Conformal hyperbolic formulation of the Einstein equations

Miguel Alcubierre,[1] Bernd Brügmann,[1] Mark Miller,[2] and Wai-Mo Suen[2,3]
[1]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Schlaatzweg 1, D-14473, Potsdam, Germany*
[2]*McDonnell Center for the Space Sciences, Department of Physics, Washington University, St. Louis, Missouri 63130*
[3]*Physics Department, Chinese University of Hong Kong, Hong Kong*
(Received 8 March 1999; published 23 August 1999)

We propose a reformulation of the Einstein evolution equations that cleanly separates the conformal degrees of freedom and the nonconformal degrees of freedom with the latter satisfying a first order strongly hyperbolic system. The conformal degrees of freedom are taken to be determined by the choice of slicing and the initial data, and are regarded as given functions (along with the lapse and the shift) in the hyperbolic part of the evolution. We find that there is a two parameter family of hyperbolic systems for the nonconformal degrees of freedom for a given set of trace free variables. The two parameters are uniquely fixed if we require the system to be ''consistently trace-free,'' i.e., the time derivatives of the trace free variables remain trace-free to the principal part, even in the presence of constraint violations due to numerical truncation error. We show that by forming linear combinations of the trace free variables a conformal hyperbolic system with only physical characteristic speeds can also be constructed. [S0556-2821(99)05516-2]

PACS number(s): 04.25.Dm

I. INTRODUCTION

With the advent of large amounts of observational data from high-energy astronomy and gravitational wave astronomy, general relativistic astrophysics—astrophysics involving gravitational fields so strong and dynamical that the full Einstein field equations are required for its accurate description—is emerging as an exciting research area. This calls for an understanding of the Einstein theory in its nonlinear and dynamical regime, in order to study the physics of general relativistic events in a realistic astrophysical environment. This in turn calls for solving the full set of Einstein equations numerically. However, the complicated set of partial differential equations present major difficulties in all of these three tightly coupled areas: the understanding of its mathematical structure, the derivation of its physical consequences, and its numerical solution. The difficulties have attracted a lot of recent effort, including two ''Grand Challenge'' [1,2] efforts on the numerical studies of black holes and neutron stars, respectively.

One major obstacle in solving the Einstein equations numerically is that we lack a complete understanding of the mathematical structure of the Einstein equations. The difficulties in numerically integrating the Einstein equations in a stable fashion have motivated intense effort in rewriting the Einstein equations into a form that is explicitly well-posed [3–18] (for an excellent overview, see [19]). The main idea has been rewriting the six space-space components of the Einstein equations into a first order hyperbolic system. These space-space parts of the Einstein equations are dynamical evolution equations, while the time-time and space-time parts of the Einstein equations are (elliptic) constraint equations.

The central question we raise in this communication is: In order to enable an accurate and stable numerical integration of the full set of the Einstein equations, what part of the system should be taken to form a hyperbolic system?

Our question is motivated by two observations. First, there are many recent proposals on reformulating the six space-space parts of the Einstein equations into a first order hyperbolic system [8–18]. Three of the hyperbolic formulations have been coded up for numerical treatment (to the best of our knowledge), namely, the York *et al.* formulation [9,14,16] (see, e.g., [20,21]), the Bona-Masso formulation [15] (see e.g., [17]), and Friedrich's formulation [3,4,22–26] (which is rather different from the first two formulations in its use of a global conformal transformation of the four-metric to compactify hyperboloidal slices). However, in all these cases, the numerical integration of the first order hyperbolic system consisting of the six space-space components of the Einstein equations so far have not lead to a substantial improvement over those using the traditional Arnowitt-Deser-Misner (ADM) [27] evolution equations. This is despite the original hope that the well-posedness of the hyperbolic formulations leads to an immediate numerical advantage.

The second observation is that there have been various attempts in rewriting the traditional ADM form of the evolution equations by separating out the conformal degree of freedom, beginning with Nakamura *et al.* [28] (see references cited therein). Lately this has received much attention with [29] reporting that a variant of the approach leads to highly stable numerical evolutions. A detailed study of the approach using gravitational wave systems carried out by our group [30] confirmed that the approach has advantages over the standard ADM formulation. We find that the approach yields results with accuracy comparable to that obtained by the standard ADM formulation with the K-driving technique [31] for weak to medium waves, and has better stability properties especially in the case of strong fields that needs high resolution with ADM [32] (see also [33]).

These two observations motivated us to study the possibility of a formulation that separates out the conformal degree of freedom in the 6 evolution equations, while requiring the remaining 5 equations governing the non-conformal degrees of freedom to form a first order hyperbolic system.

A recap of the various components of the Einstein equations is in order for a clearer discussion of our approach. In the standard ADM 3+1 formulation, the Einstein equations are broken into (a) the Hamiltonian constraint equation (the time-time part)

$$H = {}^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi\rho_{ADM} = 0, \tag{1}$$

(b) the 3 momentum constraint equations (the time-space part)

$$H^i = \nabla_j K^{ij} - \gamma^{ij}\nabla_j K - 8\pi j^i = 0, \tag{2}$$

where $\rho_{ADM}, j^i, S_{ij}, S = g^{ij}S_{ij}$ are the components of the stress energy tensor projected onto the 3-space, and (c) the 6 evolution equations (the space-space part) given as 12 first order equations

$$\partial_{\hat{t}} g_{ij} = -2\alpha K_{ij}, \tag{3}$$

$$\partial_{\hat{t}} K_{ij} = -\nabla_i\nabla_j\alpha + \alpha\bigg[R_{ij} + KK_{ij} - 2K_{im}K^m_j - 8\pi\left(S_{ij} - \frac{1}{2}g_{ij}S\right) - 4\pi\rho_{ADM}g_{ij}\bigg], \tag{4}$$

where ∇_i denotes a covariant derivative with respect to the 3-metric g_{ij}, $\partial_{\hat{t}}$ stands for $\partial_t - \mathcal{L}_\beta$ with $\mathcal{L}_\beta$ being the Lie derivative with respect to β^i, and R_{ij} is the Ricci curvature of the 3-metric. In the ADM formulation, Eqs. (3),(4) are used to evolve the 12 variables $\{K_{ij}, g_{ij}\}$ forward in time for given lapse α and shift vector β^i. The constraint equations are automatically satisfied if $\{K_{ij}, g_{ij}\}$ satisfy them on the initial time slice. However, in numerical evolutions the constraints will be violated due to truncation error. One major difficulty in numerical relativity is that the constraint violations often drive the development of instabilities, at least in the case of numerical evolution using the standard ADM equations (3),(4).

In the hyperbolic reformulations of the evolution equations [9–12,14–17], one makes use of the constraint equations (1),(2), and introduces additional variables (e.g., $d_{ijk} = g_{ij,k}$ or its linear combinations) to cast Eqs. (3),(4) into a first order strongly hyperbolic system (often the symmetric hyperbolic subclass). (More variables would have to be introduced for formulations involving higher derivatives [9,11].) However, we note that hyperbolicity is often shown only under the assumption that some of the variables involved in the evolution equations, in particular the lapse α and the shift β^j, are considered as given functions of space and time. In actual numerical evolutions with no predetermined choice of spacetime coordinates, α and β^j have to be given in terms of the variables $\{K_{ij}, g_{ij}, d_{ijk}\}$ (e.g., α, β^j determined in a set of elliptic equations involving $\{K_{ij}, g_{ij}, d_{ijk}\}$). In the Bona-Masso formulation [15], the lapse can be part of the hyperbolic system for some choice of slicings (while the inclusion of the shift into the hyperbolic system severely restricts the class of applicable shifts). In [9,11], in addition to the lapse and the shift, the trace of the extrinsic curvature, $K = g^{ij}K_{ij}$, is also regarded as a given function (K is used to specify the slicing, e.g., $K=0$ for maximal slicing). The point we want to bring out here is that in all of the existing hyperbolic reformulations of the Einstein evolution equations, part of the quantities $\{K_{ij}, g_{ij}, d_{ijk}, \alpha, \beta^j\}$ are considered to be given, while others are evolved using hyperbolic equations.

In the following, we present a formulation in which the nonconformal degrees of freedom are separated out for hyperbolic evolution.

II. FORMULATION

For the evolution of the three-geometry, the conformal degree of freedom is represented by g (the determinant of the spatial 3-metric g_{ij}), its spatial derivative $g_{,i}$ and its time derivative K [$K = -1/(2g\alpha)\partial_{\hat{t}} g$]. For the non-conformal degrees of freedom, we define

$$\tilde{g}_{ij} = g_{ij}/g^{1/3}, \tag{5}$$

$$\tilde{A}_{ij} = \left(K_{ij} - \frac{1}{3}g_{ij}K\right) \bigg/ g^{1/3}, \tag{6}$$

$$\tilde{D}^{ij}{}_k = \tilde{g}^{ij}{}_{,k}. \tag{7}$$

$\tilde{g}_{ij}$ has unit determinant, and $\tilde{A}_{ij}$ is the rescaled trace-free part of K_{ij}. All indices of tilde quantities are raised and lowered with the conformal 3-metric $\tilde{g}_{ij}$. We note that $\tilde{D}^{ij}{}_k$ is trace-free with respect to the indices (i,j). We take $\{\tilde{g}^{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$, or their covariant component counterparts, to represent the nonconformal degrees of freedom.

In the following we develop a first order hyperbolic system for the nonconformal degrees of freedom, under the simplifying assumption that the 5 conformal degrees of freedom $\{g, g_{,i}, K\}$ and the gauge choice functions $\{\alpha, \beta^i\}$ can be regarded as given functions of space and time. Note that these variables cannot be specified independently of each other. A concrete example is that of maximal slicing, $K=0$, and vanishing shift, $\beta^i=0$, in which case both g and $g_{,i}$ are part of the initial data (time independent), and are therefore truely given functions in the numerical evolution. In other cases, with K given to specify the slicing, it involves a nontrivial time integration to determine g (from the definition of K in terms of the time derivative of g).

We now discuss hyperbolicity of the evolution of the nonconformal variables, $\{\tilde{g}_{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$, by examining the principal part of the evolution equations, which is the part that decides about strong hyperbolicity of the system [34]. To obtain the principal part we drop all terms that can be expressed by (1) the variables $\{\tilde{g}^{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$ themselves, and (2) spacetime functions that are regarded as given, i.e. $\{\alpha, \beta^i, g, g_{,i}, K\}$ and their space and time derivatives. We have

$$\partial_{\hat{t}} \tilde{g}_{ij} \approx 0, \tag{8}$$

$$\partial_{\hat{t}} \tilde{D}^{ij}{}_k \approx 2\alpha \tilde{A}^{ij}{}_{,k}, \tag{9}$$

$$\partial_{\hat{t}}\tilde{A}^{ij}\approx \alpha g^{1/3}\left(R^{ij}-\frac{1}{3}g^{ij}R\right), \tag{10}$$

where $\approx$ represents ''equal up to principal part,'' and where for the evolution equation of $\tilde{D}^{ij}{}_k$ we have used that spatial derivatives ∂_i and the time derivative $\partial_{\hat{t}}$ commute.

To evaluate R^{ij} and R in Eq. (10), we use

$$R^{ij}\approx g^{-2/3}\tilde{R}^{ij} \tag{11}$$

$$\approx\frac{1}{2}g^{-2/3}(\tilde{g}^{kl}\tilde{D}^{ij}{}_{k,l}-\tilde{g}^{il}\tilde{D}^{jk}{}_{l,k}-\tilde{g}^{jl}\tilde{D}^{ik}{}_{l,k}), \tag{12}$$

where the relation

$$g_{kl}g^{kl}{}_{,i}=-g_{,i}/g\approx 0, \tag{13}$$

and the spatial derivatives of it have been used. We obtain

$$\partial_{\hat{t}}\tilde{A}^{ij}\approx\frac{1}{2}\alpha g^{-1/3}\left(\tilde{g}^{kl}\tilde{D}^{ij}{}_{k,l}-\tilde{g}^{il}\tilde{D}^{jk}{}_{l,k}-\tilde{g}^{jl}\tilde{D}^{ik}{}_{l,k}+\frac{2}{3}\tilde{g}^{ij}\tilde{D}^{kl}{}_{k,l}\right). \tag{14}$$

To make the nonconformal system strongly hyperbolic, one can add a combination of the momentum constraint to Eq. (9). To principal part the momentum constraint (2) is $H^i\approx g^{-1/3}\tilde{A}^{ij}{}_{,j}$. We obtain

$$\partial_{\hat{t}}\tilde{D}^{ij}{}_k\approx 2\alpha\tilde{A}^{ij}{}_{,k}-2\alpha g^{1/3}(\tilde{g}^i_kH^j+\tilde{g}^j_kH^i) \tag{15}$$

$$\approx 2\alpha(\tilde{A}^{ij}{}_{,k}-\tilde{g}^i_k\tilde{A}^{jl}{}_{,l}-\tilde{g}^j_k\tilde{A}^{il}{}_{,l}). \tag{16}$$

An energy norm can be constructed for the system

$$E=\int\tilde{g}^{ij}\tilde{g}_{ij}+\tilde{A}^{ij}\tilde{A}_{ij}+\frac{1}{4}g^{-1/3}\tilde{D}^{ij}{}_k\tilde{D}_{ij}{}^k. \tag{17}$$

It is straightforward to demonstrate using Eqs. (8), (14), and (16) that ∂_tE is a total derivative up to terms that can be expressed by the variables $\{\tilde{g}^{ij},\tilde{A}^{ij},\tilde{D}^{ij}{}_k\}$ themselves. One can also show directly that the characteristic metric of the system (8), (14), and (16) has a complete set of eigenvectors with real eigen values. The system is similar to but *not* contained in the one parameter family of the hyperbolic systems constructed in [10].

Next we go one step beyond hyperbolicity. We make the following observations.

(i) Since $\tilde{A}^{ij}$ and $\tilde{D}^{ij}{}_k$ are trace-free, one can add a term $\epsilon_1\alpha g^{-1/3}\tilde{g}^{ij}H$ to Eq. (14), and a term $\epsilon_2\alpha\tilde{g}^{ij}H_k$ to Eq. (16) without affecting the hyperbolicity. We have therefore a two parameter family of hyperbolic evolution equations (without making a variable change).

(ii) With these two terms added respectively to Eqs. (14) and (16), the trace of the principle parts of the RHS's of the equations are $3\epsilon_1\alpha\tilde{D}^{ks}{}_{k,s}$ (proportional to the principal part of the Hamiltonian constraint), and $\alpha(3\epsilon_2-4)\tilde{A}_k{}^l{}_{,l}$ (proportional to the principal part of the momentum constraint), respectively. On the other hand, the LHS of the equations, $\partial_{\hat{t}}\tilde{A}^{ij}$ and $\partial_{\hat{t}}\tilde{D}^{ij}{}_k$ are trace-free to the principal order. This means that truncation error in the numerical evolution which leads to a violation of the constraints will drive $\tilde{A}^{ij}$ and $\tilde{D}^{ij}{}_k$ to evolve away from being trace-free, even up to the principal order.

(iii) We therefore propose to fix the freedom in the parameters ϵ_1 and ϵ_2 by requiring the system to be ''consistently trace-free,'' i.e., $\epsilon_1=0$ and $\epsilon_2=4/3$, so that the equations are trace-free to principal order consistently. Hence Eq. (14) for $\tilde{A}^{ij}$ is left unchanged, but

$$\partial_{\hat{t}}\tilde{D}^{ij}{}_k\approx 2\alpha\tilde{A}^{ij}{}_{,k}-2\alpha g^{1/3}(\tilde{g}^i_kH^j+\tilde{g}^j_kH^i)+\frac{4}{3}\alpha\tilde{g}^{ij}H_k \tag{18}$$

$$\approx 2\alpha\left(\tilde{A}^{ij}{}_{,k}-\tilde{g}^i_k\tilde{A}^{jl}{}_{,l}-\tilde{g}^j_k\tilde{A}^{il}{}_{,l}+\frac{2}{3}\tilde{g}^{ij}\tilde{g}_{km}\tilde{A}^{ml}{}_{,l}\right). \tag{19}$$

The system [Eqs. (8),(14),(19)] forms a strongly hyperbolic system with the same energy norm (17).

(iv) The remaining freedom in constructing conformal-hyperbolic systems that are ''consistently trace-free'' is through forming linear combinations of the variables. There are clearly infinite choices. Here we show for example a linear combination that leads to a system with only physical characteristic speeds, a property advocated by York *et al.*, see e.g., [14]. Equation (14) can be written as

$$\partial_{\hat{t}}\tilde{A}^{ij}\approx(\alpha g^{-1/3})\tilde{g}^{kl}\partial_l\tilde{U}^{ij}{}_k\approx\alpha g^{kl}\partial_l\tilde{U}^{ij}{}_k, \tag{20}$$

where

$$\tilde{U}^{ij}{}_k=\frac{1}{2}\left(\tilde{D}^{ij}{}_k-\tilde{g}^i_k\tilde{D}^{il}{}_l-\tilde{g}^j_k\tilde{D}^{jl}{}_l+\frac{2}{3}\tilde{g}^{ij}\tilde{g}_{km}\tilde{D}^{ml}{}_l\right). \tag{21}$$

We can take $\tilde{U}^{ij}{}_k$ to be our basic nonconformal variables (note $\tilde{g}_{ij}\tilde{U}^{ij}{}_k=0$). Taking the time derivative of $\tilde{U}^{ij}{}_k$ and commuting time and space derivatives leads to

$$\partial_{\hat{t}}\tilde{U}^{ij}{}_k\approx\alpha\left(\tilde{A}^{ij}{}_{,k}-\tilde{g}^i_k\tilde{A}^{jl}{}_{,l}-\tilde{g}^j_k\tilde{A}^{il}{}_{,l}+\frac{2}{3}\tilde{g}^{ij}\tilde{g}_{km}\tilde{A}^{ml}{}_{,l}\right). \tag{22}$$

To make the system strongly hyperbolic, we follow the step leading to Eq. (15) and add the combination of momentum constraints $\alpha g^{1/3}(\tilde{g}^i_kH^j+\tilde{g}^j_kH^i)-2\alpha\tilde{g}^{ij}H_k/3$ to Eq. (22) to arrive at

$$\partial_{\hat{t}}\tilde{U}^{ij}{}_k\approx\alpha\tilde{A}^{ij}{}_{,k}. \tag{23}$$

Equations (20) and (23) form a conformal hyperbolic system for $\{\tilde{U}^{ij}{}_k,\tilde{A}^{ij}\}$ with only physical characteristic speeds. The system can be symmetrized by contracting Eq. (23) with g^{kl}.

III. DISCUSSION AND CONCLUSION

We raise the question of what part of the variables in Einstein theory should be evolved in a hyperbolic fashion in numerical relativity. We propose a reformulation of the Einstein evolution equations that cleanly separates the conformal degrees of freedom $\{g, g_{,i}, K\}$ and the nonconformal degrees of freedom $\{\tilde{g}^{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$ (or their linear combinations), with the latter satisfying a first order strongly hyperbolic system. The conformal degrees of freedom are taken to be determined by the choice of slicings and the initial data, and are regarded as given functions in the hyperbolic part of the evolution equations, along with the lapse and the shift.

We find a two parameter family of nonconformal hyperbolic system for $\{\tilde{g}^{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$. The two parameters are uniquely fixed if we require the system to be ''consistently trace-free,'' i.e., the time derivative of the trace-free variables $\{\tilde{g}^{ij}, \tilde{D}^{ij}{}_k, \tilde{A}^{ij}\}$ remains trace-free to principal part, even in the presence of constraint violations caused by numerical truncation error. We also show that certain linear combinations of the $\tilde{D}^{ij}{}_k$ lead to a conformal hyperbolic system with physical characteristic speed.

This formulation merges two recent trends in re-writing the Einstein evolution equations for numerical relativity: first order hyperbolicity and the separating out of the conformal degrees of freedom. We believe it will lead to many interesting investigations: Given the coordinate conditions, e.g., maximal slicing and an appropriate shift condition, can the combined elliptic hyperbolic system be shown to be well-posed analytically [9,35]? When posted as initial boundary value problem, what are the suitable boundary conditions for stability in numerical evolutions? How will the constraints propagate under this system of conformal-hyperbolic equations? One particularly interesting issue that will be reported on in a followup paper is the stability of this formulation in numerical evolution, and how the stability is related to the slicing conditions (K) one chooses.

ACKNOWLEDGMENTS

This research was supported by the NSF grant Phy 96-00507, NASA HPCC grant NCCS5-153, NRAC grant 93S025, and HKRGC grant CUHK 4189/97P. We thank Carles Bona, Helmut Friedrich, Alan Rendall, and Ed Seidel for discussions.

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