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The geometry of sigma models with twisted supersymmetry

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Abstract

We investigate the relation between supersymmetry and geometry for two-dimensional sigma models with target spaces of arbitrary signature, and Lorentzian or Euclidean world-sheets. In particular, we consider twisted forms of the two-dimensional (p,q) supersymmetry algebra. Superspace formulations of the (p,q) heterotic sigma models with twisted or untwisted supersymmetry are given. For the twisted (2,1) and the pseudo-Kähler sigma models, we give extended superspace formulations. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

In Ref. [1], the analysis of Refs. [2–4] on the geometry of (1,0) and (1,1) supersymmetric sigma models was generalised to the case in which the target space had arbitrary signature, and the conditions for the theory to be invariant under extra supersymmetries were investigated. Covariantly constant complex structures, i.e. (1,1) tensors J satisfying $J^2 = -1$, led to extra supersymmetries, each satisfying the usual superalgebra $Q^2 \sim P$, while covariantly constant real structures, i.e. (1,1) tensors S satisfying $S^2 = 1$, led to extra *twisted* supersymmetries [1], each satisfying the twisted superalgebra $Q^2 \sim -P$. The number of structures of either type depended on the target space holonomy of a certain connection which had torsion if the sigma model had a Wess–Zumino term. For example, if the holonomy is contained in $USp(2m)$, there are three complex structures I, J, K satisfying the quaternion algebra with $SU(2)$ commutation relations, while if the holonomy is contained in $Sp(2m,\mathbb{R})$, there is one complex structure J and two real structures S,T satisfying the pseudo-quaternion algebra with $SU(1,1)$ commutation relations. The aim of this paper is to give the superspace formulation of these models and to investigate their structure further.

Extended world-sheet supersymmetries have had two different uses in string theory. In the study of heterotic or type II strings, complex manifolds such as Calabi–Yau spaces have played an important role. In these cases, the string theory only has gauged

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(1,1) or (1,0) world-sheet supersymmetry, but the (1,1) or (1,0) sigma model on a suitable background can have extra rigid world-sheet supersymmetries; on a Kähler manifold, for example, $N = 1$ world-sheet supersymmmetry is extended to $N = 2$, and $N = 2$ superconformal field theory has played a central role in the study of such compactifications. There are also string theories in which an extended world-sheet supersymmetry is gauged, such as those with $N = 2$ local world-sheet supersymmetry, and in these a target space with either four Euclidean dimensions, or with two space and two time dimensions naturally arises. Our results on general signature have applications to both heterotic or type II strings in general signature [5,6] and to $(2, p)$ strings in $2 + 2$ dimensions [7–10].

In Refs. [5,6], new string theories were found in which the ten-dimensional space-time had arbitrary signature, and in some cases the world-sheet was Lorentzian (signature $1 + 1$), while in others it was Euclidean (signature $2 + 0$). All of these were linked to the usual string theories with target space signature $9 + 1$ and Lorentzian world-sheets by chains of dualities [5,6]. The world-sheet formulation of these string theories is a sigma model with target space of the appropriate signature. Target spaces that admit extra supersymmetries play an important role in the study of solutions of these theories, just as in the case of compactifying on Euclidean signature internal spaces.

Another context in which non-Lorentzian signature target spaces have played a role is in $N = 2$ strings, or more generally in strings with (2,0), (2,1) or (2,2) world-sheet supersymmetry. In these theories, the target spaces had signature $2 + 2$ (or $4 + 0$), and the heterotic theories were reduced (via a null reduction) to ones with signature $1 + 1$ or $2 + 1$. The (2,1) string is of particular interest. It was shown by Kutasov and Martinec 11], and by the same authors with O'Loughlin [12], that different vacua of the (2,1) superstring describe the D1-string or the D2-brane, and via dualities these are linked to all usual types of ten-dimensional superstrings and to the eleven-dimensional supermembrane [11,12]. This led to the suggestion that the (2,1) heterotic string may provide many of the degrees of freedom of M theory, although this approach has so far only yielded specially symmetric points in the moduli space of vacua of M-theory [13,14]. Martinec [15–17] has proposed an interpretation of the (2,1) string as describing the continuum limit of the matrix model of M-theory [18] with all spatial dimensions compactified.

The (2,1) heterotic string [10] has a four-dimensional target space-time with signature (2,2) that is required to have an isometry generated by a null Killing vector, which must be gauged. In general there are obstructions to the gauging of a given isometry [19,20], and the isometry is required to be one for which these are absent. For (2,2) signature, this null reduction yields either a space with signature (2,1) (corresponding to a membrane world-volume [11,12]) or a space with signature (1,1) (corresponding to a string world-sheet [11,12]). The theory defined on a space-time with signature (2,2) before null reduction is a theory of self-dual gravity with torsion coupled to self-dual Yang–Mills gauge fields [10]. The exact classical effective action for the gravitational, antisymmetric tensor and gauge degrees of freedom was given in [14] and derived independently in [1] using sigma-model techniques (see Refs. [13,17] for reviews). In Ref. [21], this action was simplified using an auxiliary metric and shown to be Weyl invariant at the classical level in four dimensions. A dual form of this action was found; in four dimensions, the dual geometry is self-dual gravity without torsion coupled to a scalar field.

The heterotic sigma models which describe the target spaces of (2,1) strings have been discussed in Refs. [2–4]. The geometry is Hermitean with torsion and the field equations imply that the curvature with torsion is self-dual in four dimensions, or satisfies generalised self-duality equations in higher dimensions. The conditions under which these models have isometry symmetries were analysed in Ref. [19], while the gauging of such isometries and the construction of manifestly (2,1) supersymmetric gauged actions were discussed in Refs. [19,20,22,23].

This paper is organised as follows. In Section 2 we discuss untwisted and twisted (p,q) supersymmetry in two dimensions and introduce a superspace for the general (p,q) superalgebra. In Section 3 we construct the corresponding two-dimensional non-linear sigma models on target spaces of general signature, and derive the geometric conditions imposed by supersymmetry. In Section 4 we give a superspace formulation of the models with twisted (p,q) supersymmetry and discuss their isometry symmetries. In Section 5 we review the geometry and the extended superspace formulation of the sigma model with the usual (2,1) supersymmetry. An extended superspace formulation of the sigma model with twisted (2,1) supersymmetry is given in Section 6. In Section 7 we discuss the various possible $N=2$ sigma models and in particular give superspace formulations of the pseudo-Kähler sigma models with or without torsion. We summarise the results in Section 8, and close with some remarks on a reformulation with ‘double numbers’.

2. Superalgebras and superspaces

In two-dimensional Minkowski space, the global supersymmetry algebra of type (p,q) was defined in Ref. [2]. There also exists a twisted form of this algebra [1] and the general case is

$$\left\{Q_+^I,Q_+^J\right\}=2\eta^{IJ}P_+,\qquad \left\{Q_-^{I'},Q_-^{J'}\right\}=2\eta^{I'J'}P_-,\qquad \left\{Q_+^I,Q_-^{J'}\right\}=0, \tag{1}$$

where Q_+^I, $I=1,\dots,p$, are the p positive-chirality supersymmetry charges, $Q_-^{I'}$, $I'=1,\dots,q$ are the q negative-chirality charges and $+,-$ are chiral spinor indices; our superspace conventions are as in [24]. The supercharges $Q_\pm$ are 1-component Majorana–Weyl spinors which, in our conventions, are real, $Q_\pm^*=Q_\pm$. Consider the right-handed superalgebra generated by the Q_+^I. In the conventional (untwisted) superalgebra of [2], $\eta^{IJ}=\delta^{IJ}$, while in the general case η^{IJ} in (1) can be an arbitrary symmetric matrix. If invertible, it can be brought to the form

$$\eta^{IJ}=\begin{pmatrix}1_u & 0\\ 0 & -1_t\end{pmatrix} \tag{2}$$

with $u+t=p$. Then Q_+^I for $I=1,\dots,u$ are normal supersymmetries that square to P_+, while Q_+^I for $I=u+1,\dots,p$ are twisted supersymmetries that square to $-P_+$, and we refer to the superalgebra as being twisted. This can be generalised further to allow non-invertible metrics

$$\eta^{IJ}=\begin{pmatrix}1_u & 0 & 0\\ 0 & -1_t & 0\\ 0 & 0 & 0_v\end{pmatrix} \tag{3}$$

with v zeroes as well as u $+1$'s and t -1's $(t+u+v=p)$; there would then be v nilpotent supercharges Q_+^I for $I=u+t+1,\dots,p$ (i.e. $Q^2=0$). Note that for e.g. the twisted (2,0) algebra, the supercharges $Q_+^{\pm}=Q_+^1 \pm Q_+^2$ are each nilpotent, $(Q^{\pm})^2=0$, but they do not anti-commute with each other. It would be interesting to study the cohomology associated with such nilpotent supercharges. The discussion of the left-handed superalgebra generated by the Q_-^I is similar, and there are corresponding expressions for $\eta^{I'J'}$ with p and q interchanged.

The above can be extended further to allow central charges $Z^{IJ'}$ with

$$\left\{Q_+^I, Q_-^{J'}\right\} = Z^{IJ'}, \tag{4}$$

or vectorial charges $X_+^{IJ}, X_-^{I'J'}$ with

$$\left\{Q_+^I, Q_+^J\right\} = X_+^{IJ}, \qquad \left\{Q_-^{I'}, Q_-^{J'}\right\} = X_-^{I'J'}, \tag{5}$$

but this will not be discussed further here.

Twisted superalgebras are possible in higher dimensions also; for instance the ten-dimensional type II * string theories related by timelike T-duality to the usual type II superstring theories have twisted IIA or IIB superalgebras in ten dimensions [5].

It is straightforward to introduce a superspace for the general (p,q) superalgebra (1). There are two real bosonic coordinates $\sigma^+=\sigma^1+\sigma^2$, $\sigma^-=\sigma^1-\sigma^2$, p real positive-chirality Fermi coordinates θ_I^+ and q real negative-chirality Fermi coordinates $\theta_{I'}^-$. The supersymmetry generators

$$Q_+^I = \frac{\partial}{\partial\theta_I^+} - i\eta^{IJ}\theta_J^+ \frac{\partial}{\partial\sigma^+}, \qquad Q_-^{I'} = \frac{\partial}{\partial\theta_{I'}^-} - i\eta^{I'J'}\theta_{J'}^- \frac{\partial}{\partial\sigma^+}, \tag{6}$$

satisfy the superalgebra (1); the corresponding supercovariant derivatives are

$$D_+^I = \frac{\partial}{\partial\theta_I^+} + i\eta^{IJ}\theta_J^+ \frac{\partial}{\partial\sigma^+}, \qquad D_-^{I'} = \frac{\partial}{\partial\theta_{I'}^-} + i\eta^{I'J'}\theta_{J'}^- \frac{\partial}{\partial\sigma^+}, \tag{7}$$

and satisfy the anticommutators

$$\left\{D_+^I, D_+^J\right\} = 2i\eta^{IJ}\partial_+, \qquad \left\{D_-^{I'}, D_-^{J'}\right\} = 2i\eta^{I'J'}\partial_-, \qquad \left\{D_+^I, D_-^{I'}\right\} = 0. \tag{8}$$

For Minkowski world-sheets, there are one-component Majorana–Weyl spinors, but for Euclidean signature there are no Majorana–Weyl spinors, so the analysis is different. A Dirac spinor

$$\psi_a = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{9}$$

has two complex components $\psi_{\pm}$. One can impose a Majorana condition $(\psi_+)^* = \psi_-$ or a pseudo-Majorana condition $(\psi_+)^* = -\psi_-$, or a Weyl condition $\psi_+=0$ or $\psi_-=0$; there are thus various types of minimal spinor with two real components, but none with one component.

There are then various types of superalgebras in two Euclidean dimensions. There is a (p,q) algebra with p right-handed Weyl supercharges with complex components Q_+^I and q left-handed Weyl supercharges with complex components $Q_-^{I'}$, and the superalgebra is again (1), but with all charges complex, and $P_{\pm} \equiv P_1 \pm iP_2$. For N Majorana spinors Q_a^I, $I = 1, \dots, N$, the general algebra (without central charges or extra vector charges) is

$$\left\{Q_a^I, Q_b^J\right\} = M^{IJ} P_\mu (\gamma^\mu C)_{ab} + N^{IJ} P_\mu (\gamma^3 \gamma^\mu C)_{ab}, \tag{10}$$

where C is the two-dimensional charge conjugation matrix, the γ^μ are two-dimensional Dirac matrices, $\gamma^3 = i\gamma^0\gamma^1$ and M^{IJ}, N^{IJ} are some symmetric matrices. The matrix M^{IJ} can be taken to be diagonal with eigenvalues $+1, -1$ and 0, as in (3). This can be obtained from the (N,N) algebra with N left-handed and N right-handed Weyl supercharges by imposing the Majorana condition $(Q_+^I)^* = Q_-^I$, with $M^{IJ} = \frac{1}{2}(\eta^{IJ} + \eta^{I'J'})$ and $N^{IJ} = \frac{1}{2}(\eta^{IJ} - \eta^{I'J'})$. For pseudo-Majorana supercharges, the result is similar. The general (N,M,r,s) superalgebra with N Majorana supercharges, M pseudo-Majorana supercharges, r right-handed Weyl supercharges and s left-handed Weyl supercharges can be obtained from the (p,q) superalgebra with $p = N + M + r$, $q = N + M + s$, by imposing the Majorana condition $(Q_+^I)^* = Q_-^{I'}$ for $I = I' = 1, \dots, N$ and the pseudo-Majorana condition $(Q_+^I)^* = -Q_-^{I'}$ for $I = I' = N+1, \dots, M+N$. Thus all cases are contained in the Euclidean (p,q) algebra, and much of the analysis of the Minkowski (p,q) models carries over to the Euclidean (p,q) theories; in particular, the Euclidean (p,q) superspace has two complex bosonic coordinates $\sigma^\pm = \sigma^1 \pm i\sigma^2$, p complex positive-chirality Fermi coordinates θ_I^+ and q complex negative-chirality ones $\theta_{I'}^-$, with supercharges and derivatives again given by (6) and (7).

3. (p,q) Sigma models with general target space signature

We now turn to the construction of non-linear two-dimensional sigma models with twisted or untwisted (p,q) supersymmetry on target spaces of arbitrary signature.

It is convenient to first consider the (1,1) supersymmetric sigma model with superspace action [24]

$$S_{(1,1)} = \int d^2\sigma \, d\theta^+ \, d\theta^- \left[g_{ij}(\phi) + b_{ij}(\phi) \right] D_+ \phi^i D_- \phi^j, \tag{11}$$

where the ϕ^i are superfields which can be viewed as coordinates on some D-dimensional manifold M with metric g_{ij} and torsion 3-form H given by the curl of the antisymmetric tensor b_{ij},

$$H_{ijk} = \frac{3}{2} \partial_{[i} b_{jk]}. \tag{12}$$

The action (11) is invariant under (1,1) supersymmetry, general coordinate transformations on the target manifold M and antisymmetric tensor gauge transformations

$$\delta b_{ij} = \partial_{[i} \lambda_{j]}. \tag{13}$$

This model will be conformally invariant at one loop if there is a function Φ such that

$$R_{ij}^{(+)} - \nabla_{(i} \nabla_{j)} \Phi - H_{ij}^k \nabla_k \Phi = 0, \tag{14}$$

where $R^{(+)}_{ij}$ is the Ricci tensor for a connection with torsion. We define the connections with torsion

$$\Gamma^{(\pm)i}_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \pm H^i_{jk}, \tag{15}$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ is the Christoffel connection, and the corresponding covariant derivatives $\nabla^{(\pm)}$. The curvature and Ricci tensors with torsion are

$$R^{(+)k}_{lij} = \partial_i \Gamma^{(+)k}_{jl} - \partial_j \Gamma^{(+)k}_{il} + \Gamma^{(+)k}_{im}\Gamma^{(+)m}_{jl} - \Gamma^{(+)k}_{jm}\Gamma^{(+)m}_{il}, \qquad R^{(+)}_{ij} = R^{(+)k}_{ikj}. \tag{16}$$

Eq. (14) can be obtained by varying the action

$$S = \int d^D x\, e^{-2\Phi}\sqrt{|g|}\left(R - \frac{1}{3}H^2 + 4(\nabla\Phi)^2\right). \tag{17}$$

We now seek the conditions on the target space geometry under which the (1,1) superspace action (11) is invariant under extra supersymmetries, generalising the analysis of Refs. [2–4,25] to arbitrary signature and giving a superspace derivation of the results of Ref. [1]. If there are $p-1$ right-handed and $q-1$ left-handed extra supersymmetry transformations, then they must be of the form

$$\delta\phi^i = \varepsilon^r T^i_{(+)rj} D_+ \phi^j + \varepsilon^{r'} T^i_{(-)r'j} D_- \phi^j \tag{18}$$

for some tensors $(T_{(+)r})^i_j$, $(T_{(-)r'})^i_j$ with $r = 1,\dots,p-1$ and $r' = 1,\dots,q-1$. Invariance of the action (11) requires that the tensors $T^i_{(+)rj}$, $T^i_{(-)r'j}$ satisfy

$$g_{ki}T^k_{(+)rj} + g_{kj}T^k_{(+)ri} = 0, \qquad g_{ki}T^k_{(-)r'j} + g_{kj}T^k_{(-)r'i} = 0, \tag{19}$$

and

$$\nabla^{(+)}_k T^i_{(+)rj} = \nabla^{(-)}_k T^i_{(-)r'j} = 0. \tag{20}$$

If the supersymmetry transformations (18) are to satisfy a superalgebra, which may be twisted or untwisted, then the matrices $T_{r(+)}$ and $T_{r'(-)}$ must satisfy anticommutation relations of the form

$$\{T_{(+)r}, T_{(+)s}\} = -2\eta_{rs}, \qquad \{T_{(-)r'}, T_{(-)s'}\} = -2\eta_{r's'}, \tag{21}$$

for some metrics η^{rs}, $\eta^{r's'}$. In addition, the generalised Nijenhuis concomitants $\mathcal{N}(T^r_+, T^s_+)$ and $\mathcal{N}(T^{r'}_-, T^{s'}_-)$ must vanish. For any (1,1) tensors T_1 and T_2 the generalised Nijenhuis concomitant is defined by [26]

$$\mathcal{N}(T_1,T_2)^i_{jk} = T^l_{1j}\partial_l T^i_{2k} - T^l_{1k}\partial_l T^i_{2j} - T^i_{1l}\partial_j T^l_{2k} - T^i_{1l}\partial_k T^l_{2j} + (1 \to 2) \tag{22}$$

so that $\mathcal{N}(T_1,T_2) = \mathcal{N}(T_2,T_1)$ and $\mathcal{N}(T_1,T_2)^i_{jk}$ is antisymmetric in the indices j,k. Then $\frac{1}{4}\mathcal{N}(T,T) \equiv \mathcal{N}(T)$ is the usual Nijenhuis tensor of T,

$$\mathcal{N}^k_{ij}(T) = T^l_i T^k_{[j,l]} - T^l_j T^k_{[i,l]}. \tag{23}$$

The condition $\mathcal{N}(T) = 0$ implies that T is integrable, i.e. that a coordinate system can be chosen in which it is constant. However, if there are several integrable such tensors, it will usually not be possible to choose coordinates in which they are simultaneously integrable.

If the above conditions are satisfied, then the supersymmetry transformations (18) together with the manifest (1,1) supersymmetries satisfy the algebra (1) with

$$\eta^{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & \eta^{rs} \end{pmatrix}, \qquad \eta^{I'J'} = \begin{pmatrix} 1 & 0 \\ 0 & \eta^{r's'} \end{pmatrix}. \tag{24}$$

Diagonalising η^{rs}, $\eta^{r's'}$, we find that each tensor T squares to either $+1$, -1 or 0; those satisfying $T^2 = -1$ are complex structures while those satisfying $T^2 = 1$ are sometimes referred to as real structures (as in Refs. [27,28]) and sometimes as almost product structures (as in Ref. [24]).

Consider first the case of the right-handed supersymmetries with the tensors $T_r = T_{(+)r}$. Each is Hermitean, $T_{ij} = -T_{ji}$, and covariantly constant with respect to the connection $\Gamma^{(+)}$, and so the $p-1$ tensors T_r must be singlets under the holonomy group $\mathscr{H}$ of $\Gamma^{(+)}$. We will restrict ourselves to the cases in which the holonomy is irreducible. For signature (m,n), $\mathscr{H}$ is $O(n,m)$, or a subgroup thereof, as the metric with signature (m,n) is covariantly constant. There will be a covariantly constant complex structure J, with $J^2 = -1$, if m,n are even, $n = 2n_1, m = 2n_2$, so that the signature is $(2n_1, 2n_2)$, and if $\mathscr{H} \subseteq U(n_1,n_2)$. If there are two covariantly constant complex structures, I, J, then $K = IJ$ is a third covariantly constant complex structure and the I,J,K satisfy the quaternion algebra

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J \tag{25}$$

with I,J,K satisfying $SO(3)$ commutation relations. This requires that the holonomy group is contained in $USp(2m)$ for Euclidean spaces of even complex dimension $n = 2m$ (where $USp(2m)$ is compact, with the convention that $USp(2) = SU(2)$; we use the definitions of groups and their non-compact forms given in Ref. [29]). For spaces of signature $(4n,4m)$, this requires that the holonomy is contained in $USp(2n,2m)$ (this is the subgroup of $U(2n,2m)$ preserving a symplectic structure).

For a real structure S satisfying $S^2 = 1$, the hermiticity condition implies that the metric, if it is to be non-degenerate, has to be of signature (m,m), and the holonomy group has to be in $GL(m,\mathbb{R})$. If there are two real structures, S,T with $\{S,T\} = 0$, then $J = ST$ is a complex structure and J,S,T must satisfy the pseudo-quaternion algebra

$$\begin{aligned} &J^2 = -1, \qquad S^2 = T^2 = 1, \\ &ST = -TS = -J, \qquad TJ = -JT = S, \qquad JS = -SJ = T \end{aligned} \tag{26}$$

with J,S,T satisfying $SO(2,1)$ commutation relations, so that there is a pseudo-quaternionic structure [27,28]. Similarly, if there is a complex structure J and a real structure S with $\{S,J\} = 0$, then $T = JS$ is another real structure and J,S,T again satisfy the pseudo-quaternion algebra (26). The existence of such a covariantly constant pseudo-quaternionic structure requires that m is even, $m = 2k$, and the holonomy is in $Sp(2k,\mathbb{R})$. If $p > 4$, the tensors T satisfy an octonion or pseudo-octonion algebra and the holonomy must be trivial. Similar results apply for the left-handed supersymmetries, the number of which depends on the holonomy of the connection $\Gamma^{(-)}$.

The currents

$$j_{(\pm)r} = \frac{1}{2} T^{ij}_{(\pm)r} \psi_i \psi_j \tag{27}$$

generate left- and right-handed Kač–Moody algebras. The right-handed currents $j_{(+)r}$ generate an affine $SO(2)$ or $SO(3)$ if there are $p=2$ or $p=4$ untwisted supersymmetries, and an affine $SO(1,1)$ if $p=2$ and one of the supersymmetries is twisted, and an affine $SO(2,1)$ if $p=4$ and two of the supersymmetries are twisted. In the latter case, the $SO(2,1)$ Kač–Moody algebra is part of a non-compact twisted form of the (small) $N=4$ superconformal algebra with global limit given by (1), where η^{IJ} is the $O(2,2)$ invariant metric [1].

In the special case in which the torsion vanishes, then $\Gamma^{(+)}=\Gamma^{(-)}=\Gamma$ and the number of left-handed supersymmetries is the same as the number of right-handed supersymmetries, $p=q$. For (2,2) untwisted supersymmetry the geometry is Kähler, for (4,4) untwisted supersymmetry the geometry is hyper-Kähler, while for (2,2) twisted supersymmetry we shall call the geometry pseudo-Kähler, and for (4,4) twisted supersymmetry we shall call the geometry pseudo-hyper-Kähler. The pseudo-Kähler geometry shares many of the features of Kähler geometry; in particular, the metric can in both cases be given in terms of a scalar potential, as we shall see in Section 6.

4. Extended superspace and isometries

A superspace formulation of the models with twisted (p,q) supersymmetry can be given in (p,q) superspace using a formalism which generalises that proposed by Howe and Papadopoulos in Refs. [30,31]. Let

$$\left(\sigma^{\pm},\theta^{-}_{\mu},\theta^{+}_{\mu'},\tilde{\theta}^{-}_{\tilde{\nu}},\tilde{\theta}^{+}_{\tilde{\nu}'}\right) \tag{28}$$

with $\mu=0,\ldots,u$, $\tilde{\mu}=u+1,\ldots,p-1$ and $\nu'=0,\ldots,v$, $\tilde{\nu}'=v+1,\ldots,q-1$ $(1\leqslant u\leqslant p-1,\ 1\leqslant v\leqslant q-1)$ be the superspace coordinates. The non-vanishing anticommutators of the flat superspace derivatives $D_{\mu+}$ and $D_{\mu'-}$ are

$$\begin{aligned}&\{D_{\mu+},D_{\nu+}\}=2i\delta_{\mu\nu}\partial_{+}, &&\{D_{\mu'-},D_{\nu'-}\}=2i\delta_{\mu'\nu'}\partial_{-},\\ &\{D_{\tilde{\mu}+},D_{\tilde{\nu}+}\}=-2i\delta_{\tilde{\mu}\tilde{\nu}}\partial_{+}, &&\{D_{\tilde{\mu}'-},D_{\tilde{\nu}'-}\}=-2i\delta_{\tilde{\mu}'\tilde{\nu}'}\partial_{-}.\end{aligned} \tag{29}$$

$D_{\mu+}$ and $D_{\mu'-}$ anticommute with the supercharges $Q_{\mu+}$ and $Q_{\mu'-}$, while $D_{\tilde{\mu}+}$ and $D_{\tilde{\mu}'-}$ anticommute with $Q_{\tilde{\mu}+}$ and $Q_{\tilde{\mu}'-}$. The generalised (p,q) non-linear sigma model is described by a superfield φ^i which is a map from the (p,q) superspace to M. The chirality constraints [30,31]

$$\begin{aligned}&D_{r+}\varphi^i=T^i_{(+)rj}D_{0+}\varphi^j, &&r=1,\ldots,p-1,\\ &D_{r'-}\varphi^i=T^i_{(-)r'j}D_{0-}\varphi^j, &&r'=1,\ldots,q-1,\end{aligned} \tag{30}$$

imply that the (p,q) supersymmetry transformation of either type generated by $(Q_{\mu+}, Q_{\mu'-})$ and $(Q_{\tilde{\mu}+}, Q_{\tilde{\mu}'-})$ reduce to the transformations (18) on expanding into (1,1) superfields.

The twisted or untwisted (p,q)-supersymmetric sigma model action in the corresponding (p,q) superspace is then [30,31]

$$\begin{aligned}S=-i\Big[&\int d^2\sigma\, d\theta^+_0\ d\theta^-_0\ g_{ij}D_{0+}\varphi^iD_{0-}\varphi^j\\ &+\int d^2\sigma\, dt\, d\theta^+_0\ d\theta^-_0\ H_{ijk}\partial_tD_{0+}\varphi^jD_{0-}\varphi^k\Big],\end{aligned} \tag{31}$$

where the (p,q) superfields satisfy the constraints (30). If Eqs. (19) and (20) hold, then using the constraints (30) it can be shown that the action (31) is independent of the extra supercoordinates $(\theta_r,\ \theta_{r'})$, and as a result is invariant (up to surface terms) under the non-manifest supersymmetries generated by $(Q_{r+},\ Q_{r'-})$ and $(Q_{\tilde{r}+},\ Q_{\tilde{r}'-})$.

Now consider (p,q) infinitesimal superspace transformations of the form

$$\delta\varphi^i = \lambda^a\, \xi_a^i(\varphi) \tag{32}$$

with constant parameters λ^a. These will constitute proper symmetries of the sigma model action (31) if the metric and torsion are Lie invariant,

$$(\mathscr{L}_a g)_{ij} = 0, \qquad (\mathscr{L}_a H)_{ijk} = 0, \tag{33}$$

and if in addition

$$\mathscr{L}_a T_{(+)r} = \mathscr{L}_a T_{(-)r'} = 0, \tag{34}$$

i.e. the real or complex structures are also Lie invariant. Then the ξ_a^i are Killing vectors which are holomorphic with respect to each complex structure, or 'holomorphic' in a generalised sense with respect to each real structure. This implies locally on M that

$$\xi_a^i H_{ijk} = 2\,\partial_{[j} u_{k]a}, \tag{35}$$

where u is a locally defined one-form u_{ia} which is determined in every coordinate patch of M up to an exact Lie-algebra valued one-form. It follows that there are generalised Killing potentials $X_{(+)ra},\ X_{(-)r'a}$ satisfying

$$g_{ij}\xi_a^j + u_{ia} = T^j_{(+)ri}\partial_j X_{(+)ra} = T^j_{(-)r'i}\partial_j X_{(-)r'a} \tag{36}$$

for every $r = 1,\dots,p-1$ and $r' = 1,\dots,q-1$.

5. (2,1) Sigma models

In this section we review the (2,1) sigma model with untwisted supersymmetry; the model with twisted (2,1) supersymmetry will be discussed in Section 6. The geometric conditions for the (1,1) model to have untwisted (2,1) world-sheet supersymmetry were first obtained in Ref. [2], and follow from the general discussion given above. The manifold must be complex (with dimension $D = 2n$) with metric g_{ij} of signature $(2m_1, 2m_2)$ with $m_1 + m_2 = n$ and a complex structure J_j^i which is covariantly constant with respect to the connection with torsion $\Gamma^{(+)}$ defined in (15) and with respect to which the metric is Hermitean, so that $J_{ij} = g_{ik} J_j^k$ is antisymmetric. Introducing complex coordinates $z^\alpha,\ \bar{z}^{\bar{\beta}} = (z^\beta)^*$ in which the complex structure is constant and diagonal,

$$J_j^i = i\begin{pmatrix} \delta_\alpha^\beta & 0 \\ 0 & -\delta_{\bar{\alpha}}^{\bar{\beta}} \end{pmatrix}, \tag{37}$$

any N-form can be decomposed into a set of (r,s) forms with r factors of dz and s factors of $d\bar{z}$, where $r+s=N$. The conditions above then imply that the (0,3) and (3,0) parts of the three-form H vanish and H is given in terms of the fundamental two-form

$$J=\frac{1}{2}J_{ij}\,d\phi^i\wedge d\phi^j=-ig_{\alpha\bar{\beta}}\,dz^\alpha\wedge\bar{z}^{\bar{\beta}} \tag{38}$$

by

$$H=i(\partial-\bar{\partial})\,J. \tag{39}$$

The exterior derivative decomposes in the complex coordinate system as $d=\partial+\bar{\partial}$, so the closure of the three-form H implies

$$i\partial\bar{\partial}J=0. \tag{40}$$

It follows that locally a (1,0) form $k=k_\alpha\,dz^\alpha$ exists such that

$$J=i(\partial\bar{k}+\bar{\partial}k). \tag{41}$$

The metric and torsion potential are then given (in a suitable gauge) by

$$g_{\alpha\bar{\beta}}=\partial_\alpha\bar{k}_{\bar{\beta}}+\partial_{\bar{\beta}}k_\alpha, \qquad b_{\alpha\bar{\beta}}=\partial_\alpha\bar{k}_{\bar{\beta}}-\partial_{\bar{\beta}}k_\alpha. \tag{42}$$

If $k_\alpha=\partial_\alpha K$ for some K then the torsion vanishes and the manifold is Kähler with Kähler potential K and the (2,1) supersymmetric model in fact has (2,2) supersymmetry, but if $dk\neq 0$, then M is a Hermitean manifold with torsion [2]. The metric and torsion are invariant under [19]

$$\delta k_\alpha=i\partial_\alpha\chi+\theta_\alpha, \tag{43}$$

where χ is real and θ_α is holomorphic, $\partial_{\bar{\beta}}\theta_\alpha=0$, but $b_{\alpha\bar{\beta}}$ as defined in (42) transforms as

$$\delta b_{\alpha\bar{\beta}}=-2i\partial_\alpha\partial_{\bar{\beta}}\chi, \tag{44}$$

which is an antisymmetric gauge transformation (13) with parameter $\lambda_\alpha=2i\partial_\alpha\chi$.

Much of the above structure can be found using superspace methods. We start by seeking the most general (2,1) supersymmetric sigma model that can be written in a (2,1) superspace parametrised by σ^μ, θ^+, $\bar{\theta}^+$, θ^-, where $\theta^+=\theta_1+i\theta_2$ is a complex Weyl spinor and the corresponding supercovariant derivatives are

$$D_+=\frac{\partial}{\partial\theta^+}+i\bar{\theta}^+\partial_+, \qquad \bar{D}_+=\frac{\partial}{\partial\bar{\theta}^+}+i\theta^+\partial_+, \quad D_-=\frac{\partial}{\partial\theta^-}+i\theta^-\partial_-, \tag{45}$$

so that

$$\left\{D_+,\bar{D}_+\right\}=2i\partial_+, \qquad \{D_+,D_-\}=\left\{\bar{D}_+,D_-\right\}=0. \tag{46}$$

We introduce complex (2,1) scalar superfields φ^α, $\bar{\varphi}^{\bar{\alpha}}=(\varphi^\alpha)^*$ satisfying the chiral constraint

$$\bar{D}_+\varphi^\alpha=0, \qquad D_+\bar{\varphi}^{\bar{\alpha}}=0. \tag{47}$$

The lowest components $\varphi^\alpha|_{\theta=0} = z^\alpha$ of the superfields are bosonic complex coordinates of the target space. The general sigma-model action is [32]

$$S = i\int d^2\sigma\, d\theta^+ \bar\theta^+ d\theta^- \Big(k_\alpha D_- \varphi^\alpha - \bar k_{\bar\alpha} D_- \bar\varphi^{\bar\alpha}\Big) \tag{48}$$

for some local vector potentials $k_\alpha(\varphi^\alpha,\bar\varphi^{\bar\alpha})$, $\bar k_{\bar\alpha}(\varphi^\alpha,\bar\varphi^{\bar\alpha})$, which are required to be complex conjugate if the action (48) is to be real, $\bar k_{\bar\alpha} = (k_\alpha)^*$. Expanding in components, the bosonic part of the action is a bosonic sigma model with metric $g_{\alpha\bar\beta}$ and torsion potential $b_{\alpha\bar\beta}$ given in terms of k by (42), so that we find the geometry described above. In particular, if $k_\alpha = \partial_\alpha K$ for some scalar K, then the torsion vanishes and the metric is given by

$$g_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} K \tag{49}$$

so it is Kähler.

The additional geometric conditions under which the model has isometry symmetries have been analysed in Ref. [19]. There it was shown that the geometry determines the potentials χ and θ that appear in Eq. (43). The construction of gauged (2,1) superspace actions was discussed in Refs. [19,20,22].

It will be useful to define the vector

$$w^i = H_{jkl} J^{ij} J^{kl} \tag{50}$$

together with the $U(1)$ part of the curvature

$$C^{(+)}_{ij} = J_k^{\ l} R^{(+)k}_{lij} \tag{51}$$

and the $U(1)$ part of the connection (15),

$$\Gamma^{(+)}_i = J_j^k \Gamma^{(+)j}_{ik} = i\Big(\Gamma^{(+)\alpha}_{i\alpha} - \Gamma^{(+)\bar\alpha}_{i\bar\alpha}\Big). \tag{52}$$

Note that C_{ij} is a representative of the first Chern class, and that it can be written as $C^{(+)}_{ij} = 2\partial_{[i}\Gamma^{(+)}_{j]}$ in a complex coordinate system. If the metric has Euclidean signature, the holonomy of any metric connection (including $\Gamma^{(\pm)}$) is contained in $O(2n)$, while if it has signature $(2m_1,2m_2)$ with $m_1+m_2=n$, it will be contained in $O(2m_1,2m_2)$. As the complex structure is covariantly constant, the holonomy $\mathscr{H}(\Gamma^{(+)})$ of the connection with torsion $\Gamma^{(+)}$ is contained in $U(m_1,m_2)$, but it will be contained in $SU(m_1,m_2)$ if in addition $C^{(+)}_{ij}=0$; a necessary condition for this is the vanishing of the first Chern class.

It was shown in Refs. [4,25,33] that geometries for which

$$\Gamma^{(+)}_i = 0 \tag{53}$$

in some suitable choice of coordinate system will satisfy the conditions for one-loop conformal invariance (14) provided the dilaton is chosen as

$$\Phi = -\frac{1}{2}\log|\det g_{\alpha\bar\beta}|, \tag{54}$$

which implies

$$\partial_i \Phi = v_i. \tag{55}$$

Moreover, the one-loop dilaton field equation is also satisfied for compact manifolds, or for non-compact ones in which $\nabla\Phi$ falls off sufficiently fast [1]. This implies that $\mathscr{H}(\Gamma^{(+)})$ is contained in $SU(m_1,m_2)$. These geometries generalise the Kähler Ricci-flat or Calabi–Yau geometries, and reduce to these in the special case in which $H=0$. However, they are not the most general solutions of the conditions (14) [1].

The condition that the connection $\Gamma^{(+)}$ has $SU(m_1,m_2)$ holonomy can be cast as a generalised self-duality condition on the curvature. Defining the four-form

$$\phi^{ijkl}\equiv -3J^{[ij}J^{kl]}, \tag{56}$$

the condition that $\mathscr{H}(\Gamma^{(+)})\subseteq SU(m_1,m_2)$ is equivalent to [21]

$$R^{(+)}_{ijkl}=\frac{1}{2}g_{im}g_{jn}\phi^{mnpq}R^{(+)}_{pqkl}. \tag{57}$$

For $D=4$, $\phi^{ijkl}=-\epsilon^{ijkl}$ and this is the usual anti-self-duality condition.

Eq. (53) can be viewed as a field equation for the potential k_α, and can be obtained by varying the action [1,13,14]

$$S=\int d^Dx\sqrt{|\det g_{\alpha\overline{\beta}}|}\,, \tag{58}$$

where $g_{\alpha\overline{\beta}}$ is given in terms of k_α by (42). This action can be rewritten as

$$S=\int d^Dx|\det g_{ij}|^{1/4} \tag{59}$$

which is non-covariant but is invariant under volume-preserving diffeomorphisms. This can be rewritten in the classically equivalent alternative form [21]

$$S'=T'_4\int d^Dx|\gamma|^{1/4}\left[\gamma^{ij}g_{ij}-(D-4)c\right], \tag{60}$$

where γ_{ij} is an auxiliary metric, $\gamma=\det\gamma_{ij}$ and c,T'_4 are (real) constants. In the special case of four dimensions, the constant term in the action (60) vanishes and there is a generalised Weyl symmetry under

$$\gamma_{ij}\to\omega(x)\gamma_{ij}. \tag{61}$$

The dualisation of the action (58) was discussed in Ref. [21]. This is achieved by adding a Lagrange multiplier term imposing the constraint $g_{\alpha\overline{\beta}}=\partial_\alpha\overline{k}_{\overline{\beta}}+\partial_{\overline{\beta}}k_\alpha$. The vector potentials $k_\alpha,k_{\overline{\alpha}}$ are then Lagrange multipliers for a certain constraint, and solving this leads to a dual form of the action [21]. In four dimensions, the dual geometry is self-dual gravity without torsion coupled to a scalar field, while in $D>4$ dimensions the dual geometry is Hermitean and determined by a $D-4$ form potential K which generalises the Kähler potential of the four-dimensional case. The coupling to the Yang–Mills fields is through a term $K\wedge\mathrm{tr}(F\wedge F)$ and leads to a Uhlenbeck–Yau field equation $\tilde{J}^{ij}F_{ij}=0$ [21].

6. Twisted (2,1) sigma models

Consider now the case of space-time signature (d,d), which was called Kleinian in Ref. [28]. We start by considering the (2,1) superspace formulation to obtain the geometry in a special coordinate system (the analogue of the complex coordinates of Section 5), then show how the same results can be obtained in a coordinate independent manner using the results of Section 3. Using the (2,1) superspace introduced in Section 2, we define

$$\theta_+ = \theta^1_+ + \theta^2_+ \,, \qquad \tilde{\theta}_+ = \tilde{\theta}^1_+ - \tilde{\theta}^2_+ \tag{62}$$

and the supercharges

$$Q_+ = \frac{\partial}{\partial\theta_+} - \tilde{\theta}_+ \frac{\partial}{\partial\sigma^+} \,, \qquad \tilde{Q}_+ = \frac{\partial}{\partial\tilde{\theta}_+} - \theta_+ \frac{\partial}{\partial\sigma^+} \,, \tag{63}$$

satisfying the algebra

$$\left\{Q_+, \tilde{Q}_+\right\} = 2\,\partial_+ \,, \qquad Q^2_+ = \tilde{Q}^2_+ = 0, \tag{64}$$

together with the superderivatives

$$D_+ = \frac{\partial}{\partial\theta^+} + \tilde{\theta}^+ \frac{\partial}{\partial\sigma^+} \,, \qquad \tilde{D}_+ = \frac{\partial}{\partial\tilde{\theta}^+} + \theta^+ \frac{\partial}{\partial\sigma^+} \,, \tag{65}$$

which satisfy the anticommutators

$$\{D_+, D_+\} = \left\{\tilde{D}_+, \tilde{D}_+\right\} = 0 \qquad \left\{D_+, \tilde{D}_+\right\} = 2\,\partial_+ . \tag{66}$$

Thus the structure associated with $\theta_+, \tilde{\theta}_+$ in the twisted (2,1) case is similar to that associated with $\theta_+, \bar{\theta}_+$ in the untwisted (2,1) case, with the important difference that in the usual case $\theta_+, \bar{\theta}_+$ are complex and related by $(\theta_+)^* = \bar{\theta}_+$, while in the twisted case $\theta_+, \tilde{\theta}_+$ are independent real coordinates.

The twisted (2,1) supersymmetric sigma model can be formulated in a twisted (2,1) extended superspace as follows. First we introduce chiral scalar superfields U^α and $\tilde{V}^{\tilde{\alpha}}$ satisfying

$$\tilde{D}_+ U^\alpha = 0, \qquad D_+ \tilde{V}^{\tilde{\alpha}} = 0. \tag{67}$$

Note that as $\tilde{D}_+, D_+$ are independent real derivatives, we take $U^\alpha, \tilde{V}^{\tilde{\alpha}}$ as independent real superfields. Here $\alpha = 1, \ldots, n$ and $\tilde{\alpha} = 1, \ldots, \tilde{n}$ for some $n, \tilde{n}$. The general twisted superspace action is

$$S = -\int d^2\sigma\, d\theta^+\, d\tilde{\theta}^+\, d\theta^- \left(k_\alpha D_- U^\alpha - \tilde{k}_{\tilde{\alpha}} D_- \tilde{V}^{\tilde{\alpha}} \right) \tag{68}$$

for some independent real vector potentials $k_\alpha(U^\alpha, \tilde{V}^{\tilde{\alpha}}), \tilde{k}_{\tilde{\alpha}}(U^\alpha, \tilde{V}^{\tilde{\alpha}})$. The corresponding Lagrangian in (1,1) superspace can be obtained by integrating over $\tilde{\theta}^+_2$. Up to a total derivative term, we find the action

$$S = \int d^2\sigma\, d\tilde{\theta}^+_1\, d\theta^- \left[g_{\alpha\tilde{\beta}} + b_{\alpha\tilde{\beta}} \right] D_{1+} u^\alpha D_- v^{\tilde{\beta}}, \tag{69}$$

where $u,\tilde{v}$ are the lowest components of the superfields U and $\tilde{V}$. The metric and torsion potential are given by

$$g_{\alpha\tilde{\beta}} = \partial_\alpha \tilde{k}_{\tilde{\beta}} + \partial_{\tilde{\beta}} k_\alpha , \tag{70}$$

$$b_{\alpha\tilde{\beta}} = \partial_\alpha \tilde{k}_{\tilde{\beta}} - \partial_{\tilde{\beta}} k_\alpha \tag{71}$$

with $g_{\alpha\beta} = b_{\alpha\beta} = 0$. The target space line element is

$$ds^2 = 2\, g_{\alpha\tilde{\beta}}(u,v)\, du^\alpha d\tilde{v}^{\tilde{\beta}} \tag{72}$$

so that $\partial/\partial u^\alpha$ and $\partial/\partial \tilde{v}^{\tilde{\beta}}$ are null vectors. If $n \neq \tilde{n}$, the metric constructed in this way is degenerate in general; we will not consider this case further and restrict ourselves to the case $n = \tilde{n}$.

The condition for the torsion to vanish is

$$k_\alpha = -\partial_\alpha \kappa , \qquad \tilde{k}_{\tilde{\beta}} = \partial_{\tilde{\beta}} \tilde{\kappa} \tag{73}$$

for some locally defined potentials $\kappa,\tilde{\kappa}$. If this is satisfied, then the metric is given in terms of a scalar potential $\tilde{K} = \kappa - \tilde{\kappa}$,

$$g_{\alpha\tilde{\beta}} = \frac{\partial^2}{\partial u^\alpha \partial v^{\tilde{\beta}}} \tilde{K}, \tag{74}$$

giving a real-structure analogue of Kähler geometry, pseudo-Kähler geometry.

The same geometry can be obtained using the results of Section 3 as follows. The (1,1) sigma models with target space signature (n,n) and a covariantly constant real structure S will have twisted (2,1) supersymmetry with global limit given by the supersymmetry algebra (1), where $I,J = 1,2$ and $\eta^{IJ} = \mathrm{diag}(1,-1)$. The holonomy is $\mathscr{H}(\Gamma^{(+)}) \subseteq GL(n,\mathbb{R})$. The integrable real structure S squares to $+1$,

$$S^i_k S^k_j = +\delta_j^{\ i}. \tag{75}$$

Twisted (2,1) supersymmetry requires S^i_j to be covariantly constant with respect to the connection with torsion $\Gamma^{(+)}$,

$$\nabla^{(+)}_k S^i_j = 0 \tag{76}$$

and to be antisymmetric

$$S_{ij} = -S_{ji}. \tag{77}$$

As S^i_j is integrable (i.e. its Nijenhuis tensor (23) vanishes), there is a coordinate system in which it is constant and diagonal. Choosing such adapted real coordinates u^α, $v^{\tilde{\alpha}}$ ($\alpha = 1,2$; $\tilde{\alpha} = 1,2$), the real structure takes the form

$$S^i_j = \begin{pmatrix} \delta_\beta^{\ \alpha} & 0 \\ 0 & -\delta_{\tilde{\beta}}^{\tilde{\alpha}} \end{pmatrix}. \tag{78}$$

The fundamental two-form is then

$$S = \frac{1}{2} S_{ij}\, d\phi^i \wedge d\phi^j = -g_{\alpha\tilde{\beta}}\, du^\alpha \wedge dv^{\tilde{\beta}}, \tag{79}$$

and the line element takes the form (72), so that $\partial/\partial u^{\alpha}$ and $\partial/\partial \tilde{v}^{\tilde{\beta}}$ are null vectors. Any N-form can be decomposed into a set of (r,s) forms with r factors of du and s factors of dv with $r+s=N$. The exterior derivative decomposes as $d=\partial_u+\partial_v$ where $\partial_u : H^{(r,s)} \to H^{(r+1,s)}$ and $\partial_v : H^{(r,s)} \to H^{(r,s+1)}$.

Consider first the case in which there is no torsion, $H=0$. Then the conditions (76) and (77) imply that the geometry is given in terms of some locally defined scalar potential $\tilde{K}$, and the metric takes the form (74) in adapted coordinates; the sigma model with this geometry will be considered further in the next section.

If $H \neq 0$, then the conditions (76) and (77) imply that the torsion three-form is given in terms of the fundamental two-form (79) by

$$H=(\partial_u-\partial_v)S. \tag{80}$$

The condition $dH=0$ then implies

$$\partial_u\partial_v S=0 \tag{81}$$

so that locally there is a (1,0) form $k=k_{\alpha}du^{\alpha}$ and a (0,1) form $\tilde{k}=\tilde{k}_{\tilde{\beta}}dv^{\tilde{\beta}}$ such that

$$S=\partial_u\tilde{k}+\partial_v k. \tag{82}$$

The potentials $k,\tilde{k}$ are independent real 1-forms. The metric and torsion potential are given, in a suitable gauge, by Eq. (71), so that

$$H=\partial_u\partial_v(k+\tilde{k}). \tag{83}$$

If the condition (73) holds for some locally defined potentials κ, $\tilde{\kappa}$, then the torsion vanishes and

$$S=\partial_u\partial_v(\tilde{\kappa}-\kappa), \tag{84}$$

so that (74) is satisfied with potential $\tilde{K}=\tilde{\kappa}-\kappa$. We thus recover the results obtained from extended superspace; the extended superspace approach gives the general solution to the geometric constraints immediately, without having to integrate differential equations.

If $H=0$, then the curvature two-form is a (1,1) form and the only non-vanishing components of the curvature are $R_{\alpha\tilde{\beta}\gamma\tilde{\delta}}$. It follows that the Ricci tensor $R_{\alpha\tilde{\beta}}$ is proportional to $\tilde{C}_{\alpha\tilde{\beta}}$ and is given by

$$R_{\alpha\tilde{\beta}}=\partial_{\alpha}\partial_{\tilde{\beta}}\log|\det g_{\gamma\tilde{\delta}}| \tag{85}$$

with $R_{\alpha\beta}=0$. Thus the Einstein equation $R_{ij}=0$ is equivalent to demanding $SL(d,\mathbb{R})$ holonomy and gives, with a suitable choice of coordinates,

$$|\det g_{\gamma\tilde{\delta}}|=1 \tag{86}$$

which is a Monge–Ampère equation for $\tilde{K}$,

$$\det\left|\frac{\partial^2}{\partial u^{\alpha}\partial v^{\tilde{\beta}}}\tilde{K}\right|=1. \tag{87}$$

If $H \neq 0$, then the metric and torsion are preserved by the gauge transformations

$$\delta k_{\alpha}=\partial_{\alpha}\chi+\theta_{\alpha}, \qquad \delta\tilde{k}_{\tilde{\alpha}}=-\partial_{\tilde{\alpha}}\chi+\tilde{\theta}_{\tilde{\alpha}}, \tag{88}$$

where $\partial_{\tilde\beta}\theta_\alpha = \partial_\beta\tilde\theta_{\tilde\alpha} = 0$. In analogy with Eqs. (50), (51), (52), it is useful to define the vector

$$\tilde w_i = H_{ijk}S^{jk} \tag{89}$$

together with the $GL(1,\mathbb{R})$ part of the curvature

$$\tilde C^{(+)}_{ij} = S^l_k R^{(+)k}_{lij} \tag{90}$$

and the $GL(1)$ part of the connection (15),

$$\tilde\Gamma^{(+)}_i = S^k_j \Gamma^{(+)j}_{ik} = \Gamma^{(+)\alpha}_{i\alpha} - \Gamma^{(+)\tilde\alpha}_{i\tilde\alpha}. \tag{91}$$

For Kleinian signature (n,n), the holonomy of the connection $\Gamma^{(+)}$ is contained in $GL(n,\mathbb{R})$. It will be contained in $SL(n,\mathbb{R})$ if in addition $\tilde C^{(+)}_{ij} = 0$.

If $H \neq 0$, the condition (53) of the complex case is replaced by

$$\tilde\Gamma^{(+)}_i = 0 \tag{92}$$

and this implies that the one-loop field Eq. (14) is satisfied, provided the dilaton is chosen as in (54). Furthermore, the condition (92) implies $\tilde C^{(+)}_{ij} = 0$ and so the holonomy is in $SL(n,\mathbb{R})$.

The field equation (92) can be obtained from the action (59), but where now the metric is given by (70) in terms of the potentials k, $\tilde k$ corresponding to the real structure S, and it is these that are varied to give the field equation (92).

The real 1-form potentials k, $\tilde k$ can be dualised in the same way as in the complex case to obtain a new form of the dual action as well as the dual of the real geometry presented above. The first step is to add to (58) a Lagrange multiplier term of the form

$$\frac{1}{2}\tilde\Lambda^{\alpha\tilde\beta}\Big(g_{\alpha\tilde\beta} - \partial_\alpha\tilde k_{\tilde\beta} - \partial_{\tilde\beta}k_\alpha\Big). \tag{93}$$

Eliminating $\tilde\Lambda^{\alpha\tilde\beta}$ from the resulting action, one recovers the action (58) subject to the constraint $g_{\alpha\tilde\beta} = \partial_\alpha\tilde k_{\tilde\beta} + \partial_{\tilde\beta}k_\alpha$. Integrating over the vectors k_α, $\tilde k_{\tilde\beta}$ instead yields the constraints

$$\partial_\alpha\tilde\Lambda^{\alpha\tilde\beta} = 0, \qquad \partial_{\tilde\beta}\tilde\Lambda^{\alpha\tilde\beta} = 0 \tag{94}$$

which in four dimensions are solved locally in terms of a scalar $\tilde K$ by

$$\tilde\Lambda^{\alpha\tilde\beta} = \tilde L^{\alpha\tilde\beta}, \tag{95}$$

where $\tilde L^{\alpha\tilde\beta}$ is the 'field strength' of $\tilde K$ given by

$$\tilde L^{\alpha\tilde\beta} \equiv \epsilon^{\alpha\gamma\tilde\beta\tilde\delta}\partial_\gamma\tilde\partial_{\tilde\delta}\tilde K \tag{96}$$

and $\epsilon^{\alpha\gamma\tilde\beta\tilde\delta}$ is the antisymmetric tensor density (with $\epsilon^{1\tilde1 2\tilde2} = 1$).

The solution (96) implies that the pseudo-Kähler metric $G_{\alpha\tilde\beta} = \partial_\alpha\partial_{\tilde\beta}\tilde K$ satisfies the constraint

$$\det G_{\alpha\tilde\beta} = -1 \tag{97}$$

for signature (2,2), or $\det G_{\alpha\tilde{\beta}} = +1$ for signature (4,0). Writing $G_{\alpha\tilde{\beta}} = \eta_{\alpha\tilde{\beta}} + \partial_\alpha \partial_{\tilde{\beta}} \tilde{\varphi}$ where $\eta_{\alpha\tilde{\beta}}$ is a flat background metric, the analysis of Ref. [21] then leads to the dual $D = 4$ action

$$\int \partial\tilde{\varphi}\tilde{\partial}\tilde{\varphi} + \frac{1}{3!}\tilde{\varphi}\tilde{\partial}\tilde{\varphi} \wedge \partial\tilde{\partial}\tilde{\varphi} + \int \sqrt{G}\, G^{\alpha\tilde{\beta}} \partial_\alpha \tilde{\Omega} \partial_{\tilde{\beta}} \tilde{\Omega} \tag{98}$$

for some scalar $\tilde{\Omega}$. Thus the dual geometry in four dimensions is a real form of self-dual gravity without torsion determined by the potential $\tilde{K}$ coupled to the harmonic scalar $\tilde{\Omega}$. The generalisation to dimensions $D > 4$ is straightforward, and the results are analogous to those obtained in Ref. [21] for the complex case.

7. (2,2) Supersymmetric sigma models

If the (2,2) supersymmetry closes off-shell, the sigma model can be formulated in terms of off-shell (2,2) superfields. For the usual untwisted (2,2) supersymmetry, we introduce the complex superspace coordinates z^α ($\alpha = 1, \dots, d_1$), θ_+, θ_- together with the supersymmetry generators and supercovariant derivatives

$$Q_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+ \partial_+ , \qquad Q_- = \frac{\partial}{\partial\theta^-} - i\bar{\theta}^- \partial_- \tag{99}$$

and

$$D_+ = \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+ \partial_+ , \qquad D_- = \frac{\partial}{\partial\theta^-} + i\bar{\theta}^- \partial_- . \tag{100}$$

One can either introduce chiral superfields U^α, $\overline{U}^{\bar{\beta}}$ ($\alpha, \bar{\beta} = 1, \dots, d_1$) satisfying

$$\overline{D}_\pm U^\alpha = 0, \qquad D_\pm \overline{U}^{\bar{\beta}} = 0, \tag{101}$$

or twisted chiral superfields V^i, $\overline{V}^{\bar{j}}$ ($i, \bar{j} = 1, \dots, d_2$) satisfying the constraints

$$D_+ V^i = 0, \qquad \overline{D}_- V^i = 0, \qquad D_- \overline{V}^{\bar{j}} = 0, \qquad \overline{D}_+ \overline{V}^{\bar{j}} = 0. \tag{102}$$

The action for the Kähler sigma model is

$$S = \int d^2\sigma\, d^4\theta\, K(U, \overline{U}), \tag{103}$$

where K is the Kähler potential, so that the metric is given by Eq. (49). The action and metric are invariant under the Kähler gauge transformations

$$\delta K = f(U) + \bar{f}(\overline{U}). \tag{104}$$

The action [24]

$$S = \int d^2\sigma\, d^4\theta\, K(U, \overline{U}, V, \overline{V}) \tag{105}$$

defines a supersymmetric non-linear sigma model with torsion on a target space of complex dimension $d_1 + d_2$ with coordinates $x^\mu = (u, \bar{u}, v, \bar{v})$, where u, $\bar{u}$, v and $\bar{v}$ are the lowest components of the superfields U, $\overline{U}$, V and $\overline{V}$. The action (105) is invariant under generalised Kähler gauge transformations

$$\delta K = f_1(U, V) + f_2(U, \overline{V}) + \bar{f}_1(\overline{U}, \overline{V}) + \bar{f}_2(\overline{U}, V). \tag{106}$$

The bosonic part of the component sigma model action is

$$S=\frac{1}{2}\int d^2\sigma\Big(g_{\mu\nu}\partial_\alpha x^\mu\partial^\alpha x^\nu+b_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\Big), \tag{107}$$

where the metric $g_{\mu\nu}$ and the torsion potential $b_{\mu\nu}$ are given by

$$g_{\alpha\bar\beta}=K_{\alpha\bar\beta},\qquad g_{i\bar j}=-K_{i\bar j},\qquad b_{\alpha\bar j}=K_{\alpha\bar j},\qquad b_{i\bar\beta}=K_{i\bar\beta}. \tag{108}$$

All other components of $g_{\mu\nu}$ and $b_{\mu\nu}$ not related to these by complex conjugation or symmetry vanish, and $K_{\mu\nu\ldots\rho}$ denotes the partial derivative $\partial_\mu\partial_\nu\ldots\partial_\rho K$. The geometry is that of a Hermitean locally product space with two commuting complex structures $J_\nu^{(\pm)\mu}$. In the special case in which either $d_1=0$ or $d_2=0$, the torsion vanishes and the target space is Kähler.

For twisted (2,2) supersymmetry with the superalgebra (1), we introduce the real superspace coordinates z^μ, θ_+, $\tilde\theta_+$, θ_-, $\tilde\theta_-$ together with the supersymmetry generators and supercovariant derivatives

$$Q_+=\frac{\partial}{\partial\theta^+}-\tilde\theta^+\partial_+,\qquad Q_-=\frac{\partial}{\partial\theta^-}-\tilde\theta^-\partial_-,$$
$$\tilde Q_+=\frac{\partial}{\partial\tilde\theta^+}-\theta^+\partial_+,\qquad \tilde Q_-=\frac{\partial}{\partial\tilde\theta^-}-\theta^-\partial_- \tag{109}$$

and

$$D_+=\frac{\partial}{\partial\theta^+}+\tilde\theta^+\partial_+,\qquad D_-=\frac{\partial}{\partial\theta^-}+\tilde\theta^-\partial_-,$$
$$\tilde D_+=\frac{\partial}{\partial\tilde\theta^+}+\theta^+\partial_+,\qquad \tilde D_-=\frac{\partial}{\partial\tilde\theta^-}+\theta^-\partial_-. \tag{110}$$

One can either introduce superfields U^α, $\tilde U^{\tilde\beta}$ satisfying the constraints

$$D_+\tilde U^{\tilde\beta}=0,\qquad \tilde D_-U^\alpha=0,\qquad \tilde D_+U^\alpha=0,\qquad D_-\tilde U^{\tilde\beta}=0, \tag{111}$$

or superfields V^i, $\tilde V^{\tilde j}$ satisfying the twisted constraints

$$D_+\tilde V^{\tilde j}=0,\qquad \tilde D_-\tilde V^{\tilde j}=0,\qquad \tilde D_+V^i=0,\qquad D_-V^i=0. \tag{112}$$

The pseudo-Kähler sigma model action is then

$$S=\int d^2x\,d^2\theta\,d^2\tilde\theta\,\tilde K(U,\tilde U). \tag{113}$$

The action, metric and torsion are left invariant under the pseudo-Kähler transformations

$$\delta\tilde K=f(U)+\tilde f(\tilde U). \tag{114}$$

The action

$$S=\int d^2\sigma\,d^2\theta\,d^2\tilde\theta\,\tilde K(U,\tilde U,V,\tilde V) \tag{115}$$

defines a supersymmetric non-linear sigma model with torsion on a target space of dimension $2(d_1+d_2)$ with coordinates $x^\mu=(u,\tilde{u},v,\tilde{v})$, where u, $\tilde{u}$, v and $\tilde{v}$ are the lowest components of the superfields U, $\tilde{U}$, V and $\tilde{V}$. The action (115) is invariant under generalised pseudo-Kähler gauge transformations

$$\delta K=f_1(U,V)+f_2(U,\tilde{V})+\tilde{f}_1(\tilde{U},\tilde{V})+\tilde{f}_2(\tilde{U},V). \tag{116}$$

The bosonic part of the component sigma model action is again given by (107), where the metric $g_{\mu\nu}$ and the torsion potential $b_{\mu\nu}$ are given by

$$g_{\alpha\tilde{\beta}}=K_{\alpha\tilde{\beta}}, \qquad g_{i\tilde{j}}=-K_{i\tilde{j}}, \qquad b_{\alpha\tilde{j}}=K_{\alpha\tilde{j}}, \qquad b_{i\tilde{\beta}}=K_{i\tilde{\beta}}. \tag{117}$$

All other components of $g_{\mu\nu}$ and $b_{\mu\nu}$ not related to these by 'real' conjugation or symmetry vanish. The geometry is that of a real locally product space with two commuting real structures $S_\nu^{(\pm)\mu}$; see Ref. [24] for details. In the special case in which either $d_1=0$ or $d_2=0$, the torsion vanishes and the target space is pseudo-Kähler.

The metrics and torsion potentials (49), (108) and (117) will define consistent string backgrounds if the corresponding sigma model is conformally invariant. For the Kähler model, this will be the case if the metric is Ricci-flat or equivalently if the curvature is self-dual (or anti-self-dual), i.e.

$$\star R_{\mu\nu\rho\sigma}=\frac{1}{2}\epsilon_{\mu\nu}^{\ \ \lambda\tau}R_{\lambda\tau\rho\sigma}=\pm R_{\mu\nu\rho\sigma}. \tag{118}$$

There are also generalisations of these self-dual solutions to the condition for one-loop conformal invariance with non-trivial dilaton, some of which were discussed in Ref. [23].

The sigma model with action (105) was shown in [4] to be one-loop conformally invariant provided the $U(1)$ parts of the two curvature tensors $R^{(\pm)}_{\mu\nu\rho\sigma}$ vanish,

$$C^{(\pm)}_{\mu\nu}=0, \tag{119}$$

so that both connections $\Gamma^{(\pm)}$ have $SU(d_1+d_2)$ holonomy and the first Chern class vanishes; see [34] for a discussion of higher loops. For the twisted case with action (105), the condition for one-loop conformal invariance is that both connections $\Gamma^{(\pm)}$ have $SU(d_1,d_2)$ holonomy.

Similarly, for the pseudo-Kähler sigma model with action (115), one-loop conformal invariance will hold provided the $GL(1,\mathbb{R})$ parts of the two curvature tensors (defined as in Eq. (90)) vanish,

$$\tilde{C}^{(\pm)}_{\mu\nu}=0. \tag{120}$$

If this condition holds, then both connections $\Gamma^{(\pm)}$ will have $SL(d_1+d_2,\mathbb{R})$ holonomy.

In Refs. [36,37] it was argued that all sigma models with the usual (2,2) supersymmetry can be formulated in superspace using chiral, twisted chiral and semi-chiral [38] superfields. Semi-chiral superfields have twice as many components as chiral or twisted chiral ones, half of which are auxiliary. Here we note that a real analogue of the semi-chirality condition can be imposed, viz.

$$D_+W^\alpha=0, \qquad \tilde{D}_+\tilde{W}^{\tilde{\beta}}=0, \qquad D_-\tilde{X}^{\tilde{j}}=0, \qquad \tilde{D}_-\tilde{X}^{\tilde{j}}=0. \tag{121}$$

This leads to a straightforward generalisation of many of the results of [36,37] to twisted (2,2) supersymmetric theories.

8. Summary and discussion

To summarise, the usual supersymmetry algebra of type (p,q) can be generalised to include the possibility of twisted heterotic supersymmetry, as in (1) and (2), and a superspace for this can be defined. The geometry of the heterotic sigma models which realise this algebra is a generalisation of Kähler geometry with torsion, or a further generalisation involving real structures squaring to $+1$.

A superspace formulation of the supersymmetric non-linear sigma models with untwisted or twisted (p,q) supersymmetry was given in Section 3 in a formalism in which (1,1) supersymmetry is manifest. For such sigma models, more general isometries of the form (32) can be considered, where the vectors ξ_a are Killing vectors which are holomorphic with respect to each complex structure, or 'holomorphic' in a generalised sense with respect to each real structure. The gauging of such isometries can be obtained from a straightforward extension of the results of Refs. [30,31,35].

The results concerning the amount and type of supersymmetry that can be realised can be summarised in terms of the holonomy group of the connection with torsion. The various possibilities, which depend on the signature of the target space, are listed in Table 1.

For example, in the case of target spaces of Kleinian signature (d,d) with a single real structure, the holonomy group is contained in $GL(d,\mathbb{R})$ and the model has twisted (2,1) supersymmetry. The geometry generalises that of the usual (2,1) sigma model: in particular, the metric and torsion potential are given by (70), (71) where k and $\tilde{k}$ are *independent* real forms. This model can be formulated in superspace as shown in Section 6. Sigma models with untwisted or twisted $N=2$ supersymmetry can also be formulated in superspace, and this leads to new pseudo-Kähler (without torsion) and twisted pseudo-Kähler (with torsion) sigma models whose geometry is determined by a scalar potential analogous to the twisted Kähler potential of Ref. [24]. If the torsion vanishes, then the twisted (2,1) supersymmetric model reduces to the pseudo-Kähler model. These real models are listed in Table 2.

It is remarkable how much of the geometry based on a complex structure J carries over to the case of a real structure S. Instead of using complex numbers, it is useful to

Table 1
The relation of right-handed supersymmetry to the holonomy of the connection with torsion $\Gamma^{(+)}$. We give the type of target space geometry for the case in which the torsion vanishes

Target signature	Holonomy of $\Gamma^{(+)}$	Geometry when torsion-free	Supersymmetry
(d_1,d_2)	$O(d_1,d_2)$	no restriction	(1,1)
$(2n_1,2n_2)$	$U(n_1,n_2)$	Kähler	(2,1)
$(4m_1,4m_2)$	$USp(2m_1,2m_2)$	hyper-Kähler	(4,1)
$(2n,2n)$	$GL(n,\mathbb{R})$	pseudo-Kähler	twisted (2,1)
$(4m,4m)$	$Sp(2m,\mathbb{R})$	pseudo-hyper-Kähler	twisted (4,1)

Table 2
Geometry, superfields and supersymmetry of some sigma models with real target spaces

Target geometry	Superfields	Supersymmetry
real with torsion	$U_{(2,1)},\tilde{V}_{(2,1)}$	twisted (2,1)
pseudo-Kähler	$U_{(2,2)},\tilde{U}_{(2,2)}$	twisted (2,2)
twisted-pseudo-Kähler	$U_{(2,2)},\tilde{V}_{(2,2)}$	twisted (2,2)

introduce *double numbers* in this context [28]. These are based on a real unit e which satisfies

$$e^2 = +1 \tag{122}$$

instead of the usual imaginary unit i satisfying $i^2 = -1$. It is useful to define a real conjugation taking $e \to -e$, so that $(x+ey)^* = x - ey$ for real numbers x, y.

For example, consider the formulation of the twisted (2,1) sigma model of Section 6 using double numbers. The real structure S^i_j takes the form

$$S^i_j = e\begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^{\tilde\alpha}_{\tilde\beta} \end{pmatrix} \tag{123}$$

in an adapted coordinate system. The fundamental two-form is then

$$S = \frac{1}{2} S_{ij}\, d\phi^i \wedge d\phi^j = -e g_{\alpha\tilde\beta}\, du^\alpha \wedge dv^{\tilde\beta}. \tag{124}$$

If $H \neq 0$, the torsion is given in terms of the fundamental two form by

$$H = e(\partial_u - \partial_v) S. \tag{125}$$

The closure of H then implies

$$e\partial_u \partial_v S = 0 \tag{126}$$

and the geometry is given, in a suitable gauge, by Eqs. (70)–(83). The metric and torsion are preserved by the gauge transformations

$$\delta k_\alpha = e\partial_\alpha \chi + \theta_\alpha, \qquad \delta \tilde{k}_{\tilde\alpha} = -e\partial_{\tilde\alpha}\chi + \tilde\theta_{\tilde\alpha}, \tag{127}$$

where $\partial_{\tilde\beta}\theta_\alpha = \partial_\beta \tilde\theta_{\tilde\alpha} = 0$. The superspace action is

$$S = -e\int d^2\sigma\, d\theta^+ d\tilde\theta^+ d\theta^- \left(k_\alpha D_- U^\alpha - \tilde{k}_{\tilde\alpha} D_- \tilde{V}^{\tilde\alpha} \right), \tag{128}$$

where the superfields U^α, $\tilde{V}^{\tilde\alpha}$ are chiral with respect to the superderivatives

$$D_+ = \frac{\partial}{\partial\theta^+} + e\tilde\theta^+ \frac{\partial}{\partial\sigma^+}, \qquad \tilde{D}_+ = \frac{\partial}{\partial\tilde\theta^+} + e\theta^+ \frac{\partial}{\partial\sigma^+}. \tag{129}$$

If the twisted (2,1) superspace action (128) is required to be real self-conjugate with respect to the conjugation $e \to e^* = -e$, i. e. if

$$S = S^*, \tag{130}$$

then we find that the potentials k and $\tilde{k}$ are real conjugates,

$$\tilde{k} = k^* . \tag{131}$$

This is the generalisation to the double numbers of the reality condition

$$S = S^* \tag{132}$$

on the action (48), which implies that $k = (\bar{k})^*$; in turn, this implies hermiticity of the metric and antihermiticity of the torsion potential given in (42). For the general models we have discussed, the condition (132) does not hold, the potentials k and $\tilde{k}$ are independent and the action is not real self-conjugate.

Setting $e = 1$, the formulations of previous sections are recovered, but introducing e is a useful book-keeping device. In particular, it leads to the introduction of the real conjugation operation, and makes the structure similar to that of the complex case.

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