Asymptotic Zero Energy States for SU(N $\geq$ 3)

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Abstract

Some ideas are presented concerning the question which of the harmonic wavefunctions constructed in [hep-th/9909191] may be annihilated by all supercharges.

In an attempt to extend our knowledge about the asymptotic form of zero-energy wave functions of SU(N) invariant supersymmetric matrix models beyond $N = 2$, it was recently shown [1], for $N = 3$, how to construct Weyl $\otimes$ Spin(d) invariant asymptotic states out of the Cartan-subalgebra degrees of freedom. To find out which of these harmonic wavefunctions is annihilated by the asymptotic supercharges [2]

$$Q_\beta = -i\gamma^t_{\beta\alpha}\nabla_{tk}\Theta_{\alpha k} \tag{1}$$

$$\begin{aligned} t &= 1, \ldots, d = (2), 3, 5 \text{ or } 9 \\ \alpha, \beta &= 1, \ldots, s_d = 2, 4, 8 \text{ resp. } 16 \\ k &= 1, 2 \end{aligned}$$

is non-trivial. The "guess" presented in this note will hopefully be a first step[1]. Should the answer really be that for arbitrary N, already for this "free" problem, exactly one supersymmetric state exists for $d = 9$ (and none for $d = 2, 3, 5$), this should obviously have a "simple" (deep?) mathematical explanation, of more general relevance.

Each harmonic state, constructed in [1], has the form

$$\Psi = \sum_{l,S,R,m} r^{-2l-2(d-1)} \bar{\Psi}^{S\times R}_{lm}(\mathbf{x}_1, \mathbf{x}_2)|S \times R; m\rangle \tag{2}$$

with l=0,1,…,$\bar{\Psi}^{S\times R}_{lm}$ a harmonic polynomial of degree l (in the variables $\mathbf{x}_1, \mathbf{x}_2$; $r := \sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2}$) transforming under the Weyl-group (the permutation group S_3) and Spin(d) according to the irreducible representation $S(= 1, \in \text{ or } \rho)$, cf. [1],

[1] A detailed calculation of the matrix element (13) is under investigation by J. Plefka.

resp. $R, m = 1, 2, \ldots \dim(S \times R)$, and $|S \times R; m\rangle$ is the corresponding basis-vector in a $S \times R$ representation present in the fermionic Fock-space $\mathcal{H}(d) = \mathcal{H}_{2^{\frac{1}{2}s_D}} \times \mathcal{H}_{2^{\frac{1}{2}s_D}}$ (cf. [1]). As pointed out by M. Bordemann, one way to construct a state that will be annihilated by all the supercharges, is to let

$$\Psi = \left(\prod_{\beta=1}^{s_d} Q_\beta \right) \Phi \tag{3}$$

with Φ being any of the harmonic states (2). The crucial question is: for which Φ will (3) be non-zero? Let $d = 9$ now ($s_d = 16$). The guess which I would like to discuss here, is that

$$\Phi = r^{-16} |1\rangle \tag{4}$$

will do, where $|1\rangle$ is the unique Weyl $\times$ Spin(9) invariant state in $\mathcal{H} = \mathcal{H}_{256} \otimes \mathcal{H}_{256}$. Why (4)? One simple reason(ing) is the following: As each Q_β, acting on the product of a harmonic, homogenous polynomial and a negative power of r will *increase* the degree of the polynomial, (4) is the only harmonic state which certainly (a priori!) can *not* be the image of Q_β acting on some harmonic χ. Actually, if we could show that all harmonic Φ's not containing a contribution from $l = 0$ are of the form

$$\Phi = \sum_\rho Q_\beta \Phi_\beta \tag{5}$$

with Φ_β harmonic, (4) would necessarily be the only chance left, as (3) is clearly identically zero, if Φ is of the form (5).

In any case, consider now

$$\Psi := \epsilon_{\beta_1 \cdots \beta_{16}} Q_{\beta_1} \cdot Q_{\beta_2} \cdots Q_{\beta_{16}} \frac{1}{r^{16}} |1\rangle \tag{6}$$

Is it zero? First of all, one needs to know more explicitly, what the state $|1\rangle \in \mathcal{H}$ is.

As $\mathcal{H}_{256}$ contains only 3 irreducible Spin(9) representations,

$$\mathcal{H}_{256} = 44 \oplus 84 \oplus 128$$

$\mathcal{H}$ contains only 3 Spin(9) singlets, namely

$$\begin{aligned} |1\rangle_{44} &:= \sum_{s,t} |st\rangle |st\rangle' \\ |1\rangle_{84} &:= \sum_{s,t,u} |stu\rangle |stu\rangle' \\ |1\rangle_{128} &:= \sum_{t,\alpha} |t\alpha\rangle |t\alpha\rangle' \ . \end{aligned} \tag{7}$$

For notational simplicity, the fermions $\Theta_{\alpha k=2}$ are sometimes denoted by Θ'_α, and $|st\rangle = |ts\rangle$ ($\sum_s |ss\rangle = 0$), $|stu\rangle$ (totally antisymmetric in s, t, u) and $|t\alpha\rangle$(with

$\gamma^t_{\beta\alpha}|t\alpha\rangle = 0$) stands for the basis-elements of the 44,84, resp. 128-dimensional representation.

Defining fermionic creation operators

$$\lambda_{\alpha k} := \frac{1}{\sqrt{2}}(\Theta_{\alpha k} + i\Theta_{\alpha+8,k})_{\alpha=1,\dots,8} \tag{8}$$

together with the representation

$$\gamma^9 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^8 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & i\Gamma^j \\ -i\Gamma^j & 0 \end{pmatrix},$$

$$(i\Gamma^j)_{k8} := \delta_{jk}, \quad (i\Gamma^j)_{kl} := -c_{jkl},$$

and totally antisymmetric 'octonionic structure constants' $c_{jkl} = +1$ for $(ijk) =$ 123, 147, 165, 246, 257, 354, 367, the 3 states in (7) may also be explicitely given as concrete polynomials in the creation operators $\lambda_{\alpha k}$. E.g., with

$$b_j := \frac{i}{4}\lambda_\alpha \Gamma^i_{\alpha\beta}\lambda_\beta, \quad c_j := \frac{i}{4}\lambda'_\alpha \Gamma^i_{\alpha\beta}\lambda'_\beta \tag{9}$$

one finds

$$|1\rangle_{44} = \Big((\mathbf{b}\cdot\mathbf{c})^2 - \frac{1}{9}\mathbf{b}^2\mathbf{c}^2 - \frac{2}{9}\mathbf{b}\cdot\mathbf{c}(b^2+c^2) + \frac{2}{63}(b^4+c^4)\Big)|0\rangle \tag{10}$$

while the states $|st\rangle$ are explicitly given as follows ($|8\rangle := \lambda_1 \cdots \lambda_8|0\rangle$)

$$\begin{aligned}
|i \neq j\rangle &= b_i b_j|0\rangle \\
|jj\rangle &= (b_j^2 - \frac{1}{9}\mathbf{b}^2)|0\rangle \\
|j9\rangle &= -\frac{i}{2}(b_j + \frac{2}{9}b_j\mathbf{b}^2)|0\rangle \\
|j8\rangle &= \frac{1}{2}(b_j - \frac{2}{9}b_j\mathbf{b}^2)|0\rangle \\
|89\rangle &= -\frac{i}{2}(|0\rangle - |8\rangle) && (11) \\
|88\rangle &= -\frac{1}{2}(-|0\rangle + \frac{2}{9}\mathbf{b}^2|0\rangle - |8\rangle) && (12) \\
|99\rangle &= -\frac{1}{2}(|0\rangle + \frac{2}{9}\mathbf{b}^2|0\rangle + |8\rangle)
\end{aligned}$$

In any case, as one of the Weyl-transformations changes λ'_α to -λ'_α (while leaving λ_α invariant), $|1\rangle_{128}$ can not be contained in the Weyl-invariant state $|1\rangle$, which therefore must be a linear combination of $|1\rangle_{44}$ and $|1\rangle_{84}$

Projecting (6) onto this linear combination will give some Weyl × Spin(9) invariant differential operator of degree 16 (with constant coefficients), acting on r^{-16}. While R. Suter and I checked, by using quite different methods, that a priori only 2 such independent operators, not containing the full Laplace-operator (which annihilates r^{-16}!) exist, one needs to know

$$\langle 1|\Theta_{\alpha_1 k_1}\Theta_{\alpha_2 k_2}\cdots\Theta_{\alpha_{16} k_{16}}|1\rangle \tag{13}$$

resp. the contraction with $\epsilon_{\beta_1 \cdots \beta_{16}} \gamma^{t_1}_{\beta_1 \alpha_1} \cdots \gamma^{t_{16}}_{\beta_{16} \alpha_{16}}$ (times $\nabla_{t_1 k_1} \cdots \nabla_{t_{16} k_{16}} r^{-16}$).

Should the result turn out to be non-zero, (6) will, by construction, be a non-trivial supersymmetric wave function. For general $N > 2$ the corresponding asymptotic fall off would be $r^{-((N-1)d+14)}$.

A simpler way to describe the fermionic part of the wavefunction is to define fermionic creation operators

$$\Lambda_\alpha = \frac{1}{\sqrt{2}}(\theta_{\alpha 1} + i\theta_{\alpha 2}), \quad \alpha = 1, \ldots, 16,$$

and to observe that

$$\gamma^{uv}_{\alpha_1 \alpha_2} \gamma^{vp}_{\alpha_3 \alpha_4} \gamma^{pq}_{\alpha_5 \alpha_6} \gamma^{qu}_{\alpha_7 \alpha_8} \Lambda_{\alpha_1} \Lambda_{\alpha_2} \cdots \Lambda_{\alpha_8} |0\rangle$$

is Spin(9) $\times$ Weyl invariant.

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References

[1] M. Bordemann, J. Hoppe, R. Suter; hep-th/9909191

[2] V.Kac, A. Smilga; hep-th/9908096.