex analytic properties of the Feynman propagator in $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$ are consistent with thermality in the limit of asymptotically early and late proper times: the reasoning is similar to that given for the $\mathbb{RP}^3$ geon in [9], and we shall not spell out the detail here.

The action of the Riemannian section of $\mathbb{RP}^3$dS is half of the action of the Riemannian section of dS. If one uses these actions in a semiclassical estimate to a quantum gravitational partition function, and if one associates to $\mathbb{RP}^3$dS the de Sitter Hawking temperature $H/(2\pi)$, one finds for the entropy of $\mathbb{RP}^3$dS the result $2\pi H^{-2}$, which is half of the value obtained for dS [33]. An analogous observation was made in [9] for the entropy of the $\mathbb{RP}^3$ geon. Although the entropy associated with cosmological horizons may be physically less clear than the entropy associated with black hole horizons, it should prove interesting to understand whether this naive instanton-method evaluation of the entropy of $\mathbb{RP}^3$dS could be physically justified, and, in particular, whether the factor of one half relative to dS might also arise in any state-counting approach to the entropy.

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Appendix. Stress–energy tensor via a conformal transformation

In this appendix we verify the result (3.12) for the stress–energy tensor of a massless conformally coupled field by the conformal technique of Parker [34] and zeta-function regularization.

To begin, recall [34] that any covariantly conserved symmetric tensor $K^{\mu\nu}$ in a spacetime with a conformal Killing vector† ξ^ν satisfies the relation

$$\nabla_\mu (K^{\mu\nu}\xi_\nu) = \tfrac{1}{4}\nabla_\nu \xi^\nu K^\mu_\mu. \tag{A.1}$$

Integration of (A.1) over a compact spacetime region M with spacelike boundary ∂M yields

$$\int_{\partial M} \mathrm{d}^3x\, h^{1/2} K^{\mu\nu}\xi_\nu n_\mu = \tfrac{1}{4}\int_M \mathrm{d}^4x\, (-g)^{1/2} \nabla_\nu \xi^\nu K^\mu_\mu, \tag{A.2}$$

where n_μ is the outward unit normal form on ∂M. If both the spacetime and $K^{\mu\nu}$ are spatially homogeneous, we can choose ∂M to consist of two homogeneous spatial hypersurfaces, and the spatial integration in (A.2) then factors out on both sides of the equation. One recovers a relation that relates the projection of $K^{\mu\nu}$ orthogonal to the homogeneity hypersurfaces to K^μ_μ.

† A conformal Killing vector in four spacetime dimensions satisfies $\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \frac{1}{2}\nabla_\rho\xi^\rho g_{\mu\nu}$.

We apply the above to the spacetimes dS and $\mathbb{RP}^3$dS, for both of which the metric takes the form (2.5). For the conformal Killing vector ξ^μ, we choose $\partial_\eta = H^{-1}\cosh(Ht)\partial_t$, for which $\nabla_\nu \xi^\nu = 4\sinh(Ht)$. In the coordinates $(t, \chi, \theta, \varphi)$, we obtain

$$\cosh^4(Ht_f)K^{00}(t_f) - \cosh^4(Ht_i)K^{00}(t_i) = -H\int_{t_i}^{t_f} \mathrm{d}t\, \cosh^3(Ht)\sinh(Ht)K^\mu_\mu(t). \tag{A.3}$$

We wish to use the relation (A.3) to determine the difference of the renormalized stress–energy tensors in the vacua $|0_{\mathrm{E}}\rangle$ and $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$. We denote these tensors, respectively, by $\langle T^{\mu\nu}_{S^3}\rangle$ and $\langle T^{\mu\nu}_{\mathbb{RP}^3}\rangle$. Both tensors are covariantly conserved and invariant under the isometries of the respective spacetimes. Further, we can use the projection from dS to $\mathbb{RP}^3$dS to map $\langle T^{\mu\nu}_{S^3}\rangle$ into a tensor on $\mathbb{RP}^3$dS. By the usual abuse of notation, we denote also this tensor on $\mathbb{RP}^3$dS by $\langle T^{\mu\nu}_{S^3}\rangle$. Then $\Delta T^{\mu\nu} := \langle T^{\mu\nu}_{\mathbb{RP}^3}\rangle - \langle T^{\mu\nu}_{S^3}\rangle$ is a well defined, covariantly conserved tensor on $\mathbb{RP}^3$dS, and it fully characterizes the differences in the stress–energy tensors in $|0_{\mathrm{E}}\rangle$ and $|0_{\mathbb{RP}^3\mathrm{E}}\rangle$. Equation (A.3) hence holds with $K^{\mu\nu} = \Delta T^{\mu\nu}$.

We now specialize to the conformally coupled massless field, $\mu = 0$ and $\xi = \frac{1}{6}$. As the divergences in the trace of the renormalized stress energy tensor are purely local and anomalous, they are determined entirely by the local geometry. These contributions are the same for $\mathbb{RP}^3$dS and dS; thus $\Delta T^\mu_\mu = 0$. Equation (A.3) with $K^{\mu\nu} = \Delta T^{\mu\nu}$ then implies

$$\Delta T^{00}(t) = \frac{C}{\cosh^4(Ht)}, \tag{A.4}$$

where C is a constant. Together with the tracelessness and symmetries of $\Delta T^{\mu\nu}$, this implies

$$\Delta T^\mu{}_\nu = \frac{C}{\cosh^4(Ht)}\,\mathrm{diag}\big(-1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\big). \tag{A.5}$$

Note that ΔT^{00} behaves as if it were classical radiation. In particular, it redshifts exponentially to zero at large times t.

To evaluate the constant C, and, in particular, to show that it is nonzero†, we employ a conformal transformation technique. Observe [36] that ΔT^{00} is entirely due to the nongeometrical contribution from the conformal vacuum, i.e. that reflecting the boundary conditions on the state set by the topology rather than that contributed from the anomalous trace. One can therefore compute ΔT^{00} by first finding the corresponding quantity, $\Delta\tilde{T}^{00}$, in suitable conformally related spacetimes and then performing a conformal transformation: from equation (6.129) in [36], this transformation reads

$$\Delta T^\nu_\mu = \left(\frac{\tilde{g}}{g}\right)^{1/2} \Delta\tilde{T}^\nu_\mu, \tag{A.6}$$

where g and $\tilde{g}$ are the determinants of the conformally related metrics.

Suitable spacetimes conformally related to dS and $\mathbb{RP}^3$dS are, respectively, the usual Einstein static universe, with spatial topology S^3, and the Einstein static universe with spatial topology $\mathbb{RP}^3$. Their metrics are obtained by multiplying the metric (2.8), on, respectively, dS and $\mathbb{RP}^3$dS, by $\sin^2\eta$. For the ordinary Einstein static universe with curvature radius c, the energy density is $\rho_{S^3} = 1/(480\pi^2 c^4)$ [37, 38], and a derivation of this result by zeta-function regularization methods [39] is given in [40]. For the Einstein static universe with spatial topology $\mathbb{RP}^3$, we adapt the zeta-function calculation of [40], noting that among the hyperspherical harmonics $\{Q_{nlm}\}$ on the round S^3, those that project to the round $\mathbb{RP}^3$ are

† This point is nontrivial: see Kennedy and Unwin [35] for an example where changing the boundary conditions on states does not change the energy density.

precisely those whose principal quantum number n is odd [21]. The regularized expression for the total energy on the spacelike hypersurfaces of the Einstein static universe with spatial topology $\mathbb{RP}^3$ and curvature radius c reads thus

$$E(s)_{\mathbb{RP}^3} = \frac{1}{2} \sum_{\substack{n \geqslant 1 \\ n \text{ odd}}} n^2 (n/c)^{-s} = \frac{1}{2} c^s \big(1 - 2^{2-s}\big) \zeta(s-2), \tag{A.7}$$

where s is the regularization parameter and ζ is the Riemann zeta-function. Taking $s = -1$ and dividing by the spatial volume $\pi^2 c^3$ yields the energy density $\rho_{\mathbb{RP}^3} = -7/(240\pi^2 c^4)$. Taking the difference between $\rho_{\mathbb{RP}^3}$ and ρ_{S^3}, we obtain

$$\Delta \tilde{T}^{00} = -\frac{1}{32\pi^2 c^4}. \tag{A.8}$$

From (A.6) and (A.8) we thus find

$$\Delta T^{00}(t) = -\frac{H^4}{32\pi^2 \cosh^4(Ht)}, \tag{A.9}$$

which is (A.4) with $C = -H^4/(32\pi^2)$. With this value of C, the expression (A.5) agrees with the result (3.12) obtained in the main text by point-splitting methods.

If the field is not conformally coupled and massless, ΔT^{μ}_{μ} need not vanish. It would be possible to obtain partial information about ΔT^{00}, in particular, about its falloff as $|t| \to \infty$, by first computing ΔT^{μ}_{μ} via point-split methods and then applying (A.3). This calculation would, however, not be substantially simpler than the full point-splitting evaluation of $\Delta T^{\mu\nu}$.

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